Volume as an instrument in forming spatial concepts

Introduction
The ability to foresee the result of a rotation of a solid around an axis, recognize planes or parallel straight lines in two-dimensional representations of spatial configurations, and identify orthogonal directions in a diagram are all spatial skills that involve a grasp of spatial concepts. This paper explores the relationship between the development of the concept of volume in students and the development of spatial skills.

It is our premise that volume is basic to the development of spatial concepts. While some may tend to believe that concept of volume is grafted onto a previously structured notion of space and, at best, makes use of already developed skills, we are of the opinion that the specifics of the measurement process that pertain to exercises involving volume introduce new elements in one's structuring of space. An understanding of volume, along with all the underlying skills it implies, does not merely complement the usual organizational concepts, such as parallelism, ortho-
The video-tape is entitled 'Realisé par Daniel Campeau avec la participation de Chantal Forget' and presents highlights of the experiences of the participants in a course held in the fall semester of 1990.

This paper is based on an in-class study coordinated by the author. It describes an examination of the main stages in a series of lessons, focusing on how the participants in the course used objects represented in the plane, as well as in space. The ideas developed in this paper are illustrated in a video-tape intended for teachers that presents highlights of the experience.

In activity-oriented teaching, it is not unusual for a certain number of questions to arise pertaining to spatial skills, nor is it unusual in the pursuit of the two primary objectives of having students reason out formulas and taking students' major difficulties into consideration.

**Becoming familiar with space beforehand**

Volume is a characteristic that pertains to objects in space or even to portions of space. Although it seems essential to provide students with exercises that involve real objects, since problems that deal with the concept of volume generally use objects represented in the plane, it is also important that all the available means be used to enable students to work confidently with planar representations of space objects. This concern naturally led us to start students off with a number of exercises that are variations of exercises used in cognitive psychology to test a range of spatial skills. The first few lessons consisted of exercises that involved the construction of space objects on the basis of various drawings, and exercises that involved drawing various space objects. The lesson plans were designed to allow for an alternation between these two types of activities in order to ensure a frequent shift from working in the plane (through drawing) and in space (through constructing objects).

Upon examining such exercises, which require the student to relate the various elements involved in a given task (edges, faces, orthogonality, dihedral angles, etc.), we noted that certain exercises focus on area, while others are more concerned with volume. That may point to a criterion that could be used to establish categories of spatial skills.
Area vs. volume

An a priori analysis of the concept of volume carried out with a group of students enabled us to identify a number of major difficulties, several of which have already been topics of research [2], [3]. To start, let us consider a central obstacle: the conflict between area and volume, which has its counterpart in plane geometry.

It is widely known that senior elementary-school students easily see a close link between the length (perimeter) of a closed curve and its area, to the point that they confuse the two notions in certain problems. (Piaget and Inhelder [1] conducted a series of studies on this issue.) Once that difficulty is overcome, new obstacles arise when problems involve changes in perimeter or area. For example, students may conclude that a closed curve whose form has been transformed but whose perimeter has remained unchanged will nevertheless have the same area. Such problems have been dealt with in a number of studies from a Piagetian perspective.

Where volume is concerned, there is a similar problem: students naturally tend to think that solids that have an identical lateral area are necessarily identical in volume. The first introductory lesson to the concept of volume dealt with that point of confusion.

First, the teacher instructed the students to form two cylinders from two standard-sized sheets of paper by rolling one lengthwise and the other widthwise (see Figure 1). He prompted them to reflect on whether each cylinder could contain the same quantity of sand or sugar. An in-class survey led to surprising results: only a few of the students believed that the two cylinders were different in volume. The teacher then carried out an experiment in front of the class. Much to the surprise of most students, the two volumes differed! However, how can students be convinced by other means than hands-on experimentation?

We therefore wondered whether the skill involved in avoiding that trap corresponded to a particular spatial skill and, if so, whether it could be developed. Let us examine a common reasoning strategy: the use of extreme positions. If the volume remains the same, it should remain the same for any rectangle. First of all, a square, whether rolled lengthwise or widthwise, will obviously produce the same cylinder. However, could very long, thin rectangles not re-
sult in cylinders that are perceptibly different in volume? Is there means of carrying out a mental calculation to compare the two? If we take a pair of long, thin rectangles, one could be rolled to resemble a piece of spaghetti and the other a shallow cake (Figure 2). We must be able to imagine the cake becoming flatter and wider, like a pancake, and the piece of spaghetti becoming thinner and longer as the initial rectangle stretches out.

How can one be sure through visualization that the spaghetti will fit into the pancake? We might be inclined to curl it inside the pancake, or cut it into pieces and place them inside the pancake. It is at that stage that it becomes evident that in the transformation of those objects, the linear dimensions that characterize them must be considered. Throughout the visualization the following must be borne in mind: (1) the circumference of the pancake is equal to the length of the spaghetti, and (2) the thickness of the pancake is equal to a section of spaghetti. Therefore, the thickness of the pancake is $\pi$ times the diameter of the spaghetti. It can thus be concluded that if the spaghetti is sectioned into four pieces, each piece will be smaller than the diameter of the pancake. Therefore, the quarter-length pieces of spaghetti will all fit into the pancake.

This example is useful in identifying certain spatial skills that relate to the notion of volume. It has been determined that the rectangle must first be transformed into two cylinders. Then, once mental images are formed, taking into consideration the linear dimensions of the cylinders, a comparison can be made that clears up any initial confusion.

Another interesting case is that of the bottomless box. Students initially tend to think that the volume remains unchanged when the box is transformed (Figure 3). Only when major transformations are carried out and the volume shrinks significantly do they realize that the volume has in fact changed. Why, however, would the volume suddenly change at a particular stage in the transformation? What would lead to the conclusions that volume cannot change suddenly, and that if it does change, the changes—though perhaps imperceptible—must have started at the beginning of the transformation?

Merely visualizing the various configurations of the box is insufficient; visualization must be accompanied by ‘reasoning by contradiction,’ as in the following statement: ‘Suppose that only substantial transformations produce a savoir si la compétence à résister à ce type d’obstacle correspond à une habileté spatiale particulière. Peut-on la développer? Examinons une stratégie que plusieurs utilisent à savoir: le recours aux positions extrêmes. Si le volume est conservé, il devrait l’être pour tout rectangle. D’abord, il est évident qu’un carré fournit deux cylindres identiques. Ne serait-il pas possible que des rectangles très allongés donnent des cylindres qui ont des volumes perceptivement différents? Peut-on fournir mentalement des calculs qui établissent la différence entre les deux cylindres? Pour le cas qui nous concerne, avec un rectangle très allongé donc très mince, un des cylindres ressemble à un spaghetti et l’autre à une galette. Cette galette et ce spaghetti (voir figure 2) doivent être visualisés de manière à ce que l’on puisse imaginer la galette devenant de plus en plus plate et large et le spaghetti de plus en plus fin et long, lorsque le rectangle de base s’allonge. Comment ensuite s’assurer «mentalement» que le spaghetti rentre toujours dans la galette? On peut être tenté de l’enrouler à l’intérieur de la galette ou encore de le mettre en morceaux dans la galette. C’est là que l’on constate qu’il faut conjuger ces transformations d’objets avec les dimensions linéaires qui les caractérisent. Dans toute cette visualisation, il faut se rappeler que: 1) la circonférence de la galette vaut la longueur du spaghetti; 2) l’épaisseur de la galette vaut la circonférence d’une section de spaghetti. Donc, l’épaisseur de la galette vaut $\pi$ fois le diamètre du spaghetti. On peut donc conclure que si le spaghetti est coupé en quatre, chaque bout de spaghetti est plus petit que le diamètre de la galette. On peut donc insérer nos quarts de bouts de spaghetti dans la galette.

Ce problème est un bon exemple pour cerner certaines habiletés spatiales reliées à la notion de volume. Nous avons constaté que le rectangle doit d’abord être transformé en deux cylindres. Ensuite, les images mentales élaborées lorsqu’elles sont conjuguées aux dimensions linéaires des cylindres permettent une comparaison qui écarte manifestement la confusion initiale.

Un autre cas intéressant est celui de la boîte sans fond, ni dessus. Les élèves sont d’abord portés à croire que le volume ne change pas lorsque la boîte est déformée (figure 3). Il faut faire des déformations suffisamment grandes pour que, le volume devenant très petit, ils affirment soudainement que le volume a changé. Mais pourquoi faudrait-il qu’à un certain stade de la transformation le volume change...
change in volume, and that small transformations do not. Yet how can one transformation produce a change while similar transformations do not?" The use of extremes can help to demonstrate that our perception may not be accurate and that slight transformations may not result in identical volume.

In passing, another difficulty should be noted: the comparison of the volumes of various solids is often hampered by a disproportionate influence of their linear dimensions on reasoning. Imagine a block made from several pieces of Lego and a rod made from the same number of pieces. When both solids are shown to the students, the length of the rod will incite several of them to conclude that the volume of the rod is greater than that of the block. The primary strategy for demonstrating the contrary is to dismantle the rod and use its parts to construct (either mentally or physically) a block identical to the first one. The ability to dismantle and reconstruct space objects therefore seems to be a key spatial skill. Of course, since only a close approximation of the volumes of the two solids can be put forth, no convictions can be expressed.

In the introduction, it was mentioned that we consider volume to be basic to the development of spatial concepts. The volume-comparison problem described above (the spaghetti vs. the pancake) is an interesting illustration of this point of view.

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**Figure 2**
Sheet of paper, cake, and piece of spaghetti.

**Feuille (dimension: L x ℓ)**

- D
- \( \pi \times D = L \)
- \( \pi \times d = ℓ \)

**Figure 3**
The box.

La boîte.
The concept of volume: semantic problems

In addition to the problems and reasoning described previously, an enlightened approach to the teaching of volume must take semantic problems into account. In everyday language, the term volume is sometimes used ambiguously to refer to content, and in some circumstances, weight and volume are confused. Moreover, the amount of space occupied by objects such as tables and chairs is more an issue of area. Therefore, we favour hands-on activities that allow for clarifications to be made in order to dispel any semantic confusion. Certain spatial skills are undoubtedly required; however, we will limit our discussion to an analysis of the concept of volume in terms of magnitude. (Our analysis will be saved for the end of this paper so as not to interrupt the flow of the text.)

Basic formula for a right rectangular prism

At the elementary-school level, students determined the number of units contained in a solid in a somewhat haphazard way. Teachers would generally have them find the volume of several right prisms on the basis of tables listing the values assigned to the sides, leading to the conclusion that volume was the product of three dimensions: length × width × height. This formula, which produces a result, can also be presented as a generalization of the formula for determining the area of a rectangle. Thus, for many teachers, volume is the result of calculations validated by a series of verifications. For students, the formula is a mystery, since it is only discovered once the series of verifications has been carried out. At best, they can merely confirm that the formula works. Our approach to volume diverges from that approach and requires the use of spatial skills, as we shall attempt to explain.

With students, we approach the formula as a systematic means of quantifying the units that make up the solid in question. The objective is therefore to use reasoning to discover how to quantify the units. Thus, a specific quantification procedure can take the form of an expression that describes how to determine the number of basic units contained in the solid. The expression devised will, of course, be equivalent to the basic formula. It translates into words the actions required by the formula.

At this stage, the notion of “layers” or “slices” is introduced (Figure 4). The ability to section a right prism into...
layers, or even the ability to visualize the operation and express it in terms of quantification, is the primary skill required to discover the formula for the volume of a right prism. Also involved is the notion that the sum of the units in the solid as a whole is the sum of the units in each layer—a natural property of measurement. Therefore, the first formula we arrive at is as follows:

\[
\text{Volume} = \text{number of units per layer} \times \text{number of layers}
\]

From a pedagogical point of view, we considered it expedient to avoid the introduction of volume directly in terms of standard units: cm\(^3\). Instead, we opted for a situation in which the students could reflect on the possible forms of a box of sugar containing a specific number of cubes. The use of previously memorized formulas is thus avoided. The resulting expression does not depend heavily on symbols and it enables the students to focus on the object for which the volume is sought, as well as on the procedure used.

The obstacle of round objects

In any subsequent application of the basic formula to more complex objects, the problem of breaking up units arises. A single sugar cube is not the same as two half cubes or four quarter cubes. The introduction of formulas of volume must therefore involve a stage of acceptance that the unit of measure can take on very diverse forms when a solid is dissected. This is a familiar problem when dealing with round objects. According to teachers at the secondary level, when students begin secondary school, they are very reticent to accept that round objects can have a calculable volume, as they do not see how such objects can contain cubic units—a foreseeable aftermath of the teaching dispensed in elementary school! Are there any critical experiments that can dispel such views? In our study, we felt it would be important to introduce solids with fractional linear dimensions early on. However, in doing so, we merely initiated students to smaller and smaller subunits. For students at that level, calculating the volume of a rounded solid will never approximate an abstract process such as that devised by Archimedes or, later, Newton. It is by accepting the invariance of the volume of solids that have undergone transformations that they will be able to equate the volume of a rounded solid with that of a right prism that is deemed

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Le volume comme instrument de conceptualisation de l'espace

equivalent. Nevertheless, we maintain that the early introduction of subunits is a step in the right direction.

The standard formula
The transition from the basic formula to the standard formula requires reasoning that goes beyond pure empiricism. In the formula, "number of units per layer" must be replaced by "base area." This is achieved by the teacher manipulating both a layer, and a rectangle representing the base. The manipulation illustrates a one-to-one correspondence between each cm\(^3\) of the layer and each cm\(^2\) of the base. This correspondence does not establish an equivalence between two calculations but rather suggests that both calculations, because of their similarity, will lead to the same results. In that respect, this correspondence is much more a form of reasoning than the result of evidence found through experimentation (see Figure 5).

Through a similar argument, we can demonstrate that the height accurately represents the number of layers, especially when a fraction of a layer is involved. This leads us, therefore, to the standard formula:

\[
\text{VOLUME} = \text{base area} \times \text{height}
\]

Cavalieri's principle applied to inclined solids
Experiments conducted in the classroom have demonstrated that the sectioning of objects into layers in order to determine their volume naturally leads students to use dissection to find the volume of inclined solids. Thus, students without any specific briefing are able to ascertain that the volume of an inclined solid is the base area times the height, since solids can always be sectioned into layers. In fact, students envisage very thin layers that can slide on top of one another. They implicitly use Cavalieri's principle\(^4\) in this type of analysis of inclined solids.

Cavalieri's principle: Given two solids and a plane, suppose that every plane parallel to the given plane, intersecting one of the two solids, also intersects the other, and gives cross sections with the same area. Then the two solids have the same volume.\(^4\)

**Figure 6a** illustrates Cavalieri's principle as applied to an inclined solid. As **Figure 6b** illustrates, however, there are more complex cases as well. Cavalieri's principle is based on

La formule standard
Le passage de la formule de base à la formule standard exige un autre raisonnement qui dépasse un recours exclusif à l'empirisme. Il faut, en effet, remplacer « nombre d'unités par tranche » par « aire de la base ». Ceci est réalisé grâce à une manipulation de l'enseignant avec une tranche et un carton de forme rectangulaire représentant la base. Cette manipulation met en évidence l'existence d'une correspondance bi-univoque entre les cm\(^3\) de la tranche et les cm\(^2\) de la base. Cette mise en correspondance ne conduit pas à l'égalité de deux calculs mais mène plutôt à une analyse implicite du fait que les deux démarches, par leur similitude, vont conduire à la même réponse. En ce sens, elle tient beaucoup plus du raisonnement que d'une conviction issue d'une évidence expérimentale (voir **figure 5**).

Par une argumentation semblable, on montre que la hauteur représente précisément le nombre de tranches et surtout dans le cas où on a une fraction de tranche. On aboutit à la formule standard:

\[
\text{VOLUME} = \text{aire de la base} \times \text{hauteur}
\]

Des tranches au principe de Cavalieri pour les solides inclinés
Comme l'a montré l'expérimentation faite en classe, découper les objets en tranches pour trouver leur volume conduit naturellement les élèves à recourir au découpage pour « trouver le volume » de solides inclinés. C'est ainsi que les élèves, sans préparation particulière affirment que, pour les solides inclinés, on fait toujours le produit de l'aire de la base par la hauteur ce qui peut toujours découper en tranches. En fait, les élèves imaginent des tranches qui peuvent glisser
the fact that in certain transformations, solids retain their volume. "Cavalieriian transformations appear as slide transformations of the thinnest possible layers. (Of course, a rigorous analytical definition could be formulated for such transformations.) Cavalierian transformations are, in the author's opinion, opposed to transformations of non-rigid structures that are carried out by pivoting and that generally do not conserve the same volume. Such is the case with the bottomless box that can be taken apart, as illustrated in Figure 3.

Sectioning a prism into pyramids

Another important step in applying the standard formula of volume is determining the volume of pyramids. This case, sectioning two- and three-dimensional representations is essential. The starting point can be a cube sectioned into square-based pyramids. We can work with both two- and three-dimensional representations, but eventually a planar representation of a cube will have to be broken down, as illustrated in Figure 7.

The objective is to develop students' representation skills so as to enable them to transform any triangular right pyramid into a right prism with the same base and height. Transformations involving breakdown and completion are illustrated in Figures 8a and 8b respectively. Though complementary, the two transformations must be distinguished from each other.

We should add, however, that the actual transformation skill is undoubtedly not needed in order to establish the formula. The mere knowledge that a triangular prism can always be broken down into three pyramids of the same volume and that a prism can always be constructed from a les unes sur les autres; ces tranches sont imaginées « très minces ». Ils font implicitement usage, dans ce genre d'analyse des solides inclinés, du principe de Cavalieri 4 que nous allons maintenant formuler.

**Principe de Cavalieri** Si chacun des plans d'une famille de plans parallèles à une direction donnée coupe deux solides selon des figures ayant la même aire, alors ces deux solides ont le même volume. 4

La figure 6a montre l'application du principe de Cavalieri pour un solide incliné; mais il existe des cas plus complexes comme le montre la figure 6b. Le principe de Cavalieri repose sur le fait que certaines transformations appliquées à des solides conservent leur volume. Les transformations de Cavalieri se présentent comme des glissements de tranches que l'on imagine aussi petites que possible. Elles pourraient, bien entendu, être analytiquement définies d'une manière rigoureuse. A mon sens, les transformations de Cavalieri sont à opposer à d'autres transformations de structures non rigides qui, elles, s'effectuent par pivotement et ne conservent en général pas le volume. C'est le cas de la boîte sans fond que l'on peut déformer et que nous avons présentée à la figure 3.

**Le découpage de prisme en pyramides**

Une autre étape importante dans la généralisation de la formule standard du volume est le passage au volume des pyramides. Cet, il nous apparait que l'articulation des représentations 2D et 3D est essentielle. On peut d'abord se limiter au cube à couper en pyramides à base carrée. On peut travailler sur des représentations 2D et 3D mais il faudra éventuellement bien réaliser la décomposition d'un cube représenté dans le plan, comme nous l'avons illustré à la figure 7.

Le but vise est de développer chez les étudiants leur compétence à représenter de manière à leur permettre de transformer toute pyramide droite à base triangulaire en un prisme droit ayant la même base et la même hauteur. Les transformations de décomposition et de composition sont illustrées aux figures 8a et 8b. Elles sont surtout à distinguer l'une de l'autre même si elles sont complémentaires.

Cependant, l'habileté proprement dite n'est: sans doute pas nécessaire à l'établissement de la formule. En effet, la seule conviction que l'on peut toujours **décomposer un prisme**
pyramid by adding two other pyramids of the same volume is sufficient to conclude that the volume of a pyramid can be determined by dividing the volume of a prism by 3. That knowledge may be the result of the student doing the construction himself, which is preferable, or of a demonstration by the teacher.

**Cavalieri and pyramids of identical volume**

Pyramids with the same base and height have the same volume. This assertion is used in determining the volume of a pyramid on the basis of that of a prism. The teacher can use a model of a pyramid made of elastic bands. By moving the vertex while keeping it in the same plane, the teacher can illustrate an entire family of pyramids of the same height and base and, by using Cavalieri's principle, can conclude that all the pyramids are identical in volume. In fact, each has the same volume as a reference pyramid, \( A'B'C'D' \). As Figure 9 illustrates, each intersecting plane produces shapes that are identical in area.

To demonstrate that these shapes have the same area, the teacher can simply argue on the basis of a plane \( \pi \) intersecting the height of the pyramids by a ratio 1:2 or 1:3. Thus, a ratio of 1:3 applied to the height of each (AH and A'H')\(^5\) will result in the following:

\[
\frac{AX}{AH} = \frac{A'H'}{A'H'} = 1/3
\]

(\( X \) and \( X' \) being the intersection point of \( \pi \).

If Thales' theorem is applied to the triangles whose two sides consist of a height and an edge, we can conclude that the sides will be intersected in the same ratio:

\[
\frac{AS}{AB} = \frac{AT}{AC} = \frac{AU}{AD} = 1/3
\]

and

\[
\frac{A'S'}{A'B'} = \frac{A'T'}{A'C'} = \frac{A'U'}{A'D'} = 1/3.
\]

Finally, upon considering each face of the pyramid, we can conclude using Thales' theorem once again that each side of the triangle intersected by \( \pi \) and \( \pi' \), namely \( ST \), \( TU \) and \( US \), is a third of the length of \( AB \), \( BC \), \( CA \). Similarly, \( ST' \), \( TU' \) and \( US' \) are a third of the length of \( A'B' \), \( B'C' \) and \( CA' \).

Furthermore, since

\[
AB = A'B', \quad BC = B'C' \quad \text{and} \quad CA = C'A'
\]

(the sides of the base of the pyramid), it follows that

\[
ST = S'T', \quad TU = T'U' \quad \text{and} \quad US = U'U.
\]

\(^5\) In this paper, AH refers to the measurement (AH) for all letter symbols A and H.

\(^6\) Dans le présent chapitre, AB désigne la mesure de (AB) pour toutes majuscules A et B.

triangulaire en trois pyramides de même volume et toujours construire un prisme à partir d'une pyramide en y accolant deux autres pyramides de même volume, permet d'établir la nécessité de la division par 3 pour passer du volume du prisme à celui de la pyramide. Cette conviction peut résulter du fait que l'élève a lui-même réalisé ces constructions (ce qui est préférable) ou que l'enseignant en a fait une démonstration.

**Cavalieri et les pyramides de même volume**

Les pyramides de même base et de même hauteur ont le même volume. Cette proposition est utilisée dans le passage du volume du prisme à celui de la pyramide. Grâce à un petit montage (figure 9), l'enseignant peut présenter une pyramide faite de bandes élastiques et, en déplaçant son sommet de manière à le maintenir dans un même plan, il peut illustrer toute une famille de pyramides de même hauteur et de même base. En utilisant le principe de Cavalieri, il peut conclure que toutes ces pyramides ont le même volume. En effet, chacune d'elles a le même volume qu'une pyramide de référence \( A'B'C'D' \). Comme le montre la figure 9, chaque coupe détermine des figures de même aire.

Pour montrer que ces figures ont la même aire, il peut en classe argumenter simplement en considérant un plan quelconque \( \pi \) coupant la hauteur des pyramides à la moitié ou au tiers. Alors, pour chacune des hauteurs AH et A'H' (voir figure 10), on aura, par exemple, le rapport 1/3

\[
\frac{AX}{AH} = \frac{A'H'}{A'H'} = 1/3
\]

\( X \) et \( X' \) étant les points d'intersection de \( \pi \) avec les hauteurs.

Et, en appliquant le théorème de Thalès aux triangles dont deux des côtés sont respectivement la hauteur et une arête, on déduit que chacune des arêtes sera coupée dans un même rapport. On aura donc

\[
\frac{AS}{AB} = \frac{AT}{AC} = \frac{AU}{AD} = 1/3
\]

et

\[
\frac{A'S'}{A'B'} = \frac{A'T'}{A'C'} = \frac{A'U'}{A'D'} = 1/3.
\]

Finalement, en considérant chacune des faces des pyramides, le théorème de Thalès nous permet de conclure que chaque côté de la figure découpée par \( \pi \) et \( \pi' \), soit \( ST \), \( TU \) et \( US \) vaut le tiers de \( AB \), \( BC \), \( CA \). De même, \( S'T' \), \( T'U' \) et \( U'S' \) vaut le tiers de \( A'B' \), \( B'C' \) et \( C'A' \).

Et, comme

\[
AB = A'B', \quad BC = B'C' \quad \text{and} \quad CA = C'A'
\]
Consequently, figures STU et S'T'U' are equal in area, as they are congruent. Therefore, by using Cavalieri's principle, we can conclude that both pyramids are identical in volume. Although a specific ratio of 1:3 has been used here for purposes of illustration, \( m/n \) can be used instead in order to make a generalization.

**Breakdown of a cone and sphere into pyramids**

Once the volume of the given triangular pyramids is known, the volume of polygonal-based pyramids can be established. The base simply has to be broken down into triangles. Thus, students will conclude that the volume of a pyramid with a polygonal base is the product of height times base area divided by 3.

The progression to the volume of a cone is then simple. **Figure 11** illustrates how a cone can always be approached by means of a series of pyramids, each with a regular polygonal base. As the number of sides of the polygon increases, the pyramid increasingly resembles a cone. The principle of extremes is applied to the polygon, which then becomes a circle, thus transforming the pyramid into a cone. Without hesitation, students will assert that the formula remains valid—the volume is always the product of the base area (\( \pi r^2 \)) times height divided by 3.

For the sphere, the process is similar. The starting point is a soccer ball: a truncated icosahedron. Its surface consists of 12 pentagons and 20 hexagons—polygons that can be sectioned into triangles. Since a soccer ball can be broken down into triangular pyramids, the following formula can be used to determine the approximate volume of a sphere:

\[
\text{VOLUME} = \frac{(\text{area of polygons} \times \text{height})}{3}.
\]

étant les côtés respectifs des bases des pyramides, on conclut que:

\[ ST = ST', \quad TU = TU' \quad \text{et} \quad US = U'S'. \]

Ainsi les figures STU et S'T'U' ont la même aire car elles sont congrues. On conclut donc en s'appuyant sur le principe de Cavalieri que les deux pyramides ont le même volume.

Nous avons pris pour la démonstration le rapport particulier 1/3, mais il suffit de remplacer 1/3 par \( m/n \) dans la démonstration pour la généraliser.

**Décomposition du cône et de la sphère en pyramides**

Une fois le volume de pyramides triangulaires connu, on peut passer au volume de pyramides polygonales quelconques; il suffit de décomposer la base en triangles. Ainsi, les élèves affirmeront que le volume des pyramides est le produit de la hauteur par l'aire de la base divisé par 3.

On aborde ensuite facilement le volume du cône. La figure 11 montre comment le cône peut toujours être approché par une suite de pyramides à bases polygonales régulières. Lorsque le nombre de côtés des polygones augmente, la pyramide s'approche de plus en plus d'un cône. Le passage à la limite se fait sur le polygone devenant un cercle qui amène la pyramide à se transformer en cône. Les élèves affirmeront sans hésitation que la formule est toujours valable; le volume est toujours le produit de l'aire de la base (\( \pi r^2 \)) par la hauteur divisé par 3.

Pour la sphère, la démarche est semblable. On part d'un ballon de soccer: un icosahedre tronqué. Sa surface est composée de 12 pentagones et de 20 hexagones; ces polygones sont décomposables en triangles. On peut donc décomposer un ballon de soccer en pyramides triangulaires. Pour la sphère, on a donc trouver un volume approximatif avec la formule:

\[
\text{VOLUME} = \frac{(\text{aire des polygones} \times \text{hauteur})}{3}.
\]

La discussion avec les élèves conduit à établir que l'aire des polygones à la limite devient l'aire de la sphère et la hauteur des pyramides sera le rayon de la sphère. Ainsi, le volume de la sphère peut s'écrire:

\[
\text{VOLUME} = \frac{(\text{aire de la sphère} \times \text{rayon})}{3}.
\]

Bien que le passage à la limite pose, pour la rigueur mathématique, des problèmes qui exigeraient de prendre des précautions spéciales pour le découpage en triangles, les
In terms of magnitude, volume is primarily numerical. In French, there is often confusion between a solid or space figure and its corresponding measurement of volume. For example, three-dimensional figures are sometimes referred to (incorrectly) as volumes rather than solides.

Through discussion with the students, it is determined that the area of the polygons is essentially the area of the sphere and the height of the pyramids is the radius of the sphere. Thus, the volume of the sphere can be expressed as follows:

\[ \text{Volume} = \left( \text{area} \times \text{radius of the sphere} \right) / 3. \]

Although the use of extremes, from the perspective of mathematical precision, presents certain problems that require special precautions to be taken when shapes are sectioned into triangles, students are quickly convinced of the validity of the process. The spatial skills that would enable students to detect incorrect extremes are of a higher level and would merit further study.

**Magnitude and measurement**

Before concluding, let us look at a major obstacle to the effective use of the concept of volume. Volume must be considered in terms of magnitude, a term that has fallen out of use in mathematics. As it is traditionally defined, magnitude refers to a numerical quantitative measure of an object. For instance, when considered in terms of magnitude, a segment is expressed as length, a surface as area, and a solid as volume. There is therefore a duality between the object being measured and the measurement itself, even though in everyday life, such a distinction is not always made. Therefore, a segment may commonly be referred to as a length, and a surface as an area.

It would be appropriate to bring the process of measurement to the forefront. We must make it easier to distinguish between the object being measured and the measurement itself. However, such a distinction, which is purely mathematical, eludes se laissent facilement convaincre de la validité du processus. Ici, les habiletés spatiales qui les rendraient vigilants par rapport aux passages à la limite incorrecte sont d'un niveau supérieur et mériteraient d'être mieux étudiées.

**Grandeur et mesure**

Terminons avec l'une des difficultés majeures s'opposant à l'utilisation efficace de la notion de volume. Le volume est une grandeur, terme maintenant absent du vocabulaire mathématique. Selon son acception traditionnelle, une grandeur se présente comme une caractéristique numérique d'un objet auquel on applique une mesure. Les segments, les surfaces et les solides sont autant d'objets géométriques qui, lorsque considérés comme grandeurs, deviennent des longueurs, des aires et des volumes. Les grandeurs ont donc une «double personnalité» que leur confère cette dualité objet-mesure. On constate d'ailleurs que dans la vie courante on ne fait pas toujours la distinction entre l'objet et sa mesure. On parle d'une longueur comme d'un segment, on confond souvent surface et aire. En tant que grandeur, le volume est avant tout numérique. Mais la distinction entre solide ou espace occupé et la mesure qui en est le volume n'est pas toujours faite. On dira, par exemple, que l'on a différents volumes sur la table (en se référant à des solides); ou encore que l'on cherche l'espace occupé plutôt que la mesure de l'espace occupé.

Il convient de mettre le processus par lequel on établit la mesure au premier plan; on doit faciliter la distinction entre l'objet mesuré et sa mesure. Mais cette distinction, toute mathématique, n'est ni toujours commode, ni toujours efficace. Il arrive que la formulation de problèmes, s'effectuant dans le langage courant, néglige la distinction entre l'objet et sa mesure. Cette difficulté, provenant d'une absence de précision dans l'expression, s'inscrit cependant dans la logique de l'utilisation de la métonymie qui confond ici l'objet avec sa mesure. Faut-il s'en prendre à l'utilisation métonymique du terme volume. Elle entraîne peut-être d'erreurs de raisonnement ou de calcul; ainsi, nous croyons qu'il faudra tolérer certaines ambiguités tout en prenant les précautions qui s'imposent. Et, c'est là que nous revenons aux habiletés spatiales. En effet, le volume comme la mesure de l'espace occupé par un solide est une interprétation qui est à l'occasion prise en défaut. Il arrive souvent que «l'objet»
The analysis of the concept of volume that has been presented and illustrated here is intended to demonstrate how volume can be a catalyst for the development of spatial skills. In order to find formulas, transformations of space objects must be carried out (by means of Cavalieri's principle, for instance). This involves spatial skills. Spatial skills are combined with various forms of reasoning, such as the use of extremes and limitations. We have stressed the fact that such skills require the coordination of numerical and visual elements—a deeper understanding of which will lead to more effective teaching of the concept of volume.

The reader will hopefully be convinced that the introduction of the concept of volume in academic programs provides an excellent opportunity to foster the development of various skills that can be applied to other space-related problems. This paper has nevertheless not dealt with describing basic spatial skills that should be developed by means of basic construction and drawing. There is still a great deal of work to be done in exploring basic spatial skills alongside those that relate more specifically to the concept of volume. As our studies have shown, that is where a large portion of our students' difficulties lie. Further analysis and research is certainly needed.
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<td>volume as an instrument for the formation of spatial concepts</td>
<td>volume comme instrument de conceptualisation de l'espace</td>
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