Contents

- Financial markets and derivatives
- Basic derivative pricing and hedging
- Advanced derivatives
Banking

- **Retail banking**
  - Deposit-taking and loans to individuals
- **Commercial banking**
  - Loans to companies
- **Investment banking**
  - Issuance and trading of securities
  - Corporate finance
  - Derivatives of primary instruments
Securities

- **Equity**
  - Shares (stocks) issued by companies
  - Pay dividends linked to profitability
  - Loses out if company does badly

- **Debt**
  - Bonds issued by companies, sovereigns, etc
  - Pay fixed coupon
  - Higher-priority claim on assets if entity defaults

- **Other assets**
  - Foreign exchange, eg USD/JPY, EUR/USD
  - Commodities: Oil, Gold, agriculturals, metals
Derivatives

- Derivatives are contracts, usually over-the-counter (OTC), to exchange future random cash flows which are defined in terms of the prices of primary instruments.

- Examples:
  - Forwards – agreement to buy/sell later at a price fixed now
  - Options – right to buy/sell later at a fixed price
  - Swaps – set of cash flow exchanges to swap, for instance, fixed interest payments into floating
  - Callable products – exotic swap products which can be terminated early
Suppose we are an equity derivatives desk. Our customer wants to buy a certain stock one year forward.

Here are some facts we know about the stock:
- The current price is 50
- Interest rates are 10%
- Our research analysts think the stock will be priced somewhere between 33-66 in a year
- The stock does not pay any dividends

What price should we charge?
Forward pricing

- We could charge 66. In a year’s time the customer will give us 66 which is probably (if our analysts are right) enough to buy the stock and deliver it.

- But we could be more clever. Suppose we borrow 50 units of cash and buy the stock now.
  - In one year’s time we will have to stock ready to deliver
  - Our debt will have increased to 55 (110% x 50)

- So we could charge the customer 55 for forward purchase.

- We can also reverse this for forward-selling and get the same risk-free price.
Options

- For the same stock, our customer now wants an option to buy the stock in one year for 55.
- Our research analysts are now convinced that the stock will be worth either 33 or 66, and they think these outcomes are equally likely.
- If the stock goes up, we will have to pay 11. If it goes down, we pay nothing.

- What should we charge the customer for the option?
Option pricing

• A simple answer is to charge 5.50, which is the expected payoff. (Or rather 5.00 to allow for discounting.)
• But again we can do better. Suppose we have (borrow) $\beta$ units of cash and buy $\alpha$ units of stock. The value of this portfolio will be:
  - $V_u = 66\alpha + 1.10\beta$ if the stock goes up
  - $V_d = 33\alpha + 1.10\beta$ if the stock goes down
Option pricing

- We can now try to match the option payoff in both cases by solving the linear equations:
  - $V_u = 66\alpha + 1.10\beta = 11$
  - $V_d = 33\alpha + 1.10\beta = 0$
- This has the solution $\alpha = 1/3$, $\beta = -10$
- The cost of this portfolio today is
  - $V = 50/3 - 10 = 6.667$
  - This is what we should charge for the option
General Options

- Suppose we have an arbitrary option $X$ which pays $X_u$ if the stock goes up, and $X_d$ if it goes down. Then we can solve again:
  - $V_u = 66\alpha + 1.10\beta = X_u$
  - $V_d = 33\alpha + 1.10\beta = X_d$
- This has the solution $\alpha = \frac{(X_u - X_d)}{33}$, $\beta = \frac{(2X_d - X_u)}{1.10}$
- The cost of this portfolio today is
  - $V = 50\alpha + \beta = 1.10^{-1}(\frac{2}{3}X_u + \frac{1}{3}X_d)$
  - This is the arbitrage-free price for the option
Crucial observation

- Consider the option pricing formula:
  \[ V = 1.10^{-1}( 2/3 X_u + 1/3 X_d ) \]
- It resembles the discounted expected payoff of the derivative, but using different probabilities
  - Up-chance is 2/3, down-chance is 1/3
- Allowing for interest-rates, the stock goes from 50 to either 30 or 60
  - Using these new probabilities, the expected future discounted value of the stock is exactly 50
  - Buying or selling the stock is “fair” under these probabilities
- A martingale is a process \( M_t \), for which \( E(M_t) = M_0 \) and
  - \( E(M_T | F_t) = M_t \), for \( t < T \)
  - The expected future value conditional on the past is the present value
Theory of option pricing

- Let us use some notation:
  - $S_0$, stock today, $S_T$ stock at time $T$
  - $B_0$, cash today (1), $B_T$ cash at time $T$ (1.10)
  - $X$ derivative on $S$ paying at $T$, such as $X = f(S_T)$
  - $Z$ discounted stock process, $Z_t = S_t / B_t$
  - $Q$ equivalent martingale measure such that $Z_t$ is a martingale under $Q$

- The cost of this portfolio today is
  - $V = E_Q(X/B_T)$
  - This price can be enforced by hedging
General Theory 1

- Theorem 1 (Harrison and Pliska)
  - The market is arbitrage-free if and only if there is at least one equivalent martingale measure (EMM), under which the discounted asset prices are martingales
  - In which case, every possible derivative can be replicated by hedging if and only if there is exactly one such EMM and no other.
General Theory 2

- Fundamental Theorem of Finance
  - If both parts of Theorem 1 hold, then for any numeraire asset $B_t$, there is an EMM $Q$ for that numeraire so that
  - Any asset, discounted by $B_t$, is a $Q$-martingale
  - If $X$ is a derivative paying at $T$ then its value at time $t$ is $V_t = B_t E_Q( X / B_T | F_t )$.
  - This price can be enforced by hedging
Summary

- Derived products can be hedged using vanilla instruments
- The cost of the hedge is the expected discounted payoff under a special measure
- Risk-free replication of payoffs enforces this theoretical price
- Banks need mathematicians to make this work in practice
Where to Get More Information

- *Options, Futures and Other Derivatives*, Hull, Prentice Hall 2005
- *Probability with Martingales*, Williams, CUP 1991
- *An Elementary Introduction to Mathematical Finance*, Ross, CUP 2002