Infinite families of 3-numerical semigroups with arithmetic-like links

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Abstract

Let $S = \langle a, b, N \rangle$ be a numerical semigroup generated by $a, b, N \in \mathbb{N}$ with $1 < a < b < N$ and $\gcd(a, b, N) = 1$. The conductor of $S$, denoted by $c(S)$ or $c(a, b, N)$, is the minimum element of $S$ such that $c(S) + m \in S$ for all $m \in \mathbb{N} \cup \{0\}$. Some arithmetic-like links between 3-numerical semigroups were remarked by V. Arnold. For instance he gave links of the form

\[
\frac{c(13, 32, 52)}{c(13, 33, 51)} = \frac{c(9, 43, 45)}{c(9, 42, 46)} = \frac{c(5, 35, 37)}{c(5, 34, 38)} = 2 \quad \text{or} \quad \frac{c(4, 20, 73)}{c(4, 19, 74)} = 4.
\]

In this work several infinite families of 3-numerical semigroups with similar properties are given. These families have been found using a plane geometrical approach, known as L-shaped tile, that can be related to a 3-numerical semigroup. This tile defines a plane tessellation that gives information on the related semigroup.

1 Introduction and known results

A 3-semigroup $S = \langle a, b, N \rangle$ with $a, b, N \in \mathbb{N}$ and $1 < a < b < N$, is defined as $\langle a, b, N \rangle = \{m \in \mathbb{N} \mid m = xa + yb + zN; \; x, y, z \in \mathbb{N}\}$. The values $a$, $b$ and $N$ are called the generators of $S$. The set $\overline{S} = \mathbb{N} \setminus S$ is called the set of gaps of $S$. If the cardinality of $\overline{S}$ is finite, then $S$ is a 3-numerical semigroup. It is well known that $S$ is a 3-numerical semigroup if and only if $\gcd(a, b, N) = 1$. The Frobenius Number of $S$ is the value $f(S) = \max \overline{S}$. The conductor of $S$ is the value $c(S) = f(S) + 1$. Given $m \in S \setminus \{0\}$,
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the Apéry set of $S$ with respect to $m$, $\text{Ap}(S, m) = \{s \in S \mid s - m \notin S\}$, contains significant information of $S$. In particular, it is well known that $|S| = \max\text{Ap}(S, m) - m$. A 3-numerical semigroup $S = \langle a, b, N \rangle$ is minimally generated if the semigroups $\langle a, b \rangle$, $\langle a, N \rangle$ or $\langle b, N \rangle$ are proper subsets of $S$. You can find recent results on numerical semigroups in the book of Rosales and García-Sánchez [6]. Recent results mainly related on the Frobenius number can be found in the book of Ramírez Alfonsín [4].

The equivalence class of $m$ modulo $N$ will be denoted by $[m]_N$. A weighted double-loop digraph $G(N; a, b; a, b)$ is a directed graph with set of vertices $V(G) = \{[0]_N, \ldots, [N - 1]_N\}$ and set of weighted arcs $A(G) = \{[v]_N \xrightarrow{a} [v+a]_N, [v]_N \xrightarrow{b} [v+b]_N \mid [v]_N \in V(G)\}$. The idea of using weighted double-loop digraphs as a tool in the study of the Frobenius number of 3-numerical semigroups was already used by Selmer [8] in 1977 and Rødseth [5] in 1978.

Each weighted double-loop digraph $G$ has related several minimum distance diagrams (MDD for short) that periodically tessellates the squared plane. Each vertex $[ia + jb]_N$ of $G$ is associated with the unit square of the plane $(i, j) \in \mathbb{N}^2$, that is the interval $[i, i + 1] \times [j, j + 1] \in \mathbb{R}^2$. An MDD is composed by $N$ unit squares and has a geometrical shape like the (capital) letter ‘L’ or it is a rectangle (that is considered a degenerated L-shape), see [5, 3] for more details. Sabariego and Santos [7] gave an algebraic characterization of these diagrams in any dimension. Here we include this characterization in two dimensions.

**Definition 1** [Sabariego and Santos, [7]] A minimum distance diagram is any map $D : \mathbb{Z}_N \rightarrow \mathbb{N}^2$ with the following two properties:

(a) For every $[m]_N \in \mathbb{Z}_N$, $D([m]_N) = (i, j)$ satisfies $ia + jb \equiv m$ (mod $N$) and $\|D([m]_N)\|$ is minimum among all the vectors in $\mathbb{N}^2$ with that property ($\|s, t\| = sa + tb$).

(b) For every $[m]_N$ and for every $(s, t) \in \mathbb{N}^2$ that is coordinate-wise smaller than $D([m]_N)$, we have $(s, t) = D([n]_N)$ for some $[n]_N$ (with $n \equiv sa + tb$ (mod $N$)).

An MDD $\mathcal{H}$ is denoted by the lengths of his sides, $\mathcal{H} = L(l, h, w, y)$, with $0 \leq w < l$, $0 \leq y < h$, $\gcd(l, h, w, y) = 1$ and $lh - wy = N$, as it is depicted in the Figure 1. The vectors $u$ and $v$ define the tessellation of the plane by the L-shaped tile $\mathcal{H}$. These lengths fulfill the compatibility
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Figure 1: Generic MDD tessellating the plane

equations, stated by Fiol, Yebra, Alegre and Valero [3] in 1987, related to the tessellation

\[ la - yb \equiv 0 \pmod{N}, \quad wa - hb \equiv 0 \pmod{N}. \] (1)

**Definition 2** [Tessellation related to \( S \)] Let \( S = \langle a, b, N \rangle \) be a 3-numerical semigroup. A tessellation related to \( S \) is a tessellation of the plane generated by an L-shaped MDD of the weighted double-loop digraph \( G(N; a, b; a, b) \).

Let \( D \) be the map that appears in Definition 1 associated with \( G = G(N; a, b; a, b) \), that is \( a = a \) and \( b = b \). Then

\[ \text{Ap}(S, N) = \{ D([0]_N), ..., D([N - 1]_N) \} \]

and \( D([m]_N) \) can be though as the length of a minimum path from \([0]_N\) to \([m]_N\) in \( G \). Definition 2 gives a metrical view of some properties of \( S \). A geometrical characterization of MDD related to \( S \) is needed for practical reasons. This characterization is given in the following result.

**Theorem 3** (A., Miralles and Zaragozá, [1]) The L-shaped tile \( \mathcal{H} = L(l, h, w, y) \) satisfying (1) with \( lh - wy = N \) and \( \gcd(l, h, w, y) = 1 \) is related to \( S = \langle a, b, N \rangle \) iff \( la \geq yb \) and \( hb \geq wa \) and both equalities are not satisfied.
Example 4 Consider the weighted double-loop digraph $G = G(8; 3, 7; 3, 7)$ that is depicted in the Figure 2. An L-shaped MDD related to $G$ is $H = L(5, 2, 2, 1)$. Note that the lengths of $H$, $(l, h, w, y) = (5, 2, 2, 1)$, fulfill the conditions $\gcd(l, h, w, y) = 1$ and $lh - wy = N$, the compatibility equations (1) and Theorem 3. The left-hand side of Figure 2 shows a piece of the first quadrant of the squared plane and how $H$ tessellates the plane. It also shows the periodic distribution of the equivalence classes modulo 8, where each unit square $(i, j)$ is labelled by the class $[3i + 7j]_8$. The right-hand side of this figure shows the same piece of the first quadrant, however each unit square $(i, j)$ is labelled now by $\|D([3i + 7j]_8)\| = 3i + 7j$ ($D$ is the map of Definition 1). Note that the labels inside the grey L-shape (the one that contains the unit square $(0, 0)$) form the set $\mathbb{A}p((3, 7, 8), 8)$. In particular, we have $f((3, 7, 8)) = 13 - 8 = 5$.

V. Arnold [2] in 2009 commented that his 1999 calculations of Frobenius numbers provided hundreds of empirical properties. He remarked some strange arithmetical facts like

$$\frac{c(13, 32, 52)}{c(13, 33, 51)} = \frac{c(9, 43, 45)}{c(9, 42, 46)} = \frac{c(5, 35, 37)}{c(5, 34, 38)} = 2, \quad \frac{c(4, 20, 73)}{c(4, 19, 74)} = 4. \quad (2)$$

It was shown in [1] that if $H = L(l, h, w, y)$ is related to $S = \langle a, b, N \rangle$, then the Frobenius number is

$$f(\langle a, b, N \rangle) = \max\{(l - 1)a + (h - y - 1)b, (l - w - 1)a + (h - 1)b\} - N. \quad (3)$$

Therefore, from the identities $c(S) = f(S) + 1$ and (3), arithmetic-like links between conductors as those appearing in (2) can be though as geometrical-like relations between related L-shaped MDD tiles.

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**Figure 2:** Minimum distance diagram related to $G(8; 3, 7; 3, 7)$
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When the semigroup is 2-minimally generated, that is $S = \langle a, b \rangle$ with $\gcd(a, b) = 1$, it is well known that his Frobenius number is

$$f(\langle a, b \rangle) = ab - a - b.$$  \hfill (4)

Although this result was published by Sylvester [9] in 1884, it seems to be true that (4) was given first by Frobenius in his lectures. Therefore, the conductor is given by the expression $c(a, b) = f(\langle a, b \rangle) + 1 = (a - 1)(b - 1)$.

In this work, several infinite families of pairs of 3-numerical semigroups are given such that each pair fulfills a (2)-like relation.

## 2 Computer assisted numerical remarks

Properties in (2) suggest looking for semigroups like

$$\frac{c(\alpha, n, m)}{c(\alpha, n - 1, m + 1)} = k,$$  \hfill (5)

where $\langle \alpha, n, m \rangle$ and $\langle \alpha, n - 1, m + 1 \rangle$ are 2 and 3 minimally generated numerical semigroups respectively, for different natural numbers $n$ and $m$ and fixed values of $\alpha$ and $k$.

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Table 1: Cardinalities of some sets $P(\alpha, k, 100)$

Let us consider the set

$$P(\alpha, k, \ell) = \{ \langle \alpha, n, m \rangle \mid \frac{c(\alpha, n, m)}{c(\alpha, n - 1, m + 1)} = k, \; m \leq \ell \}$$

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where \( \langle \alpha, n, m \rangle \) and \( \langle \alpha, n - 1, m + 1 \rangle \) are 2 and 3 minimally generated. A computer search reveals the cardinality of some sets \( P(\alpha, k, 100) \). These cardinalities are included in Table 1.

Let us consider now the set \( Q(\alpha, k, \ell) \), defined as \( P(\alpha, k, \ell) \) but now both semigroups \( \langle \alpha, n, m \rangle \) and \( \langle \alpha, n - 1, m + 1 \rangle \) are 3-minimally generated. The cardinalities of \( Q(\alpha, k, 100) \), with \( \alpha = 4, \ldots, 10 \), are 276, 5, 0, 15, 0, 218 and 4, respectively. We have now \( Q(\alpha, k, 100) = \emptyset \) for \((\alpha, k)\in\{4, \ldots, 10\}\times\{2, 3\}\).

Let us denote the sets
\[
P(\alpha, k) = \bigcup_{\ell \geq \alpha + 2} P(\alpha, k, \ell) \quad \text{and} \quad Q(\alpha, k) = \bigcup_{\ell \geq \alpha + 2} Q(\alpha, k, \ell).
\]

We use the numerical data of this section to search infinite families of pairs of semigroups belonging to \( P(\alpha, k) \) or \( Q(\alpha, k) \), for some values of \( \alpha \) and \( k \).

3 Infinite families

In this section we use the L-shaped tile technique included in Section 1 for finding infinite families of 3-numerical semigroups that belong to \( P(4,1) \), \( P(7,3) \) and \( Q(9,1) \).

**Theorem 5** Let us consider the 3-numerical semigroups \( S_t = \langle 4, 4t + 3, 8t + 6 \rangle \) for \( t \geq 1 \). Then \( \{S_t\}_{t \geq 1} \subset P(4,1) \).

**Proof:** Let us consider \( S_t \) and \( T_t = \langle 4, 4t + 2, 8t + 7 \rangle \). First, we check that \( S_t \) and \( T_t \) are numerical semigroups for \( t \geq 1 \), that is \( \gcd(4, 4t + 3, 8t + 6) = \gcd(4, 4t + 2, 8t + 7) = 1 \),
\[
\gcd(4, 4t + 3, 8t + 6) = \gcd(4, 3, 6) = \gcd(3, 2) = 1,
\]
\[
\gcd(4, 4t + 2, 8t + 7) = \gcd(4, 2, 7) = \gcd(2, 7) = 1.
\]

Second, we have to see that \( S_t \) and \( T_t \) are 2 and 3 minimally generated, respectively. To this end, note that \( 8t + 6 = 2 \times (4t + 3) \) and so \( S_t = \langle 4, 4t + 3, 8t + 6 \rangle = \langle 4, 4t + 3 \rangle \), that is a 2-minimally generated semigroup because \( 4t + 3 \) can not be a multiple of 4. Consider now \( T_t = \langle 4, 4t + 2, 8t + 7 \rangle \), we have that neither \( 4t + 2 \) nor \( 4t + 7 \) are multiples of 4; also \( 8t + 7 \) is not a multiple of \( 4t + 2 \). Let us see also that \( 8t + 7 \not\in \langle 4, 4t + 2 \rangle \), that is \( 8t + 7 \neq c_t \times 4 + d_t \times (4t + 2) \) with \( c_t, d_t \in \mathbb{N} \), for \( t \geq 1 \); if so, the
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even number \( c_t \times 4 + d_t \times (4t + 2) \) would equalize the odd one \( 8t + 7 \), a contradiction.

Third, we have to see the identity \( c(S_t) = c(T_t) \), for all \( t \geq 1 \). The conductor \( c(S_t) \) is easy to compute because \( S_t \) is 2-generated and we can apply (4), that is \( c(a, b) = f(a, b) + 1 = (a - 1)(b - 1) \). So, \( c(S_t) = (4 - 1)(4t + 3 - 1) = 12t + 6 \). To compute the conductor \( c(T_t) \), we use the expression (3). To this end, we have to find the related sequence of L-shaped minimum distance diagrams.

Let us see that \( T_t \) has related the L-shaped MDD \( H_t = L(5t + 4, 2t + 1, 1) \), for all \( t \geq 1 \). Obviously \( \text{gcd}(5t + 4, 2t + 1, 1) = 1 \). Set \( N_t = 8t + 7 \), \( a_t = 4 \), \( b_t = 4t + 2 \), \( l_t = 5t + 4 \), \( h_t = 2 \), \( w_t = 2t + 1 \) and \( y_t = 1 \). It is easily checked that \( l_t h_t - w_t y_t = (5t + 4) \times 2 - (2t + 1) = N_t \) and the compatibility equations (1)

\[
\begin{align*}
l_t a_t - y_t b_t &\equiv 0 \pmod{N_t} \iff 20t + 16 - 4t - 2 = 16t + 14 \equiv 0 \pmod{N_t}, \\
h_t b_t - w_t a_t &\equiv 0 \pmod{N_t} \iff 8t + 4 - 8t - 4 = 0 \equiv 0 \pmod{N_t}.
\end{align*}
\]

\( H_t \) is also an MDD because Theorem 3 is fulfilled, that is \( l_t a_t > y_t b_t \) and \( h_t b_t = w_t a_t \), for all \( t \geq 1 \). Therefore \( H_t \) is related to \( T_t \) and we can use the expression (3) to compute the conductor \( c(T_t) \)

\[
c(T_t) = f(T_t) + 1 = \max\{(5t + 3) \times 4 + 0, (3t + 2) \times 4 + 4t + 3\} - 8t - 7 + 1 = 12t + 6.
\]

Hence, \( c(S_t) = c(T_t) \) as it is stated. \( \square \)

**Theorem 6** Consider the 3-numerical semigroups \( S_t = \langle 7, 7t + 7, 14t + 9 \rangle \) for \( t \geq 1 \). Then \( \{S_t\}_{t \geq 1} \subset P(7, 3) \).

**Proof:** Consider \( S_t \) and \( T_t = \langle 7, 7t + 6, 14t + 10 \rangle \). We have \( \text{gcd}(7, 7t + 7, 14t + 9) = \text{gcd}(7, 7t + 6, 14t + 10) = 1 \), so \( S_t \) and \( T_t \) are numerical semigroups. The semigroup \( S_t \) is minimally 2-generated and \( S_t = \langle 7, 14t + 9 \rangle \), so his conductor is \( c(S_t) = (7 - 1)(14t + 9 - 1) = 84t + 48 \).

Let us see that \( T_t \) is 3-minimally generated. We have \( 7 \nmid 7t + 6, 7 \nmid 14t + 10 \) and \( 7 \nmid 6 \nmid 14t + 10 \), for all \( t \geq 1 \). We have to see now \( 14t + 10 \not\in (7, 7t + 6) \). If \( 7m_t + (7t + 6) \times n_t = 14t + 10 \) with \( m_t, n_t \in \mathbb{N} \), then \( 0 \leq n_t \leq 1 \) (if \( n_t \geq 2 \) then \( n_t \times (7t + 6) > 14t + 10 \)). If \( n_t = 0 \), the identity can not be satisfied, hence \( n_t = 1 \). So the equality turns to be \( 7m_t = 7t + 4 \) that has no solution for \( m_t \in \mathbb{N} \) because \( 7m_t \equiv 0 \pmod{7} \) and \( 7t + 4 \equiv 4 \pmod{7} \). Therefore, the semigroup \( T_t \) is 3-minimally generated.
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The semigroup \( T_t \) has related the L-shaped MDD \( \mathcal{H}_t = L(5t+4,4,2t+2,3,2) \), that is \( \gcd(5t+4,4,2t+2,3) = 1 \), his area is \( 14t+10 \) and \( \mathcal{H}_t \) fulfills the compatibility equations (1) and Theorem 3. Therefore, by using (3), his conductor is

\[ c(T_t) = \max\{ (5t+3) \times 7 + 0, (3t+1) \times 7 + 3 \times (7t+6) \} - 14t - 10 = 28t + 16. \]

So \( c(S_t) = 3c(T_t) \) as it is stated. □

**Theorem 7** Consider the 3-numerical semigroups \( S_t = \langle 9, 9t + 7, 9t + 12 \rangle \) for \( t \geq 1 \). Then \( \{S_t\}_{t \geq 1} \subset Q(9,1) \).

**Proof:** Consider \( S_t \) ad \( T_t = \langle 9, 9t + 6, 9t + 13 \rangle \). From the identities \( \gcd(9, 9t + 7, 9t + 12) = \gcd(9, 9t + 6, 9t + 13) = 1 \), the semigroups \( S_t \) and \( T_t \) are numerical semigroups. Let us see that both semigroups are 3-minimally generated.

From \( 9 \not| 9t + 7, 9t + 6, 9t + 12, 9t + 13 \) and \( 9t + 7 \not| 9t + 12 \) and \( 9t + 6 \not| 9t + 13 \), we have to see \( 9t + 12 \not\in \langle 9, 9t + 7 \rangle \) and \( 9t + 13 \not\in \langle 9, 9t + 6 \rangle \). Let us assume that \( 9 \times m_t + (9t + 7) \times n_t = 9t + 12 \) with \( m_t, n_t \in \mathbb{N} \) and \( 0 \leq n_t \leq 1 \) (if \( n_t \geq 2 \) then \( n_t \times (9t + 7) > 9t + 12 \)). Then \( n_t = 1 \) because \( 9 \not| 9t + 12 \) and so we have the identity \( 9m_t = 5 \) for \( m_t \in \mathbb{N} \), that is a contradiction. A similar argument proves that \( 9t + 13 \not\in \langle 9, 9t + 6 \rangle \).

It can be checked that \( S_t \) and \( T_t \) have related the L-shaped minimum distance diagrams \( L(3t+4,3,2t+1,0) \) and \( L(4t+5,3,3t+2,1) \), respectively. Therefore, from (3), we have \( c(S_t) = c(T_t) = 36t + 30. \) □

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