ANNEX 1. DEFINITION OF ORBITAL PARAMETERS AND IMPORTANT CONCEPTS OF CELESTIAL MECHANICS

A1.1. Kepler's laws

Johannes Kepler (1571-1630) discovered the laws of orbital motion, now called Kepler's laws.

LAW 1: The orbit of an orbiting celestial body (i.e.planet/comet/satellite) around another reference celestial body (i.e.Sun) is a conic section (ellipse, parabola, hyperbola) with the reference celestial body's center of mass at one focus.

LAW 2: A line joining an orbiting celestial body and the reference celestial body sweeps out equal areas in equal intervals of time.

LAW 3: The squares of the periods of the planets are proportional to the cubes of their semi-major axes:

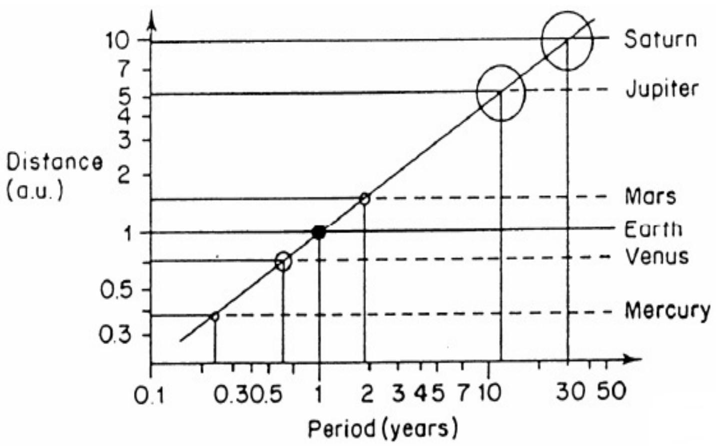

Fig. A1.1 3rd Kepler's law

A1.2. Standard orbital parameters

In this annex the orbital parameters used in the report are briefly explained for an easy understanding of the concepts. It can be noticed that, due to the shape of the described orbits, all the parameters are related to the ellipse.

Exist several parameters that let to describe an orbit that obeys Kepler's laws. While there are various ways to choose the parameters, the standard set follows:

The ones which describe the shape of the orbit:

semi-major axis (*a*)
eccentricity (*e*)

The ones which describe the orientation of the orbit:
inclination (*i*)
longitude of ascending node (Ω)
argument of the perigee (ω)

A1.2.1. The Ellipse

Before explaining any concept about the parameters, it is important to mention some important aspects of the ellipse in a general way.

A simple way to illustrate the *ellipse* is to picture a piece of string with each end fastened to fixed points called focus points or foci. The string length is arbitrarily set to 2a. If a pencil is used to pull the string tight and is then moved around the foci, the resulting shape will be an ellipse. The length of string remains constant at 2a, but the distance (S1 and S2) from the pencil to each focus will change at each point. The foci are located at *F* and *F'*.

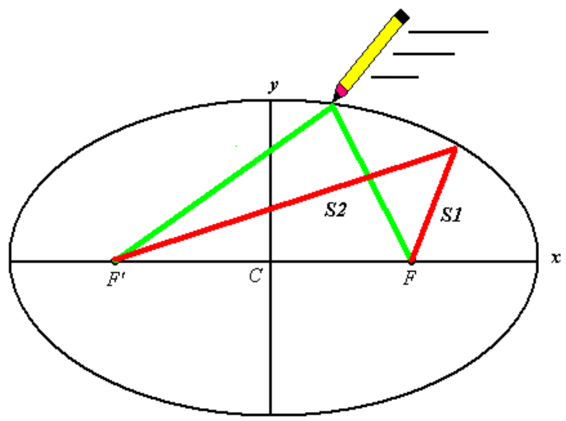

Fig. A1.2a Concept of ellipse

With this, the principal characteristic of the ellipse is that: if P is any point of the ellipse, $PF + PF' = 2a$.

The ellipse's formula in Cartesian coordinates is (centred in one focus):

$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \textbf{(A1.1)}$$

Dimension characteristics:

- Longitude of major axis $A'A = 2a$
- Longitude of minor axis $B'B = 2b$
- Distance between the centre *C* and the foci (*F* or *F'*) is

$$c = \sqrt{a^2 - b^2} \tag{A1.2}$$

- Distance between the foci $2ae$

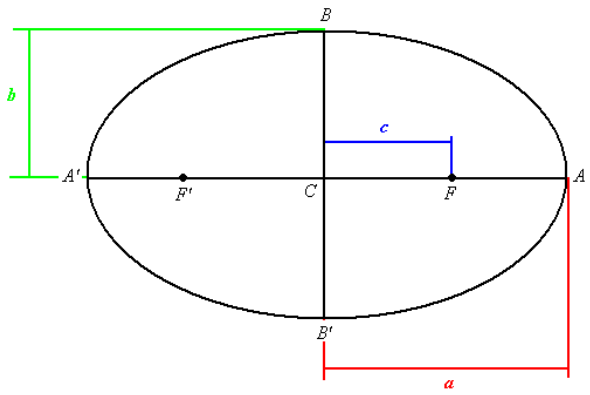

Fig. A1.2b Concept of ellipse

The radius (position vector magnitude) can be expressed as (ellipse centred in one focus):

$$r = a(1 - e\cos E) \tag{A1.3a}$$

or, in polar coordinates (ellipse centred in one focus):

$$r = \frac{a(1 - e^2)}{1 + e\cos \upsilon} \tag{A1.3b}$$

the conversion from polar to Cartesian (ellipse centred in one focus):

$$\begin{aligned} x &= r\cos\upsilon = a\cos E - c \\ y &= r\sin\upsilon = b\sin E \end{aligned} \tag{A1.4}$$

where:

- υ is the true anomaly
- E is the eccentric anomaly

- a is the semi-major axis of the ellipse
- b is the semi-minor axis of the ellipse
- c is the center-focus distance of the ellipse
- e is the eccentricity of the ellipse

NOTE: υ and E parameters are explained in the following sections.

A1.2.1.1. Apoapsis, periapsis and line of apsides

In astronomy, an *apsis* is the point of greatest or least distance of the elliptical orbit of a celestial body from its centre of attraction (the centre of mass of the system).

The point of closest approach is called the **periapsis** or **pericentre** and the point of farthest excursion is the **apoapsis** (Greek απο, from), **apocentre** or **apapsis**.

A straight line drawn through the periapsis and apoapsis is the *line of apsides*. This is the major axis of the ellipse, the line through the longest part of the ellipse.

Apsis is a general term. It is substituted to identify the body being orbited. In the following table some examples are specified.

Table A1.1. Examples of the *apsis* nomenclature

Body	Closest approach	Farthest approach
Galaxy	Perigalacticon	Apogalacticon
Star	Periastron	Apastron
Sun	Perihelion	Aphelion
Earth	Perigee	Apogee
Jupiter	Perijove	Apojove

A1.2.2. Semi-major and semi-minor axis

The *semi-major axis, a,* is an orbital element that determines the shape of the orbit. It can be derived from the Mean Motion and the Eccentricity. If the orbit is a circle, the semi-major axis is the radius of the circle. If it is an ellipse, the *major axis* is the longest diameter, while the *minor axis* is the shortest. The *semi-major axis* is half the major axis; the *semi-minor axis, b,* is half the minor axis.

Note that for elliptical orbits, the Earth is at one **focus**. These points (the *foci*) are not centered in the ellipse, so the semi-major axis does not directly describe the altitude at perigee.

$$a = \sqrt[3]{\frac{\mu}{n^2}} = \frac{c}{e} \qquad \textbf{(A1.5)}$$

where:

- μ is the standard gravitational parameter
- n is the mean motion
- c is the center-focus distance of the ellipse
- e is the eccentricity of the ellipse

A1.2.3. Eccentricity

The *eccentricity*, e, is an element that determines the shape of the orbit. It gives the information of how much the ellipse deviates from a circle. It can adopt several values, from 0 to ∞:

$e = 0$	circle
$e < 1$	ellipse
$e = 1$	parabola
$e > 1$	hyperbola

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} \qquad \textbf{(A1.6)}$$

where:

- a is the semi-major axis of the ellipse
- b is the semi-minor axis of the ellipse
- c is the center-focus distance of the ellipse

A1.2.4. Inclination

The *inclination*, i, is the angular distance of the orbital plane (in fact, the normal of the orbital plane) w.r.t a reference plane (for example, if the orbit is around the Earth, the reference plane is the equatorial or ecliptic plane). Normally stated in degrees. $0º < i < 180º$.

$i = 0º$ or $180º$	equatorial orbit
$i = 90º$	polar orbit
$0º < i < 90º$	direct orbit (or prograde)
$90º < i < 180º$	retrograde orbit

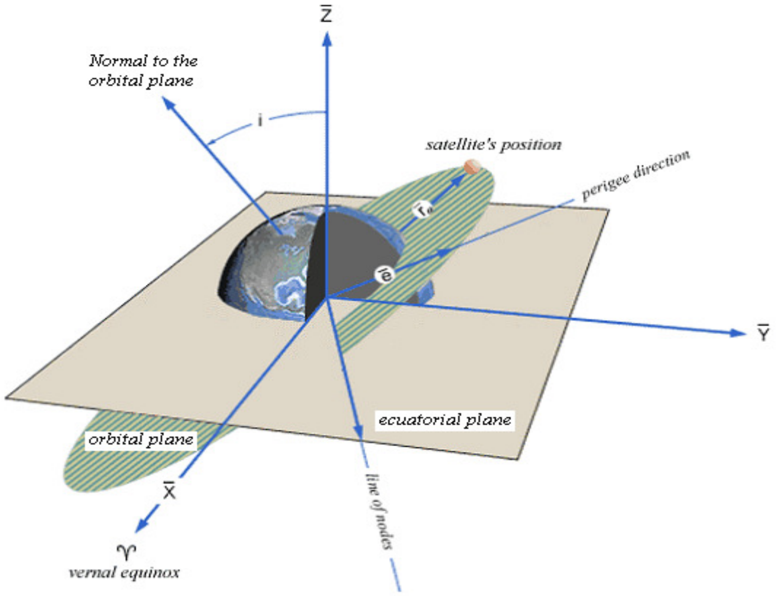

Fig. A1.3 Inclination

A1.2.5. Longitude of the ascending node

The *nodes* of an orbit are the points where it crosses a reference plane. The *ascending node* is the point at which the northbound (ascending) satellite crosses the equator (for the Earth's case), passing from the southern hemisphere to the northern hemisphere. The *longitude of the ascending node,* Ω, is the angle measured from the principal direction (longitude origin) to the ascending node in the counter clockwise direction. For a Sun-orbiting body, it is the angle formed at the Sun from the First Point of Aries (Vernal equinox) to the body's ascending node, measured in the reference plane (the ecliptic). $0° < \Omega < 360°$. It determines the orientation of the orbit.

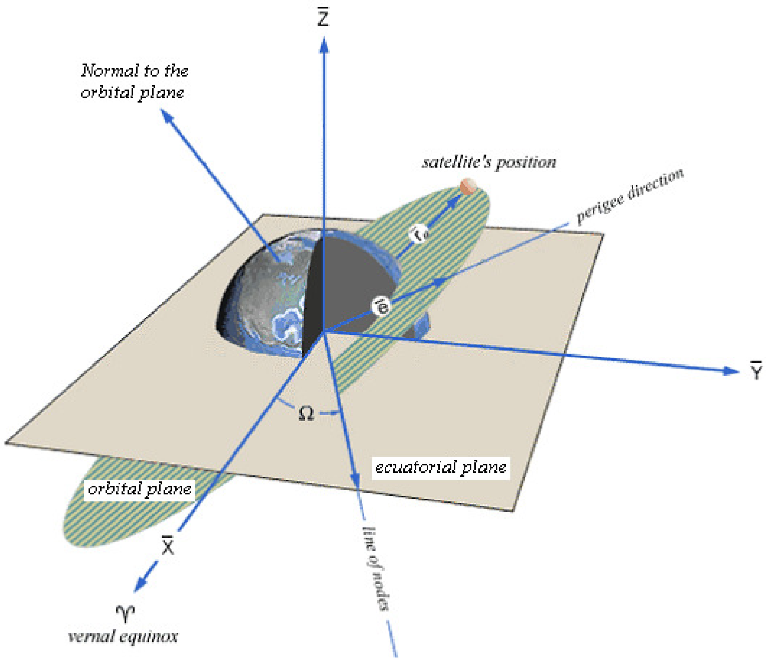

Fig. A1.4 Longitude of ascending node

A1.2.6. Argument of perigee

The *argument of perigee,* ω, is the angle measured from the ascending node (or line of nodes) to the perigee in the counter clockwise sense. $0^{\circ} < \omega < 360^{\circ}$. It determines the orientation of the orbit inside its plane.

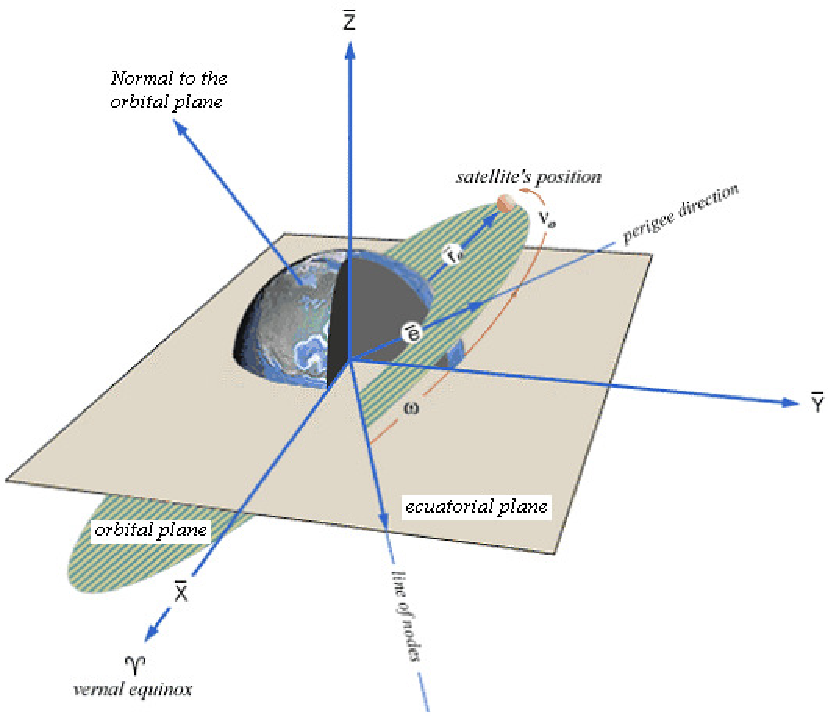

Fig. A1.5a Argument of perigee

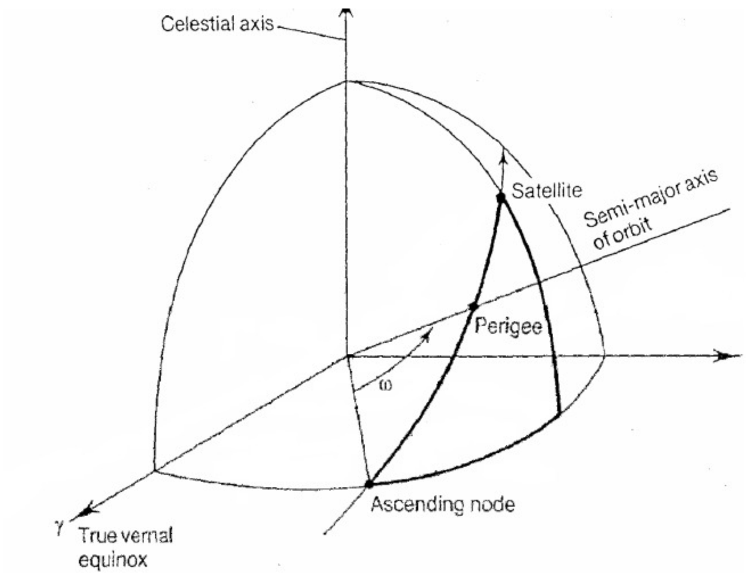

Fig. A1.5b Argument of perigee

A1.2.7. True anomaly

The *true anomaly,* υ, is the angular distance, measured in the orbital plane, from the occupied focus (i.e Earth's center) from the perigee to the current location of the satellite (orbital body). Countered in the direction of movement of the satellite. In degrees. $0^\circ < \upsilon < 360^\circ$.

$$
\begin{aligned}
\sin\upsilon &= \frac{\sqrt{1-e^2}\,\sin E}{1-e\cos E} \\
\cos\upsilon &= \frac{\cos E - e}{1-e\cos E} \\
\tan\frac{\upsilon}{2} &= \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}
\end{aligned}
\qquad \textbf{(A1.7)}
$$

where:

- E is the eccentric anomaly
- e is the eccentricity of the ellipse

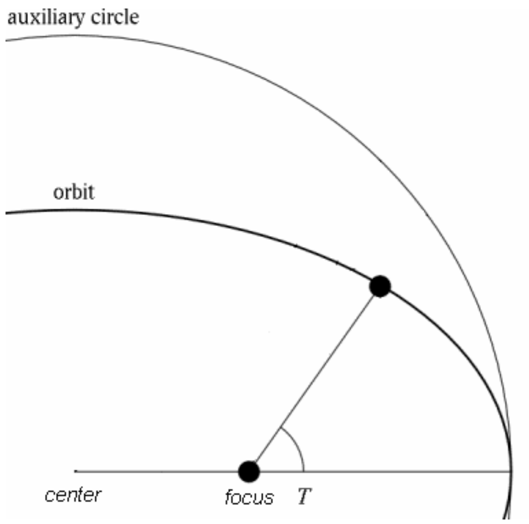

Fig. A1.6 True anomaly

A1.3. Other parameters

A1.3.1. Mean anomaly

The *mean anomaly,* M , is the angle which describes the position of a satellite in its orbit, relative to perigee. At perigee, the Mean Anomaly is zero, it increases to 180 degrees at apogee, then back to perigee at 360 degrees. Since the orbit is often not a circle, the measurement of the angle is difficult, so the angle is expressed in terms of a fractional orbit (one orbit = 360 degrees). This is the reason for the *mean* adjective. *Anomaly* is a term for an angle. For circular orbits, the *mean anomaly* is just the angle between perigee and the current satellite position.

$$M = M_0 + n\left(t - t_0\right) = E - e\sin E \qquad \textbf{(A1.8)}$$

The relation between M and E is the Kepler equation, where:

- M_0 is the mean anomaly at time t_0
- t_0 is the start time
- t is the time of interest
- n is the mean motion
- E is the eccentric anomaly
- e is the eccentricity of the ellipse

A1.3.2. Eccentric anomaly

The *eccentric anomaly*, E , is the angle between the direction of periapsis and the current position of an object on its orbit, projected onto the ellipse's circumscribing circle perpendicularly to the major axis, measured at the centre of the ellipse.

$$
\begin{aligned}
&M = E - e\sin E \\
&\sin \upsilon = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E} \\
&\cos \upsilon = \frac{\cos E - e}{1 - e\cos E} \\
&E = 2\arctan\sqrt{\frac{1-e}{1+e}}\tan \upsilon
\end{aligned}
\qquad \textbf{(A1.9)}
$$

where:

- M is the mean anomaly
- υ is the true anomaly
- e is the eccentricity of the ellipse

Usually the used relation is the first one, solved iteratively (i.e. with the Newton-Raphson numerical method)

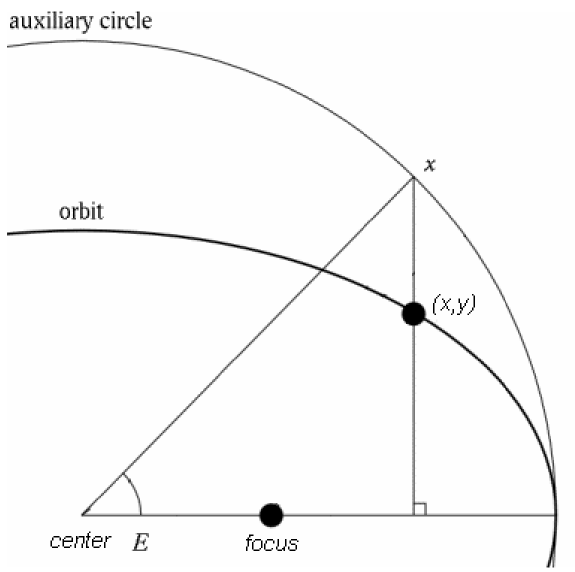

Fig. A1.7 Eccentric anomaly

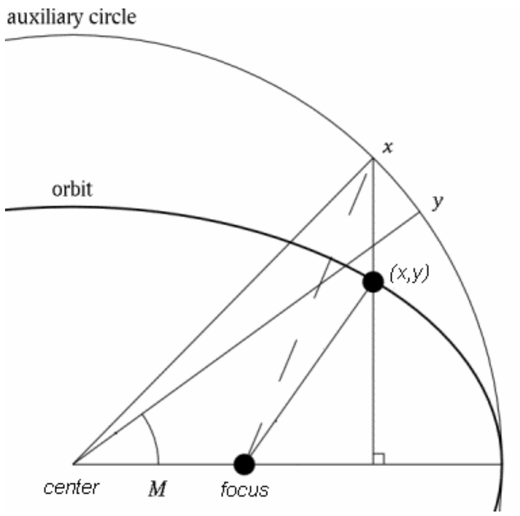

Fig. A1.8 Mean anomaly

A1.3.3. Mean motion

The *mean motion*, *n*, is a element (a number) that indicates the complete number of orbits a satellite makes in one day.

$$n = \frac{dM}{dt} = \sqrt{\frac{\mu}{a^3}} \qquad \textbf{(A1.10)}$$

where:

- M is the mean anomaly
- t is the time
- μ is the standard gravitational parameter
- a is the semi-major axis of the ellipse

A1.3.4. Period of orbit

The *period of orbit*, *P*, is time it takes a satellite to complete one revolution (orbit).

$$P = \frac{2\pi}{n} \qquad \textbf{(A1.11)}$$

where:

- n is the mean motion

A1.3.5. Standard gravitational parameter

In astrodynamics, the *standard gravitational parameter*, μ, of a celestial body is the product of the gravitational constant G and its mass M. The units of the standard gravitational parameter are km^3s^{-2}

$$\mu = GM \qquad \textbf{(A1.12)}$$

Table A1.2. Standard gravitational parameters of the Solar System

Body	**μ (km^3s^{-2})**
Sun	132,712,440,018
Mercury	22,032
Venus	324,859
Earth	398,600
Mars	42,828
Jupiter	126,686,534
Saturn	37,931,187
Uranus	5,793,947
Neptune	6,836,529
Pluto	1,001

A1.3.6. Time of perigee passage

The *time of perigee passage* is the instant time when the satellite crosses its perigee.

A1.3.7. Semilatus Rectum

The *semilatus rectum,* p*,* is the line parallel to the minor axis measured from the focus to the ellipse when the true anomaly, υ, is 90°. *Semilatus rectum* is a compound of the Latin *semi-,* meaning half, *latus*, meaning side and *rectum*, meaning straight,

$$p = a\left(1-e^2\right) = \frac{b^2}{a} \qquad \textbf{(A1.13)}$$

where:

- a is the semi-major axis of the ellipse
- b is the semi-minor axis of the ellipse
- e is the eccentricity of the ellipse

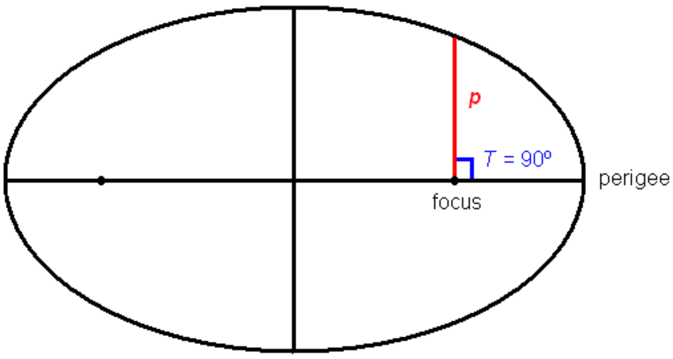

Fig. A1.9 Semilatus rectum

A1.4. Perturbations

A1.4.1. Introduction

Most of the studies for identifying potential relative motion orbits for flying formations have assumed a spherical Earth. The relative motion orbits identified from this assumption result primarily from small changes in the eccentricity and the inclination. This is satisfactory for identifying the potential relative motion orbits, but unsatisfactory for determining long term motion, fuel budgets and the best formations. Assuming that all satellites in the formation are nearly identical, the primary perturbation is the differential gravitational perturbation due to the Earth's oblateness, J2. Since the differential gravity perturbations are a function of (a,e,i) the small changes in these elements result in different drift rates for each satellite and the negation of these drifts result in different fuel requirements for each satellite. Since some satellites running out of fuel before others will degrade the system performance it would be advantageous to have the satellites have equal fuel consumption.

Satellites do not describe perfect elliptical orbits around its central body (this case is Earth). This is due to the influence of several non-keplerian effects, that ordered by importance are:

- Moon's mass
- Sun's mass
- Non sphericity of Earth
- Sun radiation

A1.4.2. Perturbations related to the Sun

Depending on if the satellite is closer to the Sun than to the Earth or not, the acceleration will be higher or lower than the acceleration received from the Earth (the force is inverse proportional to the square of the distance), and this will produce some deviations on satellite's orbit.

Perturbations related to the Sun are two:

- Curvature distortion of the elliptical orbit, making it to be shorter in the Earth-Sun line
- Curvature distortion of the orbit produced by a turn in angular momentum or in the orbit's normal. The correction of this effect means a great extra consume of fuel

Orbit's normal deviation changes with an arc behaviour, corresponding each arc to a journey of six months. The maximum deviation is at the medium point, where the Sun's declination is maximum and then decreases to values around zero, where the declination is minimum.

A1.4.3. Perturbations related to the Moon

The presence of Moon also creates the same perturbation effect as the Sun, but in this case the effect is higher and its study is more complex, because the normal does not remain constant, as it happened in the previous case. The correction of this effect also means an extra consume of fuel that must be foresighted.

A1.4.4. Perturbations related to the non spherical shape of Earth

Several effects are produced for this phenomenon:

- Pole's flattening affects to the orbit's normal
- This flattening also changes slightly the trajectory of the geostationary orbit
- The non circularicity of Equator causes a variation on the longitude of satellite

A1.4.4.1. Effects on satellite's normal

The Earth is not a perfect sphere; it actually bulges at the Equator and it is flattened at the Poles. The polar radius (6356.77 km) is 22 km shorter than the equatorial radius (6378.14 km). A satellite orbiting the Earth experiments an extra attraction when it passes through the Equator and, for this reason, the gravitational attraction force on satellite is not pointed exactly to its centre.

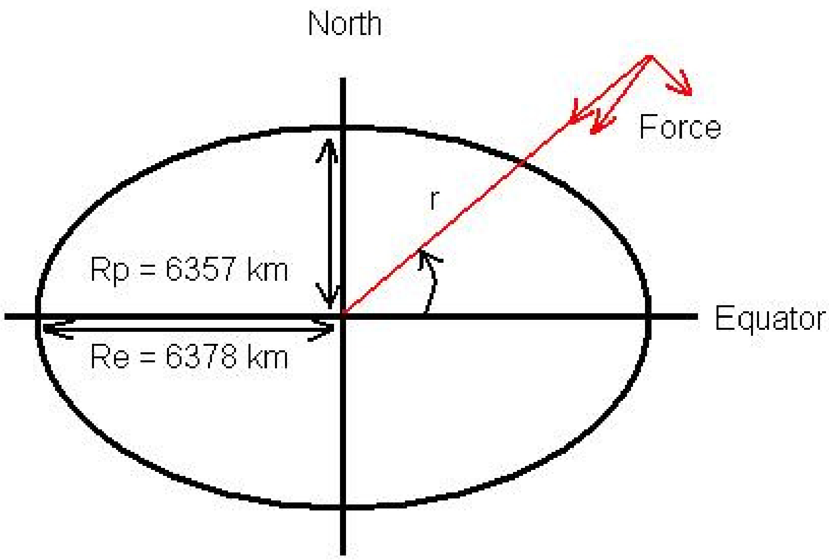

Fig. A1.10 Deviation in gravitational pointing

This force can be expressed as a sum of harmonics:

- First term, 1, corresponds to the principal law of Gravitation
- Following term, with J2 coefficient, corresponds to the flattening of the Earth. The value of this coefficient is $J_2 = 1082.63 \cdot 10^{-6}$.

The flattening of the Earth has two effects on geostationary orbit:

- The gravitational acceleration of the geostationary orbit increases, so, if the orbital period has to be the same as a sidereal day, the radius of the orbit will have to be increased slightly
- If the orbit has some inclination, the satellite will experiment a force pointed towards the equatorial plane that will cause a variation in the normal of the orbit, being this one higher as the inclination increases

A1.4.4.1.1. J2 coefficient

The J2 perturbation (first-order) accounts for secular (constant rate over time) variations in the orbit elements due to Earth oblateness, mainly nodal precession and rotation of the semi major axis of orbital elements that are otherwise those of unperturbed, Newtonian orbits.

J2 is the highest perturbation that affects to the gravitational field of the Earth and it is due to the flatness of the poles. It enables the prediction of orbit rotation an precession. The even zonal harmonic coefficients of the gravity field are the only coefficients that result in secular changes in satellite orbital elements. The effects of J4 (next coefficient) are approximately 1000 times smaller than J2.

Simulating the mission requires propagating the initial Keplerian elements forward in time, which in turn requires knowing the time rate of change in the elements due to the applied forces. Under the ideal two-body problem, only the true anomaly varies through the mean anomaly, which has a time rate of change equal to the mean motion of the satellite.

$$\dot{M} = n \quad \textbf{(A1.14)}$$

Now other disturbing forces yields will be included for a more accurate model. First, the Earth's equatorial bulge exerts a torque on the orbit changing the orientation of the orbital plane through secular changes in the right ascension of the ascending node, the argument of perigee, and the mean anomaly.

$$\dot{\Omega} = \frac{-3nR_{\oplus}^{2}J_{2}}{2p^{2}}\cos i \quad \textbf{(A1.15)}$$

$$\dot{\omega} = \frac{3nR_{\oplus}^{2}J_{2}}{4p^{2}}\left(4-5\sin^{2} i\right) \quad \textbf{(A1.16)}$$

$$\dot{M}_{0} = \frac{3nR_{\oplus}^{2}J_{2}\sqrt{1-e^{2}}}{4p^{2}}\left(3\sin^{2} i-2\right) \quad \textbf{(A1.17)}$$

A1.4.4.2. Effects on satellite’s longitude (east-west position)

As it has been said previously, the sphericity of the Earth makes the gravity vector to not be pointed directly towards the centre of the Earth. Furthermore, another force component is created that can act favourably (or not) to the velocity of the satellite, as it can be observed in Fig A1.11:

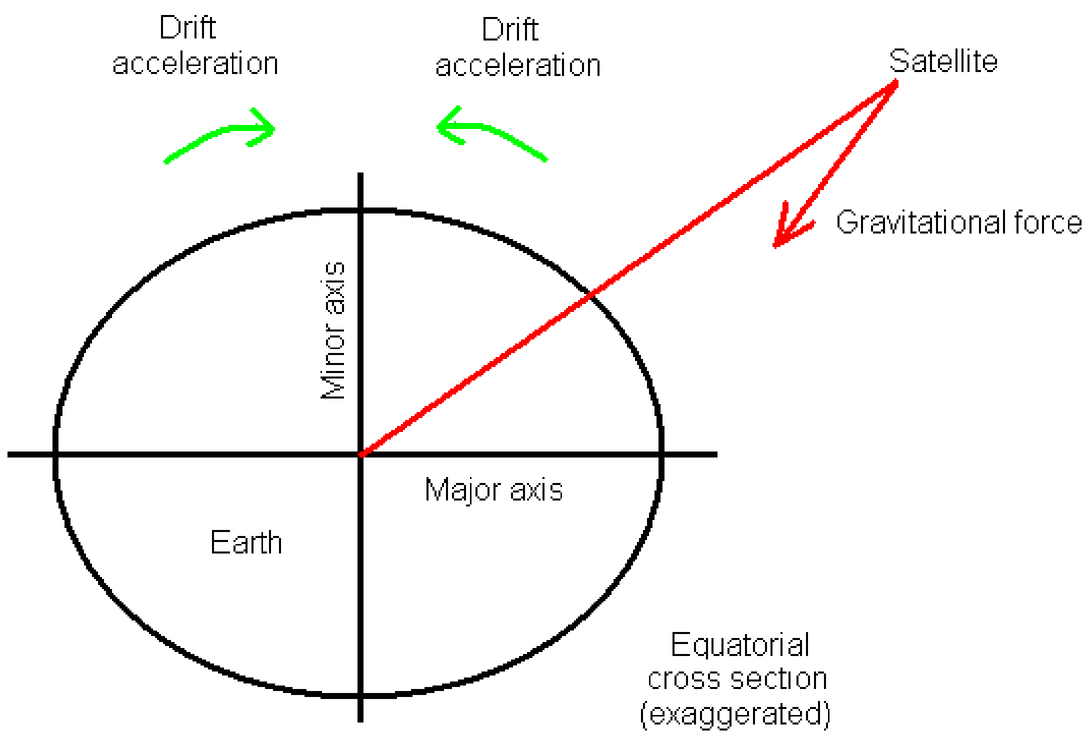

Fig. A1.11 Acceleration drift

When the acceleration is positive, the longitude increases and the satellite moves to the right. When acceleration is negative, the longitude decreases and the satellite moves to the left. Two zero acceleration points are observed (stable points, the satellite remains stationary). The rest of points are unstable, because they produce several accelerations.