

A Primer on Media Theory

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Abstract

Media theory is a new branch of discrete applied mathematics originally developed in mid-nineties to deal with stochastic evolution of preference relations in political science and mathematical psychology. However, many different examples of media can be found, ranging from learning spaces to hypercube computers, suggesting that this concept is ubiquitous. The paper presents very basic concepts and results of media theory and is aimed at a wide body of researchers in discrete applied mathematics.

1 Introduction

A medium is an algebraic structure describing a mathematical, physical, or behavioral system as it evolves from one ‘state’ to another, in a set of such states. This structure is formalized as an ordered pair $(\mathcal{S}, \mathcal{T})$ consisting of a set \mathcal{S} of states and a set \mathcal{T} of tokens and specified by constraining axioms (see Section 3). Tokens are transformations of the set \mathcal{S} ; strings of tokens are messages of the medium. States, tokens, and messages are three fundamental notions of media theory.

Media theory was introduced by Jean-Claude Falmagne in his founding paper [8] where basic concepts and results were introduced. That paper was followed by papers [16] and [11] in which the theory was further advanced.

The set \mathcal{P} of all partial orders on a given finite set X is an example of a set of states that can be casted as medium. For any two distinct partial orders P and Q , one can ‘walk’ in \mathcal{P} from P to Q by adding or removing a single ordered pair of elements of X [2]. The transformations of \mathcal{P} consisting in the addition or removal of some pair are tokens of the medium on \mathcal{P} . There are many other families of partial orders that can be casted as media, including linear orders, weak orders, and semiorders, [8, 11, 13]. Additional examples of media include learning spaces [3] and hyperplane arrangements [13, 14].

Various stochastic applications of media theory have been made in the context of opinion polls and related situations [7, 8, 9, 10, 17]. Effective algorithms for visualization [4] and enumeration [5] of media have been developed.

The paper presents a concise introduction to basic concepts and results of media theory. Our exposition differs, in some details, from those given in [8, 11] and [16]. For a more comprehensive exposition of media theory the reader is referred to the forthcoming monograph [6].

2 Token systems

Let \mathcal{S} be a set of *states*. A *token* is a transformation $\tau : S \mapsto S\tau$. By definition, the identity function τ_0 on \mathcal{S} is not a token. Let \mathcal{T} be a set of tokens. The pair $(\mathcal{S}, \mathcal{T})$ is called a *token system*. To avoid trivialities, we assume that $|\mathcal{S}| \geq 2$ and $\mathcal{T} \neq \emptyset$.

Let V and S be two states of a token system $(\mathcal{S}, \mathcal{T})$. Then V is *adjacent* to S if $S \neq V$ and $S\tau = V$ for some token $\tau \in \mathcal{T}$. A token $\tilde{\tau} \in \mathcal{T}$ is a *reverse* of a token τ if for all distinct $S, V \in \mathcal{S}$, we have

$$S\tau = V \iff V\tilde{\tau} = S.$$

Two distinct states S and V are *adjacent* if S is adjacent to V and V is adjacent to S .

Remark 2.1. In both examples of Figure 3.2, the state V is adjacent to the state S , but these two states are not adjacent in either example.

Remark 2.2. It is easy to verify that if a reverse of a token exists, then it is unique and the reverse of a reverse is the token itself; that is, $\tilde{\tilde{\tau}} = \tau$, provided that $\tilde{\tau}$ exists. In general, a token of a token system $(\mathcal{S}, \mathcal{T})$ does not necessarily have a reverse in $(\mathcal{S}, \mathcal{T})$. For instance, the token τ in Example [M1] of Figure 3.2 does not have a reverse in \mathcal{T} . It is also possible for a token to be the reverse of itself. For example, let $\mathcal{S} = \{S, V\}$, $\mathcal{T} = \{\tau\}$ where τ is the function defined by $S\tau = V$ and $V\tau = S$. Clearly, $\tilde{\tau} = \tau$.

A *message* of a token system $(\mathcal{S}, \mathcal{T})$ is a string of elements of the set \mathcal{T} . We write these strings in the form $\mathbf{m} = \tau_1\tau_2 \dots \tau_n$. If a token τ occurs in the string $\tau_1\tau_2 \dots \tau_n$, we say that the message $\mathbf{m} = \tau_1\tau_2 \dots \tau_n$ *contains* τ .

A message $\mathbf{m} = \tau_1\tau_2 \dots \tau_n$ defines a transformation

$$S \mapsto S\mathbf{m} = (((S\tau_1)\tau_2) \dots \tau_n)$$

of the set of states \mathcal{S} . By definition, the empty message defines the identity transformation τ_0 of \mathcal{S} . If $V = S\mathbf{m}$ for some message \mathbf{m} and states $S, V \in \mathcal{S}$, then we say that \mathbf{m} *produces* V from S or, equivalently, that \mathbf{m} transforms S into V . More generally, if $\mathbf{m} = \tau_1 \dots \tau_n$, then we say that \mathbf{m} *produces* a sequence of states (S_i) , where $S_0 = S$ and $S_i = S\tau_1 \dots \tau_i$ for $1 \leq i \leq n$.

If \mathbf{m} and \mathbf{n} are two messages, then \mathbf{mn} stands for the concatenation of the strings \mathbf{m} and \mathbf{n} . We denote by $\tilde{\mathbf{m}} = \tilde{\tau}_n \dots \tilde{\tau}_1$ the *reverse* of the message $\mathbf{m} = \tau_1 \dots \tau_n$, provided that the tokens in $\tilde{\mathbf{m}}$ exist. If $\mathbf{n} = \mathbf{m}\mathbf{p}\mathbf{m}'$ is a message, with \mathbf{m} and \mathbf{m}' possibly empty messages, and \mathbf{p} non empty, then we say that \mathbf{p} is a *segment* of \mathbf{n} .

The *content* of a message $\mathbf{m} = \tau_1 \dots \tau_n$ is the set $\mathcal{C}(\mathbf{m}) = \{\tau_1, \dots, \tau_n\}$ of its distinct tokens. The content of the empty message is the empty set. We write $\ell(\mathbf{m}) = n$ to denote the *length* of the message \mathbf{m} and assume that the length of the empty message is zero. It is clear that $|\mathcal{C}(\mathbf{m})| \leq \ell(\mathbf{m})$ for any message \mathbf{m} .

A message is *consistent* if it does not contain both a token and its reverse, and *inconsistent* otherwise. A message $\mathbf{m} = \tau_1 \dots \tau_n$ is *vacuous* if the set of indices $\{1, \dots, n\}$ can be partitioned into pairs $\{i, j\}$, such that τ_i and τ_j are mutual reverses.

A message \mathbf{m} is *effective* (resp. *ineffective*) for a state S if $S\mathbf{m} \neq S$ (resp. $S\mathbf{m} = S$) for the corresponding transformation \mathbf{m} . A message $\mathbf{m} = \tau_1 \dots \tau_n$ is *stepwise effective* for S if $S_k \neq S_{k-1}$, $1 \leq k \leq n$, in the sequence of states produced by \mathbf{m} from S . A message is said to be *concise* for a state S if it is stepwise effective for S , consistent, and any token occurs at most once in the message. A message is *closed* for a state S if it is stepwise effective and ineffective for S . When it is clear from the context which state is under consideration, we may drop a reference to that state.

Some properties of the concepts introduced in this section are listed below. These properties are straightforward and will be used implicitly in this paper.

1. One must distinguish messages from transformations defined by these messages. For instance, for any token τ_i , the two distinct messages $\mathbf{m} = \tau_i \tau_i$ and $\mathbf{n} = \tau_i$ of the token system displayed in Figure 3.1 define the same transformation of the set of states \mathcal{S} .
2. A consistent message may not contain a token which is identical to its reverse. Clearly, this also holds for concise messages.
3. The length of a vacuous message is an even number.
4. The reverse $\widetilde{\mathbf{m}}$ of a concise message \mathbf{m} producing a state V from a state S is a concise message for V , provided that $\widetilde{\mathbf{m}}$ exists.
5. Let $\mathbf{m} = \tau_1 \dots \tau_n$ be a stepwise effective message for a state S . For any i , the state S_{i+1} is adjacent to the state S_i in the sequence of states produced by \mathbf{m} . In general, there could be identical states in this sequence.
6. Any segment of a concise message is a concise message for some state.
7. If \mathbf{m} is a concise message for some state, then $\ell(\mathbf{m}) = |\mathcal{C}(\mathbf{m})|$.

3 Axioms for a Medium

Definition 3.1. A token system $(\mathcal{S}, \mathcal{T})$ is called a *medium* (on \mathcal{S}) if the following axioms are satisfied.

- [M1] For any two distinct states S and V in \mathcal{S} there is a concise message transforming S into V .
- [M2] A message which is closed for some state is vacuous.

A medium $(\mathcal{S}, \mathcal{T})$ is *finite* if \mathcal{S} is a finite set.

Example 3.1. Figure 3.1 displays the digraph of a medium with set of states $\mathcal{S} = \{S, V, W, X, T\}$ and set of tokens $\mathcal{T} = \{\tau_i \mid 1 \leq i \leq 6\}$. It is clear that $\tilde{\tau}_1 = \tau_2$, $\tilde{\tau}_3 = \tau_4$, and $\tilde{\tau}_5 = \tau_6$. We omit loops in digraphs representing token systems.

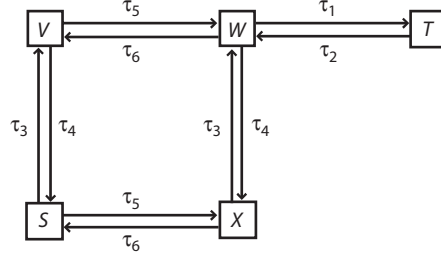


Figure 3.1: Digraph of a medium with set of states $\mathcal{S} = \{S, V, W, X, T\}$ and set of tokens $\mathcal{T} = \{\tau_i \mid 1 \leq i \leq 6\}$.

Theorem 3.1. *The axioms [M1] and [M2] are independent.*

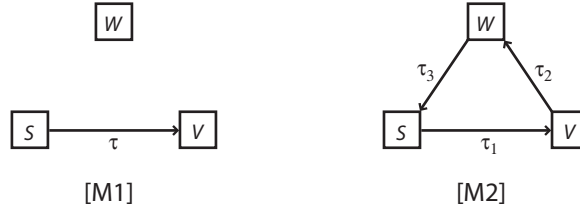


Figure 3.2: Digraphs of two token systems. Each digraph is labeled by the unique failing Axiom.

Proof. It is easy to verify that each of the two digraphs in Figure 3.2 represents a token system satisfying only one of the two axioms defining a medium. \square

4 A ‘canonical’ example of a medium

Let X be a set and \mathcal{F} be a family of subsets of X such that $|\mathcal{F}| \geq 2$. For every $x \in \cup \mathcal{F} \setminus \cap \mathcal{F}$, we define transformations γ_x and $\tilde{\gamma}_x$ of the family \mathcal{F} by

$$\gamma_x : \mathcal{S} \mapsto \mathcal{S}\gamma_x = \begin{cases} \mathcal{S} \cup \{x\}, & \text{if } \mathcal{S} \cup \{x\} \in \mathcal{F}, \\ \mathcal{S}, & \text{otherwise,} \end{cases}$$

and

$$\tilde{\gamma}_x : \mathcal{S} \mapsto \mathcal{S}\tilde{\gamma}_x = \begin{cases} \mathcal{S} \setminus \{x\}, & \text{if } \mathcal{S} \setminus \{x\} \in \mathcal{F}, \\ \mathcal{S}, & \text{otherwise,} \end{cases}$$

respectively, and denote $\mathcal{G}_{\mathcal{F}}$ the family of all these transformations. We say that the family \mathcal{F} is *connected* if, for any two sets $S, T \in \mathcal{F}$, there is a sequence

$$S_0 = S, S_1, \dots, S_n = T$$

of sets in \mathcal{F} such that $d(S_i, S_{i+1}) = 1$ for all i .

Lemma 4.1. *If \mathcal{F} is connected, then $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$ is a token system.*

Proof. We need to show that $\gamma_x \neq \tau_0$ and $\tilde{\gamma}_x \neq \tau_0$ for any given x . Since x is an element of $\cup \mathcal{F} \setminus \cap \mathcal{F}$, there are $S, T \in \mathcal{F}$ such that $x \notin S$ and $x \in T$. Let (S_i) be a sequence of sets in \mathcal{F} such that $S_0 = S$, $S_n = T$, and $d(S_i, S_{i+1}) = 1$ for all i . Clearly, there is k such that $x \notin S_k$ and $x \in S_{k+1}$. It follows that $S_{k+1} = S_k + \{x\}$, so $S_k \gamma_x = S_{k+1}$. Therefore, $\gamma_x \neq \tau_0$. Evidently, $S_{k+1} \tilde{\gamma}_x = S_k$, so $\tilde{\gamma}_x \neq \tau_0$. \square

Let $\mathcal{F} = \{\{a\}, \{b\}, \{a, b\}, \{c\}, \{d\}, \{c, d\}\}$. The family \mathcal{F} is not connected, but $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$ is a token system. Thus, the converse of the previous lemma does not hold. Note also that the connectedness condition does not guarantee that $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$ is a medium. Indeed, let $\mathcal{F} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{b, c\}\}$. Clearly, \mathcal{F} is connected but there is no concise message of the token system $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$ producing $\{b, c\}$ from \emptyset . Hence Axiom [M1] does not hold. This example motivates the following definition.

Definition 4.1. A family \mathcal{F} of subsets of a set X is *well-graded* (a *wg-family*) if, for any two distinct subsets $S, T \in \mathcal{F}$ with $d(S, T) = n$, there is a sequence $S_0 = S, S_1, \dots, S_n = T$ such that $d(S_i, S_{i+1}) = 1$ for all $0 \leq i < n$.

Remark 4.1. A family \mathcal{F} of finite subsets of X is well-graded if and only if the induced graph $\langle \mathcal{F} \rangle$ is an isometric subgraph of the cube $\mathcal{H}(X)$, that is, $\langle \mathcal{F} \rangle$ is a partial cube on X (see Section 9).

We need the following result to prove Theorem 4.1.

Lemma 4.2. *Let (S_0, S_1, \dots, S_n) be a sequence of subsets of X such that*

$$d(S_0, S_n) = n \quad \text{and} \quad d(S_{i-1}, S_i) = 1 \quad \text{for } 1 \leq i \leq n.$$

Then $d(S_i, S_j) = |i - j|$, for all $0 \leq i, j \leq n$.

Proof. We may assume that $i < j$. By the triangle inequality,

$$\begin{aligned} n = d(S_0, S_n) &\leq d(S_0, S_i) + d(S_i, S_j) + d(S_j, S_n) \\ &\leq (i - 0) + (j - i) + (n - j) = n. \end{aligned}$$

It follows that $d(S_i, S_j) = j - i$. \square

Theorem 4.1. *$(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$ is a medium if and only if \mathcal{F} is a wg-family.*

Proof. (Necessity.) Let S and T be two distinct sets in \mathcal{F} . By [M1], there is a concise message $\mathbf{m} = \tau_1 \dots \tau_n$ transforming S into T . Let (S_i) be a sequence of sets produced by \mathbf{m} from S , so $S_0 = S$ and $S_n = T$. Each τ_i is either γ_{x_i} or $\tilde{\gamma}_{x_i}$ for

some x_i . Since \mathbf{m} is a concise message, all elements x_i are distinct. Suppose first that $\tau_i = \gamma_{x_i}$ for some i . Then $S_i = S_{i-1} + \{x_i\}$. Since \mathbf{m} is a concise message, we must have $x_i \in S_j$ for all $j \geq i$ and $x_i \notin S_j$ for all $j < i$. Hence, $x_i \in T \setminus S$. Suppose now that $\tau_i = \tilde{\gamma}_{x_i}$ for some i . Then $S_i = S_{i-1} \setminus \{x_i\}$. Arguing as in the previous case, we obtain $x_i \in S \setminus T$. Therefore, $x_i \in S \Delta T$ for any i . On the other hand, it is clear that any element of $S \Delta T$ is one of the x_i 's. Thus $S \Delta T = \cup_i \{x_i\}$, so $d(S, T) = n$. Clearly, we have $d(S_{i-1}, S_i) = 1$, for all i . It follows that \mathcal{F} is a wg-family.

(Sufficiency.) Let \mathcal{F} be a well-graded family of subsets of some set X . By Lemma 4.1, $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$ is a token system. It is clear that the tokens γ_x and $\tilde{\gamma}_x$ are mutual reverses for any $x \in \cup \mathcal{F} \cap \mathcal{F}$. We need to show that Axioms [M1] and [M2] are satisfied for $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$.

Axiom [M1]. Let S and T be two distinct states in the wg-family \mathcal{F} , and let (S_i) be a sequence of states in \mathcal{F} such that $S_0 = S$, $S_n = T$, $d(S, T) = n$, and $d(S_{i-1}, S_i) = 1$. By the last equation, for any i , there is x_i such that $S_{i-1} \Delta S_i = \{x_i\}$. Suppose that $x_i = x_j$ for some $i < j$. We have

$$(S_{i-1} \Delta S_j) \Delta (S_i \Delta S_{j-1}) = (S_{i-1} \Delta S_i) \Delta (S_{j-1} \Delta S_j) = \{x_i\} \Delta \{x_j\} = \emptyset.$$

Hence, $S_{i-1} \Delta S_j = S_i \Delta S_{j-1}$, so, by Lemma 4.2,

$$j - (i - 1) = d(S_{i-1}, S_j) = d(S_i, S_{j-1}) = (j - 1) - i,$$

a contradiction. Thus, all x_i 's are distinct. Since $S_{i-1} \Delta S_i = \{x_i\}$, we have $S_{i-1}\tau_i = S_i$, where τ_i is either γ_{x_i} or $\tilde{\gamma}_{x_i}$. Clearly, the message $\tau_1 \dots \tau_n$ is concise and produces T from S .

Axiom [M2]. Let $\mathbf{m} = \tau_1 \dots \tau_n$ be a stepwise effective message for a state S which is ineffective for S . As before, (S_i) stands for the sequence of states produced by \mathbf{m} from S , so $S_0 = S_n = S$. Since $S\mathbf{m} = S$, for any occurrence of τ in \mathbf{m} there must be occurrence of $\tilde{\tau}$ in \mathbf{m} . Suppose that we have two consecutive occurrences of a token $\tau = \tau_i = \tau_j = \gamma_x$ in \mathbf{m} . Then $x \in S_i$ and $x \notin S_{j-1}$. Therefore we must have an occurrence of $\tilde{\tau} = \tilde{\gamma}_x$ between this two occurrences of τ . A similar argument shows that there is an occurrence of a token between any two consecutive occurrences of its reverse, so occurrences of token and its reverse alternate in \mathbf{m} . Finally, let τ_i be the first occurrence of τ in \mathbf{m} . We may assume that there are more than one occurrence of τ in \mathbf{m} . The message $\mathbf{n} = \tau_{i+1} \dots \tau_n \tau_1 \dots \tau_i$ is stepwise effective and ineffective for S_i . By the previous argument, occurrences of τ and its reverse alternate in \mathbf{n} . It follows that the number of occurrences of both τ and $\tilde{\tau}$ in \mathbf{m} is even, so \mathbf{m} is vacuous. \square

Theorem 4.1 justifies the following definition.

Definition 4.2. Let \mathcal{F} be a wg-family of subsets of a set X . The medium $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$ is said to be the *representing medium* of \mathcal{F} .

We will show later (Theorem 9.3) that any medium is isomorphic to the representing medium of some wg-family of sets.

The representing medium $(\mathcal{B}(X), \mathcal{G}_{\mathcal{B}(X)})$ of the family $\mathcal{B}(X)$ of all finite subsets of X has a rather special property:

For any state S and any token γ , either γ or $\tilde{\gamma}$ is effective for S .

Any medium satisfying this property is said to be *complete* (cf. [11]).

5 Tokens and messages of media

The two axioms defining a medium are quite strong. We derive a few basic consequences of these axioms. In what follows we assume that a medium $(\mathcal{S}, \mathcal{T})$ is given.

Lemma 5.1. (i) *Any token of a medium has a reverse. In particular, if S is adjacent to V , then S and V are adjacent.*

(ii) *No token can be identical to its own reverse. In particular, a single token τ is a concise message for any state S such that $S\tau \neq S$.*

(iii) *For any two adjacent states, there is exactly one token producing one state from the other.*

Proof. (i) and (ii). Let τ be a token in \mathcal{T} . Since $\tau \neq \tau_0$ (recall that τ_0 stands for the identity transformation of \mathcal{S} and is not a token), there are two distinct states S and V in \mathcal{S} such that $S\tau = V$. By Axiom [M1], there is a concise message \mathbf{m} producing S from V . The message $\tau\mathbf{m}$ is stepwise effective for S and ineffective for that state. By Axiom [M2], this message is vacuous. Hence, the message \mathbf{m} contains a reverse of τ . It follows that there is a reverse of τ in \mathcal{T} . If $\tau = \tilde{\tau}$, then \mathbf{m} contains both τ and $\tilde{\tau}$. This contradicts the assumption that \mathbf{m} is a concise message.

(iii) Suppose that $S\tau_1 = S\tau_2 = V$, so V is adjacent to S . By (i), the message $\tau_1\tilde{\tau}_2$ is stepwise effective and ineffective for S . By Axiom [M2], it is vacuous, that is, $\{\tau_1, \tilde{\tau}_2\}$ is a pair of mutually reverse tokens. Therefore, $\tau_1 = \tilde{\tilde{\tau}}_2 = \tau_2$. \square

Let τ be a token of a medium. We define

$$\mathcal{U}_\tau = \{S \in \mathcal{S} \mid S\tau \neq S\}. \quad (5.1)$$

Note that $\mathcal{U}_\tau \neq \emptyset$, since τ is a token.

Lemma 5.2. *For any given $\tau \in \mathcal{T}$ we have*

(i) $(\mathcal{U}_\tau)\tau = \mathcal{U}_{\tilde{\tau}}$.

(ii) $\mathcal{U}_\tau \cap \mathcal{U}_{\tilde{\tau}} = \emptyset$.

(iii) *The restriction $\tau|_{\mathcal{U}_\tau}$ is a bijection from \mathcal{U}_τ onto $\mathcal{U}_{\tilde{\tau}}$ with $\tau|_{\mathcal{U}_\tau}^{-1} = \tilde{\tau}|_{\mathcal{U}_{\tilde{\tau}}}$.*

(iv) τ is not a one-to-one transformation.

Proof. (i) We have

$$T \in (\mathcal{U}_\tau)\tau \Leftrightarrow S\tau = T \ (S \neq T) \Leftrightarrow T\tilde{\tau} = S \ (S \neq T) \Leftrightarrow T \in \mathcal{U}_{\tilde{\tau}}.$$

(ii) If $S \in \mathcal{U}_\tau \cap \mathcal{U}_{\tilde{\tau}}$, then there exist $T \neq S$ such that $S\tau = T$ and $V \neq S$ such that $S\tilde{\tau} = V$, so $V\tau = S$. If $V = T$, then, by [M2], the message $\tau\tau$ is vacuous, so $\tilde{\tau} = \tau$, which contradicts Lemma 5.1(ii). If $V \neq T$, then, by [M1], there is a concise message \mathbf{m} producing V from T . By [M2], the message $\tau\tau\mathbf{m}$ is vacuous, so we must have two occurrences of $\tilde{\tau}$ in \mathbf{m} a contradiction, since \mathbf{m} is a concise message. It follows that $\mathcal{U}_\tau \cap \mathcal{U}_{\tilde{\tau}} = \emptyset$.

(iii) and (iv) follow immediately from (i) and (ii). \square

Lemma 5.3. *If \mathbf{m} is a concise message for some state S , then \mathbf{m} is effective for that state.*

Proof. If $S\mathbf{m} = S$, then, by Axiom [M2], \mathbf{m} must be vacuous, which contradicts our assumption that \mathbf{m} is a concise message. \square

Lemma 5.4. *A vacuous message \mathbf{m} which is stepwise effective for a state S is ineffective for S .*

Proof. Suppose that $T = S\mathbf{m} \neq S$, and let \mathbf{n} be a concise message producing S from T . By Axiom [M2], the message $\mathbf{m}\mathbf{n}$ is vacuous, so \mathbf{n} must contain a pair of mutually reverse tokens, a contradiction. Hence, $S\mathbf{m} = S$. \square

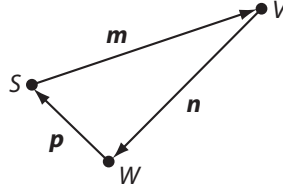


Figure 5.1: Diagram for Lemma 5.5.

Lemma 5.5. *Let S , V , and W be three states of the medium $(\mathcal{S}, \mathcal{T})$ and suppose that $V = S\mathbf{m}$, $W = V\mathbf{n}$ for some concise messages \mathbf{m} and \mathbf{n} , and $S = W\mathbf{p}$ where \mathbf{p} is either a concise message or empty (see the diagram in Figure 5.1). There is at most one occurrence of each pair of mutually reverse tokens in the closed message $\mathbf{m}\mathbf{n}\mathbf{p}$.*

Proof. Let τ be a token in $\mathcal{C}(\mathbf{m})$. Since \mathbf{m} is a concise message, there is only one occurrence of τ in \mathbf{m} and $\tilde{\tau} \notin \mathcal{C}(\mathbf{m})$. By Axiom [M2], the message $\mathbf{m}\mathbf{n}\mathbf{p}$ is vacuous, so we must have $\tilde{\tau} \in \mathcal{C}(\mathbf{n}) \cup \mathcal{C}(\mathbf{p})$. Suppose that $\tilde{\tau} \in \mathcal{C}(\mathbf{n})$ (the case when $\tilde{\tau} \in \mathcal{C}(\mathbf{p}) \neq \emptyset$ is treated similarly). Since \mathbf{n} is a concise message, there are no more occurrences of $\tilde{\tau}$ in \mathbf{n} and $\tau \notin \mathcal{C}(\mathbf{n})$. Thus there is only one occurrence of the pair $\{\tau, \tilde{\tau}\}$ in the message $\mathbf{m}\mathbf{n}$. The pair $\{\tau, \tilde{\tau}\}$ cannot occur in \mathbf{p} , since \mathbf{p} is a concise message or empty. The result follows. \square

Corollary 5.1. *Let \mathbf{m} and \mathbf{n} be two concise messages producing V from S . Then the string \mathbf{n} is a permutation of the string \mathbf{m} . In particular, $\ell(\mathbf{m}) = \ell(\mathbf{n})$.*

One can say more in the special case when $\mathbf{p} = \tau$ is a single token.

Lemma 5.6. *Let S, V and W be distinct states of a medium and suppose that*

$$V = S\mathbf{m}, \quad W = V\mathbf{n}, \quad S = W\tau$$

for some concise messages \mathbf{m} and \mathbf{n} and a token τ (see Figure 5.2). Then

$$\tau \notin \mathcal{C}(\mathbf{n}), \quad \tau \notin \mathcal{C}(\mathbf{m}),$$

and either

$$\tilde{\tau} \in \mathcal{C}(\mathbf{m}), \quad \mathbf{n}\tau \text{ is a concise message,} \quad \mathcal{C}(\mathbf{n}\tau) = \mathcal{C}(\tilde{\mathbf{m}}), \quad \ell(\mathbf{m}) = \ell(\mathbf{n}) + 1,$$

or

$$\tilde{\tau} \in \mathcal{C}(\mathbf{n}), \quad \tau\mathbf{m} \text{ is a concise message,} \quad \mathcal{C}(\tau\mathbf{m}) = \mathcal{C}(\tilde{\mathbf{n}}), \quad \ell(\mathbf{n}) = \ell(\mathbf{m}) + 1.$$

Accordingly,

$$|\ell(\mathbf{m}) - \ell(\mathbf{n})| = 1. \tag{5.2}$$

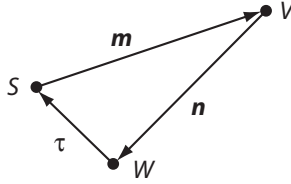


Figure 5.2: For Lemma 5.6.

Proof. By Lemma 5.5, $\tau \notin \mathcal{C}(\mathbf{n})$, $\tau \notin \mathcal{C}(\mathbf{m})$, and $\tilde{\tau}$ occurs either in \mathbf{m} or in \mathbf{n} . Suppose that $\tilde{\tau} \in \mathcal{C}(\mathbf{m})$. By the same lemma, neither τ nor $\tilde{\tau}$ occurs in \mathbf{n} . Therefore, $\mathbf{n}\tau$ is a concise message. The equality $\mathcal{C}(\mathbf{n}\tau) = \mathcal{C}(\tilde{\mathbf{m}})$ also follows from Lemma 5.5. Since \mathbf{m} is a concise message, we have

$$\ell(\mathbf{m}) = |\mathcal{C}(\mathbf{m})| = |\mathcal{C}(\tilde{\mathbf{m}})| = |\mathcal{C}(\mathbf{n}\tau)| = \ell(\mathbf{n}) + 1.$$

The case when $\tilde{\tau} \in \mathcal{C}(\mathbf{n})$ is treated similarly. □

The results of Lemma 5.5 suggest an interpretation of the length function on messages. First, by Corollary 5.1, we have $\ell(\mathbf{m}) = \ell(\mathbf{n})$ for any two concise messages \mathbf{m} and \mathbf{n} producing a state V from a state S . Therefore the function

$$\delta(S, V) = \begin{cases} \ell(\mathbf{m}), & \text{if } S\mathbf{m} = V, \\ 0, & \text{otherwise,} \end{cases} \tag{5.3}$$

where \mathbf{m} is a concise message, is well-defined. Since $\ell(\tilde{\mathbf{m}}) = \ell(\mathbf{m})$, the function δ is symmetric.

Second, for the messages in Figure 5.1 we have

$$|\mathcal{C}(\mathbf{p})| \leq |\mathcal{C}(\mathbf{n})| + |\mathcal{C}(\mathbf{m})|,$$

by Lemma 5.5. Indeed, for any token in $\mathcal{C}(\mathbf{p})$ we have a unique matching reverse either in $\mathcal{C}(\mathbf{m})$ or in $\mathcal{C}(\mathbf{n})$. Since the length of a concise message equals the cardinality of its content, we have the triangle inequality

$$\delta(S, W) \leq \delta(S, V) + \delta(V, W).$$

It is easy to verify that this inequality holds for any choice of states $S, V, W \in \mathcal{S}$. We obtained the following result.

Theorem 5.1. *The function $\delta(S, V)$ defined by (5.3) is a metric on \mathcal{S} .*

6 The graph of a medium

Definition 6.1. The *graph* of a medium $(\mathcal{S}, \mathcal{T})$ has \mathcal{S} as the set of vertices; two vertices of the graph are adjacent if and only if the corresponding states are adjacent in the medium.

By Lemma 5.1, for any two adjacent states S and T of a medium $(\mathcal{S}, \mathcal{T})$ there is a unique token τ such that $S\tau = T$ and $T\tilde{\tau} = S$. Thus, a unique pair of mutually reversed tokens $\{\tau, \tilde{\tau}\}$ is assigned to each edge $\{S, V\}$ of the graph of $(\mathcal{S}, \mathcal{T})$.

Let $(\mathcal{S}, \mathcal{T})$ be a medium and G be its graph. If $\mathbf{m} = \tau_1 \dots \tau_m$ is a stepwise effective message for a state S producing a state V , then the sequence of vertices (S_i) of G produced by \mathbf{m} , is a walk in G ; the vertex $S_0 = S$ is a tail of this walk and the vertex $S_m = V$ is its head. On the other hand, if the sequence of vertices $S_0 = S, S_1, \dots, S_m = V$ is a walk in G , then edges $\{S_{i-1}, S_i\}$ define unique tokens τ_i such that $S_{i-1}\tau_i = S_i$. Then $\mathbf{m} = \tau_1 \dots \tau_m$ is a stepwise effective message for the state S producing the state V . Thus we have a one-to-one correspondence between stepwise effective messages of the medium and walks in its graph. In particular, a closed message for some state produces a closed walk in G .

A deeper connection between media and their graphs is the result of the following theorem.

Theorem 6.1. *Let $(\mathcal{S}, \mathcal{T})$ be a medium and G be its graph. If $\mathbf{m} = \tau_1 \dots \tau_m$ is a concise message producing a state V from a state S , then the sequence of vertices (S_i) produced by \mathbf{m} forms a shortest path connecting S and V in the graph G . Conversely, if $S_0 = S, S_1, \dots, S_m = V$ is a shortest path in G , then the corresponding message is a concise message of G .*

Proof. (Necessity.) Let $P_0 = S, P_1, \dots, P_n = V$ be a path in G joining S to V and $\mathbf{n} = \mu_1 \dots \mu_n$ be the (stepwise effective) message of the medium corresponding to this path. By Axiom [M2], the message $\mathbf{m}\tilde{\mathbf{n}}$ is vacuous, so $\ell(\mathbf{m}) \leq \ell(\tilde{\mathbf{n}}) = \ell(\mathbf{n})$, since \mathbf{m} is a concise message for S . Thus the sequence (S_i) is a shortest path in G .

(Sufficiency.) Let $S_0 = S, S_1, \dots, S_m = V$ be a shortest path in G and let $\mathbf{m} = \tau_1 \dots \tau_m$ be the corresponding stepwise effective message of the medium. By Axiom

[M1], there is a concise message \mathbf{n} producing V from S . By the necessity part of the proof, the walk defined by \mathbf{n} is a shortest path from S to V , so $\ell(\mathbf{n}) = \ell(\mathbf{m})$. By Axiom [M2], the message $\mathbf{m}\tilde{\mathbf{n}}$ must be vacuous. Since the message \mathbf{n} is concise and $\ell(\mathbf{n}) = \ell(\mathbf{m})$, the message \mathbf{m} must be concise. \square

Let G be the graph of a medium. By Axiom [M1], G is connected. Let $S_0 = S, S_1, \dots, S_n = S$ be a closed walk. By Axiom [M2], the corresponding message of the medium is vacuous. Therefore it must be of even length. It follows that the graph of a medium is bipartite. Note that not every connected bipartite graph is the graph of some medium.

Example 6.1. The simplest counterexample is the complete bipartite graph $K_{2,3}$ shown in Figure 6.1. Suppose that this graph is the graph of a medium and let τ be a token producing T from S . By Axiom [M2], the closed message producing the sequence of states (S, T, P, V, S) must be vacuous and therefore must contain an occurrence of $\tilde{\tau}$. We cannot have $T\tilde{\tau} = P$ or $V\tilde{\tau} = S$, since tokens are functions. Therefore, $P\tilde{\tau} = V$, so $V\tau = P$. The same argument applied to the closed message producing the sequence (V, P, Q, S, V) shows that $S\tau = Q$. Thus $S\tau = T$ and $S\tau = Q$, a contradiction.

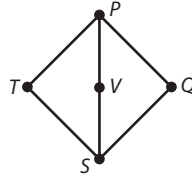


Figure 6.1: Complete bipartite graph $K_{2,3}$.

It follows from Theorem 6.1 that the metric δ on the set of states of a medium is the graph distance on the graph of that medium.

7 Contents

Definition 7.1. Let $(\mathcal{S}, \mathcal{T})$ be a medium. For any state S , the *content* of S is the set \hat{S} of all tokens each of which is contained in at least one concise message producing S . The family $\hat{\mathcal{S}} = \{\hat{S} \mid S \in \mathcal{S}\}$ is called the *content family* of \mathcal{S} .

Lemma 7.1. *The content of a state cannot contain both a token and its reverse.*

Proof. Suppose that $S\mathbf{m} = W\mathbf{n} = V$ for two concise messages \mathbf{m} and \mathbf{n} and let \mathbf{p} be a concise message producing S from W , if $W \neq S$, and empty, if $W = S$. By Lemma 5.5, there is at most one occurrence of any token τ in the message $\mathbf{m}\tilde{\mathbf{n}}\mathbf{p}$. Therefore we cannot have both $\tau \in \mathcal{C}(\mathbf{m})$ and $\tilde{\tau} \in \mathcal{C}(\mathbf{n})$. \square

Theorem 7.1. *For any token τ and any state S , we have either $\tau \in \hat{S}$ or $\tilde{\tau} \in \hat{S}$. Consequently, $|\hat{S}| = |\hat{V}|$ for any two states S and V .*

Proof. Since τ is a token, there are two states V and W such that $W = V\tau$. By Axiom [M1], there are concise messages \mathbf{m} and \mathbf{n} such that $S = V\mathbf{m}$ and $S = W\mathbf{n}$. By Lemma 5.6, there are two mutually exclusive options: either $\tilde{\tau} \in \mathcal{C}(\mathbf{n})$ or $\tau \in \mathcal{C}(\mathbf{m})$. \square

Theorem 7.2. *If S and V are two distinct states, with $S\mathbf{m} = V$ for some concise message \mathbf{m} , then $\widehat{V} \setminus \widehat{S} = \mathcal{C}(\mathbf{m})$.*

Proof. Let τ be a token in $\mathcal{C}(\mathbf{m})$, so $\tilde{\tau} \in \mathcal{C}(\widetilde{\mathbf{m}})$. Thus, $\tau \in \widehat{V}$ and $\tilde{\tau} \in \widehat{S}$. By Theorem 7.1, $\tau \notin \widehat{S}$. It follows that $\tau \in \widehat{V} \setminus \widehat{S}$, that is, $\mathcal{C}(\mathbf{m}) \subseteq \widehat{V} \setminus \widehat{S}$.

If $\tau \in \widehat{V} \setminus \widehat{S}$, then $\tau \in \widehat{V}$ and $\tau \notin \widehat{S}$, so, by Theorem 7.1, $\tilde{\tau} \in \widehat{S}$. Since $\tau \in \widehat{V}$, there is a concise message \mathbf{n} producing the state V from some state W such that $\tau \in \mathcal{C}(\mathbf{n})$, so $\tilde{\tau} \in \mathcal{C}(\widetilde{\mathbf{n}})$. Let \mathbf{p} be a concise message producing S from W (or empty if $S = W$). By Lemma 5.5, there is exactly one occurrence of the pair $\{\tau, \tilde{\tau}\}$ in the message $\mathbf{m}\widetilde{\mathbf{n}}\mathbf{p}$. Since $\tilde{\tau} \in \widehat{S}$, we have $\tau \notin \mathcal{C}(\mathbf{p})$. Hence, $\tau \in \mathcal{C}(\mathbf{m})$. In both cases we have $\widehat{V} \setminus \widehat{S} \subseteq \mathcal{C}(\mathbf{m})$. The result follows. \square

Theorem 7.3. *For any two states S and V we have*

$$S = V \quad \Leftrightarrow \quad \widehat{S} = \widehat{V}.$$

Proof. Suppose that $\widehat{S} = \widehat{V}$, $S \neq V$, and let \mathbf{m} be a concise message producing V from S . By Theorem 7.2,

$$\emptyset = \widehat{V} \setminus \widehat{S} = \mathcal{C}(\mathbf{m}),$$

a contradiction. Thus, $\widehat{S} = \widehat{V} \Rightarrow S = V$. The implication $S = V \Rightarrow \widehat{S} = \widehat{V}$ is trivial. \square

Theorem 7.4. *Let \mathbf{m} and \mathbf{n} be two concise messages transforming some state S . Then $S\mathbf{m} = S\mathbf{n}$ if and only if $\mathcal{C}(\mathbf{m}) = \mathcal{C}(\mathbf{n})$.*

Proof. (Necessity.) Suppose that $V = S\mathbf{m} = S\mathbf{n}$. By Theorem 7.2,

$$\mathcal{C}(\mathbf{m}) = \widehat{V} \setminus \widehat{S} = \mathcal{C}(\mathbf{n}).$$

(Sufficiency.) Suppose that $\mathcal{C}(\mathbf{m}) = \mathcal{C}(\mathbf{n})$ and let $V = S\mathbf{m}$ and $W = S\mathbf{n}$. By Theorem 7.2,

$$\widehat{V} \Delta \widehat{S} = \mathcal{C}(\mathbf{m}) \cup \mathcal{C}(\widetilde{\mathbf{m}}) = \mathcal{C}(\mathbf{n}) \cup \mathcal{C}(\widetilde{\mathbf{n}}) = \widehat{W} \Delta \widehat{S},$$

which implies $\widehat{V} = \widehat{W}$. By Theorem 7.3, $V = W$. \square

We will need the result of the following lemma in the proof of Theorem 9.3.

Lemma 7.2. *Let $(\mathcal{S}, \mathcal{T})$ be a medium, and suppose that: $\tau \in \mathcal{T}$ is a token; S, T, P and Q are states in \mathcal{S} such that $S\tau = T$ and $P\tau = Q$. Let \mathbf{m} and \mathbf{n} be two concise messages producing P from S and Q from T , respectively (see Figure 7.1). Then \mathbf{m} and \mathbf{n} have equal contents and lengths, that is $\mathcal{C}(\mathbf{m}) = \mathcal{C}(\mathbf{n})$ and $\ell(\mathbf{m}) = \ell(\mathbf{n})$, and $\mathbf{m}\tau$ and $\tau\mathbf{n}$ are concise messages for S .*

Proof. The message $m\tau\tilde{n}\tilde{\tau}$ is stepwise effective for S and ineffective for that state. By Axiom [M2], this message is vacuous. Hence, $\mathcal{C}(m) = \mathcal{C}(n)$ and $\ell(m) = \ell(n)$. The messages m and $\tilde{\tau}$ produce P . By Theorem 7.2, $\mathcal{C}(m) \subseteq \widehat{P}$ and $\tilde{\tau} \in \widehat{P}$. By Theorem 7.1, $\tilde{\tau} \notin \mathcal{C}(m) = \mathcal{C}(n)$. The same argument applied to the state Q shows that $\tau \notin \mathcal{C}(m) = \mathcal{C}(n)$. It follows that $m\tau$ and τn are concise messages for S . \square

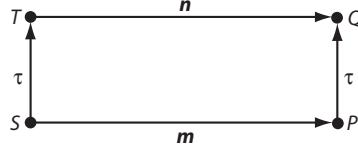


Figure 7.1: For Lemma 7.2.

8 Embeddings and isomorphisms

The purpose of combinatorial media theory is to find and examine those properties of media that do not depend on a particular structure of individual states and tokens. For this purpose we introduce the concepts of embedding and isomorphism for token systems.

Definition 8.1. Let $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{S}', \mathcal{T}')$ be two token systems. A pair (α, β) of one-to-one functions $\alpha : \mathcal{S} \rightarrow \mathcal{S}'$ and $\beta : \mathcal{T} \rightarrow \mathcal{T}'$ such that

$$S\tau = T \quad \Leftrightarrow \quad \alpha(S)\beta(\tau) = \alpha(T)$$

for all $S, T \in \mathcal{S}$, $\tau \in \mathcal{T}$ is called an *embedding* of the token system $(\mathcal{S}, \mathcal{T})$ into the token system $(\mathcal{S}', \mathcal{T}')$.

Token systems $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{S}', \mathcal{T}')$ are *isomorphic* if there is an embedding (α, β) from $(\mathcal{S}, \mathcal{T})$ into $(\mathcal{S}', \mathcal{T}')$ such that both α and β are bijections.

Clearly, if one of two isomorphic token systems is a medium, then the other one is also a medium.

If a token system $(\mathcal{S}, \mathcal{T})$ is a medium and $S\tau_1 = S\tau_2 \neq S$ for some state S , then, by Lemma 5.1(iii), $\tau_1 = \tau_2$. In particular, if (α, β) is an embedding of a medium into a medium, then $\beta(\tilde{\tau}) = \widetilde{\beta(\tau)}$. Indeed, we have

$$\begin{aligned} \alpha(S)\beta(\tilde{\tau}) = \alpha(T) &\Leftrightarrow S\tilde{\tau} = T \Leftrightarrow T\tau = S \Leftrightarrow \\ &\Leftrightarrow \alpha(T)\beta(\tau) = \alpha(S) \Leftrightarrow \alpha(S)\widetilde{\beta(\tau)} = \alpha(T), \end{aligned}$$

for $S \neq T$. We extend β to the semigroup of messages by defining

$$\beta(\tau_1 \dots \tau_k) = \beta(\tau_1) \dots \beta(\tau_k).$$

Clearly, the image $\beta(m)$ of a concise message m for a state S is a concise message for the state $\alpha(S)$.

Let $(\mathcal{S}, \mathcal{T})$ be a token system and \mathcal{Q} be a subset of \mathcal{S} consisting of more than two elements. The restriction of a token $\tau \in \mathcal{T}$ to \mathcal{Q} is not necessarily a token on \mathcal{Q} . In order to construct a medium with the set of states \mathcal{Q} , we introduce the following concept.

Definition 8.2. Let $(\mathcal{S}, \mathcal{T})$ be a token system, \mathcal{Q} be a nonempty subset of \mathcal{S} , and $\tau \in \mathcal{T}$. We define a *reduction* of τ to \mathcal{Q} by

$$S_{\tau_{\mathcal{Q}}} = \begin{cases} S_{\tau} & \text{if } S_{\tau} \in \mathcal{Q}, \\ S & \text{if } S_{\tau} \notin \mathcal{Q}, \end{cases}$$

for $S \in \mathcal{Q}$. A token system $(\mathcal{Q}, \mathcal{T}_{\mathcal{Q}})$ where $\mathcal{T}_{\mathcal{Q}} = \{\tau_{\mathcal{Q}}\}_{\tau \in \mathcal{T}} \setminus \{\tau_0\}$ is the set of all distinct reductions of tokens in \mathcal{T} to \mathcal{Q} different from the identity function τ_0 on \mathcal{Q} , is said to be the *reduction* of $(\mathcal{S}, \mathcal{T})$ to \mathcal{Q} .

We call $(\mathcal{Q}, \mathcal{T}_{\mathcal{Q}})$ a *token subsystem* of $(\mathcal{S}, \mathcal{T})$. If both $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{Q}, \mathcal{T}_{\mathcal{Q}})$ are media, we call $(\mathcal{Q}, \mathcal{T}_{\mathcal{Q}})$ a *submedium* of $(\mathcal{S}, \mathcal{T})$.

Remark 8.1. A reduction of a medium is not necessarily a submedium of a given medium. Consider, for instance, the medium shown in Figure 8.1. The set of tokens of the reduction of this medium to $\mathcal{Q} = \{P, R\}$ is empty. Thus this reduction is not a medium.

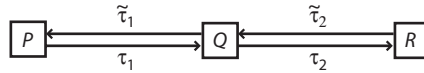


Figure 8.1: The reduction of this medium to $\{P, R\}$ is not a submedium.

We conclude this section with an example of a submedium.

Example 8.1. Let \mathcal{F} be a wg-family of finite subsets of a set X . The representing medium $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$ of \mathcal{F} is clearly the reduction of the complete medium $(\mathcal{B}(X), \mathcal{G}_{\mathcal{B}(X)})$ to \mathcal{F} . Thus, $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$ is a submedium of $(\mathcal{B}(X), \mathcal{G}_{\mathcal{B}(X)})$ for any wg-family \mathcal{F} .

9 Media and partial cubes

Definition 9.1. The *cube* $\mathcal{H}(X)$ on a set X , has the set $\mathcal{B}(X)$ of all finite subsets of X as the set of vertices; $\{S, T\}$ is an edge of $\mathcal{H}(X)$ if $|S \Delta T| = 1$. A *partial cube* is a graph that is isometrically embeddable into some cube $\mathcal{H}(X)$ (cf. [12]).

The following theorem characterizes partial cubes. For other characterizations see [1, 18, 12].

Theorem 9.1. A graph $G = (V, E)$ is a partial cube if and only if it is possible to label its edges by elements of some set J such that

- (i) Edges of any shortest path of G are of different labels.

(ii) In each closed walk of G every label appears an even number of times.

Proof. (Necessity.) Without loss of generality, we may assume that $G = (\mathcal{F}, \mathcal{E})$ is an isometric subgraph of a cube $\mathcal{H}(J)$ such that $\cap \mathcal{F} = \emptyset$ and $\cup \mathcal{F} = J$ for a wg-family \mathcal{F} . For any edge $\{S, T\}$ of G there is an element $j \in J$ such that $S \triangle T = \{j\}$, so we can label edges of G by elements of J .

(i) Let $S_0 = S, S_1, \dots, S_n = T$ be a shortest path from S to T in G . For every i , we have $S \cap T \subseteq S_i \subseteq S \cup T$. Therefore,

$$\{j_i\} = S_{i-1} \triangle S_i \subseteq S \triangle T.$$

Since (S_i) is a shortest path, $|S \triangle T| = d(S, T) = n$. It follows that all labels j_i are distinct.

(ii) Let $S_0, S_1, \dots, S_n = S_0$ be a closed walk W in G and let $E_p = \{S_{p-1}, S_p\}$ be the first edge in W labeled by j , so $S_{p-1} \triangle S_p = \{j\}$. We assume that $j \notin S_{p-1}$ and $j \in S_p$; the other case is treated similarly. Since E_p is the first edge of W labeled by j , we must have $j \notin S_0$. Since the walk W is closed and $j \in S_p$, we must have another occurrence of j in W . Let $E_q = \{S_{q-1}, S_q\}$ be the next edge of W labeled by j . We have $j \in S_{q-1}$ and $j \notin S_q$. By repeating this argument, we partition the occurrences of j in W into pairs, so the total number of these occurrences must be even.

(Sufficiency.) Let S_0 be a fixed vertex of G . For any vertex $S \in V$ and a shortest path p from S_0 to S , we define

$$J_S = \{j \in J \mid j \text{ is a label of an edge of } p\},$$

and $J_{S_0} = \emptyset$. The set J_S is well-defined. Indeed, let q be another shortest path from S_0 to S and \tilde{q} be its reverse, so $p\tilde{q}$ is a closed walk. By (i) and (ii), J_S does not depend on the choice of p .

The correspondence $\alpha : S \mapsto J_S$ defines an isometric embedding of G into the cube $\mathcal{H}(J)$. Indeed, for $S, T \in V$, let p (resp. q) be a shortest path from S_0 to S (resp. T) and let r be a shortest path from S to T . By (ii) applied to the closed walk $pr\tilde{q}$ and (i), we have

$$j \in J_S \triangle J_T \iff j \text{ is a label of an edge of } r,$$

so $\delta(S, T) = |J_S \triangle J_T| = d(J_S, J_T)$. □

Theorem 9.2. *The graph G of a medium $(\mathcal{S}, \mathcal{T})$ is a partial cube.*

Proof. The edges of G are labeled by elements of the set $J = \{\{\tau, \tilde{\tau}\}\}_{\tau \in \mathcal{T}}$. Since the shortest paths of G correspond to the concise messages of $(\mathcal{S}, \mathcal{T})$, condition (i) of Theorem 9.1 is satisfied. A closed walk W in G defines a closed message \mathbf{m} for a vertex of W . By Axiom [M2], the message \mathbf{m} is vacuous. Thus every label appears an even number of times in the walk W . The result follows from Theorem 9.1. □

Let $(\mathcal{S}, \mathcal{T})$ be a medium and G be its graph. By Theorem 9.2, G is a partial cube, so there is an isometric embedding α of G into a cube $\mathcal{H}(X)$ for some set X .

The set $\alpha(\mathcal{S})$ is a wg-family \mathcal{F} of finite subsets of X . Let $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$ be the representing medium of this wg-family. These objects are schematically shown in the diagram below, where $\langle \mathcal{F} \rangle$ is an isometric subgraph of $\mathcal{H}(X)$ induced by the family \mathcal{F} .

$$(\mathcal{S}, \mathcal{T}) \xrightarrow[\text{of } (\mathcal{S}, \mathcal{T})]{\text{graph}} G \xrightarrow{\alpha} \langle \mathcal{F} \rangle \xrightarrow[\text{medium}]{\text{representing}} (\mathcal{F}, \mathcal{G}_{\mathcal{F}}) \quad (9.1)$$

Theorem 9.3. *The media $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$ are isomorphic.*

Proof. Let τ be a token in \mathcal{T} and S and T be two distinct states in \mathcal{S} such that $S\tau = T$. Then either $\alpha(T) = \alpha(S) \cup \{x\}$ for some $x \notin \alpha(S)$ or $\alpha(T) = \alpha(S) \setminus \{x\}$ for some $x \in \alpha(S)$. We define $\beta : \mathcal{T} \rightarrow \mathcal{G}_{\mathcal{F}}$ by

$$\beta(\tau) = \begin{cases} \gamma_x, & \text{if } \alpha(T) = \alpha(S) \cup \{x\} \text{ for some } x \notin \alpha(S), \\ \tilde{\gamma}_x, & \text{if } \alpha(T) = \alpha(S) \setminus \{x\} \text{ for some } x \in \alpha(S). \end{cases}$$

Let us show that β does not depend on the choice of S and T . We consider only the case when $\beta(\tau) = \tau_x$. The other case is treated similarly.

Let P, Q be another pair of distinct states in \mathcal{S} such that $P\tau = Q$, and let $P = S\mathbf{m}$ and $Q = T\mathbf{n}$ for some concise messages \mathbf{m} and \mathbf{n} . By Lemma 7.2, $\ell(\mathbf{m}) = \ell(\mathbf{n})$. Then, by Theorem 6.1, $d(\alpha(S), \alpha(P)) = d(\alpha(T), \alpha(Q))$, and, by Lemma 7.2,

$$\begin{aligned} d(\alpha(S), \alpha(Q)) &= d(\alpha(S), \alpha(T)) + d(\alpha(T), \alpha(Q)), \\ d(\alpha(T), \alpha(P)) &= d(\alpha(T), \alpha(S)) + d(\alpha(S), \alpha(P)), \end{aligned}$$

implying

$$\begin{aligned} \alpha(S) \cap \alpha(Q) &\subseteq \alpha(T) = \alpha(S) \cup \{x\} \subseteq \alpha(S) \cup \alpha(Q), \\ \alpha(T) \cap \alpha(P) &= [\alpha(S) \cup \{x\}] \cap \alpha(P) \subseteq \alpha(S) \subseteq \alpha(S) \cup \alpha(P). \end{aligned}$$

Since $x \notin \alpha(S)$, it follows that $x \in \alpha(Q)$ and $x \notin \alpha(P)$. Then

$$\alpha(Q) = \alpha(P) \cup \{x\},$$

since $d(\alpha(P), \alpha(Q)) = 1$. Hence, the mapping $\beta : \mathcal{S} \rightarrow \mathcal{G}_{\mathcal{F}}$ is well defined.

Clearly, β is a bijection satisfying the condition

$$S\tau = T \quad \Leftrightarrow \quad \alpha(S)\beta(\tau) = \alpha(T).$$

Therefore (α, β) is an isomorphism from $(\mathcal{S}, \mathcal{T})$ onto $(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$. □

References

- [1] D.Ž. Djoković, Distance preserving subgraphs of hypercubes, *J. Combin. Theory Ser. B* 14, (1973) 263–267.
- [2] J.-P. Doignon and J.-Cl. Falmagne, Well-graded families of relations, *Discrete Math.* 173 (1997) 35–44.

- [3] J.-P. Doignon and J.-Cl. Falmagne, *Knowledge Spaces* (Springer-Verlag, Berlin, Heidelberg, New York, 1999).
- [4] D. Eppstein, Algorithms for drawing media, in: *Lecture Notes in Computer Science*, v. 3383, Springer-Verlag, 2005, 173–183.
- [5] D. Eppstein and J.-Cl. Falmagne, Algorithms for media, *Discrete Applied Mathematics* (in review), e-print: arxiv.org/abs/cs.DS/0206033.
- [6] D. Eppstein, J.-Cl. Falmagne, and S. Ovchinnikov, *Media Theory*, Springer, 2007.
- [7] J.-Cl. Falmagne, A stochastic theory for the emergence and the evolution of preference structures, *Math. Social Sci.* 31 (1996) 63–84.
- [8] J.-Cl. Falmagne, Stochastic Token Theory, *J. Math. Psych.* 41(2) (1997) 129–143.
- [9] J.-Cl. Falmagne and J.-P. Doignon, Stochastic evolution of rationality, *Theory and Decision* 43 (1997) 103–138.
- [10] J.-C. Falmagne, M. Regenwetter and B. Grofman, A stochastic model for the evolution of preferences, in: A.A.J. Marley, ed., *Choice, Decision, and Measurement: Essays in Honor of R. Duncan Luce* (Lawrence Erlbaum Associates, NJ, 1997) 111–129.
- [11] J.-Cl. Falmagne and S. Ovchinnikov, Media theory, *Discrete Appl. Math.* 121 (2002) 103–118.
- [12] W. Imrich and S. Klavžar, *Product Graphs*, John Wiley & Sons, 2000.
- [13] S. Ovchinnikov, Hyperplane arrangements in preference modeling, *J. Math. Psych.* 49 (2005) 481–488.
- [14] S. Ovchinnikov, Media theory: representations and examples, *Discrete Applied Mathematics* (to appear), doi:10.1016/j.dam.2007.05.022.
- [15] S. Ovchinnikov, Partial cubes: structures, characterizations, and constructions, *Discrete Mathematics* (in review).
- [16] S. Ovchinnikov and A. Dukhovny, Advances in media theory, *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems* 8(1) (2000) 45–71.
- [17] M. Regenwetter, J.-Cl. Falmagne, and B. Grofman, A stochastic model of preference change and its application to 1992 presidential election panel data, *Psych. Rev* 106 (2) (1999) 362–384.
- [18] P.M. Winkler, Isometric embeddings in products of complete graphs, *Discrete Applied Mathematics* 7 (1984) 221–225.