Radially Moore graphs
of radius three and large odd degree

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Abstract

Extremal graphs which are close related to Moore graphs have been defined in different ways. Radially Moore graphs are one of these examples of extremal graphs. Although it is proved that radially Moore graphs exist for radius two, the general problem remains open. Knor, and independently Exoo, gives some constructions of these extremal graphs for radius three and small degrees. As far as we know, some few examples have been found for other small values of the degree and the radius.

Here, we consider the existence problem of radially Moore graphs of radius three. We use the generalized undirected de Bruijn graphs to give a general construction of radially Moore graphs of radius three and large odd degree.

1 Introduction

Given the values of the maximum degree \( d \) and the diameter \( k \) of a graph, there is a natural upper bound for its number of vertices \( n \),

\[
    n \leq M_{d,k} = 1 + d + d(d - 1) + \cdots + d(d - 1)^{k-1}, \quad (1)
\]
Radially Moore graphs of radius three and large odd degree

N. López and J. Gómez

where $M_{d,k}$ is known as the Moore bound. Graphs attaining such a bound are referred to as Moore graphs. In the case of diameter $k = 2$, Hoffman and Singleton [4] proved that Moore graphs exist for $d = 2, 3, 7$ (being unique) and possibly $d = 57$, but for no other degrees. They also showed that for diameter $k = 3$ and degree $d > 2$ Moore graphs do not exist. The enumeration of Moore graphs of diameter $k > 3$ was concluded by Damerell [3], who used the theory of distance-regularity to prove their nonexistence unless $d = 2$, which corresponds to the cycle graph of order $2k + 1$ (an independent proof of it was given by Bannai and Ito [1]).

The fact that there are very few Moore graphs suggested the study of graphs that are in various senses ‘close’ to being Moore graphs. This ‘closeness’ has been usually measured as the difference between the (unattainable) Moore bound and the order of the considered graphs. In this sense, the existence of graphs with small ‘defect’ $\delta$ (order $n = M(d, k) - \delta$) has deserved much attention in the literature (see [8]). Another kind of approach considers relaxing some of the constraints implied by the Moore bound. From its definition, all vertices of a Moore graph have the same degree ($d$) and the same eccentricity ($k$). We could relax the condition of the degree and admit few vertices with degree $d + \delta$, as Tang, Miller and Lin [10] did for the directed case. Alternatively, we may allow the existence of vertices with eccentricity just on more than the value $k$ they should have. In this context, regular graphs of degree $d$, radius $k$, diameter $k + 1$ and order equal to $M_{d,k}$ are referred to as radially Moore graphs. Figure 1 shows all (non-isomorphic) cubic radially Moore graphs of radius $k = 2$.

Figure 1: All cubic radially Moore graphs of radius $k = 2$. Vertices with eccentricity $k$ (central vertices) are depicted in white.

It is known that radially Moore graphs of radius $k = 2$ exist for any degree (see [2]). Nevertheless, the situation for $k \geq 3$ seems to be more complicated. Knor [7] (and independently Exoo) found radially Moore
Radially Moore graphs of radius three and large odd degree

N. López and J. Gómez

graphs of radius \( k = 3 \) and small degrees. So far, no general construction for radius \( k \geq 3 \) is known. Besides, Captdevila et al. [2] give the complete enumeration of these extremal graphs for some cases (\( k = 2 \) and \( d = 3, 4 \); \( k = 3 \) and \( d = 3 \)) and rank them according to their ‘proximity’ to a theoretical Moore graph.

2 The generalized de Bruijn digraphs

The generalized de Bruijn digraphs appear in the context of the optimization problem which tries to minimize the diameter and maximize the connectivity of a digraph with \( n \) vertices, each of which has outdegree at most \( d \). The generalized de Bruijn digraph \( G_B(d, n) \) is the directed graph with \( n \) vertices labeled by the residues modulo \( n \) such that an arc from \( i \) to \( j \) exists if and only if \( j \equiv di + k \pmod{n} \), for some \( k = 0, \ldots, d - 1 \). These digraphs were first defined by Imase and Itoh [5] and independently by Reddy et al. [9] as a generalization of the well known de Bruijn digraphs. It is known that each vertex in \( G_B(d, n) \) has both indegree and outdegree \( d \) and this digraph may contain loops (cycles of length 1) and multiple arcs. The diameter of \( G_B(d, n) \) is upper bounded by \( \lceil \log_d n \rceil \) (see [5]). Essentially, the generalized de Bruijn digraphs retain all the properties of the de Bruijn digraphs, but have no restriction on the number of vertices. Next, we show the structure of the subdigraph induced by the set of vertices containing either a loop or a digon (cycle of length 2).

**Proposition 1** The digraph \( G_B(d, n) \), where \( \gcd(d - 1, n) = \gcd(d^2 - 1, n) = 1 \), has \( d \) loop vertices and \( d^2 - d \) vertices belonging to a digon.

**Proof:** A loop vertex \( i \) in \( G_B(d, n) \) satisfies the following equation

\[
i(d - 1) \equiv -k \pmod{n}
\]

where \( k \in \{0, \ldots, d - 1\} \). Since \( \gcd(d - 1, n) = 1 \), there is unique solution of equation 2 for each value of \( k \). As a consequence, there are \( d \) loop vertices in \( G_B(d, n) \), each of them of the form

\[-k(d - 1)^{-1} \pmod{n}, \quad k \in \{0, \ldots, d - 1\}.
\]

Besides, a vertex \( i \) contained in a cycle of length \( \leq 2 \) satisfies,

\[
i(d^2 - 1) \equiv -dk - k' \pmod{n}
\]

297
Radially Moore graphs of radius three and large odd degree N. López and J. Gómez

where \( k, k' \in \{0, \ldots, d-1\} \). Since \( \gcd(d^2 - 1, n) = 1 \), for every pair \( (k, k') \) there is a unique solution of equation 3. Notice that for \( k = k' \), equation 3 transforms to equation 2 which it means that there are \( d \) solutions of equation 3 corresponding to loop vertices. As a consequence, there are \( d^2 - d \) vertices contained in a digon. \( \square \)

**Proposition 2** The subdigraph of \( G_B(d, n) \), where \( \gcd(d-1, n) = \gcd(d^2 - 1, n) = 1 \) and \( n \geq d^3 \), induced by the set of vertices contained in either a loop or a digon is isomorphic to the digraph with vertex set \( V = \{(i, j) \mid i, j \in \{0, \ldots, d-1\}\} \) and where there is an arc from \((i, j)\) to \((j, i)\), for all \((i, j) \in V\).

**Proof:** From the previous proposition, every vertex in the subdigraph of \( G_B(d, n) \), induced by the set of vertices contained in either a loop or a digon is of the form

\[
(d^2 - 1)^{-1}(-di - j) \pmod{n}, \quad i, j \in \{0, \ldots, d-1\}.
\]

So, every vertex of this subdigraph can be identified by the pair \((i, j)\), where \((i, j) \in V\). Let us observe that there is an arc from \((i, j)\) to \((i', j')\) in this subdigraph if and only if,

\[
(d^2 - 1)^{-1}(-di' - j') \equiv d(d^2 - 1)^{-1}(-di - j) + k \pmod{n}
\]

for a suitable \( k \in \{0, \ldots, d-1\} \). This is equivalent to the following equation:

\[
d^2(k - i) + d(i' - j) + (j' - k) \equiv 0 \pmod{n} \quad (4)
\]

Since \( n \geq d^3 \), equation 4 holds if and only if \( k - i = i' - j = j' - k = 0 \), that is, \( k = i, i' = j \) and \( j' = i \). \( \square \)

Figure 2 shows a representation of the subdigraph of \( G_B(d, n) \) induced by the set of vertices contained in either a loop or a digon, for the particular values \( d = 6 \) and \( n = 1872 \). A vertex \((i, j)\) in the picture corresponds to vertex \((d^2 - 1)^{-1}(-di - j) \pmod{n}\) in \( G_B(d, n) \). As an example, vertex \((1, 2)\) is 1016 in \( G_B(d, n) \), this vertex has a unique arc to [from] a vertex belonging to the subdigraph itself. This special vertex is \((2, 1)\) (481 in \( G_B(d, n) \)).

298
The generalized undirected de Bruijn graphs

The generalized undirected de Bruijn graph, denoted by $UG_B(d, n)$, is the undirected graph derived from $G_B(d, n)$ by replacing arcs with edges and omitting loops and multiple edges. The diameter of $UG_B(d, n)$ is bounded above by $\lceil \log_d n \rceil$ since for any two distinct vertices $u$ and $v$ in $UG_B(d, n)$, the distance from $u$ to $v$ in the corresponding digraph $G_B(d, n)$ provides an upper bound for the distance between $u$ and $v$ in $UG_B(d, n)$, as it can be seen in [6]. From its own definition, $UG_B(d, n)$ has order $n$ and each vertex has maximum out degree $2d$. More precisely, whenever $\gcd(d - 1, n) = \gcd(d^2 - 1, n) = 1$, $UG_B(d, n)$ has $d$ vertices of degree $2d - 2$, $d^2 - d$ vertices of degree $2d - 1$ and the remaining vertices of degree $2d$. From proposition 2 we derive the following result:

**Corollary 3** The subgraph of $UG_B(d, n)$, where $\gcd(d - 1, n) = \gcd(d^2 - 1, n) = 1$ and $n \geq d^3$, induced by the set of vertices of degree $< 2d$ is isomorphic to the graph with vertex set $V = \{(i, j) \mid i, j \in \{0, \ldots, d - 1\}\}$ and where vertex $(i, j)$ is adjacent to $(j, i)$, for all $i > j$.

Whenever $d$ is even, we can add some extra edges to $UG_B(d, n)$ in order to achieve a $2d$-regular graph.
Proposition 4 $UG_B(d, n)$, where $\gcd(d - 1, n) = \gcd(d^2 - 1, n) = 1$, $n \geq d^3$ and $d$ is even, is a subgraph of a regular graph of degree $2d$ and order $n$.

Proof: We add the following adjacency relations to the subgraph of $UG_B(d, n)$ induced by the set of vertices of degree $< 2d$.

$$\begin{cases} (i, j) \sim (i, j + 1) & \text{for } j \text{ even;} \\ (i, i) \sim (i + 1, i + 1) & \text{for } i \text{ even.} \end{cases}$$

The degree of every loop vertex has been increased by two and the degree of every vertex contained in a digon has been increased by one (see figure 3). Hence, the resultant graph is regular of degree $2d$. □

Figure 3: The subdigraph of $UG_B(d, n)$ induced by the set of vertices with degree $< 2d$, for the particular values $d = 6$ and $n = 1872$. This graph is a subgraph of a 2-regular graph, as it shows the right picture. As a consequence, $UG_B(d, n)$ is a subgraph of a $2d$-regular graph.

4 Radially Moore graphs of radius three and large odd degree

Let us start with the tree $T_{d,k}$ given in Fig. 4, which corresponds to the distance preserve spanning tree of a radial Moore graph of degree $d$ and radius $k$, hanging from a central vertex $v$ (every central vertex in a radial Moore graph of degree $d$ and radius $k$ must reach any other vertex of the
Radially Moore graphs of radius three and large odd degree N. López and J. Gómez

graph in at most $k$ steps). In particular, this is the same structure that we observe in a Moore graph hanged from any of its vertices. Let $V$ be the set of vertices at maximum distance from $v$ in $T_{d,k}$. There are $N = d(d-1)^{k-1}$ of such vertices that we label by the integers modulo $d(d-1)^{k-1}$. Now, for odd $d$, we build a new graph $GM(d, k)$ taking $T_{d,k}$ as a basis and attaching to $V$ the undirected de Bruijn graph $UG_B(\Delta, N)$, where $\Delta = \frac{d-1}{2}$. That is, the subgraph of $GM(d, k)$ induced by the set of vertices at distance $k$ of $v$ is precisely $UG_B(\Delta, N)$. Obviously $GM(d, k)$ has order $M_{d,k}$ and radius $k$, since the eccentricity of the ‘root’ vertex $v$ is $k$. Next, we prove that for $k = 3$ and bigger enough $d$, the graph $GM(d, k)$ has diameter $k + 1$.

**Theorem 5** $GM(d, 3)$ has diameter four for every odd $d \geq 19$.

**Proof:** Let $v$ the root of the spanning tree $T = T_{d,3}$ of $G = GM(d, 3)$. Since the eccentricity of $v$ is 3, the maximum distance from an adjacent vertex to $v$ is at most 4. Now, we prove the following: Let $u_1$ and $u_2$ two vertices at distance three from $v$, then $\text{dist}(u_1, u_2) \leq 4$ if $d \geq 19$. Since $u_1$ and $u_2$ are at distance three from $v$, we can consider both vertices in $UG_B(\Delta, N)$. Let $D$ be the diameter of $UG_B(\Delta, N)$. Taking into account that $D \leq \log_{\Delta} N$, and $N = (2\Delta)^2(2\Delta + 1)$, then:

$$D \leq \lfloor \log_{\Delta}(2\Delta)^2(2\Delta + 1) \rfloor \leq \lfloor \log_{\Delta} \Delta^2(8\Delta + 4) \rfloor \leq \lfloor 2 + \log_{\Delta}(8\Delta + 4) \rfloor$$

Now, since $8\Delta + 4 \leq \Delta^2$ if $\Delta \geq 9$ then we derive that $D \leq 4$ if $d \geq 19$. That is, $\text{dist}(u_1, u_2) \leq 4$ whenever $d \geq 19$. Now, we prove that when $u_1$ and $u_2$ are two vertices of $GM(d, 3)$ at least one of them at distance $< 3$ from $v$, then $\text{dist}(u_1, u_2) \leq 4$. We can assume that $\text{dist}(u_1, v) = 2$ and
Radially Moore graphs
of radius three and large odd degree

N. López and J. Gómez

\[ \text{dist}(u_2, v) = 3, \] since otherwise there exist a path from \( u_1 \) to \( u_2 \) (through \( v \)) with length at most 4. Let \( u_1 \) be at distance 2 from \( v \), we will see that every vertex at distance three from \( v \) is at most at distance 4 from \( u_1 \). The set of vertices adjacent to \( u_1 \) which are at distance 3 from \( v \) is

\[ \Gamma_1(u_1) = \{ l\Delta + s \mid s = 0, \ldots, 2\Delta - 1 \} \]

for some even \( 0 \leq l \leq 4\Delta^2 + 2\Delta \). That is, \( \Gamma_1(u_1) \) is the set of vertices of \( UG_B(\Delta, N) \) hanging from \( u_1 \). Any vertex of \( UG_B(\Delta, N) \) at distance at most three from a vertex in \( \Gamma_1(u_1) \) is of the form,

\[ \Gamma_4(u_1) = \{ l\Delta^4 + \Delta^3 s + \Delta^2 k + \Delta k' + k'' \mid k, k', k'' \in \{0, \ldots, \Delta - 1\} \} \]

In particular, any vertex of \( UG_B(\Delta, N) \) belongs to \( \Gamma_4(u_1) \) whenever \( \Delta \geq 5 \), that is, \( d \geq 11 \). Hence, \( u_2 \in \Gamma_4(u_1) \) and, as a consequence, \( \text{dist}(u_1, u_2) \leq 4 \). Let us observe that the diameter of \( G \) cannot be less than four since there is no Moore graph of radius three. \( \square \)

Note that \( GM(d, 3) \) is not a regular graph, since \( UG_B(\Delta, N) \) contains vertices with degree \( 2\Delta - 1 \) and \( 2\Delta - 2 \). Nevertheless, we observe that \( \gcd(\Delta - 1, N) = \gcd(\Delta^2 - 1, N) = 1 \) if and only if \( \Delta \equiv 0, 2 \pmod{6} \). Hence, in these cases we can apply proposition 4 and derive that \( UG_B(\Delta, N) \) is a subgraph of a \( 2\Delta \)-regular graph. As a consequence, the regularity of \( GM(d, 3) \) can be completed.

**Theorem 6** Radial Moore graphs of radius three and degree \( d \) do exist for \( d = 2\Delta + 1 \geq 19 \) and \( \Delta \equiv 0, 2 \pmod{6} \).

For other values of \( \Delta \) it is not clear how to rearrange the subgraph of \( UG_B(\Delta, N) \) induced by the set of vertices with degree \( < 2\Delta \) in order to complete the regularity. We call the problem of the regularity completeness at this special situation.

**Problem 7** Solve the problem of regularity completeness for other values of \( \Delta \).

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302
References


