Algebraic characterizations of bipartite distance-regular graphs

Miquel Àngel Fiol (invited speaker)
Universitat Politècnica de Catalunya
Barcelona

Abstract

Bipartite graphs are combinatorial objects bearing some interesting symmetries. Thus, their spectra—eigenvalues of its adjacency matrix—are symmetric about zero, as the corresponding eigenvectors come into pairs. Moreover, vertices in the same (respectively, different) independent set are always at even (respectively, odd) distance. Both properties have well-known consequences in most properties and parameters of such graphs. Roughly speaking, we could say that the conditions for a given property to hold in a general graph can be somehow relaxed to guaranty the same property for a bipartite graph. In this paper we comment upon this phenomenon in the framework of distance-regular graphs for which several characterizations, both of combinatorial or algebraic nature, are known. Thus, the presented characterizations of bipartite distance-regular graphs involve such parameters as the numbers of walks between vertices (entries of the powers of the adjacency matrix $\boldsymbol{A}$), the crossed local multiplicities (entries of the idempotents $\boldsymbol{E}_i$ or eigenprojectors), the predistance polynomials, etc. For instance, it is known that a graph G, with eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ and diameter $D = d$, is distance-regular if and only if its idempotents $\boldsymbol{E}_1$ and $\boldsymbol{E}_d$ belong to the vector space $\mathcal{D}$ spanned by its distance matrices $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}_2, \ldots \boldsymbol{A}_d$. In contrast with this, for the same result to be true in the case of bipartite graphs, only $\boldsymbol{E}_1 \in \mathcal{D}$ need to be required.

1 Preliminaries

Let $G = (V, A)$ be a (simple and connected) graph with adjacency matrix $\boldsymbol{A}$, and spectrum

$$\operatorname{sp} G = \operatorname{sp} \boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}, \tag{1}$$

where the different eigenvalues of G are in decreasing order, $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$. In particular, note that when G is δ-regular, the largest eigenvalue is $\lambda_0 = \delta$ and has multiplicity $m_0 = 1$ (as G is connected). Moreover, all the multiplicities add up to $n = |V|$, the number of vertices of G.

Recall also that G is bipartite if and only if it does not contain odd cycles. Then, its adjacency matrix is of the form

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{O} \end{pmatrix}.$$

(Here and hereafter, it is assumed that the block matrices have the appropriate dimensions.) Moreover, for any polynomial $p \in \mathbb{R}_d[x]$ with even and odd parts p_0 and p_1, we have

$$p(\boldsymbol{A}) = p_0(\boldsymbol{A}) + p_1(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D} \end{pmatrix} + \begin{pmatrix} \boldsymbol{O} & \boldsymbol{M} \\ \boldsymbol{M}^\top & \boldsymbol{O} \end{pmatrix}. \tag{2}$$

Also, the spectrum of G is symmetric about zero: $\lambda_i = -\lambda_{d-i}$ and $m_i = m_{d-i}$, $i = 0, 1, \dots, d$. (In fact, a well-known result states that a connected graph G is bipartite if and only if $\lambda_0 = -\lambda_d$; see, for instance, Cvetković et al. [6].) This is due to the fact that, if $(u|v)$ is an eigenvector with eigenvalue λ_i, then $(u|-v)$ is an eigenvector for the eigenvalue $-\lambda_i$. As shown below, a similar symmetry also applies to the entries of the (*principal*) *idempotents* $\boldsymbol{E}_i$ representing the projections onto the eigenspaces $\mathcal{E}_i$, $i = 0, 1, \dots, d$. To see this, first recall that, for any graph with eigenvalue λ_i having multiplicity m_i, its corresponding idempotent can be computed as $\boldsymbol{E}_i = \boldsymbol{U}_i \boldsymbol{U}_i^\top$, where $\boldsymbol{U}_i$ is the $n \times m_i$ matrix whose columns form an orthonormal basis of $\mathcal{E}_i$. For instance, when G is δ-regular and has n vertices, its largest eigenvalue $\lambda_0 = \delta$ has eigenvector $\boldsymbol{j}$, the all-1 vector, and corresponding idempotent $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{j}\boldsymbol{j}^\top = \frac{1}{n}\boldsymbol{J}$, where $\boldsymbol{J}$ is the all-1 matrix. Alternatively, we can also compute the idempotents as $\boldsymbol{E}_i = \lambda_i^*(\boldsymbol{A})$ where λ_i^* is the Lagrange interpolating polynomial of degree d satisfying $\lambda_i^*(\lambda_j) = \delta_{ij}$. That is,

$$\lambda_i^* = \frac{1}{\phi_i}\prod_{\substack{j=0\\ j\neq i}}^{d}(x-\lambda_j) = \frac{(-1)^i}{\pi_i}\prod_{\substack{j=0\\ j\neq i}}^{d}(x-\lambda_j)$$

where $\phi_i = \prod_{j=0,j\neq i}^{d}(\lambda_i-\lambda_j)$ and $\pi_i = |\phi_i|$. Then, the idempotents of $\boldsymbol{A}$ satisfy the known properties: $\boldsymbol{E}_i\boldsymbol{E}_j = \delta_{ij}\boldsymbol{E}_i$; $\boldsymbol{A}\boldsymbol{E}_i = \lambda_i\boldsymbol{E}_i$; and $p(\boldsymbol{A}) = \sum_{j=0}^{d} p(\lambda_j)\boldsymbol{E}_j$, for any polynomial $p\in\mathbb{R}[x]$ (see, for example, Godsil [16, p. 28]). In particular, taking $p=1$ we obtain, $\sum_{j=0}^{d}\boldsymbol{E}_j = \boldsymbol{I}$ (as expected), and for $p=x$ we have the *spectral decomposition theorem* $\boldsymbol{A} = \sum_{j=0}^{d}\lambda_j\boldsymbol{E}_j$. The entries of the idempotents $m_{uv}(\lambda_i) = (\boldsymbol{E}_i)_{uv}$ has been recently called *crossed uv-local multiplicities* and satisfy

$$a_{uv}^{(j)} = (\boldsymbol{A}^j)_{uv} = \sum_{i=0}^{d} m_{uv}(\lambda_i)\lambda_i^j. \tag{3}$$

(See [15, 8, 7]). In particular, when $u=v$, $m_u(\lambda_i) = m_{uu}(\lambda_i)$ are the so-called *local multiplicities* of vertex u, satisfying $\sum_{i=0}^{d} m_u(\lambda_i) = 1$, $u\in V$, and $\sum_{u\in V} m_u(\lambda_i) = m_i$, $i=0,1,\ldots,d$ (see [12]).

From any of the above expressions of $\boldsymbol{E}_i$ we deduce that, when G is bipartite, such parameters satisfy:

- $m_{uv}(\lambda_i) = m_{uv}(\lambda_{d-i})$, $i=0,1,\ldots,d$, if $\operatorname{dist}(u,v)$ is even.
- $m_{uv}(\lambda_i) = -m_{uv}(\lambda_{d-i})$, $i=0,1,\ldots,d$, if $\operatorname{dist}(u,v)$ is odd.

In particular, the local multiplicities bear the same symmetry as the standard multiplicities: $m_u(\lambda_i) = m_u(\lambda_{d-i})$ for any vertex $u\in V$ and eigenvalue λ_i, $i=0,1,\ldots,d$.

Form the above, notice that, when G is regular and bipartite, we have $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{J}$ (as mentioned before) and

$$\boldsymbol{E}_d = \frac{1}{n}\begin{pmatrix} \boldsymbol{J} & -\boldsymbol{J}\\ -\boldsymbol{J} & \boldsymbol{J}\end{pmatrix}. \tag{4}$$

2 Polynomials and regularity

The *predistance polynomials* $p_0, p_1, \ldots, p_d$, $\deg p_i = i$, associated to a given graph G with spectrum $\operatorname{sp} G$ as in (1), are a sequence of orthogonal polynomials with respect to the scalar product

$$\langle f,g\rangle = \frac{1}{n}\operatorname{tr}(f(\boldsymbol{A})g(\boldsymbol{A})) = \frac{1}{n}\sum_{i=0}^{d} m_i f(\lambda_i)g(\lambda_i),$$

normalized in such a way that $\|p_i\|^2 = p_i(\lambda_0)$ (this makes sense as it is known that always $p_i(\lambda_0) > 0$). Notice that, in particular, $p_0 = 1$ and, if G is δ-regular, $p_1 = x$. Indeed,

- $\langle 1,x\rangle = \frac{1}{n}\sum_{i=0}^{d} m_i\lambda_i = 0$.
- $\|1\|^2 = \frac{1}{n}\sum_{i=0}^{d} m_i = 1$.
- $\|x\|^2 = \frac{1}{n}\sum_{i=0}^{d} m_i\lambda_i^2 = \delta = \lambda_0$.

Moreover, if G is bipartite, the symmetry of such a scalar product yields that p_i is even (respectively, odd) for even (respectively, odd) degree i.

In terms of the predistance polynomials, the *preHoffman polynomial* is $H = p_0 + p_1 + \cdots + p_d$, and satisfies $H(\lambda_0) = n$ (the order of the graph) and $H(\lambda_i) = 0$ for $i = 1, 2, \ldots, d$ (see Cámara et al. [5]). In [17], Hoffman proved that a (connected) graph G is regular if and only if $H(\boldsymbol{A}) = \boldsymbol{J}$, in which case H becomes the *Hoffman polynomial.* (In fact, H is the unique polynomial of degree at most d satisfying this property.) Furthermore, when G is regular and bipartite, the even and odd parts of H, H_0 and H_1, satisfy, by (2):

$$H_0(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J} \end{pmatrix} \quad \text{and} \quad H_1(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix}. \tag{5}$$

As far as we know, the following proposition is new and can be seen as the biregular counterpart of Hoffman's result. Recall that a bipartite graph $G = (V_1 \cup V_2, E)$ is called (δ_1, δ_2)-*biregular* when all the n_1 vertices of V_1 has degree δ_1, and the n_2 vertices of V_2 has degree δ_2. So, counting in two ways the number of edges $m = |E|$ we have that $n_1\delta_1 = n_2\delta_2$.

Proposition 1 *Let G be a bipartite graph with $n = n_1+n_2$ vertices, predistance polynomials $p_0, p_1, \ldots, p_d$, and consider the odd part of its preHoffman polynomial; that is, $H_1 = \sum_{i\ \mathrm{odd}} p_i$. Then, G is biregular if and only if*

$$H_1(\boldsymbol{A}) = \alpha \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix} \tag{6}$$

with $\alpha = \dfrac{n_1+n_2}{2\sqrt{n_1 n_2}}$.

Proof: Assume first that G is biregular with degrees, say, δ_1 and δ_2. Then, $\lambda_0 = -\lambda_d = \sqrt{\delta_1\delta_2}$ with respective (column) eigenvectors $\boldsymbol{u} = (\sqrt{\boldsymbol{\delta_1}}\boldsymbol{j}|\sqrt{\boldsymbol{\delta_2}}\boldsymbol{j})$ and $\boldsymbol{v} = (\sqrt{\boldsymbol{\delta_1}}\boldsymbol{j}| - \sqrt{\boldsymbol{\delta_2}}\boldsymbol{j})$, with the $\boldsymbol{j}$'s being all-1 vectors with appropriate lengths. Therefore, the respective idempotents are

$$\begin{aligned} \boldsymbol{E}_0 &= \frac{1}{\|u\|^2}uu^\top = \frac{1}{n_1\delta_1+n_2\delta_2}\begin{pmatrix} \boldsymbol{\delta_1 J} & \sqrt{\boldsymbol{\delta_1\delta_2}}\boldsymbol{J} \\ \sqrt{\boldsymbol{\delta_1\delta_2}}\boldsymbol{J} & \boldsymbol{\delta_2 J}\end{pmatrix}, \\ \boldsymbol{E}_d &= \frac{1}{\|v\|^2}vv^\top = \frac{1}{n_1\delta_1+n_2\delta_2}\begin{pmatrix} \boldsymbol{\delta_1 J} & -\sqrt{\boldsymbol{\delta_1\delta_2}}\boldsymbol{J} \\ -\sqrt{\boldsymbol{\delta_1\delta_2}}\boldsymbol{J} & \boldsymbol{\delta_2 J}\end{pmatrix}. \end{aligned}$$

As $H_1(x) = \frac{1}{2}[H(x) - H(-x)]$ and $H(\lambda_i) = n\delta_{0i}$, we have that $H_1(\lambda_0) = n/2$, $H_1(\lambda_i) = 0$ for $i \neq 0, d$, and $H_1(\lambda_d) = -n/2$. Hence, using the properties and the above expressions of the idempotents,

$$\begin{aligned} H_1(\boldsymbol{A}) &= \sum_{i=0}^{d} H_1(\lambda_i)\boldsymbol{E}_i = H_1(\lambda_0)\boldsymbol{E}_0 + H_1(\lambda_d)\boldsymbol{E}_d \\ &= \frac{n}{2}(\boldsymbol{E}_0 - \boldsymbol{E}_d) = \frac{n\sqrt{\delta_1\delta_2}}{n_1\delta_1+n_2\delta_2}\begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O}\end{pmatrix}. \end{aligned}$$

Thus, the result follows since $n_1\delta_1 = n_2\delta_2$. Conversely, if (6) holds, and $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{O}\end{pmatrix}$, the equality $\boldsymbol{A}H_1(\boldsymbol{A}) = H_1(\boldsymbol{A})\boldsymbol{A}$ yields

$$\begin{pmatrix} \boldsymbol{BJ} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{B}^\top\boldsymbol{J}\end{pmatrix} = \begin{pmatrix} \boldsymbol{JB}^\top & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{JB}\end{pmatrix}.$$

Thus, $(\boldsymbol{BJ})_{uv} = (\boldsymbol{JB}^\top)_{uv}$ implies that $\delta(u) = \delta(v)$ for any two vertices $u, v \in V_1$, whereas $(\boldsymbol{B}^\top\boldsymbol{J})_{wz} = (\boldsymbol{JB})_{wz}$ means that $\delta(w) = \delta(z)$ for any two vertices $w, z \in V_2$. Thus, G is biregular and the proof is complete. □

Notice that the constant α is the ratio between the arithmetic and geometric means of the numbers n_1, n_2. Hence, (6) holds with $\alpha = 1$ if and only if $n_1 = n_2$ or, equivalently, G is regular.

In fact, the above result could be reformulated (and proved) by saying that a (general) bipartite graph is connected and biregular if and only if there exists a polynomial satisfying (6).

3 Distance-regular graphs

Let G be a graph with diameter D, adjacency matrix $\boldsymbol{A}$ and $d+1$ distinct eigenvalues. Let $\boldsymbol{A}_i$, $i=0,1,\ldots,D$, be the distance-i matrix of G, with entries $(\boldsymbol{A}_i)_{uv}=1$ if $\mathrm{dist}(u,v)=i$ and $(\boldsymbol{A}_i)_{uv}=0$ otherwise. Then,

$$\mathcal{A}=\mathbb{R}_d[\boldsymbol{A}]=\mathrm{span}\{\boldsymbol{I},\boldsymbol{A},\boldsymbol{A}^2,\ldots,\boldsymbol{A}^d\}$$

is an algebra, with the ordinary product of matrices and orthogonal basis $\{\boldsymbol{E}_0,\boldsymbol{E}_1,\ldots,\boldsymbol{E}_d\}$ and $\{p_0(\boldsymbol{A}),p_1(\boldsymbol{A}),\ldots,p_d(\boldsymbol{A})\}$, called the *adjacency algebra*, whereas

$$\mathcal{D}=\mathrm{span}\{\boldsymbol{I},\boldsymbol{A},\boldsymbol{A}_2,\ldots,\boldsymbol{A}_D\}$$

forms an algebra with the entrywise or Hadamard product of matrices, defined by $(\boldsymbol{X}\circ\boldsymbol{Y})_{uv}=\boldsymbol{X}_{uv}\boldsymbol{Y}_{uv}$. We call $\mathcal{D}$ the *distance $\circ$-algebra*. Note that, when G is regular, $\boldsymbol{I},\boldsymbol{A},\boldsymbol{J}\in\mathcal{A}\cap\mathcal{D}$ since $\boldsymbol{J}=H(\boldsymbol{A})=\sum_{i=0}^{D}\boldsymbol{A}_i$. Thus, $\dim(\mathcal{A}\cap\mathcal{D})\geq 3$, if G is not a complete graph (in this exceptional case, $\boldsymbol{J}=\boldsymbol{I}+\boldsymbol{A}$). In this algebraic context, an important result is that G is distance-regular if and only if $\mathcal{A}=\mathcal{D}$, which is therefore equivalent to $\dim(\mathcal{A}\cap\mathcal{D})=d+1$ (and hence $d=D$); see, for instance, Biggs [2] or Brower et al. [4]. This leads to the following definitions of distance-regularity where, for types (a) and (b), p_{ji} and q_{ij}, $i,j=0,1,\ldots,d$, are constants, p_i, $i=0,1,\ldots,d$, are the predistance polynomials, and q_j, $j=0,1,\ldots,d$, are the polynomials defined by $q_j(\lambda_i)=m_j\frac{p_i(\lambda_j)}{p_i(\lambda_0)}$, $i,j=0,1,\ldots,d$:

(a) G distance-regular

$$\begin{aligned}
&\iff \boldsymbol{A}_i\boldsymbol{E}_j=p_{ji}\boldsymbol{E}_j, \qquad i,j=0,1,\ldots,d(=D),\\
&\iff \boldsymbol{A}_i=\sum_{j=0}^{d}p_{ji}\boldsymbol{E}_j, \qquad i=0,1,\ldots,d(=D),\\
&\iff \boldsymbol{A}_i=\sum_{j=0}^{d}p_i(\lambda_j)\boldsymbol{E}_j, \quad i=0,1,\ldots,d(=D),\\
&\iff \boldsymbol{A}_i\in\mathcal{A}, \quad i=0,1,\ldots,d(=D).
\end{aligned}$$

$$(b)\ G \text{ distance-regular} \iff \boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij}\boldsymbol{A}_i, \qquad i,j = 0,1,\dots,d,$$
$$\iff \boldsymbol{E}_j = \sum_{j=0}^{d} q_{ij}\boldsymbol{A}_i, \qquad j = 0,1,\dots,d,$$
$$\iff \boldsymbol{E}_j = \frac{1}{n}\sum_{i=0}^{d} q_j(\lambda_i)\boldsymbol{A}_i, \quad j = 0,1,\dots,d,$$
$$\iff \boldsymbol{E}_j \in \mathcal{D}, \quad j = 0,1,\dots,d.$$

In fact, for general graphs with $D \le d$, the conditions of type (a) are a characterization of the so-called *distance-polynomial graphs*, introduced by Weichsel [19] (see also Beezer [3] and Dalfó et al. [7]). This is equivalent to $\mathcal{D} \subset \mathcal{A}$ (but not necessarily $\mathcal{D} = \mathcal{A}$); that is, every distance matrix $\boldsymbol{A}_i$ is a polynomial in $\boldsymbol{A}$. In contrast with that, the conditions of type (b) are equivalent to $\mathcal{A} \subset \mathcal{D}$ and, hence, to $\mathcal{A} = \mathcal{D}$ (which implies $d = D$) as $\dim \mathcal{A} \ge \dim \mathcal{D}$.

Note also that in (a) (respectively, in (b)) the second implication is obtained from the first one by using that $\sum_{i=0}^{d} \boldsymbol{A}_i = \boldsymbol{J}$ (respectively, $\sum_{j=0}^{d} \boldsymbol{E}_j = \boldsymbol{I}$).

Moreover, with the $a_i^{(j)}$, $i,j = 0,1,\dots,d$, being constants, we also have:

$$(c)\ G \text{ distance-regular} \iff \boldsymbol{A}^j \circ \boldsymbol{A}_i = a_i^{(j)}\boldsymbol{A}_i, \qquad i,j = 0,1,\dots,d,$$
$$\iff \boldsymbol{A}^j = \sum_{i=0}^{d} a_i^{(j)}\boldsymbol{A}_i, \qquad j = 0,1,\dots,d,$$
$$\iff \boldsymbol{A}^j = \frac{1}{n}\sum_{i=0}^{d}\sum_{l=0}^{d} q_{il}\lambda_l^j\boldsymbol{A}_i, \quad j = 0,1,\dots,d,$$
$$\iff \boldsymbol{A}^j \in \mathcal{D}, \quad j = 0,1,\dots,d,$$

where we have used (3) with $a_{uv}(j) = a_i^{(j)}$ and $m_{uv}(\lambda_l) = q_{il}$ for vertices u, v at distance $\operatorname{dist}(u,v) = i$.

4 Characterizing bipartite distance-regular graphs

A general phenomenon is that the above conditions for being distance-regular can be relaxed giving more 'economic' characterizations (see [11]). Thus, the purpose of the following three theorems is twofold: First to show

how, for general graphs, such conditions can be relaxed if we assume some extra natural hypothesis (such as regularity) and, second, to study what happens in the case of bipartite graphs.

Theorem 2 (i) *A graph G with predistance polynomials $p_0, p_1, \ldots, p_d$ is distance-regular if and only if any of the following conditions holds:*

($a1$) $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = 2, 3, \ldots, d$.

($a2$) *G is regular and* $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = 2, 3, \ldots, d-1$.

($a3$) *G is regular and* $\boldsymbol{A}_d = p_d(\boldsymbol{A})$.

($a4$) *G is regular and* $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = d-2, d-1$.

(ii) *A bipartite graph G with predistance polynomials $p_0, p_1, \ldots, p_d$ is distance-regular if and only if*

($a5$) *G is regular and* $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = 3, 4, \ldots, d-2$.

Proof: Statement ($a1$) with $i = 0, 1, \ldots, d$ is a well-known result; see, for example, Bannai and Ito [1]. For our case, just notice that always $p_0(\boldsymbol{A}) = \boldsymbol{A}_0 = \boldsymbol{I}$ and, as $\boldsymbol{I} + \boldsymbol{A} + \sum_{i=2}^{d} p_i(\boldsymbol{A}) = \boldsymbol{J}$, G is regular and hence $p_1(\boldsymbol{A}) = \boldsymbol{A}_1 = \boldsymbol{A}$; Condition ($a2$) is a consequence of ($a1$) taking into account that, under the hypotheses, $\boldsymbol{A}_d = \boldsymbol{J} - \sum_{i=0}^{d-1} \boldsymbol{A}_i = H(\boldsymbol{A}) - \sum_{i=0}^{d-1} p_i(\boldsymbol{A}) = p_d(\boldsymbol{A})$ (see Dalfó et al. [7]); ($a3$) was first proved by Fiol et al. in [14] (see also van Dam [9] or Fiol et al. [13] for short proofs); and ($a4$) is a consequence of a more general result in [7] characterizing m-partially distance-regularity (G is called m-partially distance-regular if $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ for any $i = 0, 1, \ldots, m$). Thus, we only need to prove ($a5$). This is a consequence of ($a2$) since, if G is δ-regular, $\boldsymbol{A}_2 = p_2(\boldsymbol{A}) = \boldsymbol{A}^2 - \delta \boldsymbol{I}$. Moreover, from (5) and assuming first that d is even,

$$\boldsymbol{A}_{d-1} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix} - \sum_{\substack{i=1 \\ i \text{ odd}}}^{d-3} \boldsymbol{A}_i = H_1(\boldsymbol{A}) - \sum_{\substack{i=1 \\ i \text{ odd}}}^{d-3} p_i(\boldsymbol{A}) = p_{d-1}(\boldsymbol{A})$$

whereas, if d is odd,

$$\boldsymbol{A}_{d-1} = \begin{pmatrix} \boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J} \end{pmatrix} - \sum_{\substack{i=0 \\ i \text{ even}}}^{d-3} \boldsymbol{A}_i = H_0(\boldsymbol{A}) - \sum_{\substack{i=0 \\ i \text{ even}}}^{d-3} p_i(\boldsymbol{A}) = p_{d-1}(\boldsymbol{A}),$$

and the proof is complete. □

The above results suggest the following question:

Problem 3 Prove or disprove: A regular bipartite graph G with predistance polynomial p_{d-1} is distance-regular if and only if $\boldsymbol{A}_{d-1} = p_{d-1}(\boldsymbol{A})$.

With respect to the characterizations of type (b), we can state the following result:

Theorem 4 (i) *A graph G with idempotents $\boldsymbol{E}_0, \boldsymbol{E}_1, \ldots, \boldsymbol{E}_d$ is distance-regular if and only if any of the following conditions holds:*

$(b1)$ $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 0, 1, \ldots, d$.

$(b2)$ $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 0, 1, \ldots, d-1$.

$(b3)$ *G is regular and* $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 1, 2, \ldots, d-1$.

$(b4)$ *G is regular and* $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 1, d$.

(ii) *A bipartite graph G with idempotent $\boldsymbol{E}_1$ is distance-regular if and only if*

$(b5)$ *G is regular and* $\boldsymbol{E}_1 \in \mathcal{D}$.

Proof: Statement $(b1)$ (see also (b) in Section 3) is also well-known and comes from the fact that G is distance-regular if and only if $\mathcal{A} = \mathcal{D}$; Condition $(b2)$ is a consequence of $(b1)$ since, under the hypotheses, $\boldsymbol{E}_d = \boldsymbol{I} - \sum_{j=0}^{d-1} \boldsymbol{E}_j \in \mathcal{D}$; $(b3)$ comes from $(b2)$ since, if G is regular, then $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{J} = \frac{1}{n}H(\boldsymbol{A}) \in \mathcal{D}$; $(b4)$ was proved by the author in [10] (see also [11]). Finally, $(b5)$ can be seen as a consequence of $(b4)$ since, under the hypotheses, (4) yields

$$\boldsymbol{E}_d = \sum_{\substack{i=0 \\ i \text{ even}}}^{d} \boldsymbol{A}_i - \sum_{\substack{i=0 \\ i \text{ odd}}}^{d} \boldsymbol{A}_i \in \mathcal{D}$$

and the proof is complete. □

Now let us go to the characterizations of type (c) which are given in terms of the numbers $a_{uv}^{(j)} = (\boldsymbol{A}^j)_{uv}$ of walks of length $j \geq 0$ between vertices u, v at distance $\operatorname{dist}(u,v) = i$, $i = 0, 1, \ldots, D$. When such numbers do not depend on u, v but only on i and j, we write $a_{uv}^{(j)} = a_i^{(j)}$. In particular, notice that always $a_0^{(0)} = a_1^{(1)} = 1$ and G is δ-regular if and only if $a_2^{(2)} = \delta$.

Theorem 5 (i) *A graph G, with diameter D and $d+1$ distinct eigenvalues, is distance-regular if and only if, for any two vertices u, v at distance* $\mathrm{dist}(i,j)=i$, *any of the following conditions holds:*

($c1$) $a_{uv}^{(j)} = a_i^{(j)}$ *for* $i = 0, 1, \ldots, D$ *and* $j \geq i$.

($c2$) $a_{uv}^{(j)} = a_i^{(j)}$ *for* $i = 0, 1, \ldots, D$ *and* $j = i, i+1, \ldots, d$.

($c3$) $D = d$, *and* $a_{uv}^{(j)} = a_i^{(j)}$ *for* $i = 0, 1, \ldots, D$ *and* $j = i, i+1, \ldots, d-1$.

($c4$) *G is regular,* $D = d$, *and* $a_{uv}^{(j)} = a_i^{(j)}$ *for* $i = 0, 1, \ldots, D-1$ *and* $j = i, i+1$.

(ii) *A bipartite graph G is distance-regular if and only if*

($c5$) *G is regular,* $D = d$, *and* $a_{uv}^{(j)} = a_i^{(j)}$ *for* $i = j = 2, 3, \ldots, D-2$.

Proof: Characterization ($c1$) was first proved by Rowlinson [18]; Statement ($c2$) is a straightforward consequence of ($b1$) since $\mathcal{A} = \mathrm{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d\}$; ($c3$) comes from ($c2$) since, if G is regular and $D = d$, the number of d-walks between any two vertices u, v at distance d, is a constant:

$$a_{uv}^{(d)} = (\boldsymbol{A}^d)_{uv} = \frac{\pi_0}{n}[H(\boldsymbol{A})]_{uv} = \frac{\pi_0}{n}(\boldsymbol{J})_{uv} = \frac{\pi_0}{n} = a_d^{(d)};$$

($c4$) derives from a similar result in [10] (not requiring $D = d$) and the above reasoning on $a_{uv}^{(d)}$. Finally, ($c5$) is a consequence of ($c4$) since, when G is bipartite, there are no walks of length $j = i+1$ between vertices at distance i and, thus, $a_i^{(i+1)} = 0$. Moreover, if G is δ-regular and $D = d$, $a_{d-1}^{(d-1)} = \frac{1}{\delta} a_d^{(d)} = \frac{\pi_0}{n\delta}$. □

Problem 6 Give similar characterizations of types (a), (b) and (c) for distance biregular graphs.

Acknowledgement

Research supported by the Ministerio de Educación y Ciencia (Spain) and the European Regional Development Fund under project MTM2008-06620-C03-01, and by the Catalan Research Council under project 2009SGR1387.

References

[1] E. Bannai and T. Ito. *Algebraic Combinatorics I: Association Schemes.* Benjamin/Cummings, London, 1974, 1984.

[2] N. Biggs. *Algebraic Graph Theory.* Cambridge University Press, Cambridge, 1974, second edition, 1993.

[3] R.A. Beezer. Distance polynomial graphs, in *Proceedings of the Sixth Caribbean Conference on Combinatorics and Computing, Trinidad, 1991*, 51–73.

[4] A.E. Brouwer, A.M. Cohen, and A. Neumaier. *Distance-Regular Graphs.* Springer-Verlag, Berlin-New York, 1989.

[5] M. Cámara, J. Fàbrega, M.A. Fiol, and E. Garriga. Some families of orthogonal polynomials of a discrete variable and their applications to graphs and codes *Electron. J. Combin.* 16(1):#R83, 2009.

[6] D. M. Cvetković, M. Doob and H. Sachs. Spectra of Graphs, Theory and Application. VEB Deutscher Verlag der Wissenschaften, Berlin, second edition, 1982.

[7] C. Dalfó, E.R. van Dam, M.A. Fiol, E. Garriga, and B.L. Gorissen. On almost distance-regular graphs. *J. Combin. Theory Ser. A*, to appear.

[8] C. Dalfó, M.A. Fiol, and E. Garriga. On k-walk-regular graphs. *Electron. J. Combin.* 16(1):#R47, 2009.

[9] E.R. van Dam. The spectral excess theorem for distance-regular graphs: a global (over)view. *Electron. J. Combin.* 15(1):#R129, 2008.

[10] M.A. Fiol. On pseudo-distance-regularity. *Linear Algebra Appl.*, 323:145–165, 2001.

[11] M.A. Fiol. Algebraic characterizations of distance-regular graphs. *Discrete Math.*, 246:111–129, 2002.

[12] M.A. Fiol and E. Garriga. From local adjacency polynomials to locally pseudo-distance-regular graphs. *J. Combin. Theory Ser. B*, 71:162–183, 1997.

[13] M.A. Fiol, S. Gago, and E. Garriga. A simple proof of the spectral excess theorem for distance-regular graphs, *Linear Algebra Appl.*, 432:2418–2422, 2010.

[14] M.A. Fiol, E. Garriga, and J.L.A. Yebra. Locally pseudo-distance-regular graphs. *J. Combin. Theory Ser. B*, 68:179–205, 1996.

[15] M.A. Fiol, E. Garriga, and J.L.A. Yebra. Boundary graphs: The limit case of a spectral property. *Discrete Math.*, 226:155–173, 2001.

[16] C.D. Godsil. *Algebraic Combinatorics*. Chapman and Hall, NewYork, 1993.

[17] A.J. Hoffman. On the polynomial of a graph, *Amer. Math. Monthly*, 70:30–36, 1963.

[18] P. Rowlinson, Linear algebra, in *Graph Connections* (L.W. Beineke and R.J. Wilson, eds.), Oxford Lecture Ser. Math. Appl., Vol. 5, 86–99, Oxford Univ. Press, New York, 1997.

[19] P.M. Weichsel. On distance-regularity in graphs. *J. Combin. Theory Ser. B*, 32:156–161, 1982.