NAPOLEON, ESCHER AND TESSELLATIONS

ABSTRACT

Napoleon and Escher both have theorems about triangles named after them. It is doubtful whether Napoleon knew enough geometry to prove Napoleon's theorem [3, p.631], and Escher apparently never found a proof for the last part of Escher's theorem. The first part of Escher's theorem is a form of converse of Napoleon's theorem, and both theorems can be proved using tessellations, a method that would surely have appealed to Escher with his love of filling the plane with congruent shapes.

Given any triangle, we can tessellate the plane using congruent copies of this triangle and equilateral triangles of three sizes, as shown in Figure 1. The centres of the small equilateral triangles in this figure clearly form the vertices of an equilateral triangular lattice, shown in Figure 2 by unbroken lines. The centres of the remaining equilateral triangles lie at the centres of the triangles of the lattice; hence we see from the Figure 2 that the centres of all the equilateral triangles form the vertices of a smaller equilateral triangular lattice, shown by broken lines. Figure 3a forms just part of the

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Jean-Luc Raymond
One of the notebooks of the Dutch graphic artist M.C. Escher contains some interesting results about a special type of hexagon. Although these results were known previously, we shall group them together under the title of "Escher's theorem"; the theorem may be stated as follows.

(i) Let $ABC$ be an equilateral triangle and $E$ any point (Figure 4). Let $F$ be the point such that $AF = AE$ and $\angle FAE = 120^\circ$. Let $D$ be the point such that $BD = BF$ and $\angle DBF = 120^\circ$. Then $CE = CD$ and $\angle ECD = 120^\circ$.

(ii) Congruent copies of the hexagon $AFBDCE$ can be used to tessellate the plane.

(iii) In Figure 4 the lines $AD$, $BE$ and $CF$ are concurrent.
Escher’s notebooks have been edited by Professor Doris Schattschneider [10]. She remarked in a letter to me that “it is very likely that Escher learned about the special tiling hexagon in a paper by F. Haag [5]; this paper and an earlier one [4] were on the list of references provided to him in 1937 by his half brother B.G. Escher. He studied [5] pretty carefully, copying many diagrams, including one showing a 6-fold “rosace” sextuple of the special hexagon. As far as I can see from the articles, Haag makes no mention of the diagonals of the hexagon. The facts are that in stating his Theorem, Escher underlined his statement about the diagonals and also had no reference and no proof—that makes me pretty sure it was his own discovery. Also he wrote to his son George to ask if he could prove the result.”

To prove (i) we apply Napoleon’s theorem to the triangle DEF. The point A is the centre of the equilateral triangle erected on EF, and B is the centre of the equilateral triangle erected on FD. The centres of all three equilateral triangles erected on the sides of DEF form an equilateral triangle by Napoleon’s theorem; but ABC is equilateral, so C must be the centre of the equilateral triangle erected on DE. The result follows immediately.

(ii) Des copies congruentes de l’hexagone AFBDCE peuvent être utilisées pour construire un pavage du plan.

(iii) Dans la figure 4, les droites AD, BE et CF sont concourantes.

Les cahiers de notes d’Escher ont été édités par le professeur Doris Schattschneider [10]. Elle a noté dans une lettre qu’elle m’adressait qu’« il est très probable qu’Escher ait pris connaissance de ce pavage spécial d’hexagones dans un article de F. Haag [5]; cet article, de même qu’un autre plus ancien [4] apparaissaient sur la liste de références que lui fournissait son demi-frère B.G. Escher en 1937. Il a étudié [5] assez attentivement, copiant plusieurs diagrammes, y compris celui montrant une “rosace” sextuple de l’hexagone spécial. En autant que je puisse le voir des articles, Haag n’a pas fait mention des diagonales de l’hexagone. Le fait est qu’en énonçant son théorème, Escher souligna son énoncé concernant les diagonales et il n’avait ni référence ni démonstration. Cela m’assure presque totalement que c’était sa propre découverte. De plus, il a écrit à son fils George pour lui demander s’il pouvait démontrer le résultat.»

Pour montrer (i), on applique le théorème de Napoléon au triangle DEF. Le point A est le centre du triangle équilatéral construit sur EF, et B est le centre du triangle équilatéral construit sur FD. Les centres des trois triangles équilatéraux construits sur les côtés de DEF forment un triangle équilatéral selon le théorème de Napoléon; mais ABC est équilatéral, ainsi C doit être le centre du triangle équilatéral construit sur DE. Le résultat s’ensuit immédiatement.

Le lecteur astucieux aura remarqué qu’il existe deux triangles équilatéraux qui peuvent être construits sur EF et sur FD; il y a aussi deux triangles équilatéraux qui ont AB comme côté. Notre démonstration n’est donc pas suffisamment soignée. Mais le théorème lui-même n’a pas été énoncé de façon assez précise! La figure 5 satisfait les conditions du théorème, et on a même mesuré les angles \( \angle FAE \) et \( \angle DBF \) dans la même direction; toutefois la conclusion du théorème
The astute reader will have noticed that there are two equilateral triangles that can be erected on EF and on FD; also there are two equilateral triangles having AB as one side. Hence our proof was not sufficiently careful. But the theorem itself has not been stated carefully enough! Figure 6 satisfies the conditions of the theorem, and the angles $\angle FAE$ and $\angle DBF$ have even been measured in the same direction, yet the conclusion of the theorem is not valid. This is because the triangle ABC has "the wrong orientation". The difficulty can be resolved by a more careful statement of the theorem, but this point presumably did not worry Escher, so we shall not let it worry us.

The tessellation in (ii) can be obtained immediately from Figure 1 by joining the centre of each equilateral triangle to its three vertices, as in Figure 7.

The third part of Escher's theorem is a special case of the following result:

Let $A'EF$, $B'FD$, $C'DE$ be similar isosceles triangles erected on the sides of a triangle DEF, as in Figure 6. Then the lines $A'D$, $B'E$, $C'F$ are concurrent. It can also be shown that, as the shape of the isosceles triangles varies, the locus of the point of concurrency is a rectangular hyperbola passing through D, E and F.

A proof of these results can be found in [2]; see also [8]. Three other references have been supplied by Hans Cornet: proofs of Escher's special case can be found in [6] and [7], and a proof of concurrency in the general case, from a book by O. Bottema [1, 1st ed., p.36, 2nd ed. p.51], is so short and elegant that it is worth reproducing here.

In Figure 6, $ED'/D''F = \Delta EDA'/\Delta FDA'$ (using $\Delta$ to denote the area of a triangle) $= \frac{1}{2} DE \cdot EA \sin (E+\phi) / \frac{1}{2} FD \cdot FA \sin (F+\phi) = DE \sin (E+\phi) / FD \sin (F+\phi)$. Using this and two similar expressions we find that $(ED'/D''F) (FE'/E''D) (DF'/F''E) = 1$, and the result now follows by the converse of Ceva's theorem.

La raison en est que le triangle ABC possède "la mauvaise orientation". On peut résoudre cette difficulté par un énoncé plus précis du théorème, mais ce point n’inquiétait vraisemblablement pas Escher, et nous ne nous en ferons pas plus de souci.

On peut obtenir immédiatement la tessellation dans (ii) de la figure 1 en joignant le centre de chaque triangle équatorial à ses trois sommets, comme à figure 7.

La troisième partie du théorème d'Escher est un cas spécial du résultat suivant.

Soient $A'EF$, $B'FD$ et $C'DE$ des triangles isocèles semblables construits sur les côtés d'un triangle DEF, comme à la figure 6. Alors les droites $A'D$, $B'E$ et $C'F$ sont concourantes. On peut aussi montrer que, à mesure que la forme des triangles isocèles varie, le lieu du point d'intersection est une hyperbole rectangulaire passant par D, E et F.


Dans la figure 6, $ED'/D''F = \Delta EDA'/\Delta FDA'$ (where $\Delta$ represents the area of triangle) $= \frac{1}{2} DE \cdot EA \sin (E+\phi) / \frac{1}{2} FD \cdot FA \sin (F+\phi) = DE \sin (E+\phi) / FD \sin (F+\phi)$. En utilisant ceci et deux expressions similaires on trouve que $(ED'/D''F) (FE'/E''D) (DF'/F''E) = 1$, et le résultat s'ensuit maintenant par la réciproque du théorème de Ceva.

Voici une autre preuve du cas spécial d'Escher qui utilise les propriétés de rotation et de translation de la tessellation d'hexagones. Soit $X$ le point d'intersection de $AD$ et de $BE$ (figure 8). Alors C est le...
Here is another proof of Escher's special case, using rotational and translational properties of the tessellation of hexagons. Let AD and BE meet at X (Figure 8). Then C is the centre of the equilateral triangle XGJ; hence

\[ JG \perp CX. \]  

Also A is the centre of the equilateral triangle XKY; hence \( XY \perp AK \parallel ZF \); similarly \( ZX \perp BL \parallel YF \). Hence F is the orthocentre of triangle XYZ. Hence

\[ XF \perp YZ \parallel JG \perp CX \text{ from (*)}. \]

Hence CXF is a straight line; i.e. CF passes through X.

Further correspondence with Doris Schattschneider produced more items of interest. Escher's tessellation number 10 in his "abstract motif" notebook is amazingly similar to Figure 1, although I was not previously aware of his tessellation. Figure 9 shows the essential features of Escher's tessellation number 11 from the same notebook; it is built up of four congruent tessellations of hexagons, the original being in four colours, and the way in which the figure is built up can be described in the following way. The basic tessellation of Figure 8 is transformed into itself by three basic translations determined by the vectors \( AA_1, AA_2 \) and \( AA_3 \) (forming the angles of 60° between them).
mined by the vectors \( \vec{A}_1, \vec{A}_2 \), and \( \vec{A}_3 \) (making angles of 60° with each other). If we translate the tessellation in these same three directions, but through just half the distance, we obtain the three other tessellations of Figure 9 (the hexagons in Figure 9 are of a different shape to those in Figure 8, but this does not affect the method). The most interesting fact about this compound tessellation is that inside any hexagon of any one tessellation there are three concurrent edges from the other three tessellations. Here is a short proof of this fact.

**Figure 10** shows part of the original tessellation (with three of the equilateral triangles associated with it; compare Figure 7) and the translation of the tessellation in these same three directions, but through just half the distance, we obtain the three other tessellations of Figure 9 (the hexagons in Figure 9 are of a different shape to those in Figure 8, but this does not affect the method). The most interesting fact about this compound tessellation is that inside any hexagon of any one tessellation there are three concurrent edges from the other three tessellations. Here is a short proof of this fact.

La figure 10 montre une partie de la tessellation originale (avec trois des triangles équilatéraux qui lui sont associés ; à comparer avec la figure 7) et les trois arêtes restantes des trois autres tessellations.
three relevant edges of the other three tessellations. Let O and H be the circumcentre and orthocentre of triangle DEF, and let N be the midpoint of OH, so that N is the nine-point centre of DEF. Since C is the centre of the equilateral triangle DTE, the line TC is the perpendicular bisector of DE; hence TC passes through O. Since FW is parallel to TC, it is perpendicular to DE; hence FW is an altitude of triangle DEF, and so it passes through H. One of the basic translations mentioned earlier transforms TC to FW. A translation in the same direction but of half the magnitude transforms TC to the edge PQ of one of the other tessellations; PQ lies half-way between TC and FW, and hence PQ passes through N which lies half-way between O and H. Similarly the edges of the other two tessellations shown in the figure also pass through N.

If we form the compound tessellation using hexagons of the shape seen in Figure 8, the theorem just proved is still true, but one of the edges has to be produced before it passes through the point of concurrence. It can be shown that this happens because the angle F of the triangle DEF in Figure 8 is less than 30°.

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"Back to areas!"

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