Spatial representation and the teaching of geometry†

Abstract

After a few comments on the teaching of geometry in today's schools, we will outline some of the research results that have guided us in the development of a teaching sequence which is currently at the experimental stage. This sequence of activities, intended for ninth-grade students (14-15 years old), focuses on generating two- and three-dimensional figures, and on exploring their projective properties. The sequence will be demonstrated using symmetrical three-dimensional figures. Initial results have shown that learning activities which include the creative manipulation of three-dimensional objects foster the development of perception and representation abilities in the weakest math students, and that the students who are the weakest on the geometric spatial representation test, make the most progress in mathematics.

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Représentation de l'espace et enseignement de la géométrie†

Résumé

Après quelques remarques sur l'enseignement actuel de la géométrie dans les écoles, nous exposons quelques résultats d'une recherche qui nous ont guidé dans l'élaboration d'une séquence didactique, actuellement à l'étape de l'expérimentation. Cette séquence d'activités, dont nous présentons un exemple portant sur les volumes symétriques, à l'endroit d'élèves de secondaire III (14-15 ans), priorise la génération de formes et de figures, dans l'exploitation de leurs propriétés projectives. Les premiers résultats font voir, entre autres, que les activités d'apprentissage incluant une manipulation créatrice d'objets géométriques tridimensionnels favorisent le développement des habiletés perceptive et représentative des élèves les plus faibles en mathématiques et, réciproquement, que c'est les élèves les plus faibles au test portant sur la représentation spatiale géométrique qui progressent le plus en mathématiques.
Comments on the teaching of geometry

The États généraux de l'enseignement des mathématiques au Québec, a task force on the teaching of mathematics in Québec, adopted a resolution requesting immediate revision of the geometry components of compulsory math courses at the secondary level ([19], p.147-148). But what type of geometry is involved?

At one time, students developed logic thinking primarily through the study of plane geometry, which was followed later by the study of Aristotelian syllogism. Geometry has gradually given way to algebra, in which more mechanical, algorithmic modes of reasoning prevail. In Québec, logical reasoning is currently covered at the CEGEP level (17-19 years of age) particularly in philosophy courses.² Can nothing be done earlier? Yet are geometry courses a panacea for the development of logical reasoning in students? (For example, consider the following problem: “True or false: If a parallelogram has two symmetry axes, it must be a rectangle.”)

Though today's students know many facts about geometry, they do not seem to know how those facts interrelate, nor are they aware of how they are structured and organized.

Research on learning has demonstrated that visual aids (graphics) can be useful to understand mathematical concepts — and that is not limited to geometry [4]. But students must be able to understand the graphic representations we use, and we must be able to teach specific techniques for developing their spatial-representation abilities. What better opportunity to do so than in a mathematics course — the only compulsory subject which formally deals with spatial concepts?

Furthermore, students who have not covered projective geometry or spatial representation, have a great deal of difficulty applying their limited math knowledge to situations in which they have to draw upon their visual/spatial abilities — e.g., in drafting courses at the CEGEP level or architecture courses at university, or when dealing with spatial-representation problems in everyday situations (which are much more common than problems requiring number abilities), such as wide-range travel in a city, mid-range movement through a parking lot or tennis court, or simply when examining projective diagrams in textbooks (with projections that are hopefully correct).

Students are generally presented with a few ready-made solids (whether polyhedra or other), such as a cube, a parallelepiped prism, or a regular-based pyramid. This, however,
Perception and representation of a structural geometric space

Spatial abilities may be the underlying key to mathematical abilities. Research carried out by Bishop [3] led to the hypothesis that mathematical concepts could perhaps be developed not just through the teaching of mathematics, but also through the development of certain basic abilities, including spatial abilities. That hypothesis has not, to the best of our knowledge, been confirmed; however, it could have interesting implications for mathematics learning.

Our research over the last few years may provide a partial answer to Bishop’s questions concerning spatial abilities. A study carried out among several hundred subjects of various ages has enabled us to identify certain factors whose significance lies in the foundations of our research method. We developed a new instrument of measure, which has been validated and is geometrically complete, and is designed to assess geometric development in spatial representation [15], [16], [18]. The instrument of measure consists of 40 items, each of which contains two-dimensional geometric representations of three-dimensional objects—either graphs, or central, affine or orthogonal projections. These items are in keeping with the tradition of psychometric spatial-representation tests using paper and pencil. They allow for an assessment of the subjects’ performance in various operations that correspond to spatial abilities and various geometric modes.

Our assessment model, which is of the developmental type, was developed in large part on an empirical basis. It very specifically considers spatial abilities in the context of an essentially geometric (as opposed to a physical, social, or other) space. The matrix of the development of geometric

La perception et la représentation d’un espace géométrique structural

Il est possible que l’habileté spatiale soit l’habileté-clé sous-jacente aux habiletés mathématiques. Le travail de synthèse effectué par Bishop [3] l’a amené à intuitionner que les concepts mathématiques pourraient être développés, non seulement par l’enseignement des mathématiques, mais aussi en développant certaines habiletés fondamentales, dont les habiletés spatiales. À notre connaissance, cette hypothèse n’a pas encore été confirmée, bien qu’elle pourrait entraîner des conséquences intéressantes pour l’apprentissage des mathématiques.

Le travail que nous avons effectué ces dernières années peut nous permettre de répondre partiellement au questionnement de Bishop au niveau des habiletés spatiales. Afin d’évaluer le développement géométrique de la représentation spatiale, nous avons élaboré un nouvel instrument de mesure basé sur différents modes géométriques et nous l’avons validé auprès de plusieurs centaines de sujets d’âge varié [15], [16], [18]. Cet instrument comporte 4D items, tous constitués de représentations géométriques bidimensionnelles d’objets tridimensionnels, soit des graphes, soit des projections centrales, affines ou orthogonales. Ce type d’items s’inscrit dans la tradition des tests psychométriques de représentation spatiale de type papier-crayon et permet l’évaluation de la performance des sujets portant sur différentes opérations correspondant à des habiletés spatiales et divers modes géométriques.

Élaboré en bonne partie sur des bases empiriques, notre modèle, de type développemental, tient compte de façon
Figure 1
Matrix of the development of geometric spatial representation.
Matrice du développement de la représentation spatiale géométrique.

1 The topological mode corresponds primarily to the study of the properties of adjacency and connectedness of spatial structures, which are maintained following one or several continuous transformations, such as extending, contracting, folding, and torsion. The projective mode primarily refers to the study of the properties of incidence of straight lines and planes, which are maintained following a central projection. The affine mode corresponds primarily to the study of the properties of parallelism and convexity, which are maintained following a parallel projection. The metric mode corresponds primarily to the study of the properties of distance and angulation.

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Spatial representation (Figure 1) takes into account the primary elements of our model. The geometric modes are dealt with according to an axiomatic hierarchy. In addition, we have identified five intellectual operations: classifying and structuring, which are related to the analytical spatial ability known as "form recognition" (spatial relationships); determining and generating, which are related to an operational spatial ability, "spatial visualization" (form transformation). The last intellectual operation, transposing, seems to be linked to both the analytical and operational factors; it could therefore have an important role as a bridge between the two types of abilities with respect to the development of geometric spatial-representation abilities.

An analysis of test results has enabled us to identify a number of characteristics of the development of spatial abilities according to a typology of geometric modes (see Figure 1). Significant improvements in test results become evident in subjects between approximately 10 and 15 years of age. A marked improvement in performance occurs between childhood and adolescence, after which performance stabilizes.

In a slightly different context, Mariotti's findings are similar. We advance the hypothesis that the age span from 10 to 15 is a favourable time for activities designed to adequately structure students' capacity for spatial representation, as spatial-representation abilities develop naturally and significantly (albeit in a limited way) during that period. We maintain, as do other authors (Baracs [2], Bishop [3], Lean et al. [9], Lunkenbein [11], Mitchelmore [14], Pellegrino et al. [20], Van Hiele [23]), that if no efforts are made to impart certain concepts and properties of geometry, students' men-

très spécifique des habiletés spatiales dans le contexte d'un espace essentiellement géométrique (plutôt que physique, social ou autre). La matrice du développement de la représentation spatiale (voir la figure 1) rend compte des principaux éléments de notre modèle. Les modes géométriques sont traités selon leur hiérarchie axiomatic [22], alors que nous avons réussi à circonscrire cinq opérations intellectuelles [1], [17], [15]. Deux sont reliées à une habileté spatiale de type analytique, connue dans la littérature sous le vocable "relation spatiale" (ou reconnaissance des formes), à savoir la classification et la structuration. Deux autres sont reliées à une habileté spatiale de type opératoire, la «visualisation spatiale» (ou transformation des formes); la détermination et la génération. Une dernière opération intellectuelle, la transposition, semble être reliée aux deux facteurs analytique et opératoire, ce qui lui confère un rôle potentiel précieux pour favoriser le passage d'une habileté vers l'autre, eu égard au développement de la représentation spatiale géométrique chez l'individu.

L'analyse des performances des sujets au test spatial permet d'identifier plusieurs caractéristiques du développement des habiletés spatiales, eu égard à une typologie selon les modes géométriques (voir la figure 1). Les majorations significatives dans les performances apparaissent entre 10 et 15 ans environ. En effet, la performance des individus s'accroît brusquement entre l'enfance et l'adolescence, puis se stabilise ensuite. Mariotti [12], dans un contexte un peu différent, avait trouvé des résultats semblables. Nous faisons l'hypothèse que cet intervalle d'âge est tout indiqué pour des interventions visant à structurer adéquatement la capacité de représentation spatiale des individus, puisque celle-ci s'y développe alors de façon naturelle et significative, bien que de façon encore modeste. En effet, en accord avec d'autres auteurs (Baracs [2], Bishop [3], Lean et al. [9], Lunkenbein [11], Mitchelmore [14], Pellegrino et al. [20], Van Hiele [23]), nous estimons que sans activités favorisant l'acquisition de certaines notions et propriétés géométriques, les images internes élaborées par le sujet peuvent très bien demeurer la vie durant au niveau de ce que Piaget appelle les groupements infralinguiques [21].
Further studies point to certain optimal pathways in the development of spatial-representation abilities. The progression seems to go from the analytical to the operational in terms of psychometric factors—in other words, from form recognition to form transformation. Spatial ability development seems to begin with the topological mode and progress towards the metric mode (with a degree of overlapping), according to the model based on the axiomatic hierarchy of geometric properties; however, in an environment which is virtually devoid of representations other than affine and metric, a hiatus may occur at the projective level. Activities that involve crafts or mechanics (low-level activities) seem to be useful for individuals who have inherent difficulties in visualizing three-dimensional forms, whereas activities that require the progressive use of deductive reasoning (high-level activities), such as those involved in using geometric parameters to determine the specific elements of a spatial structure, are more pertinent to those who have natural facility, previous experience, or particular interest. This partially answers the question raised by Bishop in 1980 ([3], p.266) concerning the different teaching strategies that can be used with different types of clientele. The development of spatial-representation abilities implies a gradual shift from activities that involve topological and projective figures to creating and working with spatial representations through various means (physical, graphic, algebraic and linguistic).

When restricted to a micro-space—i.e., an environment suited to handling small objects, which can be represented on paper or on a computerized substratum—the concepts that we associate with geometric spatial abilities encompass the following elements: graphs (isomorphic, contiguous, homomorph, isotopic, heterotopic, planar, non-planar); intersecting lines; areas; surfaces (right, left); projections (central, parallel); polyhedra (convex, concave, enantioromorphous, truncated, regular, semi-regular, homogeneous, dual); parallelism, symmetry (reflection, rotation, rotary reflection); etc.

As for intellectual mathematical operations, we refer to the main activities of mathematician-geometers with respect to graphs, transformational geometry, and the study of conic sections...—e.g., comparing, discriminating, classifying, constructing, structuring, perceiving invariants, analyzing, synthesizing, representing, and creating models. In doing so, students will likely never develop beyond the stage that Piaget refers to as inferalogical groupings [21].

Furthermore, statistics point to certain optimal pathways in the development of spatial-representation abilities. The progression seems to go from the analytical to the operational in terms of psychometric factors—in other words, from form recognition to form transformation. Spatial ability development seems to begin with the topological mode and progress towards the metric mode (with a degree of overlapping), according to the model based on the axiomatic hierarchy of geometric properties; however, in an environment which is virtually devoid of representations other than affine and metric, a hiatus may occur at the projective level. Activities that involve crafts or mechanics (low-level activities) seem to be useful for individuals who have inherent difficulties in visualizing three-dimensional forms, whereas activities that require the progressive use of deductive reasoning (high-level activities), such as those involved in using geometric parameters to determine the specific elements of a spatial structure, are more pertinent to those who have natural facility, previous experience, or particular interest. This partially answers the question raised by Bishop in 1980 ([3], p.266) concerning the different teaching strategies that can be used with different types of clientele. The development of spatial-representation abilities implies a gradual shift from activities that involve topological and projective figures to creating and working with spatial representations through various means (physical, graphic, algebraic and linguistic).

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however, we limit ourselves to the concepts that pertain to teaching and learning geometry at the secondary level: measurement, angles, distance, area, volume, polygons, lines (bisecting, parallel, perpendicular, and diagonal lines; secants; axes of symmetry), similarity, congruence, reflection, rotation, translation, scaling, isometry, coordinates, planes and polyhedra (prisms, pyramids) (Ministère de l’Éducation du Québec, [13]).

Our goal is to refine the theory while perfecting our pedagogical tools through the development of better tests and more effective teaching materials in order to achieve greater student success in mathematics. “... In mathematics education much more emphasis should be put on various types of plane representations of three-dimensional shapes and relations...” [5].

“Generative” geometry

In the spring of 1991, we developed and pre-tested a “teaching sequence” that includes a conceptual outline (Figure 1) of the contents, activity situations, learning activities, application methods, teacher guidelines, and of the results expected from the students in terms of performance and evaluation. The sequence consists of eight sessions of approximately one hour. These sessions are based on various concepts pertaining to the perception of a structural or operative geometric space: intersection and space figures, isometric space figures, projections of space figures, and measurement of space figures.

The pedagogical approach underlying the teaching sequence includes:

- the manipulation and transformation of three-dimensional figures (made materials such as polystyrene);
- reinforced by representations in different media or different types of projections;
- small-group work involving three-dimensional objects;
- the use of figurative and non-figurative stimuli [8], [9];
- problem solving requiring effective perception and representation of geometric space;
- the verbalization and communication of spatial ideas among peers and between student and teacher; including the voluntary refutation of those ideas, which necessarily involves forming more highly developed concepts of spatial representation.

This type of relatively systematic approach, involving spatial representation and the teaching of geometry Dominique Dion, agent de recherche.

4 The learning activities were designed in part by research assistant Dominique Dion.

4 A participé à la conception de ces activités, Dominique Dion, agent de recherche.
teacher participation, allows for the results of the teaching sequence to be measured in terms of the transfer of a general spatial abilities to more geometric abilities, in terms of prerequisite abilities, and in terms of a more satisfactory means of integrating geometry into the school environment. The actual testing was carried out in 1991.

Activities involving perception and representation abilities were centred on the students’ ability to generate two- and three-dimensional shapes using hands-on learning materials. The other operations in our conceptual outline—determining, transposing, structuring and classifying—are presented as secondary to generating. Although seemingly contradictory, as generation is considered a high-level activity, we worked on the premise that students about 15 years of age have acquired intuitive perception through experience, even though they may never have been required to develop their spatial abilities as such. When we administered the perception and spatial-representation tests, the analytical or relational tasks (57.6%) were systematically completed more successfully than the operational or visual tasks (39.4%), which coincides with the operational inadequacies involved in academic programs. Furthermore, in view of the dynamism involved in generating two- and three-dimensional shapes in semi-concrete problem solving (three-dimensional objects and diagrams), it seemed appropriate to use that as a springboard for our other activities (see Table 1).

### Table 1 / Tableau 1

<table>
<thead>
<tr>
<th>Activities / Activités</th>
<th>Operation / Opérations</th>
<th>Geometric Mode / Géométries</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Intersection and space figure / Intersections et volumes</td>
<td>G D T S</td>
<td>T P</td>
</tr>
<tr>
<td>2. Isometric space figures / Volumes isométriques</td>
<td>G D T S C</td>
<td>P M</td>
</tr>
<tr>
<td>a) rotation / par rotation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) reflection / par réflexion</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c) rotary reflection / par composée</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Projections of space figures / Projections de volumes</td>
<td>G D T S C</td>
<td>P</td>
</tr>
<tr>
<td>a) central / centrales</td>
<td></td>
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<tr>
<td>b) parallel / parallèles</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Measuring space figures / Mesures de volumes</td>
<td>G D T S C</td>
<td>T A</td>
</tr>
</tbody>
</table>
An example of an activity involving isometric space figures

The objective of this activity is to explore the generation of space figures by means of isometric transformations: rotation, reflection and rotary reflection. At this stage, the teacher's presentation would include the following: an explanation of the distinction between a pair of congruent space figures and a space figure and its reflection image; a description of isometric transformations in space carried out on a rotation axis and on a plane of reflection symmetry; a demonstration of the relationship between the positions of the rotation axis and the plane of reflection symmetry and various other planes (perpendicular, parallel or oblique); and an overview of the possible combinations of various types of symmetry.

The teacher explains to the students how to progress from working with planes that are considered unlimited, as in the first activity (Intersection and space figures), to working with the edges and vertices of space figures in the second activity. The intersecting planes (lines and points) become the edges and vertices of space figures; the planes are then called faces. Thus, the construction sticks and joints used in the activity are the edges and vertices of the figure, and the intersections of infinite planes.

a) Reflection

In this activity, an imaginary plane of reflection, represented by a circular plate of transparent plastic (see Figure 2), is used to construct a triangular pyramid characterized by the isometry of a plane reflection.

In theory, the pyramid is constructed as follows. The first plane ($\triangle ABE$) intersects the plane of reflection obliquely; the plane of reflection reproduces plane $ABE$ on the other side ($\triangle ABD$). The intersection of the resulting dihedron (where two planes meet) is in the plane of reflection ($\triangle ABC$). Two other planes ($\triangle ADE$ and $\triangle BDE$), which are perpendicular to the plane of reflection but cannot be reproduced by that plane, complete pyramid $ABDE$ with intersections that are common to the four planes. In this isometric transformation, all the points of the space figure are reproduced at an equal distance from plane of reflection and are perpendicular to that plane.

Note: $\triangle ABC$ is in the plane of reflection; $\triangle ADE$ and $\triangle BDE$ are isosceles triangles, but unequal ones (otherwise there would be a second plane of reflection intersecting $DE$ and the centre of $AB$); $\triangle ADB$ and $\triangle BAE$ are scalene triangles (for

Un exemple d’activité sur les volumes isométriques

L’objectif de cette activité est d’explorer la génération de volumes dans l’espace à partir des transformations isométriques : la rotation, la réflexion et la rotation-réflexion. La présentation de l’enseignant ou de l’enseignante inclut ici des explications sur la distinction entre les formes égales congruentes et énantiomorphes, une description des transformations isométriques dans l’espace, opérées à partir d’un axe de rotation et d’un plan de réflexion, une démonstration des rapports de positions entre ces éléments et un plan quel-conque (perpendiculaire, parallèle et oblique), et se termine par un aperçu des combinaisons possibles entre les éléments de symétrie.

On explique aux élèves le passage de la construction avec des plans considérés comme illimites exécutée au cours de l'activité précédente (Intersections et volumes), à la construction à partir des arêtes et des sommets du volume dans la présente activité. En fait, les intersections aux croisements des plans (droites et points) deviennent alors les arêtes et les sommets du volume et les plans prennent le nom de faces. Ainsi les tiges et les noeuds d’assemblage sont à la fois les arêtes et les sommets du volume, à la fois les intersections entre des plans infinis.

a) La réflexion

En utilisant un plan imaginaire, représenté dans cette activité par une plaque circulaire en plastique transparent (voir la figure 2), on veut construire une pyramide triangulaire qui possède une isométrie de réflexion.

Théoriquement, un premier plan ($\triangle ABE$) croise dans une position oblique le plan de réflexion qui le reproduit une fois de l’autre côté ($\triangle ABD$). Le dièdre (ou deux plans qui se rencontrent) ainsi formé a son intersection située dans le plan de réflexion ($\triangle ABC$). Deux autres plans ($\triangle ADE$ et $\triangle BDE$) placés perpendiculairement au plan de réflexion, qui ne peut les doubler dans cette position, complètent la pyramide $ABDE$ par des intersections communes entre les quatre plans. Dans cette transformation isométrique, tous les points de la forme sont reportés perpendiculairement et à égale distance du plan de réflexion.

Note: $\triangle ABC$ est dans le plan de symétrie, les $\triangle ADE$ et $\triangle BDE$ sont isocèles, mais inégaux (sans quoi il y aurait un second plan de symétrie passant par $DE$ et le milieu de $AB$), les $\triangle ADB$ et $\triangle BAE$ sont scalènes (même raison que ci-de-
the same reason that $\Delta ADE$ and $\Delta BDE$ are isosceles; $BCD$ and $BCE$ are right angles; and $DC = CE$.

In practice, triangular pyramids are constructed with sticks and joints (hinges) around the plastic plate, which represents the plane of reflection. The students start by constructing two congruent scalene triangles, with segment $AB$ as a base. Deductively, they determine the last edge of the tetrahedron that joins the two vertices of the two triangles, with $C$ as its median.

The final edge poses an interesting problem: Is the final edge determined uniquely by the beginning of the structure? The tetrahedron which is generated has three different types of triangular faces: two equal scalene triangles and two unequal isosceles triangles.

**b) Rotary-reflection**

This activity involves constructing a triangular pyramid by means of an isometry which combines a rotation axis with a plane of symmetry.

In theory, a single plane is reproduced three times as the result of reflections combined with successive $90^\circ$ rotations around the axis. In the diagram of Figure 3, axis of symmetry $A$ is perpendicular to plane of symmetry $\pi$. A first plane, which we shall call $FGH$, is oblique in relation to plane of symmetry $\pi$ and axis of rotation $A$; a rotation of $90^\circ$ around axis $A$ (counter-clockwise) and a reflection in plane $\pi$ will generate plane $FGI$. This second plane will, in turn, generate plane $FHI$ in the same way. This third plane will then generate plane $GHI$. The intersection of these four planes results in tetrahedron $FGHI$. Square $BCDE$ is the intersection of tetrahedron $FGHI$ and plane of symmetry $\pi$. The vertices of square $BCDE$ are the centres of edges $FG$, $GH$, $HI$ and $IF$ respectively. Axis of symmetry $A$ goes through the centre of edges $FH$ and $GI$ and is perpendicular to them.

In practice, the four vertices of the pyramid, $F$, $G$, $H$ and $I$, can be positioned in the previous diagram (Figure 3) in relation to the isometric transformations: $F$ and $H$ are on one side of the plane of symmetry, while $G$ and $I$ are on the other side. Sticks of equal length are used to represent congruent segments $FG$, $FI$, $HI$ and $HG$; the two last edges of the tetrahedron can then be identified: $GI$ and $FH$. The students are once again asked to draw the different faces of the tetrahedron that has been generated.

**Figure 3**

Tetrahedron with a combined symmetry of rotation-reflection.

Tétraèdre composé d'une symétrie composée rotation-réflexion.

**En pratique,** on construit la pyramide triangulaire avec des tiges et des noeuds, autour de la plaque transparente, qui représente le plan de réflexion. Les élèves commencent par construire deux triangles scalènes congrus ayant pour base le segment $AB$. Ils déduisent la dernière arête du tétraèdre qui joint les deux sommets de ces triangles, en ayant $C$ comme point milieu.

Un beau problème de détermination se pose concernant la dernière arête : est-elle déterminée de façon unique par le début de la construction ? Le tétraèdre généré possède cette fois-ci trois types différents de faces triangulaires, à savoir deux triangles scalènes égaux, et deux triangles isocèles inégaux.

**b) La rotation-réflexion**

A partir de l'élément de symétrie composé d'un axe de rotation combiné à un plan de réflexion, l'idée est toujours de construire une pyramide triangulaire qui possède une isométrie de rotation-réflexion.

**Théoriquement,** un seul plan est reproduit trois fois après des opérations de réflexion combinées à des rotations successives de $90^\circ$ autour de l'axe.

L'axe de symétrie $A$, situé dans le plan du dessin ci-dessous (voir la figure 3), est perpendiculaire au plan de symétrie $\pi$. Un premier plan, disons $FGH$, oblique par rapport au plan de symétrie $\pi$ et l'axe de rotation $A$, après une rotation de $90^\circ$ autour de l'axe $A$ (dans le sens trigonométrique ou anti-horaire) et une réflexion par rapport au plan $\pi$, génère le plan $FGI$. Ce second plan, génère à son tour, de la même manière, le plan $FHI$. Et ce troisième plan génère enfin le plan $GHI$. L'intersection de ces quatre plans forme le tétraèdre $FGHI$. Le carré $BCDE$ est l'intersection du tétraèdre $FGHI$ et du plan de symétrie $\pi$. Les sommets du carré $BCDE$ sont les milieux des arêtes $FG$, $GH$, $HI$ et $IF$ respectivement. L'axe de symétrie $A$ passe par les milieux des arêtes $FH$ et $GI$ et leur est perpendiculaire.

**En pratique,** on peut positionner à l'aide du croquis de la figure 3 les quatre sommets $F$, $G$, $H$ et $I$ de la pyramide autour des éléments de symétrie : $F$ et $H$ sont d'un côté du plan de réflexion, alors que $G$ et $I$ sont de l'autre côté. On utilise des tiges de longueur égale pour représenter les segments congrus $FG$, $FI$, $HI$ et $HG$, et on peut alors déduire les deux dernières arêtes du tétraèdre, à savoir $GI$ et $FH$. De nouveau les élèves sont invités à dessiner les différentes faces du tétraèdre généré.
Results

In terms of contents, we deliberately chose to concentrate on projective properties in geometry—an area in which our previous research had shown major shortcomings with respect to topological, affine and metric properties.

Though incomplete, the results obtained thus far have not shown significant changes in the subjects' perception and representation abilities. In standardized tests of perception and spatial abilities that were administered to a test group (30 subjects) and to a control group (25 subjects), the performance of both groups remained unchanged in the perception test and increased in the spatial abilities test. However, the mathematics scores of the students in the test group increased significantly ("t" paired = 2.85, p = 0.01), while the results of the control group remained unchanged.

These results, though not necessarily caused by our activities, are nevertheless in keeping with Bishop's hypothesis [3] that the development of basic abilities other than mathematical abilities (e.g., spatial abilities) can contribute significantly to the improvement of mathematics abilities. The teacher who led the sequence of activities in the pre-test stated the following in her report [6, p.19]: "I find that the students who participated in the activities have developed a number of cognitive abilities. When they encounter a problem, they think of solutions more quickly, and even see that there is more than one way of solving it. They demonstrate better analysis and synthesis abilities. We do not have to follow explanations through to the end—they are now able to finish problems on their own".

Finally, the analysis of the "half split group" enabled us to see that the learning activities that include creative manipulation of three-dimensional geometric objects foster the development of spatial perception and representation abilities in the weakest mathematics students, and that the students who are the weakest on the geometric spatial representation test, make the most progress in mathematics (Table 2).

Sub-tests from the General Aptitude Test Battery (GATB, form B - 1002 B), Institute of Psychological Research, Montréal.

Quelques résultats

Au niveau des contenus géométriques, nous avons choisi volontairement d'accentuer le travail sur les propriétés projectives, où nos précédentes investigations nous avaient indiqué de graves lacunes, par rapport aux propriétés topologiques, affines et métriques.

Les résultats partiels dont nous disposons ne nous permettent pas de mettre en évidence des changements significatifs au niveau des habiletés perceptives et représentatives des sujets. En effet, aux tests standardisés d'habiletés spatiales et de perception que nous avons fait passer au groupe expérimental (30 sujets) et à un groupe contrôle (25 sujets), les performances sont soit restées stationnaires dans les deux groupes (test de perception), soit ont augmentées dans les deux groupes (test d'habiletés spatiales). Par contre les résultats scolaires en mathématiques des élèves du groupe expérimental se sont améliorés de façon significative ("t" pairé = 2.85, p = 0.01), alors que ceux du groupe contrôle sont demeurés stationnaires.

Ces résultats, même si l'influence de nos interventions n'est pas démontrée, vont malgré tout dans le sens de l'hypothèse de Bishop [3], à l'effet que le développement d'habiletés fondamentales autres que mathématiques, comme les habiletés spatiales, peuvent contribuer significativement à l'amélioration des compétences mathématiques. L'enseignante qui animait la séquence d'activités pendant la pré-expérimentation mentionne d'ailleurs dans son rapport [6, p.19]: "Je trouve que les élèves ayant participé à l'activité ont développé plusieurs habiletés cognitives. Lorsqu'ils rencontrent un problème, ils pensent plus rapidement à des solutions et même, voient qu'il y a plus d'une façon de le solutionner. Ils sont maintenant en mesure de finir le reste du problème par eux-mêmes."

Enfin une analyse dichotomique ("half split group") nous fait voir que les activités d'apprentissage incluant une manipulation créatrice d'objets géométriques tridimensionnels favorisent le développement des habiletés perceptives et représentatives des élèves les plus faibles en mathématiques et, réciproquement, que ce sont les élèves les plus faibles au test portant sur la représentation spatiale géométrique qui progressent le plus en mathématiques (Tableau 2).
Table 2 / Tableau 2
Results of the paired (two headed) student "t" test, comparing the results at the test on geometric spatial representation (GSR) before and after the experimentation, for weak and strong students in mathematics, and comparing the results in mathematics before and after the experimentation, of the weak and strong students in GSR.

<table>
<thead>
<tr>
<th></th>
<th>t_{paired} / t_{paire}</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weak in math / Faibles en maths</td>
<td>2.52</td>
<td>0.03</td>
</tr>
<tr>
<td>Strong in math / Forts en maths</td>
<td>0.38</td>
<td>0.71</td>
</tr>
<tr>
<td>Weak in GSR / Faibles en RSG</td>
<td>2.09</td>
<td>0.06</td>
</tr>
<tr>
<td>Strong in GSR / Forts en RSG</td>
<td>1.26</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Conclusion
We have observed problems in current programs, practices that contradict to a large extent the problem-solving approach, and a failure to transfer mathematical abilities to geometry in situations where visual/spatial abilities are required. We do not have a cure-all formula, but it seems that transformational geometry can be redefined through a generative approach to plane and space figures, based on isometric transformations. Such an approach allows for the mastery of various types of projections, as well as spatial abilities and the corresponding mental operations.

“The most effective mental images are geometric images: the classic images of plane geometry—triangles and circles—with their unlimited wealth of properties; the basic images of space geometry—cubes and spheres; isometry—the displacement and transformation of geometric figures; and similarities that allow for changes in scale to be translated into a sort of ‘intellectual close-up.’” [7, p.33]
Bibliography / Bibliographie