Introduction to Mathematical Finance

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Contents

- Financial markets and derivatives
- Basic derivative pricing and hedging
- Advanced derivatives

Banking

- Retail banking
  - Deposit-taking and loans to individuals
- Commercial banking
  - Loans to companies
- Investment banking
  - Issuance and trading of securities
  - Corporate finance
  - Derivatives of primary instruments

Securities

- Equity
  - Shares (stocks) issued by companies
  - Pay dividends linked to profitability
  - Loses out if company does badly
- Debt
  - Bonds issued by companies, sovereigns, etc
  - Pay fixed coupon
  - Higher-priority claim on assets if entity defaults
- Other assets
  - Foreign exchange, eg USD/JPY, EUR/USD
  - Commodities: Oil, Gold, agriculturals, metals

Derivatives

- Derivatives are contracts, usually over-the-counter (OTC), to exchange future random cash flows which are defined in terms of the prices of primary instruments.

- Examples:
  - Forwards – agreement to buy/sell later at a price fixed now
  - Options – right to buy/sell later at a fixed price
  - Swaps – set of cash flow exchanges to swap, for instance, fixed interest payments into floating
  - Callable products – exotic swap products which can be terminated early

Forwards

- Suppose we are an equity derivatives desk. Our customer wants to buy a certain stock one year forward.

- Here are some facts we know about the stock:
  - The current price is 50
  - Interest rates are 10%
  - Our research analysts think the stock will be priced somewhere between 33-66 in a year
  - The stock does not pay any dividends

- What price should we charge?

Forward pricing

- We could charge 66. In a year's time the customer will give us 66 which is probably (if our analysts are right) enough to buy the stock and deliver it.
- But we could be more clever. Suppose we borrow 50 units of cash and buy the stock now.
  - In one year's time we will have to stock ready to deliver
  - Our debt will have increased to 55 (110% x 50)
- So we could charge the customer 55 for forward purchase.
- We can also reverse this for forward-selling and get the same risk-free price.

Options

- For the same stock, our customer now wants an option to buy the stock in one year for 55.
- Our research analysts are now convinced that the stock will be worth either 33 or 66, and they think these outcomes are equally likely.
- If the stock goes up, we will have to pay 11. If it goes down, we pay nothing

- What should we charge the customer for the option?

Option pricing

- A simple answer is to charge 5.50, which is the expected payoff. (Or rather 5.00 to allow for discounting.)
- But again we can do better. Suppose we have (borrow) β units of cash and buy α units of stock. The value of this portfolio will be:
  - $V_u = 66\alpha + 1.10\beta$ if the stock goes up
  - $V_d = 33\alpha + 1.10\beta$ if the stock goes down

Option pricing

- We can now try to match the option payoff in both cases by solving the linear equations:
  - $V_u = 66\alpha + 1.10\beta = 11$
  - $V_d = 33\alpha + 1.10\beta = 0$
- This has the solution $\alpha = 1/3$, $\beta = -10$
- The cost of this portfolio today is
  - $V = 50/3 - 10 = 6.667$
  - This is what we should charge for the option

General Options

- Suppose we have an arbitrary option X which pays X_u if the stock goes up, and X_d if it goes down. Then we can solve again:
  - $V_u = 66\alpha + 1.10\beta = X_u$
  - $V_d = 33\alpha + 1.10\beta = X_d$
- This has the solution $\alpha = (X_u - X_d)/33$, $\beta = (2X_d - X_u)/1.10$
- The cost of this portfolio today is
  - $V = 50\alpha + \beta = 1.10^{-1}(2/3\ X_u + 1/3\ X_d)$
  - This is the arbitrage-free price for the option

Crucial observation

- Consider the option pricing formula:
  - $V = 1.10^{-1}(2/3\ X_u + 1/3\ X_d)$
- It resembles the discounted expected payoff of the derivative, but using different probabilities
  - Up-chance is 2/3, down-chance is 1/3
- Allowing for interest-rates, the stock goes from 50 to either 30 or 60
  - Using these new probabilities, the expected future discounted value of the stock is exactly 50
  - Buying or selling the stock is "fair" under these probabilities
- A martingale is a process M_t, for which $E(M_t)=M_0$ and
  - $E(M_T|F_t) = M_t$, for $t < T$
  - The expected future value conditional on the past is the present value

Theory of option pricing

- Let us use some notation:
  - S_0, stock today, S_T stock at time T
  - B_0, cash today (1), B_T cash at time T (1.10)
  - X derivative on S paying at T, such as $X=f(S_T)$
  - Z discounted stock process, $Z_t = S_t / B_t$
  - Q equivalent martingale measure such that Z_t is a martingale under Q
- The cost of this portfolio today is
  - $V = \mathrm{E}_{\mathrm{Q}}(X/B_T)$
  - This price can be enforced by hedging

General Theory 1

- Theorem 1 (Harrison and Pliska)
  - The market is arbitrage-free if and only if there is at least one equivalent martingale measure (EMM), under which the discounted asset prices are martingales
  - In which case, every possible derivative can be replicated by hedging if and only if there is exactly one such EMM and no other.

General Theory 2

- Fundamental Theorem of Finance
  - If both parts of Theorem 1 hold, then for any numeraire asset B_t, there is an EMM Q for that numeraire so that
  - Any asset, discounted by B_t, is a Q-martingale
  - If X is a derivative paying at T then its value at time t is $V_t = B_t \, E_Q(X / B_T \mid F_t)$.
  - This price can be enforced by hedging

Summary

- Derived products can be hedged using vanilla instruments
- The cost of the hedge is the expected discounted payoff under a special measure
- Risk-free replication of payoffs enforces this theoretical price
- Banks need mathematicians to make this work in practice

Where to Get More Information

- *Financial Calculus*, Baxter and Rennie, CUP 1996.
- *Options, Futures and Other Derivatives*, Hull, Prentice Hall 2005
- *Probability with Martingales*, Williams, CUP 1991
- *An Elementary Introduction to Mathematical Finance*, Ross, CUP 2002