

The planetary N -body problem*

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1 The N -body problem: a continuing mathematical challenge

The problem of the motion of $N \geq 2$ point-masses (i.e., ideal bodies with no physical dimension identified with points in the Euclidean three-dimensional space) interacting only through Newton’s law of mutual gravitational attraction, has been a central issue in astronomy, physics and mathematics since the early developments of modern calculus. When $N = 2$ the problem has been completely solved (“integrated”) by Kepler: the motion takes place on conics, whose focus is occupied by the center of mass of the two bodies; but for $N \geq 3$ a complete understanding of the problem is still far away.

While the original impulse, coming from astronomy, has been somehow shaded by the massive use of machines for computing orbits of celestial bodies or satellites, the mathematical richness and beauty of the N -body problem has retained most of its original attraction; for a selection of recent contributions, see, e.g., [6], [15], [17], [5], [16], [11].

Here, we will be concerned with the *planetary N -body problem*, which, as the name says, deals with the case of one body (the “Sun” or the “Star”) having mass much bigger than the remaining bodies (“planets”). The main question is then to determine “general” conditions under which the planets revolve around the Sun without collisions and in an “regular way” so that, in particular, no planet crashes on another planet or onto the Sun, nor it escapes away from such “solar system”.

Despite the efforts of Newton, Euler, d’Alembert, Lagrange, Laplace and, especially, Henri Poincaré and G.D. Birkhoff, such question remained essentially unanswered for centuries. It is only with the astonishing work of a 26-year-old mathematician, V.I. Arnold (1937–2010), that a real breakthrough was achieved. Arnold, continuing and extending fundamental analytical discoveries of his advisor A.N. Kolmogorov on so called “small divisors” (singularities appearing in the

perturbative expansions of orbital trajectories), stated in 1963 [1] a result, which may roughly formulated as follows¹.

If the masses of the planets are small enough compared to the mass of the Sun, there exists, in the phase space of the planetary N -body problem, a bounded set of positive Lebesgue measure corresponding to planetary motions with bounded relative distances; such motions are well approximated by Keplerian ellipses with small eccentricities and small relative inclinations.

Arnold gave a brilliant proof in a special case, namely, the planar three-body problem (two planets), giving some suggestions on how to generalize his proof to the general case (arbitrary number of planets in space). However, a complete generalization of his proof turned out to be quite a difficult task, which took nearly another fifty years to be completed: the first complete proof, based on work by M.R. Herman, appeared in [14] and a full generalization of Arnold's approach in [11].

The main reason beyond the difficulties which arise in the general spatial case, is related to the presence of certain “secular degeneracies” which do not allow a *tout court* application of Arnold's “fundamental theorem” (see § 3.2 below) to the general planetary case.

In this article we shall give a brief account (avoiding computations) of these results trying to explain the main ideas and technical tools needed to overcome the difficulties involved.

2 The classical Hamiltonian structure

2.1 Newton equations and their Hamiltonian version

The starting point are the Newton's equations for $1 + n$ bodies (point masses), interacting only through gravitational attraction:

$$\ddot{u}^{(i)} = \sum_{\substack{0 \leq j \leq n \\ j \neq i}} m_j \frac{u^{(j)} - u^{(i)}}{|u^{(i)} - u^{(j)}|^3}, \quad i = 0, 1, \dots, n, \quad (1.1)$$

where $u^{(i)} = (u_1^{(i)}, u_2^{(i)}, u_3^{(i)}) \in \mathbb{R}^3$ are the cartesian coordinates of the i^{th} body of mass $m_i > 0$, $|u| = \sqrt{u \cdot u} = \sqrt{\sum_i u_i^2}$ is the standard Euclidean norm, “dots” over functions denote time derivatives, and the gravitational constant has been set to one (which is possible by rescaling time t).

¹ *Verbatim* formulations are given in § 3.1 below.

Equations (1.1) are invariant by change of “inertial frames”, i.e., by change of variables of the form $u^{(i)} \rightarrow u^{(i)} - (a + ct)$ with fixed $a, c \in \mathbb{R}^3$. This allows to restrict the attention to the manifold of “initial data” given by²

$$\sum_{i=0}^n m_i u^{(i)}(0) = 0, \quad \sum_{i=0}^n m_i \dot{u}^{(i)}(0) = 0. \quad (1.2)$$

The total linear momentum $M_{\text{tot}} := \sum_{i=0}^n m_i \dot{u}^{(i)}$ does not change along the flow of (1.1), i.e., $\dot{M}_{\text{tot}} = 0$ along trajectories; therefore, by (1.2), $M_{\text{tot}}(t)$ vanishes for all times. But, then, also the position of the barycenter $B(t) := \sum_{i=0}^n m_i u^{(i)}(t)$ is constant ($\dot{B} = 0$) and, again by (1.2), $B(t) \equiv 0$. In other words, the manifold of initial data (1.2) is invariant under the flow (1.1).

Equations (1.1) may be seen as the Hamiltonian equations associated to the Hamiltonian function³

$$\widehat{\mathcal{H}}_{\text{N}} := \sum_{i=0}^n \frac{|U^{(i)}|^2}{2m_i} - \sum_{0 \leq i < j \leq n} \frac{m_i m_j}{|u^{(i)} - u^{(j)}|}, \quad (1.3)$$

where $(U^{(i)}, u^{(i)})$ are standard symplectic variables ($U^{(i)} = m_i \dot{u}^{(i)}$ is the momentum conjugated to $u^{(i)}$) and the phase space is the “collisionless” open domain in $\mathbb{R}^{6(n+1)}$ given by

$$\widehat{\mathcal{M}} := \{U^{(i)}, u^{(i)} \in \mathbb{R}^3 : u^{(i)} \neq u^{(j)}, 0 \leq i \neq j \leq n\}$$

endowed with the standard symplectic form

$$\sum_{i=0}^n dU^{(i)} \wedge du^{(i)} := \sum_{\substack{0 \leq i \leq n \\ 1 \leq k \leq 3}} dU_k^{(i)} \wedge du_k^{(i)}. \quad (1.4)$$

²Replace the coordinates $u^{(i)}$ by $u^{(i)} - (a + ct)$ with

$$a := m_{\text{tot}}^{-1} \sum_{i=0}^n m_i u^{(i)}(0) \quad \text{and} \quad c := m_{\text{tot}}^{-1} \sum_{i=0}^n m_i \dot{u}^{(i)}(0), \quad m_{\text{tot}} := \sum_{i=0}^n m_i.$$

³We recall that the Hamiltonian equations associated to a Hamiltonian function $H(p, q) = H(p_1, \dots, p_n, q_1, \dots, q_n)$, where (p, q) are standard symplectic variables (i.e., the associated symplectic form is $dp \wedge dq = \sum_{i=1}^n dp_i \wedge dq_i$) are given by

$$\begin{cases} \dot{p} = -\partial_q H \\ \dot{q} = \partial_p H \end{cases} \quad \text{i.e.} \quad \begin{cases} \dot{p}_i = -\partial_{q_i} H, \\ \dot{q}_i = \partial_{p_i} H, \end{cases} \quad (1 \leq i \leq n). \quad (*)$$

We shall denote the standard Hamiltonian flow, namely, the solution of (*) with initial data p_0 and q_0 , by $\phi_H^t(p_0, q_0)$. For general information, see [2].

2.2 The Linear momentum reduction

In view of the invariance properties discussed above, it is enough to consider the submanifold

$$\widehat{\mathcal{M}}_0 := \{(U, u) \in \widehat{\mathcal{M}} : \sum_{i=0}^n m_i u^{(i)} = 0 = \sum_{i=0}^n U^{(i)}\},$$

which corresponds to the manifold described in (1.2).

The submanifold $\widehat{\mathcal{M}}_0$ is symplectic, i.e., the restriction of the form (1.4) to $\widehat{\mathcal{M}}_0$ is again a symplectic form⁴.

Following Poincaré, one can perform a *symplectic reduction* (“reduction of the linear momentum”) allowing to lower the number of degrees of freedom⁵ by three units. Indeed, let $\phi_{\text{he}} : (R, r) \rightarrow (U, u)$ be the linear transformation given by

$$\phi_{\text{he}} : \begin{cases} u^{(0)} = r^{(0)}, & u^{(i)} = r^{(0)} + r^{(i)}, \quad (i = 1, \dots, n) \\ U^{(0)} = R^{(0)} - \sum_{i=1}^n R^{(i)}, & U^{(i)} = R^{(i)}, \quad (i = 1, \dots, n); \end{cases} \quad (1.5)$$

such transformation is symplectic⁶, i.e.,

$$\sum_{i=0}^n dU^{(i)} \wedge du^{(i)} = \sum_{i=0}^n dR^{(i)} \wedge dr^{(i)}. \quad (1.6)$$

Letting

$$m_{\text{tot}} := \sum_{i=0}^n m_i$$

one sees that, in the new variables, $\widehat{\mathcal{M}}_0$ reads

$$\left\{ (R, r) \in \mathbb{R}^{6(n+1)} : R^{(0)} = 0, \quad r^{(0)} = -m_{\text{tot}}^{-1} \sum_{i=1}^n m_i r^{(i)} \right. \\ \left. \text{and } 0 \neq r^{(i)} \neq r^{(j)} \forall 1 \leq i \neq j \leq n \right\}.$$

The restriction of the 2-form (1.4) on $\widehat{\mathcal{M}}_0$ is simply $\sum_{i=1}^n dR^{(i)} \wedge dr^{(i)}$ and

$$\begin{aligned} \mathcal{H}_N &:= (\widehat{\mathcal{H}}_N \circ \phi_{\text{he}})|_{\mathcal{M}_0} \\ &= \sum_{i=1}^n \left(\frac{|R^{(i)}|^2}{2 \frac{m_0 m_i}{m_0 + m_i}} - \frac{m_0 m_i}{|r^{(i)}|} \right) + \sum_{1 \leq i < j \leq n} \left(\frac{R^{(i)} \cdot R^{(j)}}{m_0} - \frac{m_i m_j}{|r^{(i)} - r^{(j)}|} \right). \end{aligned}$$

⁴Indeed: $\left(\sum_{i=0}^n dU^{(i)} \wedge du^{(i)} \right) \Big|_{\widehat{\mathcal{M}}_0} = \sum_{i=1}^n \frac{m_0 + m_i}{m_0} dU^{(i)} \wedge du^{(i)}$.

⁵The number of degree of freedom of an autonomous Hamiltonian system is half of the dimension of the phase space (classically, the dimension of the configuration space).

⁶We recall that this means, in particular, that in the new variables the Hamiltonian flow is again standard: more precisely, one has that $\phi_{\widehat{\mathcal{H}}_N}^t \circ \phi_{\text{he}} = \phi_{\text{he}} \circ \phi_{\widehat{\mathcal{H}}_N \circ \phi}^t$.

Thus, the dynamics generated by $\widehat{\mathcal{H}}_N$ on $\widehat{\mathcal{M}}_0$ is equivalent to the dynamics generated by the Hamiltonian $(R, r) \in \mathbb{R}^{6n} \rightarrow \mathcal{H}_N(R, r)$ on

$$\mathcal{M}_0 := \left\{ (R, r) = (R^{(1)}, \dots, R^{(n)}, r^{(1)}, \dots, r^{(n)}) \in \mathbb{R}^{6n} : \right. \\ \left. 0 \neq r^{(i)} \neq r^{(j)} \forall 1 \leq i \neq j \leq n \right\}$$

with respect to the standard symplectic form $\sum_{i=1}^n dR^{(i)} \wedge dr^{(i)}$; to recover the full dynamics on $\widehat{\mathcal{M}}_0$ from the dynamics on \mathcal{M}_0 one will simply set $R^{(0)}(t) \equiv 0$ and $r^{(0)}(t) := -m_{\text{tot}}^{-1} \sum_{i=1}^n m_i r^{(i)}(t)$.

Since we are interested in the planetary case, we perform the trivial rescaling by a small positive parameter μ :

$$m_0 := m_0, \quad m_i = \mu m_i \quad (i \geq 1), \quad X^{(i)} := \frac{R^{(i)}}{\mu}, \quad x^{(i)} := r^{(i)}, \\ \mathcal{H}_{\text{plt}}(X, x) := \frac{1}{\mu} \mathcal{H}_N(\mu X, x),$$

which leaves unchanged Hamilton's equations. Explicitly, if

$$M_i := \frac{m_0 m_i}{m_0 + \mu m_i}, \quad \text{and} \quad \bar{m}_i := m_0 + \mu m_i,$$

then

$$\mathcal{H}_{\text{plt}}(X, x) := \sum_{i=1}^n \left(\frac{|X^{(i)}|^2}{2M_i} - \frac{M_i \bar{m}_i}{|x^{(i)}|} \right) + \mu \sum_{1 \leq i < j \leq n} \left(\frac{X^{(i)} \cdot X^{(j)}}{m_0} - \frac{m_i m_j}{|x^{(i)} - x^{(j)}|} \right) \\ =: \mathcal{H}_{\text{plt}}^{(0)}(X, x) + \mu \mathcal{H}_{\text{plt}}^{(1)}(X, x), \quad (1.7)$$

the phase space being

$$\mathcal{M} := \left\{ (X, x) = (X^{(1)}, \dots, X^{(n)}, x^{(1)}, \dots, x^{(n)}) \in \mathbb{R}^{6n} : \right. \\ \left. 0 \neq x^{(i)} \neq x^{(j)} \forall 1 \leq i \neq j \leq n \right\}, \quad (1.8)$$

endowed with the standard symplectic form $\sum_{i=1}^n dX^{(i)} \wedge dx^{(i)}$.

Notice that while $\sum_{i=1}^n X^{(i)}$ is obviously *not* an integral⁷ for \mathcal{H}_{plt} , the transformation (1.5) does preserve the total *angular momentum* $\sum_{i=0}^n U^{(i)} \times u^{(i)}$, “ \times ” denoting the standard vector product in \mathbb{R}^3 , so that the total angular momentum

$$C = (C_1, C_2, C_3) := \sum_{i=1}^n C_i, \quad C_i := x^{(i)} \times X^{(i)}, \quad (1.9)$$

⁷We recall that $F(X, x)$ is an integral for $\mathcal{H}(X, x)$ if $\{F, \mathcal{H}\} = 0$ where $\{F, G\} = F_X \cdot G_x - F_x \cdot G_X$ denotes the (standard) Poisson bracket.

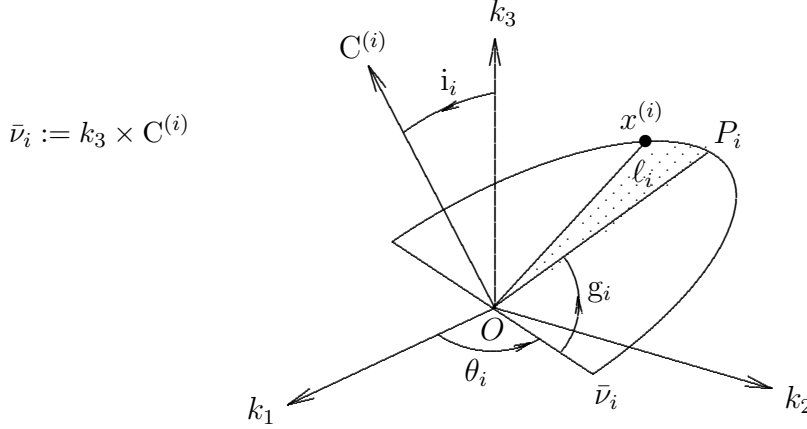


Figure 1: Spatial Delaunay angle variables.

is still a (vector-valued) integral for \mathcal{H}_{plt} . The integrals C_i , however, do not commute (i.e., their Poisson brackets do not vanish):

$$\{C_1, C_2\} = C_3, \quad \{C_2, C_3\} = C_1, \quad \{C_3, C_1\} = C_2,$$

but, for example, $|C|^2$ and C_3 are two commuting, independent integrals.

2.3 Delaunay variables

The Hamiltonian $\mathcal{H}_{\text{plt}}^{(0)}$ in (1.7) governs the motion of n decoupled two-body problems with Hamiltonian

$$h_{2\text{B}}^{(i)} = \frac{|X^{(i)}|^2}{2M_i} - \frac{M_i \bar{m}_i}{|x^{(i)}|}, \quad (X^{(i)}, x^{(i)}) \in \mathbb{R}^3 \times \mathbb{R}_*^3 := \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}). \quad (1.10)$$

Such two-body systems are, as well known, integrable. The explicit “symplectic integration” is done by means of the *Delaunay variables*, whose construction we, now, briefly, recall (for full details and proofs, see, e.g., [4, §3.2]).

Assume that $h_{2\text{B}}^{(i)}(X^{(i)}, x^{(i)}) < 0$ so that the Hamiltonian flow $\phi_{h_{2\text{B}}^{(i)}}^t(X^{(i)}, x^{(i)})$ evolves on a Keplerian ellipse \mathfrak{E}_i and assume that the eccentricity $e_i \in (0, 1)$.

Let a_i, P_i denote, respectively, the *semimajor axis* and the *perihelion* of \mathfrak{E}_i .

Let $C^{(i)}$ denote the i^{th} angular momentum $C^{(i)} := x^{(i)} \times X^{(i)}$.

Let us, also, introduce the “Delaunay nodes”

$$\bar{v}_i := k^{(3)} \times C^{(i)} \quad 1 \leq i \leq n, \quad (1.11)$$

where $(k^{(1)}, k^{(2)}, k^{(3)})$ is the standard orthonormal basis in \mathbb{R}^3 . Finally, for $u, v \in \mathbb{R}^3$ lying in the plane orthogonal to a non-vanishing vector w , let $\alpha_w(u, v)$ denote the positively oriented angle (mod 2π) between u and v (orientation follows the “right hand rule”).

The *Delaunay action–angle variables* $(\Lambda_i, \Gamma_i, \Theta_i, \ell_i, \mathbf{g}_i, \theta_i)$ are, then, defined as

$$\begin{cases} \Lambda_i := M_i \sqrt{\bar{m}_i} a_i \\ \ell_i := \text{mean anomaly of } x^{(i)} \text{ on } \mathfrak{E}_i \end{cases}$$

$$\begin{cases} \Gamma_i := |\mathbf{C}^{(i)}| = \Lambda_i \sqrt{1 - e_i^2} \\ \mathbf{g}_i := \alpha_{\mathbf{C}^{(i)}}(\bar{\nu}_i, P_i) \end{cases}$$

$$\begin{cases} \Theta_i := \mathbf{C}^{(i)} \cdot \mathbf{k}^{(3)} \\ \theta_i := \alpha_{\mathbf{k}^{(3)}}(\mathbf{k}^{(1)}, \bar{\nu}_i) \end{cases}$$
(1.12)

Notice that the Delaunay’s variables are defined on the set where $e_i \in (0, 1)$ and the nodes $\bar{\nu}_i$ in (1.11) are well defined; on such set the “Delaunay inclinations” i_i defined through the relations

$$\cos i_i := \frac{\mathbf{C}^{(i)} \cdot \mathbf{k}^{(3)}}{|\mathbf{C}^{(i)}|} = \frac{\Theta_i}{\Gamma_i},$$
(1.13)

are well defined and we choose the branch of \cos^{-1} so that $i_i \in (0, \pi)$.

The Delaunay variables become singular when $\mathbf{C}^{(i)}$ is vertical (the Delaunay node is no more defined) and in the circular limit (the perihelion is not unique). In these cases different variables have to been used (see below).

On the set where Delaunay variables are well posed, they define a symplectic set of action–angle variables, meaning that⁸

$$\sum_{i=1}^n dX^{(i)} \wedge dx^{(i)} = \sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\mathbf{g}_i + d\Theta_i \wedge d\theta_i.$$
(1.14)

In Delaunay action–angle variables $((\Lambda, \Gamma, \Theta), (\ell, \mathbf{g}, \theta))$ the Hamiltonian $\mathcal{H}_{\text{plt}}^{(0)}$ takes the form

$$- \sum_{i=1}^n \frac{M_i^3 \bar{m}_i^2}{2\Lambda_i^2} =: h_{\text{K}}(\Lambda).$$
(1.15)

We shall restrict our attention to the phase space

$$\mathcal{M}_{\text{plt}} := \left\{ (\Lambda, \Gamma, \Theta) \in \mathbb{R}^{3n} : \right.$$

$$\left. \Lambda_i > \Gamma_i > \Theta_i > 0, \quad \frac{\Lambda_i}{M_i \sqrt{\bar{m}_i}} \neq \frac{\Lambda_j}{M_j \sqrt{\bar{m}_j}}, \quad \forall i \neq j \right\} \times \mathbb{T}^{3n},$$
(1.16)

endowed with the standard symplectic form

$$\sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\mathbf{g}_i + d\Theta_i \wedge d\theta_i.$$

⁸For a proof, see [4, §3.2].

Notice that the $6n$ -dimensional phase space \mathcal{M}_{plt} is foliated by $3n$ -dimensional $\mathcal{H}_{\text{plt}}^{(0)}$ -invariant tori $\{\Lambda, \Gamma, \Theta\} \times \mathbb{T}^3$, which, in turn, are foliated by n -dimensional tori $\{\Lambda\} \times \mathbb{T}^n$, expressing geometrically the degeneracy of the integrable Keplerian limit of the $(1+n)$ -body problem.

2.4 Poincaré variables and the truncated secular dynamics

A regularization of the Delaunay variables in their singular limit was introduced by Poincaré, in such a way that the set of action-angle variables $((\Gamma, \Theta), (g, \theta))$ is mapped onto cartesian variables regular near the origin, which corresponds to co-circular and co-planar motions, while the angles conjugated to Λ_i , which remains invariant, are suitably shifted.

More precisely, the *Poincaré variables* are given by

$$(\Lambda, \lambda, z) := (\Lambda, \lambda, \eta, \xi, p, q) \in \mathbb{R}_+^n \times \mathbb{T}^n \times \mathbb{R}^{4n} ,$$

with the Λ 's as in (1.12) and

$$\begin{aligned} \lambda_i &= \ell_i + g_i + \theta_i \\ \begin{cases} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)} \cos(\theta_i + g_i) \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)} \sin(\theta_i + g_i) \end{cases} \\ \begin{cases} p_i = \sqrt{2(\Gamma_i - \Theta_i)} \cos \theta_i \\ q_i = -\sqrt{2(\Gamma_i - \Theta_i)} \sin \theta_i \end{cases} \end{aligned} \quad (1.17)$$

Notice that $e_i = 0$ corresponds to $\eta_i = 0 = \xi_i$, while $i_i = 0$ corresponds to $p_i = 0 = q_i$; compare (1.12) and (1.13).

On the domain of definition, the Poincaré variables are symplectic⁹

$$\sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge dg_i + d\Theta_i \wedge d\theta_i = \sum_{i=1}^n d\Lambda_i \wedge d\lambda_i + d\eta_i \wedge d\xi_i + dp_i \wedge dq_i . \quad (1.18)$$

As phase space, we shall consider a collisionless domain around the “secular origin” $z = 0$ (which correspond to co-planar, co-circular motions) of the form

$$(\Lambda, \lambda, z) \in \mathcal{M}_{\text{p}}^{6n} := \mathcal{A} \times \mathbb{T}^n \times B^{4n} \quad (1.19)$$

where \mathcal{A} is a set of well separated semimajor axes

$$\mathcal{A} := \left\{ \Lambda : \underline{a}_j < a_j < \bar{a}_j \quad \text{for } 1 \leq j \leq n \right\} \quad (1.20)$$

⁹For a proof, see [3, Appendix C].

where $\underline{a}_1, \dots, \underline{a}_n, \bar{a}_1, \dots, \bar{a}_n$, are positive numbers verifying $\underline{a}_j < \bar{a}_j < \underline{a}_{j+1}$ for any $1 \leq j \leq n$, $\bar{a}_{n+1} := \infty$, and B^{4n} is a small $4n$ -dimensional ball around the secular origin $z = 0$.

In Poincaré coordinates, the Hamiltonian \mathcal{H}_{plt} (1.7) takes the form

$$\mathcal{H}_{\text{P}}(\Lambda, \lambda, z) = h_{\text{K}}(\Lambda) + \mu f_{\text{P}}(\Lambda, \lambda, z), \quad z := (\eta, p, \xi, q) \in \mathbb{R}^{4n} \quad (1.21)$$

where the ‘‘Kepler’’ unperturbed term h_{K} is as in (1.15).

Because of rotation (with respect the $k^{(3)}$ -axis) and reflection (with respect to the coordinate planes) invariance of the Hamiltonian (1.7), the perturbation f_{P} in (1.21) satisfies well known symmetry relations called *d’Alembert rules*, namely, f_{P} is invariant under the following transformations¹⁰

$$\left\{ \begin{array}{l} (\Lambda, \lambda, \eta, \xi, p, q) \rightarrow (\Lambda, \pi - \lambda, -\eta, \xi, p, -q) \\ (\Lambda, \lambda, \eta, \xi, p, q) \rightarrow (\Lambda, -\lambda, \eta, -\xi, -p, q) \\ (\Lambda, \lambda, \eta, \xi, p, q) \rightarrow (\Lambda, \lambda, \eta, \xi, -p, -q) \\ (\Lambda, \lambda, \eta, \xi, p, q) \rightarrow (\Lambda, \frac{\pi}{2} - \lambda, -\xi, -\eta, q, p) \\ (\Lambda, \lambda, z) \rightarrow \mathcal{R}^g(\Lambda, \lambda, z) \end{array} \right. \quad (1.22)$$

\mathcal{R}^g being a *symplectic rotation by an angle $-g \in \mathbb{T}$* , i.e., a symplectic map of the form

$$\mathcal{R}^g : (\Lambda, \lambda, z) \rightarrow (\Lambda', \lambda', z') \quad \text{with} \quad \Lambda'_i = \Lambda_i, \quad \lambda'_i = \lambda_i + g, \quad z' = \mathcal{S}^g(z), \quad (1.23)$$

and

$$\mathcal{S}^g : z \rightarrow \mathcal{S}^g(z) = \left(\mathcal{S}_g(z_1), \dots, \mathcal{S}_g(z_{4n}) \right) \quad (1.24)$$

acts as synchronous clock-wise rotation by the angle g in the symplectic z_i -planes:

$$\mathcal{S}_g(z_i) = \begin{pmatrix} \cos g & \sin g \\ -\sin g & \cos g \end{pmatrix} z_i. \quad (1.25)$$

By such symmetries, in particular, the averaged perturbation

$$f_{\text{P}}^{\text{av}}(\Lambda, z) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f_{\text{P}}(\Lambda, \lambda, z) d\lambda, \quad (1.26)$$

which is called the *secular Hamiltonian*, is even around the origin $z = 0$ and its expansion in powers of z has the form¹¹

$$f_{\text{P}}^{\text{av}} = C_0(\Lambda) + \mathcal{Q}_h(\Lambda) \cdot \frac{\eta^2 + \xi^2}{2} + \mathcal{Q}_v(\Lambda) \cdot \frac{p^2 + q^2}{2} + O(|z|^4), \quad (1.27)$$

¹⁰Compare [11, Eq. (6.6)] or [10, Lemma 3.4]. Here, $\pi - \lambda$ means $(\pi - \lambda_1, \dots, \pi - \lambda_n)$, etc.

¹¹ $\mathcal{Q} \cdot u^2$ denotes the 2-indices contraction $\sum_{i,j} \mathcal{Q}_{ij} u_i u_j$ (\mathcal{Q}_{ij} , u_i denoting the entries of \mathcal{Q} , u).

where $\mathcal{Q}_h, \mathcal{Q}_v$ are suitable quadratic forms, showing that $z = 0$ is an *elliptic equilibrium for the secular dynamics* (i.e, the dynamics generated by f_p^{av}). The explicit expression of such quadratic forms can be found, *e.g.*, in [14, (36), (37)] (revised version).

The *truncated averaged Hamiltonian*

$$h_K + \mu \left(C_0(\Lambda) + \mathcal{Q}_h(\Lambda) \cdot \frac{\eta^2 + \xi^2}{2} + \mathcal{Q}_v(\Lambda) \cdot \frac{p^2 + q^2}{2} \right) \quad (1.28)$$

is integrable, with $3n$ commuting integrals given by

$$\Lambda_i, \quad \rho_i = \frac{\eta_i^2 + \xi_i^2}{2}, \quad r_i = \frac{p_i^2 + q_i^2}{2}, \quad (1 \leq i \leq n);$$

the general trajectory fills a $3n$ -torus with n fast frequencies $\partial_{\Lambda_i} h_K(\Lambda_i)$ and $2n$ slow frequencies given by

$$\mu\Omega = \mu(\sigma, \varsigma) = \mu(\sigma_1, \dots, \sigma_n, \varsigma_1, \dots, \varsigma_n), \quad (1.29)$$

σ_i and ς_i being the real eigenvalues of $\mathcal{Q}_h(\Lambda)$ and $\mathcal{Q}_v(\Lambda)$, respectively; such tori surround the n -dimensional elliptic tori given by $\{\Lambda\} \times \{z = 0\}$, corresponding to n -coplanar and co-circular planets rotating around the Sun with Keplerian frequencies $\partial_{\Lambda_i} h_K(\Lambda_i)$.

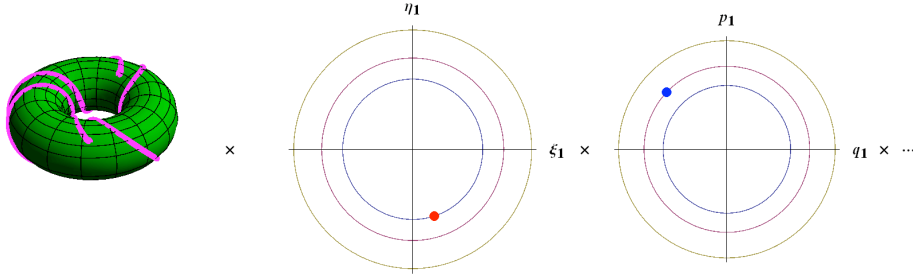


Figure 2: The truncated averaged planetary dynamics

3 Arnold's planetary Theorem

In the following section, we report some of Arnold's statements concerning the existence of regular quasi-periodic motions for the planetary $(1+n)$ -body problem. We recall that, in general, a "quasi-periodic" (or "conditionally periodic") orbit $\zeta(t)$ with (rationally independent) frequencies $(\omega_1, \dots, \omega_d) = \omega \in \mathbb{R}^d$ is a solution of the Hamilton equations of the form $\zeta(t) = Z(\omega_1 t, \dots, \omega_d t)$ for a suitable smooth function $Z(\theta_1, \dots, \theta_d)$ 2π -periodic in each variable θ_i .

3.1 Arnold’s Statements (1963)

At p. 87 of [1] Arnold says:

Conditionally periodic motions in the many–body problem have been found. If the masses of n “planets” are sufficiently small in comparison with the mass of the central body, the motion is conditionally periodic for the majority of initial conditions for which the eccentricities and inclinations of the Kepler ellipses are small. Further, the major semiaxis perpetually remain close to their original values and the eccentricities and inclinations remain small.

Later, [1, p. 125]:

With the help of the fundamental theorem of Chapter IV¹², we investigate in this chapter the class of “planetary” motions in the three–body and many–body problems. We show that, for the majority of initial conditions under which the instantaneous orbits of the planets are close to circles lying in a single plane, perturbation of the planets on one another produces, in the course of an infinite interval of time, little change on these orbits provided the masses of the planets are sufficiently small.

In particular, it follows from our results that in the n -body problem there exists a set of initial conditions having a positive Lebesgue measure and such that, if the initial positions and velocities of the bodies belong to this set, the distances of the bodies from each other will remain perpetually bounded.

Finally, [1, p. 127]:

Our basic result is that if the masses, eccentricities and inclinations of the planets are sufficiently small, then for the majority of initial conditions the true motion is conditionally periodic and differs little from Lagrangian motion¹³ with suitable initial conditions throughout an infinite interval of time $-\infty < t < +\infty$.

As mentioned in the introduction, Arnold provides a full detailed proof, checking the applicability (non–degeneracy conditions) of his fundamental theorem, only for the two–planet model ($n = 2$) in the planar regime. As for generalizations, he states:

The plane problem of $n > 2$ planets. *The arguments of §2 and 3 easily carry over to the case of more than two planets. [⋯] We shall not dwell on the details of the calculations which lead to the results of §1, 4. [1, p. 139].*

¹² The “fundamental theorem” is a KAM (Kolmogorov–Arnold–Moser) theorem for properly–degenerate nearly–integrable Hamiltonian systems: it will be discussed in § 3.2 below. For generalities on KAM theory, see, e.g., [2] or [7].

¹³ Arnold defines the “Lagrangian motions”, at p. 127 as follows: *the Lagrangian motion is conditionally periodic and to the n “rapid” frequencies of the Kepler motion are added n (in the plane problem) or $2n - 1$ (in the space problem) “slow” frequencies of the secular motions.* This dynamics corresponds, essentially, to the above “truncated integrable planetary dynamics”; the missing frequency in the space problem is related to the fact that one of the spacial secular frequency, say, ς_n vanishes identically; compare § 3.5 below.

As for the spacial general case:

The rather lengthy calculations involved in the solution of (3.5.9), the construction of variables satisfying conditions 1)–4), and the verification of non-degeneracy conditions analogous to the arguments of § 4 will not be discussed here. [1, p. 142].

In the next section we shall discuss Arnold’s strategy.

3.2 Proper degeneracies and the “Fundamental Theorem”

The main technical tool is a KAM theorem for properly degenerate systems.

A nearly-integrable system with Hamiltonian

$$H_\mu(I, \varphi) := h(I) + \mu f(I, \varphi) , \quad (I, \varphi) \in \mathbb{R}^d \times \mathbb{T}^d, \quad (1.30)$$

for which h does not depend upon all the actions I_1, \dots, I_d is called *properly degenerated*. This is the case of the many-body problem since $h_K(\Lambda)$ in (1.15) depends only on n actions $\Lambda_1, \dots, \Lambda_n$, while the number of degrees of freedom is $d = 3n$.

In general, maximal quasi-periodic solutions (i.e., quasi-periodic solutions with d rationally-independent frequencies) for properly degenerate systems do not exist¹⁴ but they may exist under further conditions on the perturbation f .

In [1, Chapter IV] Arnold overcome for the first time this problem proving the following result which he called “the fundamental theorem”.

Let \mathcal{M} denote the phase space

$$\mathcal{M} := \left\{ (I, \varphi, p, q) : (I, \varphi) \in V \times \mathbb{T}^n \text{ and } (p, q) \in B \right\} ,$$

where V is an open bounded region in \mathbb{R}^n and B is a ball around the origin in \mathbb{R}^{2m} ; \mathcal{M} is equipped with the standard symplectic form

$$dI \wedge d\varphi + dp \wedge dq = \sum_{i=1}^n dI_i \wedge d\varphi_i + \sum_{i=1}^m dp_i \wedge dq_i .$$

Let, also, H_μ be a real analytic Hamiltonian on \mathcal{M} of the form

$$H_\mu(I, \varphi, p, q) := h(I) + \mu f(I, \varphi, p, q) , \quad (1.31)$$

and denote by f^{av} the average of f over the “fast angles” φ :

$$f^{\text{av}}(I, p, q) := \int_{\mathbb{T}^n} f(I, \varphi, p, q) \frac{d\varphi}{(2\pi)^n} . \quad (1.32)$$

¹⁴Trivially, any unperturbed properly-degenerate system on a $2d$ dimensional phase space with $d \geq 2$ will have motions with frequencies not rationally independent over \mathbb{Z}^d .

Theorem 3.1 (Arnold 1963) *Assume that f^{av} is of the form*

$$f^{\text{av}} = f_0(I) + \sum_{j=1}^m \Omega_j(I) r_j + \frac{1}{2} \tau(I) r \cdot r + o_4, \quad r_j := \frac{p_j^2 + q_j^2}{2}, \quad (1.33)$$

where τ is a symmetric $(m \times m)$ -matrix and $\lim_{(p,q) \rightarrow 0} |o_4|/|(p,q)|^4 = 0$. Assume, also, that $I_0 \in V$ is such that

$$\det h''(I_0) \neq 0, \quad (1.34)$$

$$\det \tau(I_0) \neq 0. \quad (1.35)$$

Then, in any neighborhood of $\{I_0\} \times \mathbb{T}^d \times \{(0,0)\} \subseteq \mathcal{M}$ there exists a positive measure set of phase points belonging to analytic “KAM tori” spanned by maximal quasi-periodic solutions with $n + m$ rationally-independent (Diophantine¹⁵) frequencies, provided μ is small enough.

Let us make some remarks.

- (i) Actually, Arnold requires that f^{av} is in Birkhoff normal form up to order 6, which means that

$$f^{\text{av}} = f_0(I) + \sum_{j=1}^m \Omega_j(I) r_j + \frac{1}{2} \tau(I) r \cdot r + P_3(r; I) + o_6 \quad (1.37)$$

where P_3 is a homogeneous polynomial of degree 3 in the variables r_i (with I -dependent coefficients); but such condition can be relaxed and (1.33) is sufficient: compare [8], where Arnold’s properly degenerate KAM theory is revisited and various improvements obtained.

- (ii) Condition (1.34) is immediately seen to be satisfied in the general planetary problem¹⁶.
- (iii) Condition (1.35) is a “twist” or “torsion” condition. It is actually possible to develop a weaker KAM theory where no torsion is required. This theory is due to Rüssmann [20], Herman and Féjoz [14], where f^{av} is assumed to be in Birkhoff normal form up to order 2, $f^{\text{av}} = f_0(I) + \sum_{j=1}^m \Omega_j(I) r_j + o_2$, and the secular frequency map $I \rightarrow \Omega(I)$ is assumed to be *non-planar*, meaning that no neighborhood of I_0 is mapped into an hyperplane.

¹⁵We recall that $\omega \in \mathbb{R}^d$ is Diophantine if there exist positive constants γ and τ such that

$$|\omega \cdot k| \geq \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}. \quad (1.36)$$

¹⁶The correspondence with the planetary Hamiltonian in Poincaré variables (1.21) is the following: $m = 2n$, $I = \Lambda$, $\varphi = \lambda$, $z = (p, q)$, $h = h_\kappa$, $f = f_\nu$.

- (iv) Indeed, the torsion assumption (1.35) implies stronger results. First, it is possible to give explicit bounds on the measure of the the “Kolmogorov set”, i.e., the set covered by the closure of quasi-periodic motions; see [8]. Furthermore, the quasi-periodic motions found belong to a smooth family of *non-degenerate Kolmogorov tori*, which means, essentially, that the dynamics can be linearized in a neighborhood of each torus; see § 6.1 for more information.

On the base of Theorem 3.1, Arnold’s strategy is to compute the Birkhoff normal form (1.33) of the secular Hamiltonian f_P^{av} in (1.26) and to check the non-vanishing of the torsion (1.35).

3.3 Birkhoff normal forms

Before proceeding, let us recall a few known and less known facts about the general theory of Birkhoff normal forms.

Consider as phase space a $2m$ ball B_δ^{2m} around the origin in \mathbb{R}^{2m} and a real-analytic Hamiltonian of the form

$$H(w) = c_0 + \Omega \cdot r + o(|w|^2) , \quad (1.38)$$

where

$$\begin{cases} w = (u_1, \dots, u_m, v_1, \dots, v_m) \in \mathbb{R}^{2m} , \\ r = (r_1, \dots, r_m) , \quad r_j = \frac{u_j^2 + v_j^2}{2} . \end{cases}$$

The components Ω_j of Ω are called the first order Birkhoff invariants. The following is a classical by G.D. Birkhoff.

Proposition 3.1 *Assume that the first order Birkhoff invariants Ω_j verify, for some $a > 0$ and positive integer s ,*

$$|\Omega \cdot k| \geq a > 0, \quad \forall k \in \mathbb{Z}^m : 0 < |k|_1 := \sum_{i=1}^m |k_i| \leq 2s . \quad (1.39)$$

Then, there exists $0 < \delta' \leq \delta$ and a symplectic transformation $\check{\phi} : \check{w} \in B_{\delta'}^{2m} \rightarrow w \in B_\delta^{2m}$ which puts H into Birkhoff normal form up to the order $2s$, i.e.,

$$H \circ \check{\phi} = c_0 + \Omega \cdot \check{r} + \sum_{2 \leq h \leq s} P_h(\check{r}) + o(|\check{w}|^{2s}) , \quad (1.40)$$

where P_h are homogeneous polynomials in $\check{r}_j = |\check{w}_j|^2/2 := (\check{u}_j^2 + \check{v}_j^2)/2$ of degree h .

Less known is that the hypotheses of this theorem may be loosened in the case of *rotation invariant Hamiltonians*: this fact, for example, has not been used neither in [1] nor in [14].

First, let us generalize the class of Hamiltonian function so as to include the secular Hamiltonian (1.27): let us consider an open, bounded, connected set $U \subseteq \mathbb{R}^n$ and consider the phase space $\mathcal{D} := U \times \mathbb{T}^n \times B_\delta^{2m}$, endowed with the standard symplectic form $dI \wedge d\varphi + du \wedge dv$.

We say that a Hamiltonian $H(I, \varphi, w)$ on \mathcal{D} is *rotation invariant* if $H \circ \mathcal{R}^g = H$ for any $g \in \mathbb{T}$, where \mathcal{R}^g is the symplectic rotation defined in (1.23) (replacing Λ, λ, z with, respectively, I, φ, w).

Now, consider a φ -independent real-analytic Hamiltonian $H : (I, \varphi, w) \in \mathcal{D} \rightarrow H(I, w) \in \mathbb{R}$ of the form¹⁷

$$H(I, w) = c_0(I) + \Omega(I) \cdot r + o(|w|^2; I) . \quad (1.41)$$

Then, it can be proven the following

Proposition 3.2 *Assume that H is rotation-invariant and that the first order Birkhoff invariants Ω_j verify, for all $I \in U$, for some $a > 0$ and positive integer s*

$$|\Omega \cdot k| \geq a > 0, \quad \forall 0 \neq k \in \mathbb{Z}^m : \sum_{i=1}^n k_i = 0 \quad \text{and} \quad |k|_1 \leq 2s . \quad (1.42)$$

Then, there exists $0 < \delta' \leq \delta$ and a symplectic transformation $\check{\phi} : (I, \check{\varphi}, \check{w}) \in \check{\mathcal{D}} := U \times \mathbb{T}^n \times B_{\delta'}^{2m} \rightarrow (I, \varphi, w) \in \mathcal{D}$ which puts H into Birkhoff normal form up to the order $2s$ as in (1.40) with the coefficients of P_h and the remainder depending also on I . Furthermore, $\check{\phi}$ leaves the I -variables fixed, acts as a $\check{\varphi}$ -independent shift on $\check{\varphi}$, is $\check{\varphi}$ -independent on the remaining variables and is such that

$$\check{\phi} \circ \mathcal{R}^g = \mathcal{R}^g \circ \check{\phi} . \quad (1.43)$$

We shall call (1.39) the Birkhoff non-resonance condition (up to order s) and (1.42) the “reduced” Birkhoff non-resonance condition. The proof of Proposition 3.2 may be found in [11, §7.2].

3.4 The planar three-body case (1963)

In the planar case the Poincaré variables become simply

$$(\Lambda, \lambda, z) := (\Lambda, \lambda, \eta, \xi) \in \mathbb{R}_+^n \times \mathbb{T}^n \times \mathbb{R}^{2n} ,$$

with the Λ 's as in (1.12) and

$$\lambda_i = \ell_i + g_i , \quad \begin{cases} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)} \cos g_i \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)} \sin g_i \end{cases} . \quad (1.44)$$

¹⁷ $f = o(|w|^2; I)$ means that $f = f(I, w)$ and $|f|/|w|^2 \rightarrow 0$ as $w \rightarrow 0$.

The planetary, planar Hamiltonian, is then given by

$$\mathcal{H}_{\text{P,pln}}(\Lambda, \lambda, z) = h_{\text{K}}(\Lambda) + \mu f_{\text{P,pln}}(\Lambda, \lambda, z) , \quad z := (\eta, \xi) \in \mathbb{R}^{2n} \quad (1.45)$$

and

$$\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f_{\text{P,pln}} =: f_{\text{P,pln}}^{\text{av}} = C_0(\Lambda) + \mathcal{Q}_h(\Lambda) \cdot \frac{\eta^2 + \xi^2}{2} + \mathcal{O}(|z|^4) . \quad (1.46)$$

In [1, p.138, Eq. (3.4.31)], Arnold computed the first and second order Birkhoff invariants finding, in the asymptotics $a_1 \ll a_2$:

$$\begin{cases} \Omega_1 = -\frac{3}{4} m_1 m_2 \left(\frac{a_1}{a_2}\right)^2 \frac{1}{a_2 \Lambda_1} \left(1 + \mathcal{O}\left(\frac{a_1}{a_2}\right)\right) \\ \Omega_2 = -\frac{3}{4} m_1 m_2 \left(\frac{a_1}{a_2}\right)^2 \frac{1}{a_2 \Lambda_2} \left(1 + \mathcal{O}\left(\frac{a_1}{a_2}\right)\right) \end{cases} \quad (1.47)$$

$$\tau = m_1 m_2 \frac{a_1^2}{a_2^3} \begin{pmatrix} \frac{3}{4\Lambda_1^2} & -\frac{9}{4\Lambda_1 \Lambda_2} \\ -\frac{9}{4\Lambda_1 \Lambda_2} & -\frac{3}{\Lambda_2^2} \end{pmatrix} \left(1 + \mathcal{O}\left(\frac{a_1}{a_2}\right)^{\frac{5}{4}}\right) , \quad (1.48)$$

which shows that the Ω_j 's are non resonant up to any finite order (in a suitable Λ -domain), so that planetary, planar Hamiltonian can be put in Birkhoff normal form up to order 4 and that the second order Birkhoff invariants are non-degenerate in the sense that¹⁸

$$\begin{aligned} \det \tau &= -(m_1 m_2)^2 \frac{117}{16} \frac{a_1^4}{a_2^6 (\Lambda_1 \Lambda_2)^2} (1 + o(1)) \\ &= -\frac{117}{16} \frac{1}{m_0^2} \frac{a_1^3}{a_2^7} (1 + o(1)) \neq 0 . \end{aligned} \quad (1.50)$$

This allow to apply Theorem 3.1 and to prove Arnold's planetary theorem in the planar three-body ($n = 2$) case.

An extension of this method to the *spatial three-body problem*, exploiting Jacobi's reduction of the nodes and its symplectic realization, is due to P. Robutel [19].

3.5 Secular Degeneracies

In the general spatial case it is customary to call σ_i the eigenvalues of $\mathcal{Q}_h(\Lambda)$ and ς_i the eigenvalues of $\mathcal{Q}_v(\Lambda)$, so that $\Omega = (\sigma, \varsigma)$; compare (1.29).

It turns out that such invariants satisfy identically the following two *secular resonances*

$$\varsigma_n = 0 , \quad \sum_{i=1}^n (\sigma_i + \varsigma_i) = 0 \quad (1.51)$$

¹⁸In [1] the τ_{ij} are defined as 1/2 of the ones defined here.

and, actually, it can be shown that *these are the only resonances identically satisfied by the first order Birkhoff invariants*; compare [14, Proposition 78, p. 1575].

The first resonance was well known to Arnold, while the second one was apparently discovered by M. Herman in the 90's and is now known as *Herman resonance*.

Notice that:

- the first resonance ($\varsigma_n = 0$), which is a resonance of order one, violates the usual Birkhoff non-resonance condition (1.39) for any $s \geq 1$ but does not violate (1.42);
- Herman resonance is a resonance of order $(2n - 1)$ and violates (1.39) when $(2s + 1)/2 \geq n$; while it does not violate (1.42);
- combining the two resonances, also (1.42) is violated, for $(s + 1)/2 \geq n$, by taking

$$k = \left(\underbrace{1, \dots, 1}_{2n-1}, -(2n - 1) \right). \quad (1.52)$$

Another serious problem for Arnold's approach is that the matrix τ indeed is degenerate, as clarified in [10], since

$$\tau = \begin{pmatrix} \bar{\tau} & 0 \\ 0 & 0 \end{pmatrix} \quad (1.53)$$

$\bar{\tau}$ being a matrix of order $(2n - 1)$.

3.6 Herman–Fejóz proof (2004)

In 2004 J. Fejóz published the first complete proof of a general version of Arnold's planetary theorem [14]. As mentioned above (remark (ii), §3.2), in order to avoid fourth order computations (and also because M. Herman seemed to suspect the degeneracy of the matrix of the second order Birkhoff invariant¹⁹), Herman's approach was to use a first order KAM condition based on the non-planarity of the frequency map. But, the resonances (1.51) show that the frequency map lie in the intersection of two planes, violating the non-planarity condition. To overcome this problem Herman and Féjóz use a trick by Poincarè, consisting in modifying the Hamiltonian by adding a commuting Hamiltonian, so as to remove the degeneracy: by a Lagrangian intersection theory argument, commuting Hamiltonians have the same maximal transitive invariant tori, so that the KAM tori constructed for the modified Hamiltonian are indeed invariant tori also for

¹⁹compare the Remark towards the end of p. 24 in [18].

the original system. Now, the expression of the vertical component of the total angular momentum C_3 has a particular simple expression in Poincaré variables, since

$$C_3 := \sum_{j=1}^n \left(\Lambda_j - \frac{1}{2}(\eta_j^2 + \xi_j^2 + p_j^2 + q_j^2) \right),$$

so that the modified Hamiltonian

$$\mathcal{H}_\delta := \mathcal{H}_p(\Lambda, \lambda, z) + \delta C_3$$

is easily seen to have a non-planar frequency map (Keplerian frequencies + first order Birkhoff invariants), and the above abstract remark applies²⁰.

3.7 Chierchia–Pinzari proof (2011)

In [11] Arnold’s original strategy is reconsidered and full torsion of the planetary problem is shown by introducing new symplectic variables (called RPS-variables²¹), which allow for a symplectic reduction of rotations eliminating one degree of freedom (i.e., lowering by two units the dimension of the phase space). In such reduced setting the first resonance in (1.51) disappears and the question about the torsion is reduced to study the determinant of $\bar{\tau}$ in (1.53), which, in fact, is shown to be non-singular; compare [11, §8] and [10] (where a precise connection is made between the Poincaré and the RPS-variables).

The rest of this article is devoted to explain the main ideas beyond this approach.

4 Symplectic reduction of rotations

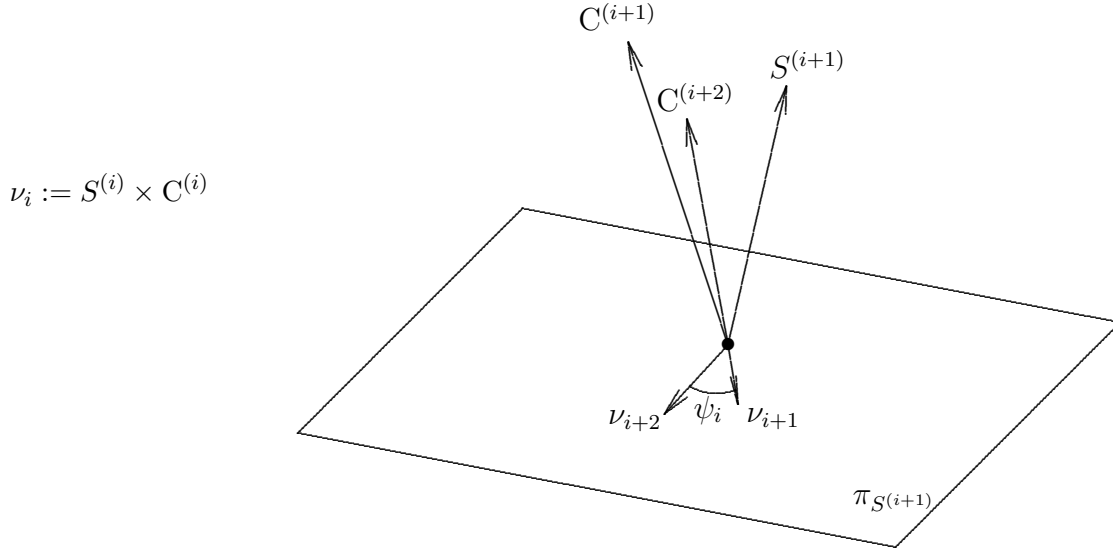
We start by describing the new set of symplectic variables, which allow to have a new insight on the symplectic structure of the phase space of the planetary model, or, more in general, of any rotation invariant model.

The idea is to start with action–angle variables having, among the actions, two independent commuting integrals related to rotations, for example, the Euclidean length of the total angular momentum C and its vertical component C_3 , and then (imitating Poincaré) to regularize around co-circular and co-planar configurations.

The variables that do the job are an action–angle version of certain variables introduced by A. Deprit in 1983 [13] (see also [9]), which generalize to an arbitrary number of bodies Jacobi’s reduction of the nodes; the regularization has been done in [11].

²⁰Actually, this idea is close to Herman’s original argument, while Fejóz uses a somewhat more abstract argument.

²¹Regularized Planetary Symplectic variables; see § 4.1 below.


 Figure 4: The angle ψ_i for $1 \leq i \leq n-2$

The ‘‘Deprit inclinations’’ ι_i are defined through the relations

$$\cos \iota_i := \begin{cases} \frac{C^{(i+1)} \cdot S^{(i+1)}}{|C^{(i+1)}| |S^{(i+1)}|}, & 1 \leq i \leq n-1, \\ \frac{C \cdot k^{(3)}}{|C|}, & i = n. \end{cases} \quad (1.57)$$

Similarly to the case of the Delaunay variables, the Deprit action–angles variables are not defined when the Deprit nodes ν_i vanish or $e_i \notin (0, 1)$; on the domain where they are well defined they define a real–analytic set of symplectic variables, i.e.,

$$\sum_{i=1}^n dX^{(i)} \wedge dx^{(i)} = \sum_{i=1}^n d\Lambda_i \wedge dl_i + d\Gamma_i \wedge d\gamma_i + d\Psi_i \wedge d\psi_i. \quad (1.58)$$

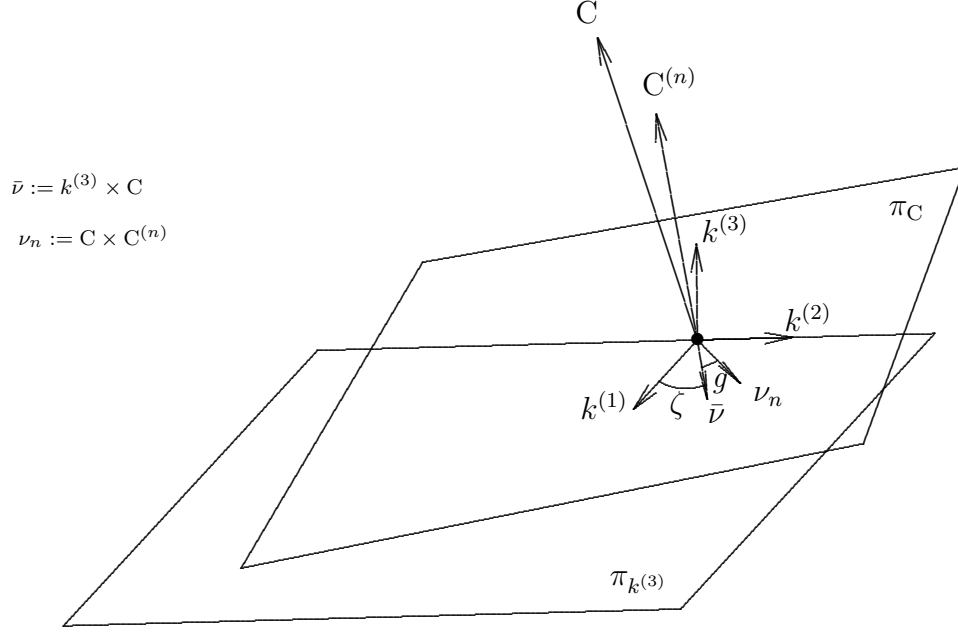
The RPS variables are given by $(\Lambda, \lambda, z) := (\Lambda, \lambda, \eta, \xi, p, q)$ with (again) the Λ ’s as in (1.12) and

$$\lambda_i = l_i + \gamma_i + \psi_{i-1}^n \quad \begin{cases} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)} \cos(\gamma_i + \psi_{i-1}^n) \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)} \sin(\gamma_i + \psi_{i-1}^n) \end{cases} \quad (1.59)$$

$$\begin{cases} p_i = \sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)} \cos \psi_i^n \\ q_i = -\sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)} \sin \psi_i^n \end{cases}$$

where

$$\Psi_0 := \Gamma_1, \quad \Gamma_{n+1} := 0, \quad \psi_0 := 0, \quad \psi_i^n := \sum_{i \leq j \leq n} \psi_j. \quad (1.60)$$

Figure 5: The angles $\psi_{n-1} =: g$ and $\psi_n =: \zeta$

The phase space in the RPS variables has the same form as in (1.19)–(1.20), namely

$$(\Lambda, \lambda, z) \in \mathcal{M}_{\text{RPS}}^{6n} := \mathcal{A} \times \mathbb{T}^n \times B^{4n} \quad (1.61)$$

with B a $4n$ -dimensional ball around the origin.

The relation between Poincaré variables and the RPS variables is rather simple. Indeed, if we denote by

$$\phi_{\text{P}}^{\text{RPS}} : (\Lambda, \lambda, z) \rightarrow (\Lambda, \lambda, z) \quad (1.62)$$

the symplectic transformation between the RPS and the Poincaré variables, one has the following

Theorem 4.1 ([10]) *The symplectic map $\phi_{\text{P}}^{\text{RPS}}$ in (1.62) has the form*

$$\lambda = \lambda + \varphi(\Lambda, z) \quad z = \mathcal{Z}(\Lambda, z) \quad (1.63)$$

where $\varphi(\Lambda, 0) = 0$ and, for any fixed Λ , the map $\mathcal{Z}(\Lambda, \cdot)$ is 1:1, symplectic²³ and its projections verify, for a suitable $\mathcal{V} = \mathcal{V}(\Lambda) \in \text{SO}(n)$,

$$\Pi_{\eta} \mathcal{Z} = \eta + O_3, \quad \Pi_{\xi} \mathcal{Z} = \xi + O_3, \quad \Pi_{\text{p}} \mathcal{Z} = \mathcal{V}p + O_3, \quad \Pi_{\text{q}} \mathcal{Z} = \mathcal{V}q + O_3, \quad (1.64)$$

where $O_3 := O(|z|^3)$,

²³I.e., it preserves the two form $d\eta \wedge d\xi + dp \wedge dq$.

4.2 Partial reduction of rotations

Recalling that

$$\Gamma_{n+1} = 0, \quad \Psi_{n-1} = |S^{(n)}| = |C|, \quad \Psi_n = C_3, \quad \psi_n = \alpha_{k^{(3)}}(k^{(1)}, k_3 \times C) \quad (1.65)$$

one sees that

$$\begin{cases} p_n = \sqrt{2(|C| - C_3)} \cos \psi_n \\ q_n = -\sqrt{2(|C| - C_3)} \sin \psi_n, \end{cases} \quad (1.66)$$

showing the *conjugated variables* p_n and q_n are both integrals and hence both cyclic for the planetary Hamiltonian, which, therefore, in such variables, will have the form

$$\mathcal{H}_{\text{RPS}}(\Lambda, \lambda, \bar{z}) = h_{\text{K}}(\Lambda) + \mu f_{\text{RPS}}(\Lambda, \lambda, \bar{z}), \quad (1.67)$$

where \bar{z} denote the set of variables

$$\bar{z} := (\eta, \xi, \bar{p}, \bar{q}) := ((\eta_1, \dots, \eta_n), (\xi_1, \dots, \xi_n), (p_1, \dots, p_{n-1}), (q_1, \dots, q_{n-1})). \quad (1.68)$$

In other words, *the phase space* $\mathcal{M}_{\text{RPS}}^{6n}$ *in* (1.61) *is foliated by* $(6n-2)$ -*dimensional invariant manifolds*

$$\mathcal{M}_{p_n, q_n}^{6n-2} := \mathcal{M}_{\text{RPS}}^{6n} |_{p_n, q_n = \text{const}}, \quad (1.69)$$

and since the restriction of the standard symplectic form on such manifolds is simply

$$d\Lambda \wedge d\lambda + d\eta \wedge d\xi + d\bar{p} \wedge d\bar{q},$$

such manifolds are symplectic and the planetary flow is the standard Hamiltonian flow generated by \mathcal{H}_{RPS} in (1.67). We shall call the symplectic, invariant submanifolds $\mathcal{M}_{p_n, q_n}^{6n-2}$ “symplectic leaves”. They depend upon a particular orientation of the total angular momentum: in particular, the leaf \mathcal{M}_0^{6n-2} correspond to the total angular momentum parallel to the vertical k_3 -axis. Notice, also, that *the analytic expression of the planetary Hamiltonian* \mathcal{H}_{RPS} *is independent of the leaves.*

In view of these observations, it is enough to study the planetary flow of \mathcal{H}_{RPS} on, say, the vertical leaf \mathcal{M}_0^{6n-2} .

5 Planetary Birkhoff normal forms and torsion

The RPS variables share with Poincaré variables the D’Alembert symmetries (i.e. invariance under (1.22)); compare [10, Lemma 3.4]. As for Poincaré variables, this implies that the averaged perturbation

$$f_{\text{RPS}}^{\text{av}} := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f_{\text{RPS}} d\lambda$$

also enjoys D'Alembert rules and thus has an expansion analogue to (1.27), but independent of (p_n, q_n) :

$$f_{\text{RPS}}^{\text{av}}(\Lambda, \bar{z}) = C_0(\Lambda) + \mathcal{Q}_h(\Lambda) \cdot \frac{\eta^2 + \xi^2}{2} + \bar{\mathcal{Q}}_v(\Lambda) \cdot \frac{\bar{p}^2 + \bar{q}^2}{2} + O(|\bar{z}|^4) \quad (1.70)$$

with \mathcal{Q}_h of order n and $\bar{\mathcal{Q}}_v$ of order $(n-1)$. Notice that the matrix \mathcal{Q}_h in (1.70) is the same as in (1.27), since, when $p = (\bar{p}, p_n) = 0$ and $q = (\bar{q}, q_n) = 0$, Poincaré and RPS variables coincide.

Using Theorem 4.1, one can also show that

$$\mathcal{Q}_v := \begin{pmatrix} \bar{\mathcal{Q}}_v & 0 \\ 0 & 0 \end{pmatrix} \quad (1.71)$$

is conjugated (by a unitary matrix) to \mathcal{Q}_v in (1.27), so that the eigenvalues $\bar{\zeta}_i$ of $\bar{\mathcal{Q}}_v$ coincide con $(\zeta_1, \dots, \zeta_{n-1})$, as one naively would expect.

In view of the remark after (1.51), and of the rotation-invariant Birkhoff theory (Proposition 3.2), one sees that, one can construct, in an open neighborhood of co-planar and co-circular motions, the Birkhoff normal form of $f_{\text{RPS}}^{\text{av}}$ up to any finite order.

More precisely, for $\epsilon > 0$ small enough, denoting

$$\mathcal{P}_\epsilon := \mathcal{A} \times \mathbb{T}^n \times B_\epsilon^{4n-2}, \quad B_\epsilon^{4n-2} := \{\bar{z} \in \mathbb{R}^{4n-2} : |\bar{z}| < \epsilon\},$$

an ϵ -neighborhood of the co-circular, co-planar region, one can find, for μ small enough, a real-analytic symplectic transformation

$$\phi_\mu : (\Lambda, \check{\lambda}, \check{z}) \in \mathcal{P}_\epsilon \rightarrow (\Lambda, \lambda, \bar{z}) \in \mathcal{P}_\epsilon$$

such that

$$\check{\mathcal{H}} := \mathcal{H}_{\text{RPS}} \circ \phi_\mu = h_{\text{K}}(\Lambda) + \mu f(\Lambda, \check{\lambda}, \check{z}) \quad (1.72)$$

with

$$\check{f}_{\text{av}}(\Lambda, \check{z}) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f \, d\check{\lambda} = C_0(\Lambda) + \Omega \cdot \check{\mathbf{R}} + \frac{1}{2} \bar{\tau} \check{\mathbf{R}} \cdot \check{\mathbf{R}} + O(|\check{z}|^6) \quad (1.73)$$

where

$$\begin{cases} \Omega = (\sigma, \bar{\zeta}) \\ \check{z} := (\check{\eta}, \check{\xi}, \check{p}, \check{q}) \in B_\epsilon^{4n-2}, \\ \check{\mathbf{R}} = (\check{\rho}, \check{r}), \quad \check{\rho} = (\check{\rho}_1, \dots, \check{\rho}_n), \quad \check{r} = (\check{r}_1, \dots, \check{r}_{n-1}), \\ \check{\rho}_i := \frac{\check{\eta}_i^2 + \check{\xi}_i^2}{2}, \quad \check{r}_i = \frac{\check{p}_i^2 + \check{q}_i^2}{2}. \end{cases}$$

With straightforward (but not trivial) computations, one can then show full torsion for the planetary problem. More precisely, one finds ([11, Proposition 8.1])

Proposition 5.1 For $n \geq 2$ and $0 < \delta_\star < 1$ there exist²⁴ $\bar{\mu} > 0$,

$$0 < \underline{a}_1 < \bar{a}_1 < \cdots < \underline{a}_n < \bar{a}_n$$

such that, on the set \mathcal{A} defined in (1.20) and for $0 < \mu < \bar{\mu}$, the matrix $\bar{\tau} = (\tau_{ij})$ is non-singular:

$$\det \bar{\tau} = d_n(1 + \delta_n) , \quad \text{with } |\delta_n| < \delta_\star$$

and

$$d_n = (-1)^{n-1} \frac{3}{5} \left(\frac{45}{16} \frac{1}{m_0^2} \right)^{n-1} \frac{m_2}{m_1 m_0} a_1 \left(\frac{a_1}{a_n} \right)^3 \prod_{2 \leq k \leq n} \left(\frac{1}{a_k} \right)^4 . \quad (1.74)$$

6 Dynamical consequences

6.1 Kolmogorov tori for the planetary problem

At this point one can apply to the planetary Hamiltonian in normalized variables $\check{\mathcal{H}}(\Lambda, \check{\lambda}, \check{z})$ Arnold's Theorem 3.1 above completing Arnold's project on the planetary N -body problem.

Indeed, by using the refinements of Theorem 3.1 as given in [8], from Proposition 5.1 there follows

Theorem 6.1 *There exists positive constants ϵ_\star , c_\star and C_\star such that the following holds. If*

$$0 < \epsilon < \epsilon_\star , \quad 0 < \mu < \frac{\epsilon^6}{(\log \epsilon^{-1})^{c_\star}} , \quad (1.75)$$

then each symplectic leaf $\mathcal{M}_{p_n, q_n}^{6n-2}$ (1.69) contains a positive measure \mathcal{H}_{RPS} -invariant Kolmogorov set \mathcal{K}_{p_n, q_n} , which is actually the suspension of the same Kolmogorov set $\mathcal{K} \subseteq \mathcal{P}_\epsilon$, which is \mathcal{H} -invariant.

Furthermore, \mathcal{K} is formed by the union of $(3n-1)$ -dimensional Lagrangian, real-analytic tori on which the \mathcal{H} -motion is analytically conjugated to linear Diophantine quasi-periodic motions with frequencies $(\omega_1, \omega_2) \in \mathbb{R}^n \times \mathbb{R}^{2n-1}$ with $\omega_1 = O(1)$ and $\omega_2 = O(\mu)$.

Finally, \mathcal{K} satisfies the bound

$$\text{meas } \mathcal{P}_\epsilon \geq \text{meas } \mathcal{K} \geq \left(1 - C_\star \sqrt{\epsilon}\right) \text{meas } \mathcal{P}_\epsilon . \quad (1.76)$$

In particular, $\text{meas } \mathcal{K} \simeq \epsilon^{4n-2} \simeq \text{meas } \mathcal{P}_\epsilon$.

²⁴ $\bar{\mu}$ is taken small only to simplify (1.74), but a similar evaluation hold with $\bar{\mu} = 1$.

6.2 Conley–Zehnder stable periodic orbits

Indeed, the tori $\mathcal{T} \in \mathcal{K}$ form a (Whitney) smooth family of *non-degenerate Kolmogorov tori*, which means the following. The tori in \mathcal{K} can be parameterized by their frequency $\omega \in \mathbb{R}^{3n-1}$ (i.e., $\mathcal{T} = \mathcal{T}_\omega$) and there exists a real-analytic symplectic diffeomorphism

$$\nu : (y, x) \in B^m \times \mathbb{T}^m \rightarrow \nu(y, x; \omega) \in \mathcal{P}_\epsilon, \quad m := 3n - 1, \quad (1.77)$$

uniformly Lipschitz in²⁵ ω such that, for each ω

- a) $\check{\mathcal{H}} \circ \nu = E + \omega \cdot y + Q;$ (Kolmogorov’s normal form)
- b) $E \in \mathbb{R}$ (the energy of the torus); $\omega \in \mathbb{R}^m$ is a Diophantine vector;
- c) $Q = O(|y|^2)$
- d) $\det \int_{\mathbb{T}^m} \partial_{yy} Q(0, x) dx \neq 0,$ (nondegeneracy)
- e) $\mathcal{T}_\omega = \nu(0, \mathbb{T}^m).$

Now, in the first paragraph of [12] Conley and Zehnder, putting KAM theory (and in particular exploiting Kolmogorov’s normal form for KAM tori) together with Birkhoff–Lewis fixed–point theorem show that long–period periodic orbits cumulate densely on Kolmogorov tori so that, in particular, *the Lebesgue measure of the closure of the periodic orbits can be bounded below by the measure of the Kolmogorov set*. Notwithstanding the proper degeneracy, this remark applies also in the present situation and as a consequence of Theorem 6.1 and of the fact that the tori in \mathcal{K} are non–degenerate Kolmogorov tori it follows that

in the planetary model the measure of the closure of the periodic orbits in \mathcal{P}_ϵ can be bounded below by a constant times ϵ^{4n-2} .

References

- [1] V. I. Arnold. Small denominators and problems of stability of motion in classical and celestial mechanics. *Uspehi Mat. Nauk*, 18(6 (114)):91–192, 1963. English translation in: *Russian Math. Surveys*, 18(6):85–191, 1963.
- [2] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt. *Mathematical aspects of classical and celestial mechanics*, volume 3 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, third edition, 2006. [Dynamical systems. III], Translated from the Russian original by E. Khukhro.

²⁵Even more: C^∞ in the sense of Whitney.

- [3] L. Biasco, L. Chierchia, and E. Valdinoci. Elliptic two-dimensional invariant tori for the planetary three-body problem. *Arch. Rational Mech. Anal.*, 170:91–135, 2003. See also: Corrigendum. *Arch. Ration. Mech. Anal.*. 180: 507–509, 2006.
- [4] A. Celletti and L. Chierchia. KAM stability and celestial mechanics. *Mem. Amer. Math. Soc.*, 187(878):viii+134, 2007.
- [5] Kuo-Chang Chen. Existence and minimizing properties of retrograde orbits to the three-body problem with various choices of masses. *Ann. of Math. (2)*, 167(2):325–348, 2008.
- [6] Alain Chenciner and Richard Montgomery. A remarkable periodic solution of the three-body problem in the case of equal masses. *Ann. of Math. (2)*, 152(3):881–901, 2000.
- [7] L. Chierchia. Kolmogorov-Arnold-Moser (KAM) Theory. In *Encyclopedia of Complexity and Systems Science*. Editor-in-chief: Meyers, Robert A. Springer, 2009.
- [8] L. Chierchia and G. Pinzari. Properly-degenerate KAM theory (following V.I. Arnold). *Discrete Contin. Dyn. Syst. Ser. S*, 3(4):545–578, 2010.
- [9] L. Chierchia and G. Pinzari. Deprit’s reduction of the nodes revisited. *Celest. Mech. Dyn. Astr.*, 109(3):285–301, 2011.
- [10] L. Chierchia and G. Pinzari. Planetary Birkhoff normal forms. *Journal of Modern Dynamics*, 5(4), 2011.
- [11] Luigi Chierchia and Gabriella Pinzari. The planetary N -body problem: symplectic foliation, reductions and invariant tori. *Invent. Math.*, 186(1):1–77, 2011.
- [12] C. Conley and E. Zehnder. An index theory for periodic solutions of a Hamiltonian system. In *Geometric dynamics (Rio de Janeiro, 1981)*, volume 1007 of *Lecture Notes in Math.*, pages 132–145. Springer, Berlin, 1983.
- [13] A. Deprit. Elimination of the nodes in problems of n bodies. *Celestial Mech.*, 30(2):181–195, 1983.
- [14] J. Féjoz. Démonstration du ‘théorème d’Arnold’ sur la stabilité du système planétaire (d’après Herman). *Ergodic Theory Dynam. Systems*, 24(5):1521–1582, 2004.
Revised version (2007) available at people.math.jussieu.fr/~fejoz/articles.html.

- [15] Davide L. Ferrario and Susanna Terracini. On the existence of collisionless equivariant minimizers for the classical n -body problem. *Invent. Math.*, 155(2):305–362, 2004.
- [16] G. Fusco, G. F. Gronchi, and P. Negrini. Platonic polyhedra, topological constraints and periodic solutions of the classical N -body problem. *Invent. Math.*, 185(2):283–332, 2011.
- [17] Marshall Hampton and Richard Moeckel. Finiteness of relative equilibria of the four-body problem. *Invent. Math.*, 163(2):289–312, 2006.
- [18] M.R. Herman. Torsion du problème planétaire, edited by J. Fejóz in 2009. Available in the electronic ‘Archives Michel Herman’ at http://www.college-de-france.fr/default/EN/all/equ_dif/archives_michel_herman.htm.
- [19] P. Robutel. Stability of the planetary three-body problem. II. KAM theory and existence of quasiperiodic motions. *Celestial Mech. Dynam. Astronom.*, 62(3):219–261, 1995. See also: Erratum, *Celestial Mech. Dynam. Astronom.*, 84(3):317, 2002.
- [20] H. Růßmann. Invariant Tori in Non-Degenerate Nearly Integrable Hamiltonian Systems. *R. & C. Dynamics*, 2(6):119–203, March 2001.