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Ordinary CM forms and local Galois representations

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Abstract

The p-adic Galois representation ρ_f attached to a p-ordinary newform f of weight $k \geq 2$ is known to be reducible when restricted to a decomposition group D_p at p. Ralph Greenberg asked for a characterization of those f for which ρ_f actually splits when restricted to D_p. In 2004, Eknath Ghate suggested this happens exactly when f has CM, and in collaboration with Vinayak Vatsal, he has obtained strong, though not definitive, results towards his guess. Here, we describe some of these results and introduce a line of thought about Greenberg's question that is in accordance with the $R = \mathbb{T}$ philosophy, and would confirm new instances of the solution proposed by Ghate.

Contents

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Introduction

A question of Ralph Greenberg

Let $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(\Gamma_0(N), \chi)$ be a normalized newform of weight $k \geq 2$, level $N \geq 1$ and Nebentypus character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. We take the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. Fix an odd prime p, a choice of algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, which we use to single out a decomposition group D_p at p inside the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$. Let $K_f \subset \overline{\mathbb{Q}}$ be the number field generated by the set of Hecke eigenvalues $a_n(f)$ of f. Denote by $\wp$ the prime above p in K_f induced by ι_p, and let $K_\wp \subset \widehat{\overline{\mathbb{Q}}}_p$ be the completion of K_f at $\mathfrak{p}$, where $\widehat{\overline{\mathbb{Q}}}_p$ denotes the completion of $\overline{\mathbb{Q}}_p$ at the prime $\mathfrak{P} \subset \overline{\mathbb{Q}}_p$ above p. Denote by c the complex conjugation in $\mathrm{Aut}(\mathbb{C})$.

Following the work of Shimura and Deligne, one can attach to f a two-dimensional continuous (irreducible) Galois representation

$$\rho_f : G_{\mathbb{Q}} \to \mathbf{GL}_2(K_\wp)$$

which is unramified away from pN and such that for every prime ℓ not dividing pN,

$$\det(\mathrm{Id} - \rho_f(\mathrm{Frob}_\ell)X) = 1 - a_\ell(f)X + \chi(\ell)\ell^{k-1}X^2, \tag{1}$$

where Frob_ℓ denotes an *arithmetic* Frobenius element at ℓ, i.e. any element $\sigma_{\mathfrak{L}} \in G_{\mathbb{Q}}$, depending on the choice of a prime $\mathfrak{L} \subset \overline{\mathbb{Q}}$ above ℓ, determined modulo the inertia group $I_{\mathfrak{L}} \subset G_{\mathbb{Q}}$ corresponding to this choice, by the requirement that $\sigma_{\mathfrak{L}} x \equiv x^\ell \pmod{\mathfrak{L}}$ for all x in the ring of integers of the completion of $\overline{\mathbb{Q}}$ at $\mathfrak{L}$.

We say that f is *ordinary* (or p-ordinary) if the Hecke eigenvalue $a_p(f) \in K_f \subset K_\wp$ is a $\wp$-adic unit. (Note that this depend on $\wp$ rather than p, but we will follow this usual terminology, since once fixed the above choices no confusion is likely to arise.) Then it follows from the work of Wiles [Wil88] that the restriction of ρ_f to the decomposition group $D_p \subset G_{\mathbb{Q}}$ at p is reducible, and that its module of coinvariants by the inertia group $I_p \subset D_p$ is free of rank one. Thus after a suitable choice of basis of the representation space for ρ_f,

$$\rho_f|_{D_p} = \begin{pmatrix} \eta' & a \\ & \eta \end{pmatrix}, \tag{2}$$

for certain continuous characters $\eta', \eta : D_p \to K_\wp^\times$ and a continuous function $a : D_p \to K_\wp$. The character η is unramified, and maps Frob_p to the unique p-adic unit root of the equation

$$X^2 - a_p(f)X + \chi(p)p^{k-1} = 0.$$

Here we convey $\chi(p) = 0$ if $p|N$, so that $\eta(\mathrm{Frob}_p) = a_p(f)$ in that case.

The representation ρ_f is said to be *(locally) split* at p if $\rho_f|_{D_p}$ is semi-simple (or equivalently, since it is reducible, if it is in fact decomposable).

Ralph Greenberg asked for a characterization of those ordinary newforms f of weight $k \geq 2$ for which the associated representation ρ_f is split at p. Eknath Ghate suggests in [Gha04] that a natural "guess" seems to be the following:

$$\rho_f \text{ splits at } p \overset{?}{\Longleftrightarrow} f \text{ has complex multiplication.} \tag{3}$$

Recall that the newform f is said to have *complex multiplication* (or to be *of CM type*) if there exists a non-trivial Dirichlet character φ such that

$$\varphi(\ell)a_\ell(f) = a_\ell(f) \tag{4}$$

for all but finitely many primes ℓ. It is easy to see that there exists at most one non-trivial character φ satisfying (4) for all such ℓ, and that φ is then odd and quadratic (cf. [Rib77]). We call φ the CM character of f, and the quadratic imaginary field cut out by φ (thought of as a finite order Galois character), the CM field of f.

It is easy to see that ρ_f splits at p if f is ordinary and of CM type. Indeed, let φ be the CM character of f, M the CM field of f, and set $G_M = \mathrm{Gal}(\overline{\mathbb{Q}}/M)$. Then

$$\rho_f = \mathrm{Ind}_M^{\mathbb{Q}}\psi \tag{5}$$

for a certain p-adic Hecke character $\psi : G_M \to K_\wp^\times$ of M, and it follows that ρ_f is semi-simple when restricted to G_M. Concretely,

$$\rho_f|_{G_M} = \psi \oplus \psi^c, \tag{6}$$

for the complex conjugate Hecke character ψ^c of ψ given by $g \mapsto \psi(cgc)$. Considering the determinant of ρ_f, it follows from the last two equalities above that, as Galois characters from G_M to $K_\wp^\times$,

$$\chi\varepsilon_{\mathrm{cyc}}^{k-1}|_{G_M} = \psi\psi^c, \tag{7}$$

where $\varepsilon_{\mathrm{cyc}}$ denotes the p-adic cyclotomic character $G_\mathbb{Q} \to \mathbb{Z}_p^\times \subset K_\wp^\times$, giving the Galois action on the group of p-power roots of unity in $\overline{\mathbb{Q}}$. Denote by $D_\mathfrak{p} \subset D_p$ the decomposition group in G_M attached to the prime $\mathfrak{p}$ above p in M determined by our fixed embedding ι_p. Due to the unramifiedness of the character η appearing in (2) we see that the characters ψ, ψ^c cannot both be ramified at $\mathfrak{p}$, and since $\varepsilon_{\mathrm{cyc}}$ is ramified at p, by (7) we may assume that ψ actually ramifies at $\mathfrak{p}$, and that ψ^c does not. But then p can only possibly split in M, and therefore $D_p = D_\mathfrak{p} \subset G_M$ and the claim follows from (6).

The notion of the representation ρ_f being split at p can be rephrased in cohomological terms. With the previous notations, let $\alpha : D_p \to K_\wp^\times$ be the continuous character on

D_p given by η'/η, and let $K_\wp(\alpha)$ be the one-dimensional D_p-representation space $K_\wp$ with D_p-action via the character α. Denote by V_f the representation space of ρ_f. After twisting by η^{-1} it follows from (2) that one has the following exact sequence of $K_\wp[D_p]$-modules:

$$0 \to K_\wp(\alpha) \to V_f(\eta^{-1}) \to K_\wp \to 0. \tag{8}$$

This sequence splits (i.e. $V_f(\eta^{-1})$ is D_p-equivariantly isomorphic to the trivial extension $K_\wp(\alpha) \oplus K_\wp$ of $K_\wp$ by $K_\wp(\alpha)$ with diagonal D_p-action) if and only if the sequence (8) remains exact after taking D_p-invariants, or equivalently, iff the coboundary map

$$\delta : K_\wp^{D_p} = K_\wp \to \mathrm{H}^1(D_p, K_\wp(\alpha))$$

induced by (8) is identically zero. If the extension (8) is non-trivial, then δ is injective and the line $\delta(K_\wp) \subset \mathrm{H}^1(D_p, K_\wp(\alpha))$ depends only on the isomorphism class of $V_f(\eta^{-1})$. This line is spanned by the *extension class* $\delta(1)$ of $V_f(\eta^{-1})$, and one can explicitly produce a cocycle in this class using the matrix representation (2). Indeed, the map $\xi_f : D_p \to K_\wp$ defined by $\xi_f(g) = a(g)\eta(g)^{-1}$ is a 1-cocycle on D_p with values in $K_\wp(\alpha)$ (as one can check using the multiplicativity of the matrix representation) and its cohomology class $[\xi_f] \in \mathrm{H}^1(D_p, K_\wp(\alpha))$ coincides with $\delta(1)$. A different choice of matrix representation affects the resulting ξ_f by a coboundary. One can thus restate the guess in (3) as

$$[\xi_f] = 0 \in \mathrm{H}^1(D_p, K_\wp(\alpha)) \overset{?}{\iff} f \text{ has complex multiplication},$$

and we see from the above discussion that the difficult problem is in the forward direction. Note also that $[\xi_f]$ is non-zero in $\mathrm{H}^1(D_p, K_\wp(\alpha))$ if and only if its restriction to the inertia subgroup $I_p \subset D_p$ is non-zero. Indeed, in the exact sequence

$$0 \to \mathrm{H}^1(D_p/I_p, K_\wp(\alpha)^{I_p}) \to \mathrm{H}^1(D_p, K_\wp(\alpha)) \to \mathrm{H}^1(I_p, K_\wp(\alpha))$$

coming from the inflation-restriction exact sequence, the first non-trivial term vanishes, since $K_\wp(\alpha|_{I_p}) = K_\wp(\eta'|_{I_p})$ is ramified if $k \geq 2$.

Until the work of Ghate and Vatsal, almost all known evidence in support of (3) came from the weight two case, and when the newform f has rational Fourier coefficients, thus being attached to an elliptic curve over $\mathbb{Q}$. Recall that if $E_{/\mathbb{Q}}$ is a rational elliptic curve with CM then E was already known to be modular (cf. [Shi71]), attached to a CM newform of weight 2. In fact, the more recent proof of the Shimura-Taniyama conjecture tells us that nothing more than the solution to Greenberg's question for that class of newforms (and for a prime p not dividing the level) is contained in the following result (cf. [Ser89] IV. A.2.4):

Theorem 1 (Serre)**.** *Let $E_{/\mathbb{Q}}$ be an elliptic curve of conductor N, with good ordinary reduction at the prime p. Denote by $\rho_{E,p}$ be p-adic Galois representation of the local Galois group $D_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ given by the Tate module $\mathrm{Ta}_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of E. Then $\rho_{E,p}$ is semi-simple if and only if E has CM.*

A Λ-adic analogue

Let f be an ordinary newform of level Np, with $(N,p)=1$, and of weight $k \geq 2$. Denote by $\Gamma = 1 + p\mathbb{Z}_p = \mathrm{Ker}(\mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^\times)$ the group units in $\mathbb{Z}_p^\times$ congruent to 1 modulo p, and by Λ the completed group ring $\mathbb{Z}_p[[\Gamma]]$, identified with the power series ring $\mathbb{Z}_p[[X]]$ after setting $1 + X = 1 + p$. It follows from the work of Hida recalled in Section 1.3, that there exists a unique *ordinary* Λ*-adic form*

$$\mathscr{F}_{\mathbb{I}} = \sum_{n=1}^{\infty} a_{\mathbb{I}}(n;\mathscr{F})q^n \in \mathbb{I}[[q]],$$

for $\mathbb{I}$ a finite flat extension of Λ, that *specializes* to f at weight k. Under the simplifying assumption that $\mathbb{I}$ equals Λ itself, this means that the formal q-expansion obtained by applying the algebra homomorphism

$$\begin{array}{rcl} \varphi_k : \mathbb{I} & \rightarrow & \mathbb{Z}_p \\ X & \mapsto & (1+p)^k - 1 \end{array}$$

to the coefficients of $\mathscr{F}_{\mathbb{I}}$ coincides with the q-expansion of f. Note that if we endow $\mathbb{Z}_p$ with the Γ-module structure induced by letting $1+p$ act via multiplication by $(1+p)^k - 1$, then φ_k becomes a Λ-algebra homomorphism. When f has complex multiplication, and therefore is the theta series

$$\theta(\psi) = \sum_{(\mathfrak{a},\mathfrak{c})=1} \psi(\mathfrak{a})q^{\mathrm{Norm}(\mathfrak{a})}$$

attached to a Hecke character ψ of an imaginary quadratic field M, where $\mathfrak{c}$ denotes the integral ideal of M that is the conductor of ψ, then the Λ-adic form $\mathscr{F}_{\mathbb{I}}$ specializing to f at weight k admits a more explicit description which is recalled in Section 1.5. We then say that $\mathscr{F}_{\mathbb{I}}$ itself has complex multiplication or that it is of CM type, and one has the easy but crucial fact that *all* its arithmetic specializations are ordinary CM forms.

Hida also attaches to any ordinary Λ-adic form $\mathscr{F}_{\mathbb{I}}$ a continuous Galois representation

$$\rho_{\mathbb{I}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathbb{I}),$$

which is unramified outside Np, and such that $\mathrm{tr}\rho_{\mathbb{I}}(\mathrm{Frob}_\ell) = a_{\mathbb{I}}(\ell;\mathscr{F})$ for every prime $\ell \nmid Np$. Here the work of Mazur and Wiles [MW86] shows that the restriction of $\rho_{\mathbb{I}}$ to the decomposition group D_p is also reducible with free rank one $\mathbb{I}$-module of I_p-coinvariants. When $\mathscr{F}_{\mathbb{I}}$ is of CM type, the corresponding representation $\rho_{\mathbb{I}}$ can be shown to be locally split at p using the above arguments. Therefore, a Λ-adic analogue of Greenberg's question naturally arises, and it makes sense to wonder whether Ghate's guess (3) has some chance to be true in this setting. In fact, by studying the weight one specializations of ordinary Λ-adic forms, Ghate and Vatsal were lead to the following result:

Theorem 2 (Ghate-Vatsal)**.** *Let $p > 2$ be a prime and $N > 0$ an integer with $(N, p) = 1$. Let $\mathscr{F}_{\mathbb{I}}$ be an ordinary Λ-adic form attached to a local component $\mathbb{I}$. Denote by $\mathfrak{m}$ the maximal ideal of $\mathbb{I}$, and by $\rho_{\mathfrak{m}}$ the reduction modulo $\mathfrak{m}$ of the Galois representation $\rho_{\mathbb{I}}$ associated to $\mathscr{F}_{\mathbb{I}}$, and assume that $\rho_{\mathfrak{m}}$ satisfies the following properties:*

- $\rho_{\mathfrak{m}}|_{D_p}$ *has non-scalar semi-simplification;*
- $\rho_{\mathfrak{m}}$ *is absolutely irreducible when restricted to $G_{\mathbb{Q}(\sqrt{p^*})}$, where $p^* = (-1)^{(p-1)/2}p$.*

Then

$$\rho_{\mathbb{I}} \text{ splits at } p \Longleftrightarrow \mathscr{F}_{\mathbb{I}} \text{ has complex multiplication.}$$

This theorem is the main result in [GV04], and by a descend argument it allows Ghate and Vatsal to show that under the corresponding technical hypothesis, Ghate's guess (3) is "generically true":

Corollary 3. *Fix a prime $p > 2$ and an integer $N > 0$ with $(N, p) = 1$. Denote by $\mathscr{S}_{\text{bad}} = \mathscr{S}_{\text{bad}}(p^\infty)$ the set of ordinary non-CM newforms of weight k and level Np^r, for any integers $k \geq 2$ and $r \geq 1$, satisfying the following properties:*

- $\rho_f|_{D_p}$ *splits;*
- $\overline{\rho}_f|_{D_p}$ *has non-scalar semi-simplification;*
- $\overline{\rho}_f$ *is absolutely irreducible when restricted to $G_{\mathbb{Q}(\sqrt{p^*})}$.*

Then $\mathscr{S}_{\text{bad}}$ is a finite set.

Note that this solution to Greenberg's question, though almost complete (when coupled with the above analysis for ordinary CM newforms), has the obvious drawback of giving no indication of the truth or falsity of Ghate's guess (3) for *a given* newform f, since one has almost no control at all over the set of possible exceptional non-CM newforms that appear in the corollary. These results are discussed in Section 1.6.

Deformation theoretic approaches

Given an ordinary newform f, it is a hard problem to decide (i.e. to prove) whether the local representation $\rho_f|_{D_p}$ is indecomposable. Ghate and Vatsal were able in [GV08] to show the first nontrivial examples of non-CM ordinary newforms of weight strictly > 2 whose associated Galois representation is non-split at p, actually proving that the local indecomposability at p occurs for the Galois representation attached to *each* of the classical specializations of certain Hida families of non-CM type. This was achieved by means of the study of what they termed the *universal (ordinary) locally split deformation ring.*

Let f be an ordinary newform as in the beginning of this introduction, $R_\wp$ the ring of integers of $K_\wp$ with maximal ideal $\mathfrak{m}_\wp$, and denote by κ the residue field $R_\wp/\mathfrak{m}_\wp$. In general, we say that a (continuous) Galois representation $\rho : G_{\mathbb{Q}} \to \mathbf{GL}_2(R)$ over a (topological) ring R is *ordinary* if it is unramified outside a finite set of primes and is realized over a free R-module of rank 2 with R-module of I_p-coinvariants free of rank one. Choose a $G_{\mathbb{Q}}$-stable lattice $T_f \subset V_f$ and consider the ordinary residual representation

$$\overline{\rho}_f : G_{\mathbb{Q}} \to \mathbf{GL}_2(\kappa)$$

obtained by reducing modulo $\mathfrak{m}_{\wp}$ the resulting ordinary representation on T_f, which we still denote by ρ_f.

Assumption 1. *$\overline{\rho}_f$ is absolutely irreducible.*

Then the lattice T_f is well-defined up to homothety, and the representation $\overline{\rho}_f$ does not depend on the choice of T_f. Fix a finite set S of primes containing ∞, p, and the prime divisors of N, so that the representation ρ_f factors through $G_{\mathbb{Q},S}$, the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S. Then under Assumption 1 the work of Mazur in [Maz89] gives the existence of a complete Noetherian local ring $(R^o = R^o(\overline{\rho}_f), \mathfrak{m}_{R^o})$ equipped with an ordinary Galois representation

$$\rho^o : G_{\mathbb{Q},S} \to \mathbf{GL}_2(R^o),$$

reducing modulo $\mathfrak{m}_{R^o}$ to $\overline{\rho}_f$, which *parametrizes* all isomorphism classes of ordinary Galois representations ρ_A with values in an Artinian local ring $(A, \mathfrak{m}_A)$ with residue field κ and reducing modulo $\mathfrak{m}_A$ to $\overline{\rho}_f$, in the sense that, roughly, each such ρ_A (unramified outside S) is obtained as the push-forward of ρ^o by a unique specialization map $\varphi_A \in \mathrm{Spec}R^o(A)$.

Assumption 2. *$\overline{\rho}_f$ is p-distinguished (i.e. $\overline{\rho}_f|_{D_p}$ has non-scalar semi-simplification).*

Assumption 3. *$\overline{\rho}_f$ is locally split at p: $\overline{\rho}_f|_{D_p} = \begin{pmatrix} \overline{\eta}' & \\ & \overline{\eta} \end{pmatrix}$.*

Under assumptions 1 and 2, it is easy to see (cf. [Gha05], Prop. 6) that Assumption 3 is necessary for the existence of liftings of $\overline{\rho}_f$ to characteristic 0 coming from modular forms with split at p local Galois representation. We note however that even under this assumption, $\overline{\rho}_f$ may well have char 0 liftings coming from newforms whose attached Galois representation is non-split at p. This phenomenon was shown by Ghate and Vatsal in [GV08] for a number of “dihedral” ordinary primes of certain level 1 eigenforms. The indecomposability of the local Galois representations attached to these eigenforms (which is expected by (3), since level one eigenforms are necessarily of non-CM type) was proved by the following method.

Consider the functor $\mathscr{D}^{\mathrm{split}}_{\overline{\rho}_f}$, from the category $\mathscr{A}\mathscr{R}_\kappa$ of Artinian local rings with residue field κ into sets, that attaches to every $A \in \mathscr{A}\mathscr{R}_\kappa$ the set of *strict* isomorphism classes of ordinary representations

$$\rho_A : G_{\mathbb{Q},S} \to \mathbf{GL}_2(A),$$

unramified outside our fixed finite set of primes S, satisfying the following two properties:

- $(\rho_A \bmod \mathfrak{m}_A) = \overline{\rho}_f$;
- $\rho_A|_{D_p} = \begin{pmatrix} \eta'_A & \\ & \eta_A \end{pmatrix}$, with $\eta'_A, \eta_A : G_{\mathbb{Q},S} \to A^\times$ reducing modulo $\mathfrak{m}_A$ to $\overline{\eta'}, \overline{\eta}$, respectively.

The original arguments of Mazur can be readily adapted to this *deformation problem* to show the pro-representability of $\mathscr{D}^{\mathrm{split}}_{\overline{\rho}_f}$, thus yielding a complete Noetherian local ring $R^{\mathrm{split}}_{\overline{\rho}_f}$ with residue field κ, together with an ordinary Galois representation

$$\rho^{\mathrm{split}}_{\overline{\rho}_f} : G_{\mathbb{Q},S} \to \mathbf{GL}_2(R^{\mathrm{split}}_{\overline{\rho}_f})$$

satisfying the above two properties, and such that $\mathscr{D}^{\mathrm{split}}_{\overline{\rho}_f}(A) = \mathrm{Spec} R^{\mathrm{split}}_{\overline{\rho}_f}(A)$ for every $A \in \mathscr{A}\mathscr{R}_\kappa$ via $[\rho_A] \mapsto \varphi_A$, where $[\rho_A]$ denotes the strict equivalence class of ρ_A and φ_A is the unique local algebra homomorphism $R^{\mathrm{split}}_{\overline{\rho}_f} \to A$ such that $[\rho_A] = [\varphi_A \circ \rho^{\mathrm{split}}_{\overline{\rho}_f}]$.

Suppose for the rest of this subsection that f has level 1, and therefore does not have CM. One can then take $S = \{p, \infty\}$ above. Each p-stabilized newform g obtained as a classical specialization of the Λ-adic form $\mathscr{F}_{\mathbb{I}}$ of tame level 1 containing f has an attached Galois representation ρ_g which is ordinary, unramified outside S, and reduces to $\overline{\rho}_g = \overline{\rho}_f$. Therefore, if such ρ_g is locally split at p, it gives rise to a point in the *locally split space* $\mathrm{Spec} R^{\mathrm{split}}_{\overline{\rho}_f}$. One can thus obviously show that a representation $\rho_g|_{D_p}$ is indecomposable by proving that no characteristic zero point in $\mathrm{Spec} R^{\mathrm{split}}_{\overline{\rho}_f}$ arises from it. Indeed, Ghate and Vatsal were able to obtain, for some of their explicit examples, a sufficient control on the size of $R^{\mathrm{split}}_{\overline{\rho}_f}$ to arrive to such a conclusion, obtaining in particular the following result:

Theorem 4 (Ghate-Vatsal)**.** *Let $f = \Delta = \sum_{n=1}^{\infty} \tau(n) q^n$ be the unique normalized newform of weight 12 and level 1 (hence of non-CM type). Then for every prime p with $7 < p < 2411$, which are all known to be ordinary for Δ, the Galois representation attached to each classical member in the $\mathbb{Z}_p[[X]]$-adic form containing Δ is non-split at p.*

This is obtained by the following analysis. For every (ordinary) prime for Δ in the above range, the local residual representation $\overline{\rho}_\Delta|_{D_p}$ is non-split, except for the primes $p = 23$ and $p = 691$. In the first exceptional case, $\overline{\rho}_\Delta = (\rho_\Delta \bmod 23)$ is induced from a character ψ_o of the imaginary quadratic field $\mathbb{Q}(\sqrt{-23})$, and the fact that $\mathrm{Spec} R^{\mathrm{split}}_{\overline{\rho}_\Delta}$ has a unique characteristic zero point, corresponding to the induction from G_M to $G_{\mathbb{Q}}$ of the Teichmüller lift of ψ_o (cf. [GV08] Thm. 4.1.7), allows to conclude that ρ_Δ does not lie in the split locus of $\overline{\rho}_\Delta$. In the second one, $\overline{\rho}_\Delta = (\rho_\Delta \bmod 691)$ is globally reducible, and by a straightforward argument using Ribet's converse to Herbrand's Theorem one deduces the indecomposability of $\rho_\Delta|_{D_{691}}$.

Approach to the problem

Ghate and Vatsal worked in [GV08] with explicit examples of ordinary newforms f of level one for which one expected $R^{\text{split}}_{\bar{\rho}_f}$ to be "small"; more concretely, not to contain points corresponding to ordinary modular forms of weight $k \geq 2$. In a quite orthogonal direction, one may try to put oneself in a situation where one knows a priori the existence of many locally split classical points, and try to show that all of them are of the type one expects. More precisely, suppose now that f is an ordinary newform of weight $k \geq 2$ satisfying Assumptions 1—3 above and the following

Assumption 4. *ρ_f is locally split at p.*

Assumption 5. *There exists a CM form f' of weight $k' \geq 2$ such that $f \equiv f' \pmod{\mathfrak{P}}$.*

Now Assumption 3 becomes redundant due to the result of Ghate alluded to above, so we actually just need to suppose that f satisfies Assumptions 1, 2, 4 and 5. Note also that by the preceding discussion, ordinary CM forms of weight $k \geq 2$ fullfill all these four requirements (of course, one can then take $f' = f$ in Assumption 5), and that Ghate's guess (3) predicts that these are, among ordinary newforms of weight $k \geq 2$, *the only ones* that satisfy those requirements (in fact, that satisfy Assumption 5).

Let f' be as in Assumption 5, so that f' is the theta series $\theta(\psi)$ attached to a Hecke character ψ of an imaginary quadratic field M. In Section 1.5 it is recalled how to attach to the pair M, ψ, a 2-dimensional Noetherian local domain $\mathbb{I}$ (in fact, an irreducible component of the universal ordinary Hida-Hecke algebra discussed in Section 1.2) equipped with an ordinary Galois representation

$$\rho_{\mathbb{I}} : G_{\mathbb{Q},S} \to \mathbf{GL}_2(\mathbb{I}).$$

The representation $\rho_{\mathbb{I}}$ is such that for any p-stabilized newform g belonging to the CM type Λ-adic form containing f', there exists a unique specialization φ_g such that

$$\rho_g \cong \varphi_g \circ \rho_{\mathbb{I}},$$

where ρ_g denotes the p-adic representation associated to the CM form g. Furthermore, the representation $\rho_{\mathbb{I}}$ itself is the induction from G_M to $G_{\mathbb{Q}}$ of a Λ-adic character $\Psi : G_M \to \mathbb{I}^{\times}$ attached to ψ. Thus $\rho_{\mathbb{I}}$ is locally split at p, and by the universal property of $R^{\text{split}}_{\bar{\rho}_f}$, there exists a unique local ring homomorphism

$$\varphi_{\Psi} : R^{\text{split}}_{\bar{\rho}_f} \to \mathbb{I}$$

such that $\rho_{\mathbb{I}} = \varphi_{\Psi} \circ \rho^{\text{split}}_{\bar{\rho}_f}$. This homomorphism can be easily seen to be surjective, and Ghate's guess leads one to believe the following

Conjecture 5. *φ_Ψ is an isomorphism.*

We remark the similarity with the conjecture raised by Mazur and Tilouine in [MT90]. There the rings involved are the ordinary deformation ring $R^{\mathrm{ord}}_{\rho_o}$ attached to an absolutely irreducible ordinary modular residual representation

$$\rho_o : G_{\mathbb{Q},S} \to \mathbf{GL}_2(\kappa),$$

and a local component R of the universal ordinary Hida-Hecke algebra $\mathbb{T}^o_N$ obtained as the localization of $\mathbb{T}^o_N$ at the maximal ideal of $\mathbb{T}^o_N$ with residue field κ. A natural local homomorphism

$$\varphi : R^{\mathrm{ord}}_{\rho_o} \to R$$

is similarly deduced from the universality of $R^{\mathrm{ord}}_{\rho_o}$ and the properties of the representation attached to the local component R. Of course, after the pioneering work of Wiles, this older conjecture is now known to be true, and thus we have $R^{\mathrm{ord}}_{\overline{\rho}_f} = R$ for our residual representation $\overline{\rho}_f$. Note that we have a natural surjection $R^{\mathrm{ord}}_{\overline{\rho}_f} \twoheadrightarrow R^{\mathrm{split}}_{\overline{\rho}_f}$. We may then view Conjecture 5 as asserting that the locally split locus $\mathrm{Spec} R^{\mathrm{split}}_{\overline{\rho}_f}$ sits as an *irreducible component* in the ordinary deformation space $\mathrm{Spec} R^{\mathrm{ord}}_{\overline{\rho}_f}$ of $\overline{\rho}_f$.

We end this introduction stating the following conjecture. This would be a direct consequence of Conjecture 5, and both statements are in accordance with Ghate's guess.

Conjecture 6. *Let $p > 2$ be a prime, and f a p-ordinary CM newform of level Np, with $(N,p) = 1$, and weight $k \geq 2$. Let $\mathcal{O}$ be the ring of integers of a finite extension $K/\mathbb{Q}_p$ with residue field κ. Let $\rho_f = \mathrm{Ind}^{\mathbb{Q}}_M \psi : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathcal{O})$ be the Galois representation attached to f, and suppose that the residual representation $\overline{\rho}_f = \mathrm{Ind}^{\mathbb{Q}}_M \overline{\psi} : G_{\mathbb{Q}} \to \mathbf{GL}_2(\kappa)$ is absolutely irreducible, and that it is p-distinguished (i.e. $\overline{\psi^{1-c}}|_{D_p} \neq 1$). Let f' be a p-stabilized newform of weight $k \geq 2$ satisfying the following two hypothesis:*

- *$\rho_{f'}$ splits at p;*
- *$\overline{\rho}_{f'} \cong \overline{\rho}_f = \mathrm{Ind}^{\mathbb{Q}}_M \overline{\psi}$.*

Then f' has complex multiplication.

Remark 1. Obviously, the conclusion of the conjecture is equivalent to the following (keeping with the same notations): if f' does not have CM and $\overline{\rho}_{f'} \cong \overline{\rho}_f$ (in particular, if $f' \equiv f \pmod{\mathfrak{P}}$, for the prime $\mathfrak{P} \subset \overline{\mathbb{Q}}_p$ above p), then $\rho_{f'}|_{D_p}$ is non-split.

Remark 2. Note that it would follow from this conjecture, together with Theorem 2, that for the irreducible components of the local ring of the universal ordinary Hecke algebra corresponding to $\mathrm{Ind}^{\mathbb{Q}}_M \overline{\psi}$, no classical specialization with split at p representation can come from an irreducible component $\mathbb{I}$ with $\rho_{\mathbb{I}}|_{D_p}$ non-split.

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1 Hida deformation theory

1.1 The space of p-adic modular cusp forms

Let p be an odd prime, and $N \geq 1$ an integer relatively prime to p. For each pair of integers $k \geq 2$ and $r > 0$, denote by $S_k(Np^r)$ the space

$$S_k(\Gamma_1(Np^r); \overline{\mathbb{Q}}_p) := S_k(\Gamma_1(Np^r); \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_p,$$

where the tensor product is taken along the fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. The q-expansion of cusp forms furnishes an inclusion $S_k(Np^r) \subset \overline{\mathbb{Q}}_p[[q]]$. For a $\mathbb{Z}_p$-algebra $R \subset \overline{\mathbb{Q}}_p$ we let $S_k(Np^r; R) \subset S_k(Np^r)$ be the submodule consisting of cusp forms whose q-expansion lies in $R[[q]]$. Let R be the ring of integers in a finite extension $K/\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. We first describe an object which collects all R-rational cusp forms with fixed prime to p *tame* level N and arbitrary p-power in the level. We let the group $(\mathbb{Z}/Np^r\mathbb{Z})^\times$ act on $S_k(Np^r)$ via the *weight zero* action $\langle\ \rangle_0$ (cf.[Til88]), i.e. the product of the usual Nebentypus action $\langle\ \rangle$ and the character $\ell \mapsto \ell_p^k$, where $\ell = (\ell_N, \ell_p) \in (\mathbb{Z}/Np^r\mathbb{Z})^\times = (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/p^r\mathbb{Z})^\times$, so that

$$a_n(f|\langle\ell\rangle_0) = \ell_p^k a_n(f|\langle\ell\rangle).$$

Let $T_\ell = T_{\ell,r}^{(k)}$ and $U_\ell = U_{\ell,r}^{(k)}$, for $\ell \nmid Np$ and $\ell | Np$ respectively, be the Hecke operators on $S_k(Np^r; R)$ whose action on the Fourier coefficients of $f = \sum_{n=1}^\infty a_n(f) q^n$ is given by the formulas

$$\begin{aligned} a_n(f|T_\ell) &= a_{n\ell}(f) + \ell^{-1} a_{n/\ell}(f|\langle\ell\rangle_0); \\ a_n(f|U_\ell) &= a_{n\ell}(f), \end{aligned}$$

with the convention that $a_{n/\ell}(f|\langle\ell\rangle_0) = 0$ if $\ell \nmid n$. These operators preserve the subspace $S_k(Np^r; R)$, and we denote by $\mathbb{T}_{Np^r}^{(k)}$ the subalgebra of $\mathrm{End}_R(S_k(Np^r; R))$ generated over R by the resulting Hecke operators T_ℓ and U_ℓ, together with the operators $\langle\ell\rangle$ for $(\ell, N) = 1$, acting on $S_k(Np^r; R)$. For every pair of integers $r' \geq r > 0$, there is a commutative diagram

$$\begin{array}{ccc} S_k(Np^r) & \hookrightarrow & S_k(Np^{r'}) \\ \downarrow & & \downarrow \\ S_k(Np^r) & \hookrightarrow & S_k(Np^{r'}), \end{array}$$

where the horizontal arrows are the natural inclusion maps, and the vertical arrows are given by the same (with a slight abuse of terminology) Hecke operator. This diagram remains commutative when restricted R-rational cusp forms, thus showing that the space

$$S_k(Np^\infty; R) := \varinjlim_r S_k(Np^r; R)$$

is naturally endowed with an action of the Hecke algebra

$$\mathbb{T}^{(k)}_{Np^\infty} := \varprojlim_r \mathbb{T}^{(k)}_{Np^r},$$

where the projective limit is taken along the restriction homomorphisms.

We can complete $S_k(Np^\infty; R)$ with respect to the natural norm $\|\ \|$ on $R[[q]]$ defined, for $f = \sum_{n_1}^{\infty} a_n(f)q^n$, by the formula

$$\| f \| := \sup_n |a_n(f)|_p \tag{9}$$

where $|\ |_p$ is the valuation on R normalized with $|p|_p = 1/p$. We denote the resulting space with the script notation $\mathscr{S}_k(Np^\infty; R)$. Note that it follows readily from the above description of the action of the Hecke operators on cusp forms, that $\mathbb{T}^{(k)}_{Np^\infty}$ acts uniformly continuously on $S_k(Np^\infty; R)$, and therefore the Hecke action extends to a continuous action on $\mathscr{S}_k(Np^\infty; R)$.

The weight zero action of $(\mathbb{Z}/Np^r\mathbb{Z})^\times$ on $S_k(Np^r; R)$ is also compatible with the natural inclusion map, and thus $\mathscr{S}_k(Np^\infty; R)$ is also equipped with an action of the profinite group

$$\mathbb{Z}^\times_{p,N} := \varprojlim_r (\mathbb{Z}/Np^r\mathbb{Z})^\times.$$

Note that since $(p, N) = 1$, we have the decomposition

$$\mathbb{Z}^\times_{p,N} = (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times = (\mathbb{Z}/Np\mathbb{Z})^\times \times \Gamma,$$

where $\Gamma = 1 + p\mathbb{Z}_p$ is the group on p-adic 1-units in $\mathbb{Z}_p^\times$.

For every integer $r > 0$, denote by $\mathscr{S}(Np^r; R)$ the subspace of $R[[q]]$ given by the completion under the norm $\|\ \|$ defined by (9) of

$$S_\infty(Np^r; R) := \varinjlim_j \left(\bigoplus_{k=1}^{j} S_k(Np^r; R) \right).$$

Note that $\bigcup_{k\geq 1} S_k(Np^r; R) \subset R[[q]]$ is naturally contained in $\mathscr{S}(Np^r; R)$. One easily checks that $S_\infty(Np^r; R)$ is a continuous module over the Hecke algebra

$$\mathbb{T}^{(\infty)}_{Np^r} = \varprojlim_j \mathbb{T}^{(j)}_{Np^r},$$

where $\mathbb{T}^{(j)}_{Np^r}$ is the R-subalgebra of $\mathrm{End}_R(\bigoplus_{k=1}^{j} S_k(Np^r;R))$ generated by the Hecke operators via the diagonal action, and the projective limit is taken along the natural restriction maps induced for every pair of integers $j' \geq j \geq 1$ by the obvious commutative diagram. As before, the Hecke action extends to a continuous action on $\mathscr{S}(Np^r;R)$.

It is a result due to Katz that for every pair of integers $r' \geq r > 0$, the natural inclusion

$$S_\infty(Np^r;R) \subset S_\infty(Np^{r'};R),$$

when extended to the completion under the topology defined by $\| \ \|$, allows us to identify

$$\mathscr{S}(Np^r;R) = \mathscr{S}(Np^{r'};R).$$

We denote by $\mathscr{S}(Np^\infty;R)$ this common space $\mathscr{S}(Np^r;R)$ for every $r > 0$, and by $\mathbb{T}_N$ the corresponding common algebra $\mathbb{T}^{(\infty)}_{Np^r} \subset \mathrm{End}_R(\mathscr{S}(Np^r;R))$ for every $r > 0$. We refer to $\mathscr{S}(Np^\infty;R)$ as the space of *p-adic modular cusp forms of tame level N* (and arbitrary weight). Note that for every integer $k \geq 1$, we have

$$S_k(Np^r;R) \subset \mathscr{S}(Np^\infty;R)$$

naturally for every $r > 0$, and therefore also $\mathscr{S}_k(Np^\infty;R) \subset \mathscr{S}(Np^\infty;R)$.

1.2 The universal ordinary Hecke algebra

Let $\mathscr{S}(Np^\infty;R)$ be the space of p-adic cusp forms $\mathbb{T}_N$ its corresponding Hecke algebra as defined in Section 1.1. The pairing

$$\begin{array}{rcl} \mathscr{S}(Np^\infty;R) \times \mathbb{T}_N & \to & R \\ (f,T) & \mapsto & a_1(f|T) \end{array}$$

is a perfect R-duality, thus inducing a linear isomorphism

$$\mathbb{T}_N \cong \mathrm{Hom}_R(\mathscr{S}(Np^\infty;R),R).$$

We use the superscript "ord" to denote the maximal quotient of a Hecke algebra in which the Hecke operator U_p becomes a unit. One then has an algebra direct sum decomposition (and similarly for $\mathbb{T}^{(k)}_{Np^\infty}$):

$$\mathbb{T}_N = \mathbb{T}_N^{\mathrm{ord}} \oplus \mathbb{T}_N^{\mathrm{ss}},$$

with U_p having topologically nilpotent image in the second direct summand. Denote by e_{ord} the idempotent attached to the *ordinary part*. The image

$$\mathscr{S}^o(Np^\infty;R) := e_{\mathrm{ord}}\mathscr{S}(Np^\infty;R)$$

is called the space of *p-adic ordinary cusp forms of tame level* N (and arbitrary weight), and one has that an eigenform $f \in S_k(Np^r; R)$, with $r \geq 0$, is ordinary (i.e. f is contained in $\mathscr{S}^o(Np^\infty; R)$) if and only if its p-th Fourier coefficient satisfies $|a_p(f)|_p = 1$. The restriction of operators via $\mathscr{S}(Np^\infty; R) \supset \mathscr{S}_k(Np^\infty; R)$ furnishes the commutative diagram

$$\begin{array}{ccc} \mathbb{T}_N & \longrightarrow & \mathbb{T}^{(k)}_{Np^\infty} \\ \cup\big| & & \big|\cup \\ \mathbb{T}^{\mathrm{ord}}_N & \longrightarrow & \mathbb{T}^{(k),\mathrm{ord}}_{Np^\infty}, \end{array}$$

where the horizontal arrows put in correspondence the respective Hecke operators, and are therefore surjective, since these operators topologically generate the Hecke algebras.

Proposition 1.1 ([H86b] Thm. 1.1)**.** *The above algebra homomorphism* $\mathbb{T}^{\mathrm{ord}}_N \to \mathbb{T}^{(k),\mathrm{ord}}_{Np^\infty}$ *is an isomorphism, independently of the integer* $k \geq 2$*.*

We denote by $\mathbb{T}^o_N$ the *universal ordinary Hecke algebra* $\mathbb{T}^{\mathrm{ord}}_N \subset \mathrm{End}_R(\mathscr{S}^o(Np^\infty; R))$, and we see that by the above proposition it is naturally isomorphic, for every integer $k \geq 1$, to the Hecke subalgebra of $\mathrm{End}_R(\mathscr{S}_k(Np^\infty; R))$.

Remark 3. For each integer $r > 0$, let $X_1(Np^r)_{/\mathbb{Q}}$ be the usual modular curve, compactification of the affine curve $Y_1(Np^r)$, and consider its jacobian variety $J_1(Np^r) = \mathrm{Jac}(X_1(Np^r))$. Denote by $\mathbb{T}_{Np^r} = \mathbb{T}^{(2)}_{Np^r}(\mathbb{Z})$ the ring generated over $\mathbb{Z}$ by the Hecke operators acting as rational endomorphisms on $J_1(Np^r)$. For every pair of integers $r' \geq r > 0$ the covering maps $X_1(Np^{r'}) \to X_1(Np^r)$ induce by Picard (contravariant) functoriality Hecke-equivariant endomorphisms $J_1(Np^r) \to J_1(Np^{r'})$. Thus we see that

$$J_\infty := \varinjlim_r J_1(Np^r)$$

is naturally endowed with an action of the Hecke algebra $\mathbb{T}_{Np^\infty} = \varprojlim_r \mathbb{T}_{Np^r}$. Then we denote by J^o_∞ the maximal submodule of J_∞ on which the ordinary part $\mathbb{T}^{\mathrm{ord}}_{Np^\infty}$ of $\mathbb{T}_{Np^\infty}$ acts faithfully. Then

$$\mathbb{T}^o_N = \mathbb{T}^{\mathrm{ord}}_{Np^\infty} \otimes_{\mathbb{Z}} R.$$

1.3 Ordinary Λ-adic forms: p-adic families

Let $f = \sum a_n(f) q^n \in S_k(Np^r; R)$, for certain integers $k > 1$ and $r > 0$, be a normalized Hecke eigenform; or equivalently, an R-algebra homomorphism

$$\begin{array}{rcl} \lambda_f : \mathbb{T}^{(k)}_{Np^r} & \to & R \\ T & \mapsto & \lambda_f(T) = a_1(f|T). \end{array}$$

Suppose that f is ordinary, so that $a_p(f) = \lambda_f(U_p)$ is a p-adic unit. We say that f is an *ordinary p-stabilized newform* if either it is a newform of level Np^r, or it is obtained from an ordinary newform g of level N by the following process of *p-stabilization*: let α_p and β_p be the reciprocal of the equation

$$\det(\mathrm{Id} - \rho_g(\mathrm{Frob}_p)X) = 0,$$

where ρ_g denotes the p-adic representation attached to g; the ordinarity of g and the assumption that $k > 1$ implies that exactly one of these roots is a p-adic unit; we suppose that this is α_p, and then we have

$$f(z) = g(z) - \beta_p g(pz).$$

Note that in the latter case, f is an old form (despite the terminology) of level Np, f and g have the same eigenvalues $a_\ell(f) = a_\ell(g)$ for all primes $\ell \neq p$, g is uniquely determined by this property, and $f|U_p = \alpha_p f$.

The weight zero action of $(\mathbb{Z}/Np^r\mathbb{Z})^\times$ on $S_k(Np^r; R)$ can be expressed in terms of Hecke operators, giving rise to an homomorphism

$$(\ell \mapsto \langle \ell \rangle_0) : (\mathbb{Z}/Np^r\mathbb{Z})^\times \to \mathbb{T}^{(k)}_{Np^r}.$$

Thus the universal ordinary Hecke algebra $\mathbb{T}^o_N$ adopts the structure of a module over the completed group ring $R[[\mathbb{Z}^\times_{p,N}]]$; so, in particular $\mathbb{T}^o_N$ is naturally a module over the Iwasawa algebra

$$\Lambda_R := R[[\Gamma]] = R[[1 + p\mathbb{Z}_p]].$$

Theorem 1.2 ([H86a] Thm. 3.1)**.** *$\mathbb{T}^o_N$ is finite free over Λ_R.*

Fix for definiteness the topological generator $\gamma_o = 1 + p$ of Γ; then Λ_R is identified with the power series ring $R[[X]]$ via $\gamma_o = 1 + X$. Recall there is a natural identification

$$\mathrm{Hom}_{\mathrm{gp}}(\Gamma, \overline{\mathbb{Q}}_p^\times) = \mathrm{Hom}_{R-\mathrm{alg}}(\Lambda_R, \overline{\mathbb{Q}}_p);$$

specifically, if $\varphi : \Gamma \to \overline{\mathbb{Q}}_p^\times$ is a continuous group homomorphism, then $\varphi(\gamma_o)$ is a principal unit if a finite extension inside $\overline{\mathbb{Q}}_p^\times$, and one defines the corresponding R-algebra homomorphism $\varphi \in \mathrm{Hom}(\Lambda_R, \overline{\mathbb{Q}}_p)$ by setting

$$\varphi(\phi(T)) := \phi(\varphi(\gamma_o) - 1)$$

for every $\phi(T) \in \Lambda_R = R[[X]]$. The elements in $\mathrm{Hom}(\Lambda_R, \overline{\mathbb{Q}}_p)$ are also called *specializations* (of Λ_R), and we will occasionally use the notation $\mathrm{Spec}(\Lambda_R)(\overline{\mathbb{Q}}_p)$. If φ is a specialization, its kernel P_φ is a height one prime in Λ_R, and the reduction maps on Λ_R modulo a height one prime exhaust all specializations. We say that a specialization φ, or that its corresponding height one prime P_φ, is *arithmetic* if $\varphi \in \mathrm{Hom}(\Gamma, \overline{\mathbb{Q}}_p^\times)$ is of the form

$$\gamma \mapsto \varepsilon(\gamma)\gamma^k,$$

for a certain finite order character $\varepsilon : \Gamma \to \overline{\mathbb{Q}}_p^\times$ and a certain integer $k \geq 1$; if further $k \geq 2$ we may refer to φ as a *classical* specialization. We note that ε is necessarily of p-power order, and sometimes we will write ε_r to indicate that it has exact order p^r. We say that ε is the *character* of P_φ and that k is the *weight* of P_φ; we then also denote φ by $\varphi_{k,\varepsilon}$, and P_φ by $P_{k,\varepsilon}$. More generally, if T is a finite Λ_R-algebra, and P is a height 1 prime of T, we say that P is an arithmetic specialization if $P \cap \Lambda_R = P_{k,\varepsilon}$ for some k and ε.

Note the if ε has exact order p^r, $r \geq 0$, so that it (optimally) factors through $\Gamma/\Gamma^{p^r} \cong \mathbb{Z}/p^r\mathbb{Z}$, where $\Gamma^{p^r} = 1+p^{r+1}\mathbb{Z}_p$, then ε is completely determined by the data of the primitive p^r-th root of unity $\zeta_r = \varepsilon(\gamma_o)$ and $\varphi_{k,\varepsilon}$ obviously restricts to $\gamma \mapsto \gamma^k$ on Γ^{p^r}. Conversely, note that for every positive integer $r \geq 1$ one has a canonical decomposition

$$(\mathbb{Z}/p^{r+1}\mathbb{Z})^\times = (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}/p^r\mathbb{Z}, \tag{10}$$

resulting form writing $m = \omega(m) \cdot \frac{m}{\omega(m)}$, where $\omega(m) = m \bmod p$. The residue class of $1+p$ modulo p^r gives a generator of the factor $\mathbb{Z}/p^r\mathbb{Z}$, and for each primitive p^r-th root of unity $\zeta_r \in \mu_{p^r}(\overline{\mathbb{Q}}_p)$ one can define a primitive character ε_{ζ_r} modulo p^{r+1} of exact order p^r by

$$\varepsilon_{\zeta_r}(\lambda, \mu) = \zeta_r^{s(\mu)},$$

where (λ, μ) corresponds to the decomposition (10) and $s(\mu)$ is an integer, well defined modulo p^r, such that $\mu = s(\mu)(1+p)$ (or equivalently, such that $\mu = (\gamma_o \bmod p^{r+1})^{s(\mu)}$). Then for any finite index subgroup Γ' of Γ, which will be of the form $\Gamma' = \Gamma^{p^r}$ for some $r \geq 0$, one can construct a unique specialization of weight k and primitive character ε_{ζ_r} for each choice of a primitive p^r-th root of unity ζ_r, by just setting $\gamma \mapsto \varepsilon_{\zeta_r}(\gamma)\gamma^k$, where one uses the decomposition (10) and the isomorphism $\Gamma/\Gamma^{p^r} \cong \mathbb{Z}/p^r\mathbb{Z}$ to define $\varepsilon_{\zeta_r}(\gamma)$ for $\gamma \in \Gamma$.

Let $P_{k,\varepsilon}$ be a classical specialization of Λ_R for some integer $k \geq 2$ and a finite order character ε on $\Gamma/\Gamma^{p^{r-1}}$, seen as a Dirichlet charater modulo Np^r. Consider the congruence subgroup $\Delta_r = \Gamma_1(Np) \cap \Gamma_0(p^r)$. Note that

$$\Gamma_1(Np^r) \;\subset\; \Delta_r \;\subset\; \Gamma_0(Np^r),$$

where the latter inclusion holds because N and p are assumed to be relatively primes, and that ε may be naturally regarded as a character on Δ_r via the chain

$$\begin{array}{rl} \Delta_r & \twoheadrightarrow \;\; \Delta_r/\Gamma_1(Np^r) \subset \Gamma_0(Np^r)/\Gamma_1(Np^r) \cong (\mathbb{Z}/Np^r\mathbb{Z})^\times \twoheadrightarrow (\mathbb{Z}/p^r\mathbb{Z})^\times \twoheadrightarrow \\ & \twoheadrightarrow \;\; \mathbb{Z}/p^{r-1}\mathbb{Z} \cong \Gamma/\Gamma^{p^{r-1}} \xrightarrow{\varepsilon} \overline{\mathbb{Q}}_p^\times. \end{array}$$

Consider the subspace $S_k(\Delta_r, \varepsilon) \subset S_k(Np^r)$, where

$$S_k(\Delta_r, \varepsilon) := \{f \in S_k(Np^r) \ \mid \ f|\langle \ell \rangle_0 = \varepsilon(\ell) f \quad \text{for all } \ell \in \Delta_r\},$$

and denote by $S_k(\Delta_r, \varepsilon; R) \subset S_k(Np^r; R)$ the corresponding submodule of R-rational cusp forms. The restriction of Hecke operators via $\mathscr{S}^o(Np^\infty; R) \supset S_k^{\text{ord}}(\Delta_r, \varepsilon; R)$ gives rise to a surjective algebra homomorphism

$$\varphi_{k,\varepsilon} : \mathbb{T}_N^o / P_{k,\varepsilon} \mathbb{T}_N^o \twoheadrightarrow \mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon),$$

where $\mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon)$ denotes the ordinary part of the quotient of $\mathbb{T}_{Np^r}^{(k)}$ that acts faithfully on $S_k(\Delta_r, \varepsilon; R)$. Indeed, from the definition of the action of $\mathbb{Z}_{p,N}^\times$ we readily see that $\mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon)$ is the maximal R-submodule of $\mathbb{T}_{Np^r}^{\text{ord}}$ where Γ acts via $\gamma \mapsto \varepsilon(\gamma)\gamma^k$.

Theorem 1.3 (Hida's Control Theorem, [H86b] Thm. 1.2)**.** *For every integer $k \geq 2$ and every character $\varepsilon : \Gamma \to \overline{\mathbb{Q}}_p^\times$ of exact order p^r, the R-algebra homomorphism $\varphi_{k,\varepsilon}$ is an isomorphism.*

Let $\mathscr{L}_R$ be the fraction field of Λ_R. By Theorem 1.2, $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R$ is a finite dimensional Artinian $\mathscr{L}_R$-algebra, and therefore decomposes as an algebra direct sum of the finitely many local components obtained as the localization of $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R$ at each one of its maximal ideals. Therefore, singling out one such local component $\mathbb{I}_{\mathscr{L}}$, we can write

$$\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R = \mathbb{I}_{\mathscr{L}} \oplus \mathbb{J}_{\mathscr{L}} \tag{11}$$

as $\mathscr{L}_R$-algebras. We denote by $\mathbb{T}_N^{o,\text{new}}$ the quotient of the Hecke algebra $\mathbb{T}_N^o$ acting faithfully on the space of newforms; it is a reduced finite torsion-free Λ_R-algebra. We say that the local component $\mathbb{I}_{\mathscr{L}}$ as above is *primitive* if it corresponds by pullback via $\mathbb{T}_N^o \twoheadrightarrow \mathbb{T}_N^{o,\text{new}}$ to a local component in $\mathbb{T}_N^{o,\text{new}} \otimes_{\Lambda_R} \mathscr{L}_R$. If the local component $\mathbb{I}_{\mathscr{L}}$ is primitive, then it is a finite dimensional $\mathscr{L}_R$-vector space. Denote by $\mathbb{I}$ the projected image of $\mathbb{T}_N^o$ on $\mathbb{I}_{\mathscr{L}}$, i.e. the image of the composite map

$$\mathbb{T}_N^o \hookrightarrow \mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R \twoheadrightarrow \mathbb{I}_{\mathscr{L}},$$

and denote by $\widetilde{\mathbb{I}}$ the normalization of $\mathbb{I}$ in $\mathbb{I} \otimes_{\Lambda_R} \mathscr{L}_R = \mathbb{I}_{\mathscr{L}}$, so that we have the chain of inclusions

$$\mathbb{I} \subset \widetilde{\mathbb{I}} \subset \mathbb{I}_{\mathscr{L}},$$

with $\widetilde{\mathbb{I}}$ being finite flat as a Λ_R-algebra with Krull dimension 2. We remark that when $\mathbb{I}_{\mathscr{L}} = \mathscr{L}_R$, then $\mathbb{I} = \Lambda_R$, and therefore we also have $\widetilde{\mathbb{I}} = \Lambda_R$.

Recall that there is a bijection between ordinary cuspidal eigenforms $f \in S_k^{\text{ord}}(\Delta_r, \varepsilon)$ and R-algebra homomorphisms $\lambda_f : \mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon) \to \overline{\mathbb{Q}}_p$, the correspondence being determined by the requirement that

$$f|T = \lambda_f(T) f$$

holds for every $T \in \mathbb{T}^{(k),\mathrm{ord}}_{Np^r}(\varepsilon)$.

If $P_{k,\varepsilon}$ is a classical specialization of Λ_R, then there is an R-algebra homomorphism $\mathbb{T}^o_N/P_{k,\varepsilon}\mathbb{T}^o_N \to \widetilde{\mathbb{I}}/P_{k,\varepsilon}\widetilde{\mathbb{I}}$ induced by the natural map $\mathbb{T}^o_N \twoheadrightarrow \mathbb{I} \subset \widetilde{\mathbb{I}}$. Thus, by Theorem 1.3 we have an R-algebra homomorphism

$$\mathbb{T}^{(k),\mathrm{ord}}_{Np^r}(\varepsilon) \to \widetilde{\mathbb{I}}/P_{k,\varepsilon}\widetilde{\mathbb{I}}.$$

Theorem 1.4 (Hida, [H86b] Cor. 1.3)**.** *Let k, ε and $\varphi_{k,\varepsilon}$ be as in Theorem 1.3.*

- *Let $\mathbb{I}_{\mathscr{L}}$ be a primitive local component of $\mathbb{T}^o_N \otimes_{\Lambda_R} \mathscr{L}_R$ and let $d = [\mathbb{I}_{\mathscr{L}} : \mathscr{L}_R]$. Then there are exactly normalized ordinary eigenforms $f \in S^{\mathrm{ord}}_k(\Delta_r, \varepsilon)$ belonging to $\mathbb{I}_{\mathscr{L}}$, in the sense that they correspond to the R-algebra homomorphisms λ_f and $\widetilde{\lambda}_f$ fitting into the commutative diagram*

$$\begin{array}{ccc} & \nearrow & \widetilde{\mathbb{I}}/P_{k,\varepsilon}\widetilde{\mathbb{I}} \\ & & \downarrow{\scriptstyle \widetilde{\lambda}_f} \\ \mathbb{T}^{(k),\mathrm{ord}}_{Np^r}(\varepsilon) & \xrightarrow{\lambda_f} & \overline{\mathbb{Q}}_p, \end{array}$$

and each such f is the ordinary p-stabilized newform attached to a unique ordinary eigenform (depending on f) of conductor dividing N.

- *Conversely, for every ordinary p-stabilized newform $f \in S^{\mathrm{ord}}_k(\Delta_r, \varepsilon)$ there exists a unique local algebra $\mathbb{I}_{\mathscr{L}}$ direct summand of $\mathbb{T}^o_N \otimes_{\Lambda_R} \mathscr{L}_R$ which is primitive and corresponds to f by the above process.*

We next discuss how this theorem due to Hida allows one to construct *p-adic analytic families* of ordinary p-stabilized newforms. Let the notations be as in Theorem 1.4, and assume for simplicity that $[\mathbb{I}_{\mathscr{L}} : \mathscr{L}_R] = 1$, so that $\widetilde{\mathbb{I}} = \Lambda_R = R[[\Gamma]]$, and therefore $\widetilde{\mathbb{I}} = R[[X]]$ after the identification $1 + p = 1 + X$. Denote by $\lambda_{\mathbb{I}}$ the composite map

$$\mathbb{T}^o_N \hookrightarrow \mathbb{T}^o_N \otimes_{\Lambda_R} \mathscr{L}_R \twoheadrightarrow \mathbb{I} = \widetilde{\mathbb{I}} = \Lambda[[X]],$$

and form the formal q-expansion

$$\mathscr{F}_{\mathbb{I}} = \sum_{n=1}^{\infty} a_{\mathbb{I}}(n;\mathscr{F}) q^n := \sum_{n=1}^{\infty} \lambda_{\mathbb{I}}(T_n) q^n \in R[[X]][[q]].$$

For each integer $k \geq 2$ and finite order character ε on $\Gamma/\Gamma^{p^{r-1}}$, the specialization map $\varphi_{k,\varepsilon} : \gamma \mapsto \varepsilon(\gamma)\gamma^k$ corresponds to the evaluation homomorphism

$$\begin{array}{rcl} R[[X]] & \to & \overline{\mathbb{Q}}_p \\ X & \mapsto & \varepsilon(\gamma_o)\gamma_o^k - 1. \end{array}$$

Thus we see that in virtue of Theorem 1.3, for every $k \geq 2$ and ε as above, the formal q-expansion

$$P_{k,\varepsilon}(\mathscr{F}_{\mathbb{I}}) := \sum_{n=1}^{\infty} \varphi_{k,\varepsilon}(a_{\mathbb{I}}(n;\mathscr{F}))q^n \in \overline{\mathbb{Q}}_p[[q]]$$

resulting from specializing the coefficients $a_{\mathbb{I}}(n;\mathscr{F})$ to $P_{k,\varepsilon} = \mathrm{Ker}(\varphi_{k,\varepsilon})$ is in fact the q-expansion of an ordinary p-stabilized newform $f_{k,\varepsilon} \in S_k^{\mathrm{ord}}(\Delta_r, \varepsilon; R)$, and we will usually identify the formal q-expansion $P_{k,\varepsilon_r}(\mathscr{F}_{\mathbb{I}})$ with f_{k,ε_r} itself. We call $\mathscr{F}_{\mathbb{I}}$ an ordinary primitive Λ*-adic form* (and often simply a Λ-adic form) and refer to $f_{k,\varepsilon}$ as a *classical specialization* of $\mathscr{F}_{\mathbb{I}}$ of weight k. Note that it follows in particular from the second assertion in Theorem 1.4 that for any ordinary newform of weight $k \geq 2$ and level Np^r there exists a unique Λ-adic form which specializes to it.

Recall that $\mathbb{T}_N^o$ has a module structure over the completed group ring $R[[\mathbb{Z}_{p,N}^\times]]$, and hence one may also regard $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R$ as a $R[[\mathbb{Z}_{p,N}^\times]]$-module considering the action on the left factor. Since $\mathbb{Z}_{p,N}^\times = \Gamma \times (\mathbb{Z}/Np\mathbb{Z})^\times$, one has that

$$R[[\mathbb{Z}_{p,N}^\times]] = \Lambda_R \otimes_R R[(\mathbb{Z}/Np\mathbb{Z})^\times],$$

and therefore (assuming all characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$ take values contained in R) we see that, as $\mathscr{L}_R$-algebras,

$$\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R = \bigoplus_{\chi \in \widehat{(\mathbb{Z}/Np\mathbb{Z})^\times}} (\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R)^{(\chi)},$$

where $(\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R)^{(\chi)}$ denotes the maximal $\mathscr{L}_R$-algebra direct summand of $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R$ on which $(\mathbb{Z}/Np\mathbb{Z})^\times$ acts via the character χ. We say that the local component $\mathbb{I}_{\mathscr{L}}$ of $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R$ has *character* χ if $\mathbb{I}_{\mathscr{L}}$ is a direct summand of $(\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R)^{(\chi)}$. Then, if an ordinary p-stabilized newform $f \in S_k^{\mathrm{ord}}(\Delta_r, \varepsilon; R)$ belongs to $\mathbb{I}_{\mathscr{L}}$, we see that

$$f \in S_k^{\mathrm{ord}}(\Gamma_0(Np^r), \chi\varepsilon\omega^{-k}),$$

where $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \xrightarrow{\sim} \mu_{p-1} \subset \mathbb{Z}_p^\times$ denotes the Teichmüller character, since for

$$\ell = (\omega(\ell), \ell_N) \in (\mathbb{Z}/Np\mathbb{Z})^\times \cong \mu_{p-1} \times (\mathbb{Z}/N\mathbb{Z})^\times,$$

we have $f|\langle\ell\rangle_0 = \chi(\ell)f$ for the weight zero action, and therefore $f|\langle\ell\rangle = \omega^{-k}(\ell)\chi(\ell)f$ for the usual Nebentypus action, and it follows that $f|\langle\ell\rangle = \omega^{-k}(\ell)\chi(\ell)\varepsilon(\ell)f$ for $\ell \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$.

1.4 p-adic families of Galois representations

Recall that one can attach to every newform $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(Np^r, \psi)$ an irreducible Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathbf{GL}_2(\overline{\mathbb{Q}}_p)$ unramified outside Np and such that the relation

$$\det(\mathrm{Id} - \rho_f(\mathrm{Frob}_\ell)X) = 1 - a_\ell(f)X + \psi(\ell)\ell^{k-1}X^2$$

holds for every prime $\ell \nmid Np$.

Let $\mathbb{I}_{\mathscr{L}}$ be a primitive local algebra direct summand of $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R$, and let $\widetilde{\mathbb{I}}$ be the integral closure of Λ_R in $\mathbb{I}_{\mathscr{L}}$. We know that $\mathbb{I}_{\mathscr{L}}$ is a field extension of $\mathscr{L}_R$ of finite degree, and that $\mathbb{I}_{\mathscr{L}}$ is an integrally closed complete Noetherian local R-algebra of Krull dimension 2. We denote by $\mathfrak{m}$ the maximal ideal of $\mathbb{I}$, and say that a Galois representation $\rho : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathbb{I}_{\mathscr{L}})$ is *continuous* if it is realizable over a $\mathbb{I}_{\mathscr{L}}$-vector space V of dimension 2 having a $G_{\mathbb{Q}}$-stable $\widetilde{\mathbb{I}}$-lattice of rank 2 such that the resulting representation $G_{\mathbb{Q}} \to \mathbf{GL}_2(\widetilde{\mathbb{I}})$ is continuous for the $\mathfrak{m}$-adic topology of $\widetilde{\mathbb{I}}$. As before, denote by $\lambda_{\mathbb{I}}$ the composite map

$$\mathbb{T}_N^o \hookrightarrow \mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R \twoheadrightarrow \mathbb{I} \subset \widetilde{\mathbb{I}}.$$

Theorem 1.5 ([H86b] Thm. 2.1)**.** *There exists a continuous irreducible Galois representation*

$$\rho_{\mathbb{I}} : G_{\mathbb{Q}} \to \mathbf{GL}_2(\widetilde{\mathbb{I}})$$

uniquely determined by the following properties:

- $\rho_{\mathbb{I}}$ *is unramified outside* Np*, and for every prime* $\ell \nmid Np$*,*

$$\det(\mathrm{Id} - \rho_{\mathbb{I}}(\mathrm{Frob}_\ell)X) = 1 - \lambda_{\mathbb{I}}(T_\ell)X + \lambda_{\mathbb{I}}(\ell^{-1}\langle \ell \rangle)X^2;$$

- *For every ordinary* p*-stabilized newform* $f \in S_k^{\mathrm{ord}}(\Delta_r, \varepsilon)$ *belonging to* $\mathbb{I}_{\mathscr{L}}$*, and for the unique* R*-algebra homomorphism* $\widetilde{\lambda}_f : \widetilde{\mathbb{I}}/P_{k,\varepsilon}\widetilde{\mathbb{I}} \to \overline{\mathbb{Q}}_p$ *attached to it,*

$$\det(\mathrm{Id} - (\widetilde{\lambda}_f \circ \rho_{\mathbb{I}})(\mathrm{Frob}_\ell)X) = \det(\mathrm{Id} - \rho_f(\mathrm{Frob}_\ell)X), \tag{12}$$

for every prime $\ell \nmid Np$*; or equivalently, if* P_f *denotes the height one prime in* $\widetilde{\mathbb{I}}$ *above* $P_{k,\varepsilon} \in \mathrm{Spec}(\Lambda_R)(\overline{\mathbb{Q}}_p)$ *given by* $\mathrm{Ker}(\widetilde{\lambda}_f)$*, then*

$$(\rho_{\mathbb{I}} \bmod P_f) \cong \rho_f,$$

as Galois representations into $\mathbf{GL}_2(\overline{\mathbb{Q}}_p)$*.*

We note that in (12) above, we denote by $\widetilde{\lambda}_f$ the homomorphism $\mathbf{GL}_2(\widetilde{\mathbb{I}}) \to \mathbf{GL}_2(\overline{\mathbb{Q}}_p)$ given by applying the R-algebra homomorphism $\widetilde{\lambda}_f : \widetilde{\mathbb{I}} \to R$ to each matrix entry.

Corollary 1.6. *If the local component* $\mathbb{I}$ *has character* χ *modulo* Np*, then*

$$\det(\rho_{\mathbb{I}}) = \chi\kappa\varepsilon_{\mathrm{cyc}}^{-1}, \tag{13}$$

where χ *denotes the finite Galois character induced by* χ*, and* κ *denotes the Galois character* $G_{\mathbb{Q}} \to \mathbb{I}^\times$*, factoring through* $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$*, given by* $\sigma \mapsto (1+X)^{s(\sigma)}$*,* $s(\sigma) \in \mathbb{Z}_p$ *being uniquely determined by the condition*

$$\langle \varepsilon_{\mathrm{cyc}}(\sigma) \rangle = \gamma_o^{s(\sigma)}.$$

Note that in fact $s(\sigma) = \frac{\log(\langle \varepsilon_{\mathrm{cyc}}(\sigma)\rangle)}{\log(\gamma_o)}$.

Proof. By Theorem 1.5, we just need to check that the left hand side in (13) interpolates $\det(P_{k,\varepsilon}(\rho_{\mathbb{I}})) = \det(\rho_{P_{k,\varepsilon}(\mathscr{F}_{\mathbb{I}})}) = \chi\varepsilon\omega^{-k}\varepsilon_{\mathrm{cyc}}^{k-1}$ as k varies, which is a straightforward calculation. Indeed,

$$(1+X)^{s(\sigma)}|_{X=\varepsilon(\gamma_o)\gamma_o^k-1} = (\varepsilon(\gamma_o)\gamma_o^k)^{s(\sigma)} = \varepsilon(\langle \varepsilon_{\mathrm{cyc}}(\sigma)\rangle)\langle \varepsilon_{\mathrm{cyc}}(\sigma)\rangle^k,$$

and the claim follows when we note that the composition $G_{\mathbb{Q}} \twoheadrightarrow \Gamma \xrightarrow{\varepsilon} \Lambda^\times$ is given by $\varepsilon(\sigma) = \varepsilon(\langle \varepsilon_{\mathrm{cyc}}(\sigma)\rangle)$. □

Mazur and Wiles studied the geometry of the big Galois representations constructed by Hida, obtaining in particular a control on the action of the inertia group at p in these representations.

Theorem 1.7 ([MW86] §8, Prop. 2)**.** *Let $\mathscr{F}_{\mathbb{I}} = \sum a_{\mathbb{I}}(n;\mathscr{F})q^n \in \mathbb{I}[[q]]$ be a Λ-adic form attached to a primitive local component $\mathbb{I}_{\mathscr{L}}$ of $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R$ of character χ. Let*

$$\rho_{\mathbb{I}} : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathbb{I}_{\mathscr{L}})$$

be the Λ-adic representation attached to $\mathscr{F}_{\mathbb{I}}$ by Theorem 1.5, and denote by V a representation space for $\rho_{\mathbb{I}}$. Then V has a D_p-stable line with unramified quotient. More precisely, after a suitable choice of basis for V,

$$\rho_{\mathbb{I}}|_{D_p} = \begin{pmatrix} \chi\kappa\varepsilon_{\mathrm{cyc}}^{-1}\nu^{-1} & * \\ & \nu \end{pmatrix}, \tag{14}$$

where

$$\nu : D_p \to \mathbb{I}^\times$$

is the unramified character given by $\nu(\mathrm{Frob}_p) = a_{\mathbb{I}}(p;\mathscr{F})$, and $\kappa : G_{\mathbb{Q}} \to \mathbb{I}^\times$ is as in Corollary 1.6.

1.5 Local components of CM type

Fix a prime p, and maintain the assumption that R is the ring of integers of a finite extension $K/\mathbb{Q}_p$. Let M be an imaginary quadratic field of discriminant $-D$, $\mathcal{O}_M$ its ring of integers, and suppose that p splits in M as $p\mathcal{O}_M = \mathfrak{p}\overline{\mathfrak{p}}$, with $\mathfrak{p} \neq \overline{\mathfrak{p}}$. Let $\mathfrak{c}$ be an integral ideal of M relatively prime to $\mathfrak{p}$, and set

$$N = \mathrm{Norm}(\mathfrak{c}) \cdot D$$

throughout this section. For each prime $\mathfrak{l}$ in $\mathcal{O}_M$ denote by $\mathcal{O}_{\mathfrak{l}}$ its $\mathfrak{l}$-adic completion $\varprojlim \mathcal{O}_M/\mathfrak{l}^n\mathcal{O}_M$. Let $\mathbb{A}_M^\times$ be the idèle group of M and consider the compact quotient

$$W(\mathfrak{c}) := \mathbb{A}_M^\times / M^\times \mathbb{C}^\times U(\mathfrak{c}),$$

where $U(\mathfrak{c})$ is defined to be the product $\prod_{\mathfrak{l}\neq\mathfrak{p}} U_{\mathfrak{l}}(\mathfrak{c})$, with $U_{\mathfrak{l}}(\mathfrak{c}) = \{x \in \mathcal{O}_{\mathfrak{l}}^{\times} \mid x \equiv 1 \pmod{\mathfrak{c}}\}$. Note that since p splits in M, there is an isomorphism $\Gamma = 1 + p\mathbb{Z}_p \cong 1 + \mathfrak{p}\mathcal{O}_{\mathfrak{p}}$ induced by the inclusion $\mathbb{Z} \subset R$, and thus a natural injection $\Gamma \hookrightarrow W(\mathfrak{c})$. Denote by W a maximal p-profinite torsion free subgroup of $W(\mathfrak{c})$ containing Γ; such a subgroup is independent of $\mathfrak{c}$ in a clear sense, and the index $(W : \Gamma)$ is finite. The quotient $\Delta_{\mathfrak{c}} = W(\mathfrak{c})/W$ is a finite group of order prime to p, and we have a non-canonical decomposition

$$W(\mathfrak{c}) = W \times \Delta_{\mathfrak{c}}.$$

Denote by $I(\mathfrak{p}\mathfrak{c})$ the set of integral ideals of M relatively prime to $\mathfrak{p}\mathfrak{c}$ and note that we have a natural injective map $I(\mathfrak{p}\mathfrak{c}) \hookrightarrow W(\mathfrak{c})$ with dense image. Let $\mathscr{C}(W(\mathfrak{c}); K)$ be the Banach space of continuous functions from $W(\mathfrak{c})$ into K, and put

$$\mathscr{S}(Np^{\infty}; K) := \mathscr{S}(Np^{\infty}; R) \otimes_R K.$$

Consider the *base change map* form $\mathbf{GL}_{1/M}$ to $\mathbf{GL}_{2/\mathbb{Q}}$ given by the linear form

$$\begin{array}{rcl} \theta : \mathscr{C}(W(\mathfrak{c}); K) & \rightarrow & \mathscr{S}(Np^{\infty}; K) \\ \phi & \mapsto & \theta(\phi) = \displaystyle\sum_{\mathfrak{a} \in I(\mathfrak{p}\mathfrak{c})} \phi(\mathfrak{a}) q^{\mathrm{Norm}(\mathfrak{a})}. \end{array}$$

That the image of θ actually lies in $\mathscr{S}(Np^{\infty}; K)$ follows from the density of $I(\mathfrak{p}\mathfrak{c})$ in $W(\mathfrak{c})$, a well-known theorem due to Hecke and Shimura, and the density in $\mathscr{C}(I(\mathfrak{p}\mathfrak{c}); K)$ of the K-subspace spanned by the Hecke characters $\phi : I(\mathfrak{p}\mathfrak{c}) \to K^{\times}$ (with the topology on $I(\mathfrak{p}\mathfrak{c})$ induced form that of $W(\mathfrak{c})$).

The Hecke action on $\theta(\phi)$ is expressed in terms of the natural translation action of $W(\mathfrak{c})$ on $\mathscr{C}(W(\mathfrak{c}); K)$ by the following formulas:

$$\theta(\phi)|T_{\ell} = \begin{cases} \theta(\phi|\mathfrak{l}) + \theta(\phi|\bar{\mathfrak{l}}) & \text{if } \ell\mathcal{O}_M = \mathfrak{l}\bar{\mathfrak{l}},\ \mathfrak{l} \neq \bar{\mathfrak{l}} \\ 0 & \text{if } \mathrm{Norm}(\mathfrak{l}) = \ell^2 \\ \theta(\phi|\mathfrak{l}) & \text{if } \ell\mathcal{O}_M = \mathfrak{l}^2 \text{ or } \ell\mathcal{O}_M = \mathfrak{l}\bar{\mathfrak{l}} \text{ with } \bar{\mathfrak{l}} \notin I(\mathfrak{p}\mathfrak{c}). \end{cases}$$

And further we have that $\theta(\phi)|\langle z\rangle_0 = z\theta(\phi|z)$ for $z \in \Gamma \subset W(\mathfrak{c})$. Thus by the duality between modular forms and the corresponding Hecke algebra on the one side, and Mahler's theorem on the other, giving a duality between $\mathscr{C}(W(\mathfrak{c}); R)$ and $R[[W(\mathfrak{c})]]$, we see that θ induces an R-algebra homomorphism

$$\varphi = \theta^{*} : \mathbb{T}_N \to R[[W(\mathfrak{c})]]. \tag{15}$$

Note that it follows from the third of the above formulas that the following relations holds for every $\phi \in \mathscr{C}(W(\mathfrak{c}); R)$:

$$\phi|\varphi(U_p) = \theta(\phi)|U_p = \theta(\phi|\overline{\mathfrak{p}}),$$

and thus $\varphi(U_p)$ is the image of $\overline{\mathfrak{p}} \in I(\mathfrak{p}\mathfrak{c})$ in $R[[W(\mathfrak{c})]]$, which is invertible, and therefore φ factors through the ordinary part $\mathbb{T}_N^o$.

Recall that there is a bijection between complex-valued ideal Hecke characters

$$\psi : I(\mathfrak{p}\mathfrak{c}) \to \overline{\mathbb{Q}}^\times$$

such that $\psi(a\mathcal{O}_M) = a^{\kappa-1}$ for all $a \in M^\times$ with $a \equiv 1 \pmod{^\times \mathfrak{p}\mathfrak{c}}$, and continuous p-adic Hecke characters

$$\psi_{\mathfrak{P}} : W(\mathfrak{c}) \to \overline{\mathbb{Q}}_p^\times$$

such that $\psi_{\mathfrak{P}}(w) = w^{\kappa-1}$ for all $w \in 1 + \mathfrak{p}\mathcal{O}_{\mathfrak{p}} \cong \Gamma$. (Here $\mathfrak{P}$ denotes the prime in $\overline{\mathbb{Q}}$ above p induced by $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.) The relation $\psi_{\mathfrak{P}}|_{I(\mathfrak{p}\mathfrak{c})} = \psi$ holds.

Note that the decomposition $W(\mathfrak{c}) = W \times \Delta_{\mathfrak{c}}$ induces an isomorphism

$$R[[W(\mathfrak{c})]] = R[[W]] \otimes_R R[\Delta_{\mathfrak{c}}],$$

and that, since the cardinality of $\Delta_{\mathfrak{c}}$ is prime to p, every primitive R-valued character of $\Delta_{\mathfrak{c}}$ determines a unique irreducible component of $R[[W(\mathfrak{c})]]$, isomorphic to $R[[W]]$, in which $\Delta_{\mathfrak{c}}$ acts via χ. We denote it by $R[[W]]^{(\chi)}$.

Theorem 1.8 ([H86b] Prop. 2.3)**.** *Fix a p-adic Hecke character $\lambda : I(\mathfrak{p}) \to \overline{\mathbb{Q}}_p^\times$ such that $\lambda(a\mathcal{O}_M) = a$ for all $a \in M^\times$ with $a \equiv 1 \pmod{^\times \mathfrak{p}}$, and assume that R contains all the values of λ. Then for each primitive character $\chi : \Delta_{\mathfrak{c}} \to R^\times$, there is a unique primitive local ring $\mathbb{I}_{\mathscr{L}}$ of $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R$ characterized by the following properties:*

- *The algebra homomorphism* (15) *induces an isomorphism:*

$$\mathbb{I}_{\mathscr{L}} \cong R[[W]]^{(\chi)} \otimes_{\Lambda_R} \mathscr{L}_R;$$

- *For every finite character $\varepsilon_r : W \to \overline{\mathbb{Q}}_p^\times$ of exact order p^r, and every integer $k \geq 1$, the theta series*

$$\theta(\chi\lambda^{k-1}\varepsilon_r) = \sum_{\mathfrak{a} \in I(\mathfrak{p}\mathfrak{c})} \chi(\mathfrak{a})\lambda(\mathfrak{a})^{k-1}\varepsilon_r(\mathfrak{a}) q^{\mathrm{Norm}(\mathfrak{a})},$$

  is a cusp form in $S_k^{\mathrm{ord}}(Np^r)$ which belongs to $\mathbb{I}_{\mathscr{L}}$.

One can give an explicit description of the Galois representation attached to local components $\mathbb{I}_{\mathscr{L}}$ of the Hecke algebra coming form imaginary quadratic fields. Let the notations and assumptions be as in Theorem 1.8. Consider the character

$$\begin{array}{rcl} \Psi : W(\mathfrak{c}) & \to & R[[W]]^\times \\ w = (w_o, w_t) & \mapsto & \chi(w_t)\lambda(w_o)^{-1}.[w_o], \end{array}$$

where $[w_o]$ is the image of w_o in $R[[W]]$ under the tautological character $W \to R[[W]]^\times$. For each integer $k \geq 1$, denote by $\varphi_{k,\varepsilon_r}$ the algebra homomorphism $R[[W]] \to \overline{\mathbb{Q}}_p^\times$ induced by $w_o \mapsto \varepsilon_r(w_o)\lambda(w_o)^k$, and let P_{k,ε_r} be the height one prime ideal of $R[[W]]$ given by $\mathrm{Ker}(\varphi_{k,\varepsilon_r})$. We put ψ_{k,ε_r} to denote the Hecke character $\chi\lambda^{k-1}\varepsilon_r$, although it is clear that it actually depends on χ, as Ψ does. Note that by construction we have a commutative diagram

$$\begin{array}{ccc} W(\mathfrak{c}) & \xrightarrow{\Psi} & R[[W]]^\times \\ & \searrow^{\psi_{k,\varepsilon_r}} & \downarrow{\scriptstyle \varphi_{k,\varepsilon_r}} \\ & & \overline{\mathbb{Q}}_p^\times . \end{array}$$

The height one prime P_{k,ε_r} of $R[[W]]$ lies above the arithmetic prime P_{k,ε'_r} of Λ_R, where $\varepsilon'_r = \varepsilon_r|_\Gamma$, and we see that the Λ-adic character Ψ specializes to every ψ_{k,ε_r}, and that these arithmetic specializations in turn uniquely determine Ψ.

Recall that by Class Field Theory, $W(\mathfrak{c})$ is naturally identified with a quotient of G_M^{ab}, and therefore Ψ may be regarded as a Galois character on G_M.

Proposition 1.9. *Let $\mathbb{I}_{\mathscr{L}}$ be a local component of $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R$ attached to the imaginary quadratic field M. The Galois representation $\rho_{\mathbb{I}} : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathbb{I}_{\mathscr{L}})$ attached to $\mathbb{I}$ by Theorem 1.5 is given by*

$$\rho_{\mathbb{I}} = \mathrm{Ind}_M^{\mathbb{Q}} \Psi.$$

Since the argument in the introduction explaining the splitting of the local Galois representation attached to an ordinary newform applies verbatim to the present situation, we obtain the following result:

Corollary 1.10. *If $\mathscr{F}_{\mathbb{I}}$ is a primitive ordinary Λ-adic form of CM type, then its associated Galois representation $\rho_{\mathbb{I}}$ splits at p.*

1.6 Greenberg's question for Λ-adic forms

Let $\mathscr{F} = \sum_{n=1}^\infty a(n;\mathscr{F})$ be a Λ-adic form attached to a primitive local component $\mathbb{I}_{\mathscr{L}}$ of the field of fractions $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathscr{L}_R$ of the universal ordinary Hecke algebra $\mathbb{T}_N^o$. After Theorem 1.5 we know that there is attached to $\mathscr{F}$ a continuous Λ-adic representation

$$\rho_{\mathscr{F}} : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathbb{I}_{\mathscr{L}}),$$

and by Theorem 1.7, this representation becomes reducible when restricted to a decomposition group D_p at p:

$$\rho_{\mathscr{F}}|_{D_p} = \begin{pmatrix} \varepsilon_{\mathscr{F}} & a_{\mathscr{F}} \\ & \delta_{\mathscr{F}} \end{pmatrix}. \qquad (16)$$

On the other hand, if $\mathscr{F}$ is of CM type, Corollary 1.10 asserts that $\rho_{\mathscr{F}}$ is actually semi-simple when restricted to D_p. The following result of Ghate and Vatsal answers in many cases the natural extension of Greenberg's question to the Λ-adic setting:

Theorem 1.11 (Ghate-Vatsal). *Let p an odd prime and $N > 0$ an integer with $(N, p) = 1$. Let $\mathscr{F} = \mathscr{F}_{\mathbb{I}}$ be a Λ-adic form of tame level N attached to a local component $\mathbb{I}$ with maximal ideal $\mathfrak{m}$, and denote by $\rho_{\mathscr{F}}$ its associated Galois representation into $\mathbf{GL}_2(\mathbb{I})$. Define $\overline{\rho}_{\mathscr{F}} := (\rho_{\mathscr{F}} \bmod \mathfrak{m})$ and suppose that the following properties are satisfied:*

- *$\overline{\rho}_{\mathscr{F}}|_{D_p}$ has non-scalar semi-simplification;*
- *$\overline{\rho}_{\mathscr{F}}$ is absolutely irreducible when restricted to $G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$.*

Then the following are equivalent:

(i) $\rho_{\mathscr{F}}|_{D_p}$ splits;

(ii) $\mathscr{F}$ has infinitely many weight one classical specializations;

(iii) $\mathscr{F}$ has infinitely many weight one classical specializations of CM type;

(iv) $\mathscr{F}$ is of CM type.

Remark 4. There are explicit examples of Λ-adic forms $\mathscr{F}$ which indicate that the second assumption is not necessary for the the properties (i) and (iv) being both satisfied by $\mathscr{F}$.

Recall that by Hida's Theorem 1.4 the specialization of the Λ-adic form $\mathscr{F}$ at any arithmetic point of $\mathbb{I}$ of weight *at least* 2 is a classical cusp form, but that no information is given about the specialization at an arithmetic point of weight one. In fact, Mazur and Wiles first showed in [MW86] explicit examples of Λ-adic forms with weight one arithmetic specializations that do not give rise to the q-expansion of any classical cusp form.

We now give a sketch of the proof of Theorem 1.11. Suppose that $\mathscr{F}$ has character χ modulo Np. By (the proof of) Corollary 1.6, $\det(\rho_{\mathscr{F}})$ specializes at P_{k,ε_r} to $\chi\varepsilon_r\omega^{-k}\varepsilon_{\rm cyc}^{k-1}$. Thus if $\rho_{\mathscr{F}}$ splits at p, by Theorem 1.7 we see that

$$P_{1,\varepsilon_r}(\rho_{\mathscr{F}})|_{I_p} \cong \begin{pmatrix} \psi\varepsilon_r\omega^{-1} & \\ & 1 \end{pmatrix} \tag{17}$$

The finite order Galois character $\chi\varepsilon_r\omega^{-1}$ has conductor divisible by p^{r+1} for each integer $r \geq 1$; if for varying r infinitely many of the representations $P_{1,\varepsilon_r}(\rho_{\mathscr{F}})$ are isomorphic to the p-adic Galois representation attached by Deligne-Serre to a weight one cusp form, it follows from this and the fact that

$$P_{1,\varepsilon_r}(\rho_{\mathscr{F}}) \cong \rho_{P_{1,\varepsilon_r}(\mathscr{F})}$$

that $\mathscr{F}$ has infinitely many weight one classical specializations. More precisely, this is indeed the case, since the two assumptions made on $\overline{\rho}_{\mathscr{F}}$ allows us to directly invoke the following result of Buzzard [Bu03]:

Theorem 1.12 (Buzzard)**.** *Let $\rho : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathcal{O})$ be a continuous representation unramified outside a finite set of primes, with $\mathcal{O}$ being the ring of integers of a finite extension of $\mathbb{Q}_p$ with maximal ideal $\mathfrak{m}_{\mathcal{O}}$. Suppose that*

$$\rho|_{D_p} \cong \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$$

for two continuous characters $\alpha, \beta : D_p \to \mathcal{O}^\times$ with finite image by the inertia group $I_p \subset D_p$ at p, and assume that the following conditions are satisfied:

- $(\alpha \bmod \mathfrak{m}_{\mathcal{O}}) \neq (\beta \bmod \mathfrak{m}_{\mathcal{O}})$*;*
- $(\rho \bmod \mathfrak{m}_{\mathcal{O}})$ *is modular;*
- $(\rho \bmod \mathfrak{m}_{\mathcal{O}})$ *is absolutely irreducible when restricted to* $G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$.

Then there exists a newform f of weight one and an embedding $\iota : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ such that $\rho_f \cong \iota \circ \rho$, where $\rho_f : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathbb{C})$ is the complex representation attached to f by Deligne and Serre.

Thus we know that (ii) follows from (i). Now suppose that (ii) holds, and fix an embedding $\iota : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. Then $\mathscr{F}$ has infinitely many classical weight one specializations whose Deligne-Serre representation has image a finite subgroup of $\mathbf{PGL}_2(\mathbb{C})$ obtained from the composition of the reduction of $\rho_{\mathscr{F}}$ modulo P_{1,ε_r} with the composite map

$$\pi \circ \iota : \mathbf{GL}_2(\overline{\mathbb{Q}}_p) \to \mathbf{GL}_2(\mathbb{C}) \to \mathbf{PGL}_2(\mathbb{C}).$$

Any finite subgroup of $\mathbf{PGL}_2(\mathbb{C})$ is well-known to be either cyclic, dihedral or one of A_4, A_5 or S_4. One immediately rules out the first possibility due to irreducibility, and a previous observation about the conductor of the specialization $P_{1,\varepsilon_r}(\rho_{\mathscr{F}})$ allows us to exclude the alternating and symmetric possibilities for r sufficiently large, noting that at least the image of I_p by any of the specialized representations has trivial intersection with the scalar matrices, and therefore injects into $\mathbf{PGL}_2(\mathbb{C})$. Thus for r large enough, all possibilities for the finite group $\mathrm{Im}((\pi \circ \iota)(P_{1,\varepsilon_r}(\rho_{\mathscr{F}}))) \subset \mathbf{PGL}_2(\mathbb{C})$ but the dihedral one, which can actually hold, are precluded. Therefore the projectivization $\pi \circ \iota(\rho_{f_{\varepsilon_r}}) : G_{\mathbb{Q}} \to \mathbf{PGL}_2(\mathbb{C})$ is dihedral for infinitely many weight one specializations $f_{\varepsilon_r} = P_{1,\varepsilon_r}(\mathscr{F})$ of $\mathscr{F}$. It follows that

$$\rho_{f_{\varepsilon_r}} = \mathrm{Ind}_K^{\mathbb{Q}} \phi_{\varepsilon_r}$$

for a Hecke character ϕ_{ε_r} of finite order of the quadratic field K cut out by $(\pi \circ \iota(\rho_{f_{\varepsilon_r}}))^{-1}(\langle c \rangle)$ for any element c of order two in the image of $\pi \circ \iota(\rho_{f_{\varepsilon_r}})$. We may (and do) fix a quadratic field K which appears for infinitely many weight one specializations, since by the conductor formula $\mathrm{disc}(K)$ must divide Np^r, and therefore for a fixed N and varying $r > 0$ there are only finitely many possibilities for K. The ordinarity of $\mathscr{F}$ forces p to split in K, say

$p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, where $\overline{\mathfrak{p}} = \sigma\mathfrak{p}$ for the non-trivial automorphism $\sigma \in G_{\mathbb{Q}} \smallsetminus G_K$, and ϕ to ramify at exactly one of $\mathfrak{p}$ or $\overline{\mathfrak{p}}$. We thus assume that infinitely many weight one specializations of $\rho_{\mathscr{F}}$ are of the form

$$P_{1,\varepsilon_r}(\rho_{\mathscr{F}}) = \mathrm{Ind}_K^{\mathbb{Q}} \phi_{\varepsilon_r},$$

for a our fixed quadratic field K, and certain (varying) Hecke character ϕ_{ε_r} of K ramified at $\mathfrak{p}$ and unramified at $\overline{\mathfrak{p}}$. Then K can not be *real* quadratic, since for a fixed N and varying $r > 0$ a real quadratic field only admits *finitely* many Hecke characters of conductor dividing $N\mathfrak{p}^r\infty_1\infty_2$, as it follows from the following:

Lemma 1.13. *Let F be a real quadratic field with infinite places ∞_1 and ∞_2. Let $\mathfrak{p}$ be a split prime of F, and $\mathfrak{n}$ and integral ideal of F relatively prime to $\mathfrak{p}$. Then the ray class field of F modulo $\mathfrak{n}\mathfrak{p}^r\infty_1\infty_2$ has bounded order as r tends to infinity.*

Indeed, the exact sequence

$$1 \to (\mathcal{O}_F/\mathfrak{p}^r)^\times/U_F \to \mathrm{Cl}_F(\mathfrak{p}^r) \to \mathrm{Cl}_F \to 1,$$

where U_F denote the image of to unit group $\mathcal{O}_F^\times$ under the natural projection $\mathcal{O}_F^\times \to (\mathcal{O}_F/\mathfrak{p}^r)^\times$, shows that

$$\#\mathrm{Cl}_F(\mathfrak{p}^r) = [(\mathcal{O}_F/\mathfrak{p}^r)^\times : U_F] \cdot h_F,$$

for the class number $h_F = \#\mathrm{Cl}_F$ of F. Thus it suffices to bound the index appearing in the above formula, since we can easily reduce to prove the claim corresponding to a conductor of the form $\mathfrak{p}^r$. Let $U_{\mathfrak{p},1} \subset U_{\mathfrak{p}}$ be the subgroup of local principal one-units in $U_{\mathfrak{p}}$, and denote by $\overline{U}_{F,1}$ the closure of the image $U_{F,1} \subset U_{\mathfrak{p},1}$ of the global units in $\mathcal{O}_F$ mapping to $U_{\mathfrak{p},1}$ under the natural inclusion map $U_F \hookrightarrow U_{\mathfrak{p}}$. It follows from Leopoldt's conjecture (which is trivial for real quadratic F) that $[U_{\mathfrak{p},1} : \overline{U}_{F,1}]$ is finite, and thus $[U_{\mathfrak{p}} : \overline{U}_F]$ is finite as well, say bounded by M. Since $U_{\mathfrak{p}} = \varprojlim_r (\mathcal{O}_F/\mathfrak{p}^r)^\times$, we see that $[(\mathcal{O}_F/\mathfrak{p}^r)^\times : U_F]$ is bounded by M independently of r, as needed.

We finally assume (iii) and show that (iv) follows. Thus we let $\mathscr{F}$ be a primitive Λ-adic form of level N with infinitely many weight one specializations of CM type,

$$f_{\varepsilon_r} = P_{1,\varepsilon_r}(\mathscr{F}) = \sum \phi_{\varepsilon_r}(\mathfrak{a}) q^{\mathrm{Norm}(\mathfrak{a})},$$

coming from finite order Hecke characters ϕ_{ε_r}, of a fixed imaginary quadratic field K, that are assumed to be ramified at $\mathfrak{p}$ and unramified at $\overline{\mathfrak{p}}$, where $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$. We will construct a Λ-adic form $\mathscr{F}'$ of CM type which interpolates infinitely many of these f_{ε_r}, and we will conclude by the following lemma:

Lemma 1.14. *Let $\mathscr{F}$ and $\mathscr{F}'$ be two Λ-adic forms of level N such that*

$$P_{k,\varepsilon_r}(\mathscr{F}) = P_{k,\varepsilon_r}(\mathscr{F}') \quad \text{for infinitely many } P_{k,\varepsilon_r} \text{ with } k, r \geq 1.$$

Then $\mathscr{F}$ and $\mathscr{F}'$ are Galois conjugates.

Proof. Denote by $\mathbb{T}_N^o$ the universal ordinary Hecke algebra of tame level N constructed by Hida. The Λ-adic form $\mathscr{F}$ corresponds to an algebra homomorphism $\lambda_{\mathscr{F}} : \mathbb{T}_N^o \to \widetilde{\mathbb{I}}$, with $\widetilde{\mathbb{I}}$ being the normalization of a local component $\mathbb{I}$ of $\mathbb{T}_N^o$ in its fraction field. We know that $\widetilde{\mathbb{I}}$ is of Krull dimesion 2 and that it corresponds to the localization of $\mathbb{T}_N^o$ at a minimal prime $\mathfrak{p}$. Absolutely analogous assertions also hold for $\mathscr{F}'$ replacing $\mathscr{F}$. By hypothesis, there exist infinitely many height one primes $P = \mathrm{Ker}(\varphi : \widetilde{\mathbb{I}} \to \overline{\mathbb{Q}}_p)$, $P' = \mathrm{Ker}(\varphi' : \widetilde{\mathbb{I}}' \to \overline{\mathbb{Q}}_p)$ such that

$$\lambda_{\mathscr{F}} \circ \varphi = \lambda_{\mathscr{F}'} \circ \varphi' : \mathbb{T}_N^o \to \overline{\mathbb{Q}}_p,$$

thus corresponding to infinitely many height one primes of $\mathbb{T}_N^o$ containing both $\mathfrak{p}$ and $\mathfrak{p}'$. If $\mathfrak{p} \neq \mathfrak{p}'$, such primes of $\mathbb{T}_N^o$ correspond bijectively minimal primes of $\mathbb{T}_N^o/(\mathfrak{p}+\mathfrak{p}')$, but these are finite in number. Thus $\mathfrak{p} = \mathfrak{p}'$, and it follows that $\mathscr{F}$ and $\mathscr{F}'$ are Galois conjugates. □

This lema allows us to conclude that (iv) holds one $\mathscr{F}'$ is constructed, since it is clear that if some conjugate of $\mathscr{F}$ is of CM type, then $\mathscr{F}$ itself is of CM type.

We proceed now to the construction of the above mentioned $\mathscr{F}'$. Write ϕ for some ϕ_{ε_r} as before. Since there are only a finite number of ideals in K with norm dividing N, we may assume that all such ϕ have conductor a fixed ideal $\mathfrak{c}$, and therefore

$$\phi = (\phi_w, \phi_t) : W(\mathfrak{c}) = W \times \Delta_{\mathfrak{c}} \to \overline{\mathbb{Q}}_p^{\times}.$$

The component ϕ_t has order prime to p, and since the reduction of all $\theta(\phi)$ are the same, it does not depend on ϕ. On the other hand, since $(p, N) = 1$, for each integral ideal $\mathfrak{q}$ of K dividing N, $\phi_w|_{I_{\mathfrak{q}}}$ cuts out a finite and tamely ramified p-extension of K, and there are only finitely many possibilities for it; we therefore assume that $\phi_w|_{I_{\mathfrak{q}}}$ does not depend on ϕ neither. Fixing one such $\phi = \phi_o$, and letting $\phi' = \phi\phi_o^{-1}$ for each ϕ, we restrict first our attention to characters

$$\phi' = (\phi'_w, \phi'_t) : W(1) = W \times \Delta_1 \to \overline{\mathbb{Q}}_p^{\times}$$

of p-power order. By a previous reasoning, we assume now that ϕ'_t does not depend on ϕ'.

Let Λ be the completed group ring $\mathbb{Z}_p[[W]]$, identified with the power series ring $\mathbb{Z}_p[[Y]]$ via $w_o = 1 + Y$ for a fixed choice of topological generator $w_o \in W$, and consider the character $G_K \to \Lambda^{\times}$ obtained by composition of the Artin map $[\ , K_{\mathfrak{p}^\infty}/K] : G_K \twoheadrightarrow W(1)$ with the tautological character $W(1) \twoheadrightarrow W \to \Lambda^{\times}$. Then Φ factors through $\mathrm{Gal}(K\mathbb{Q}_\infty/K)$ for the cyclotomic $\mathbb{Z}_p$-extension $K\mathbb{Q}_\infty$ of K, and the image by Φ of the inertia group $I_{\mathfrak{p}}$ is a finite index closed subgroup $\Gamma_{\mathbb{Q}} \subset \Gamma$ which is identified with $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. Write $p^\gamma = [\Gamma : \Gamma_{\mathbb{Q}}]$, and put $\Lambda_{\mathbb{Q}} = \mathbb{Z}_p[[\Gamma_{\mathbb{Q}}]] \cong \mathbb{Z}_p[[X]]$, where $u_o = 1 + X$ for a fixed choice of topological generator $u_o \in \Gamma_{\mathbb{Q}}$. Then Λ can be regarded as a $\Lambda_{\mathbb{Q}}$-algebra. Since $\phi' = \phi'_{\varepsilon_r}$ is uniquely determined by the image $\zeta_{\varepsilon_r} = \phi'_{\varepsilon_r}(w) \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$, we see that the character Φ_o simultaneously interpolates all ϕ'_{ε_r} under the specialization $Y = \zeta_{\varepsilon_r} - 1$.

We now turn to the general case ϕ_{ε_r}. Let f_o be a CM form, weight one specialization of $\mathscr{F}$, associated to the finite order Hecke character ϕ_o of K fixed previously, and write the character of f_o as $\psi\omega^{-1}\chi_{\nu_o}$, for $\nu_o = \varepsilon_1(u_o) \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ containing the values of ϕ_o, β_T, $\zeta_o := \nu_o^{1/p^\gamma}$ and $t_o := u_o^{1/p^\gamma}$, and let L be the power series ring $\mathcal{O}[[Y]]$. Define $\Phi := \phi_o\beta_T\tau(\Phi_o)$, where τ is the automorphism of L determined by the change of variables $1+Y \mapsto \zeta_o^{-1}t_o^{-1}(1+Y)$. Now we see that $\Phi|_{Y=\zeta_{\varepsilon_r}t_o-1}$ interpolates the characters ϕ_{ε_r} attached to the weight one CM forms $f_{\varepsilon_r} = P_{1,\varepsilon_r}(\mathscr{F})$ of character $\psi\omega^{-1}\chi_{\nu_{\varepsilon_r}}$, where $\zeta_{\varepsilon_r}^{p^\gamma} = \nu_{\varepsilon_r}$. Since $(Y-(\zeta_{\varepsilon_r}t_o-1)) \subset L$ clearly lies over $(X-(\nu_o u_o - 1)) \subset \Lambda$, we deduce that the formal q-expansion

$$\mathscr{F}' := \sum \Phi(\mathfrak{a})q^{\mathrm{Norm}(\mathfrak{a})}$$

is a Λ-adic CM form interpolating infinitely weight one CM specializations of $\mathscr{F}$, thus concluding that (iv) holds after Lemma 1.14. This finishes the proof of Theorem 1.11.

One can use Theorem 1.11 to study Greenberg's question for p-stabilized newforms, seen as classical specializations of Λ-adic forms. Recall that the expectation (3) is that a p-ordinary newform f has locally split at p Galois representation ρ_f exactly when f has CM. A descend argument from the Λ-adic setting allows to conclude that this is indeed the case for all the p-stabilized newforms obtained as classical specializations of a Λ-adic form, except possibly for finitely many *unspecified* classical points. In fact, Ghate and Vatsal deduced from their Theorem 1.11 a slightly stronger result:

Theorem 1.15. *Let p be an odd prime number and $N>0$ an integer with $(N,p)=1$. Let $\mathscr{S}(N)$ denote the set of all ordinary newforms f of weight $k \geq 2$ and level Np^r for some $r \geq 0$ satisfying the following two properties:*

- *f is p-distinguished; and*
- *$\overline{\rho}_f$ is absolutely irreducible when restricted to $G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$.*

Then, for all but finitely many $f \in \mathscr{S}(N)$, if $\rho_f|_{D_p}$ splits, then f has CM.

Proof. Each $f \in \mathscr{S}(N)$ is an arithmetic specialization of a unique Λ-adic form $\mathscr{F} \in \mathbb{I}[[q]]$ of level N, with $\mathbb{I}$ being a finite flat extension of Λ with fraction field $\mathbb{I}_{\mathscr{L}}$. Therefore, if we denote by $\mathscr{S}_{\mathscr{F}}$ the set of those ordinary newforms f that are an arithmetic specialization of the Λ-adic forms $\mathscr{F}$, then it suffices to restrict our attention to $f \in \mathscr{S}_{\mathscr{F}}$ for a fixed primitive $\mathscr{F}$ of level N.

Fix such a Λ-adic form $\mathscr{F}$, and suppose that $f \in \mathscr{S}_{\mathscr{F}}$ has no CM. Then $\mathscr{F}$ itself has no CM. Furthermore, by the hypothesis in Theorem 1.15, $\mathscr{F}$ verifies the analogous assumptions in Theorem 1.11, and therefore $\rho_{\mathscr{F}}|_{D_p}$ is non-split. Then ρ_f can possibly split at p only for finitely many $f \in \mathscr{S}_{\mathscr{F}}$. Indeed, the decomposition

$$\widetilde{\mathbb{I}}^\times = \boldsymbol{\mu}_{q-1} \times 1 + \mathfrak{m},$$

for the maximal ideal $\mathfrak{m}$ of $\widetilde{\mathbb{I}}$, and $\boldsymbol{\mu}_{q-1} \cong \mathbb{F}^\times$ for the residue field $\mathbb{F}$ of $\widetilde{\mathbb{I}}$, allows one to deduce that the characters $\delta_{\mathscr{F}}$ and $\varepsilon_{\mathscr{F}}$ appearing in

$$\rho_{\mathscr{F}}|_{D_p} = \begin{pmatrix} \delta_{\mathscr{F}} & a_{\mathscr{F}} \\ & \varepsilon_{\mathscr{F}} \end{pmatrix}$$

factor through $\mathrm{Gal}(E/\mathbb{Q}_p)$, where E is the composite of E_t, the union of the finitely many tamely ramified abelian extensions of $\mathbb{Q}_p$ of order dividing $q-1$, and E_w, the maximal abelian pro-p-extension of $\mathbb{Q}_p$. Thus if H denotes the group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E) \subset D_p$, then

$$\rho_{\mathscr{F}}|_H \cong \begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix},$$

for a certain continuous homomorphism $\lambda : H \to \widetilde{\mathbb{I}}^\times$. Since $\rho_{\mathscr{F}}|_{D_p}$ is non-split, the cohomology class $[\xi_{\mathscr{F}}] \in \mathrm{H}^1(D_p, \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1}))$ is nonzero. Write $\Delta = \mathrm{Gal}(E_t/\mathbb{Q}_p)$ and $\Gamma = \mathrm{Gal}(E_w/\mathbb{Q}_p) \cong \mathbb{Z}_p^2$, so that $\mathrm{Gal}(E/\mathbb{Q}_p) \cong \Delta \times \Gamma$. Consider the inflation-restriction exact sequence

$$0 \to \mathrm{H}^1(\mathrm{Gal}(E/\mathbb{Q}_p), \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})^H) \to \mathrm{H}^1(D_p, \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})) \to \mathrm{H}^1(H, \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})). \quad (18)$$

The decomposition $\mathrm{Gal}(E/\mathbb{Q}_p) = \Delta \times \Gamma$ gives rise to the decomposition

$$\mathrm{H}^1(\mathrm{Gal}(E/\mathbb{Q}_p), \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})^H) = \mathrm{H}^1(\Delta, \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})^\Gamma) \times \mathrm{H}^1(\Gamma, \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})^\Delta). \quad (19)$$

The first group in the right of (19) vanishes, and the second also does, unless Δ acts trivially on $\mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})$; we therefore assume that $\mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})^\Delta = \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})$ and use the decomposition $\Gamma = \Gamma_1 \times \Gamma_2$ with $\Gamma_i \cong \mathbb{Z}_p$ to decompose further

$$\mathrm{H}^1(\Gamma, \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})) = \mathrm{H}^1(\Gamma_1, \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})^{\Gamma_2}) \times \mathrm{H}^1(\Gamma_2, \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})^{\Gamma_1}). \quad (20)$$

Since p is odd, the tame part of the characters $\delta_{\mathscr{F}}$ and $\varepsilon_{\mathscr{F}}$ must reduce modulo $\mathfrak{m}_{\mathcal{O}}$ to the reduction of the tame part of the characters δ_f and ε_f corresponding to any of the specializations f of $\mathscr{F}$. Using the p-distinguishedness hypothesis we see that $\delta_t \neq \varepsilon_t \pmod{\mathfrak{m}}$, and we may thus assume that $\mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})^{\Gamma_2} = 0$, so that the first cohomology group in the right of (20) vanishes. If Γ_1 also acts nontrivially on $\mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})$ we conclude by (19) and (20) that

$$\mathrm{H}^1(\mathrm{Gal}(E/\mathbb{Q}_p), \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})^H) = 0; \quad (21)$$

otherwise we have $\mathrm{H}^1(\Gamma_2, \mathbb{I}_{\mathscr{L}}(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1})) \cong \mathbb{I}_{\mathscr{L}}/(\delta_{\mathscr{F}}\varepsilon_{\mathscr{F}}^{-1}(\gamma)-1)\mathbb{I}_{\mathscr{L}} = 0$, where the isomorphism is induced by the evaluation of cocycles at the topological generator $\gamma \in \Gamma_2$, and we conclude that (21) holds anyway, and therefore that the restriction map in (18) is injective. It follows that $\rho_{\mathscr{F}}|_H$ is non-split, and therefore λ cannot be identically zero. Finally, if I denotes the nonzero ideal of $\widetilde{\mathbb{I}}$ generated by the image of λ, then I is contained in only finitely many height one primes P of $\widetilde{\mathbb{I}}$, and it is clear that if $f = P(f_{\mathscr{F}}) \in \mathscr{S}_{\mathscr{F}}$, then $\rho_f|_H$ splits if and only if $I \subseteq P$. Therefore, $\rho_f|_H$ is non-split for all but finitely many $f \in \mathscr{S}_{\mathscr{F}}$, and in particular the same assertion holds for $\rho_f|_{D_p}$. □

2 The $R = \mathbb{T}$ philosophy

2.1 Modular Galois representations

It follows from Theorem 1.2 that the universal ordinary Hecke algebra $\mathbb{T}_N^o$ can be decomposed

$$\mathbb{T}_N^o = \bigoplus_{\mathfrak{m}} \mathbb{T}_{\mathfrak{m}}$$

as the direct sum of local rings finite free over Λ, with $\mathfrak{m}$ running over the finite set of maximal ideals in $\mathbb{T}_N^o$. For each $\mathbb{T}_{\mathfrak{m}}$, the residue ring $\mathbb{T}_{\mathfrak{m}}/P_{k,\varepsilon}\mathbb{T}_{\mathfrak{m}}$ is a local ring direct summand of the Hecke algebra $\mathbb{T}_{Np^r}^{(k),\mathrm{ord}}(\varepsilon)$, and if $f \in S_k^{\mathrm{ord}}(\Delta_r, \varepsilon)$ is a normalized cuspidal eigenform attached to $\mathbb{T}_{\mathfrak{m}}^{(k)} := \mathbb{T}_{\mathfrak{m}}/P_{k,\varepsilon}\mathbb{T}_{\mathfrak{m}}$, i.e. an algebra homomorphism

$$\lambda_f : \mathbb{T}_{Np^r}^{(k),\mathrm{ord}}(\varepsilon) \to R \subset \overline{\mathbb{Q}}_p,$$

with denoting the ring of integers of a finite extension $K/\mathbb{Q}_p$ containing the Hecke eigenvalues of f, factoring through $\mathbb{T}_{\mathfrak{m}}^{(k)}$, the composition with the reduction map $R \to \kappa = R/\pi R$, where π denotes a uniformizer of R, induces a field embedding

$$\begin{array}{rcl} \mathbb{T}_{\mathfrak{m}}^{(k)}/\mathfrak{m}\mathbb{T}_{\mathfrak{m}}^{(k)} & \hookrightarrow & \kappa \\ T \bmod \mathfrak{m} & \mapsto & \lambda_f(T) \bmod \pi, \end{array}$$

with $\mathfrak{m}$ denoting here the maximal ideal in $\mathbb{T}_{\mathfrak{m}}^{(k)}$. Therefore, once fixed an algebraic closure $\overline{\mathbb{F}}_p$ of the prime field with p elements and an embedding $\kappa \hookrightarrow \overline{\mathbb{F}}_p$, we see that for each pair of such forms f, f' attached to $\mathbb{T}_{\mathfrak{m}}^{(k)}$ either one of the following two conditions holds:

- $a_n(f') \equiv {}^{\sigma}a_n(f) \pmod{\pi}$ for all $n > 0$, for a certain $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}_p/K)$;
- $a_n(f') \equiv a_n(f) \pmod{\pi}$ for all $n > 0$,

and further, that one can use each such f to attach a *residual representation*

$$\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \to \mathbf{GL}_2(\kappa)$$

unramified outside Np with semi-simplification uniquely determined by the requirement that

$$\mathrm{tr}\rho_{\mathfrak{m}}(\mathrm{Frob}_\ell) = T_\ell \bmod \mathfrak{m} \tag{22}$$

for every prime $\ell \nmid Np$. If $\rho_{\mathfrak{m}}$ turns out to irreducible, then $\rho_{\mathfrak{m}}$ itself is uniquely determined by (22). If we assume $\mathrm{char}(\kappa) \neq 2$, then $\rho_{\mathfrak{m}}$ is automatically *absolutely* irreducible, if irreducible, since it is odd. This is highly desirable due to the following result.

Theorem 2.1. *Let $R_{\mathfrak{m}}$ be a local component of $\mathbb{T}_N^o$ with residue field κ, and denote by $\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \to \mathbf{GL}_2(\kappa)$ a residual representation attached to $\mathbb{T}_{\mathfrak{m}}$. If $\rho_{\mathfrak{m}}$ is absolutely irreducible, then:*

- *The* Λ*-module* $\mathrm{Hom}_\Lambda(\mathbb{T}_\mathfrak{m}, \Lambda)$ *is free over* $\mathbb{T}_\mathfrak{m}$*, and therefore* $\mathbb{T}_\mathfrak{m}$ *is a Gorenstein ring;*
- *The Pontryagin dual* $J_\infty(\mathbb{T}_\mathfrak{m})^\wedge = \mathrm{Hom}(J_\infty(\mathbb{T}_\mathfrak{m}), \mu_{p^\infty})$ *is free of rank two over* $\mathbb{T}_\mathfrak{m}$*, and the* $G_\mathbb{Q}$*-action on* $J_\infty(\mathbb{T}_\mathfrak{m})^\wedge$ *gives rise to a Galois representation:*

$$\rho^{\mathrm{mod}} : G_\mathbb{Q} \to \mathbf{GL}_2(\mathbb{T}_\mathfrak{m})$$

which is unramified outside Np*, and such that for every prime* $\ell \nmid Np$*,*

$$\det(\mathrm{Id} - \rho^{\mathrm{mod}}(\mathrm{Frob}_\ell)X) = 1 - T_\ell X + \ell^{-1}\langle \ell \rangle X^2.$$

Furthermore, the module of coinvariants of $J_\infty(\mathbb{T}_\mathfrak{m})^\wedge$ *by any inertia group at* p *if free of rank one over* $R_\mathfrak{m}$*.*

Since $\mathbb{T}_\mathfrak{m}/\mathfrak{m}\mathbb{T}_\mathfrak{m} \cong \mathbb{T}_\mathfrak{m}^{(k)}/\mathfrak{m}\mathbb{T}_\mathfrak{m}^{(k)}$, the above discussion allows us to note that the pair $(\mathbb{T}_\mathfrak{m}, \rho^{\mathrm{mod}})$ consisting of the local Noetherian Λ-algebra $\mathbb{T}_\mathfrak{m}$ and the degree 2 Galois representation ρ^{mod} over $\mathbb{T}_\mathfrak{m}$ satisfies the following universal property:

> "for any p-stabilized ordinary cuspidal eigenform $f \in S_k^{\mathrm{ord}}(\Delta_r, \varepsilon)$ whose associated Galois representation $\rho_f : G_\mathbb{Q} \to \mathbf{GL}_2(\overline{\mathbb{Q}}_p)$ is such that
>
> $$\rho_f \bmod \pi = \rho_\mathfrak{m}$$
>
> as representation into $\mathbf{GL}_2(\overline{\mathbb{F}}_p)$, therefore corresponding to an algebra homomorphism $\lambda_f : \mathbb{T}_\mathfrak{m}^{(k)} \to \overline{\mathbb{Q}}_p$, the exists a unique algebra homomorphism
>
> $$\varphi_f : \mathbb{T}_\mathfrak{m} \to \mathbb{T}_\mathfrak{m}^{(k)}$$
>
> such that $\rho_f = (\lambda_f \circ \varphi_f) \circ \rho^{\mathrm{mod}}$."

Considerations of this sort lead Mazur to develop his foundational work [Maz89] on the systematic study of deformation of Galois representations.

2.2 The theorem by Greenberg and Stevens

Let $E_{/\mathbb{Q}}$ be an elliptic curve of conductor Np with $(N,p) = 1$, and assume that p is a prime of split multiplicative reduction for E. Thus if $f_E = \sum a_n(E)q^n \in \mathbb{Z}[[q]]$ is the newform attached to E, then $a_p(E) = 1$ and therefore f_E is ordinary at p. Under these circumstances, Mazur and Swinnerton-Dyer attached to f_E a p-adic analytic L-function which we denote by $L_p(f_E, s)$. It is defined in terms of a (nontrivial) power series $\mathscr{L}(f_E, T) \in \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by means of the formula

$$L_p(f_E, s) := \mathscr{L}(f_E, \kappa(\gamma_o)^{1-s} - 1),$$

for any choice of topological generator $\gamma_o \in \Gamma = 1 + p\mathbb{Z}_p$, and where κ denotes the canonical isomorphism $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p)) \cong \Gamma$ induced by the cyclotomic character $\varepsilon_{\mathrm{cyc}} : \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \to \mathbb{Z}_p^\times$. Here s denotes a p-adic variable in $\mathbb{Z}_p$. For a character $\eta \in \mathrm{Hom}(\Gamma, \mu_{p^\infty})$, denote by $L(f_E, \eta, s)$ the L-function of f_E twisted by the Dirichlet character of p-power conductor associated to η. Then it is known that for the positive Néron period Ω_E of E the special values $L(f_E, \eta, 1)/\Omega_E$ for η running in $\mathrm{Hom}(\Gamma, \mu_{p^\infty})$ are all algebraic, and the above power series $\mathscr{L}(f_E, T)$ is then characterized, for $\eta \neq 1$, by the interpolation property

$$\mathscr{L}(f_E, \eta(\gamma_o) - 1) = \tau(\eta^{-1}) \cdot \frac{L(f_E, \eta, 1)}{\Omega_E} \quad \text{for all } \eta \in \mathrm{Hom}(\Gamma, \mu_{p^\infty}),$$

where $\tau(\eta^{-1})$ denotes the Gauss sum of η. One then has the evaluation formula

$$L_p(f_E, 1) = (1 - a_p(E)^{-1}) \cdot \frac{L(f_E, 1)}{\Omega_E}. \tag{23}$$

Since E is assumed to be split multiplicative, we have $a_p(E) = 1$, and (23) becomes the identity $0 = 0$ due to the factor $(1 - a_p(E)^{-1})$. Then if $L(f_E, 1) \neq 0$, one expects to "arithmetically" relate the algebraic value $L(f_E, 1)/\Omega_E$ to the derivative of $L_p(E, s)$ at $s = 1$, i.e. to give an arithmetic meaning to the scalar $\mathcal{L}_p^{\mathrm{an}}(E)$ defined by the identity

$$L_p'(E, 1) = \mathcal{L}_p^{\mathrm{an}}(E) \cdot \frac{L(E, 1)}{\Omega_E}. \tag{24}$$

Indeed, it follows form Tate's uniformization theory that one can write

$$E(\overline{\mathbb{Q}}_p) = \overline{\mathbb{Q}}_p^\times / q_E^{\mathbb{Z}} \tag{25}$$

for a certain Tate period $q_E \in p\mathbb{Z}_p$, and Mazur, Tate and Teitelbaum [MTT86] conjectured that the expression

$$\mathcal{L}_p^{\mathrm{arith}}(E) := \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}$$

is precisely the error term $\mathcal{L}_p^{\mathrm{an}}(E)$ in (24). This was proved by Greenberg and Stevens [GS93], and an important point in the argument was a purely cohomological description of $\mathcal{L}_p^{\mathrm{arith}}(E)$ in terms of the isomorphism class of the Tate module $V_p(E) = \mathrm{Ta}_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of E as a (non-trivial) extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$.

More generally, Greenberg proposed in [Gr94a] a similar cohomological definition of an $\mathcal{L}$-invariant attached to a wide class of p-adic ordinary representations V, which is conjectured by him to account for the error between the algebraic L-value and the leading term of the archimedean and p-adic L-functions attached to V, in the presence of an exceptional zero phenomenon. Unfortunately, the existence and even the precise interpolation properties that p-adic L-functions attached to more general p-adic (motivic) representations should satisfy is still largely conjectural.

A crucial point in the proof by Greenberg and Stevens (another equally important one being the construction of a suitable 2-variable p-adic L-function) is the formula

$$\mathcal{L}(E) = -2 \cdot \frac{\mathrm{d}a_p(\kappa)}{\mathrm{d}\kappa}\bigg|_{\kappa=2}, \tag{26}$$

where $a_p(\kappa)$ is the p-adic analytic function of the p-adic variable $\kappa \in U$, for a certain open disc $U \subset \mathbb{Z}_p$, given by

$$a_p(\kappa) := a_{\mathbb{I}}(p; \gamma_o^{2-\kappa} - 1),$$

where $\mathscr{F}_{\mathbb{I}} = \sum a_{\mathbb{I}}(n; X)q^n \in \mathbb{I}[[q]]$, for $\mathbb{I}$ a finite flat extension of $\mathbb{Z}_p[[X]]$, is the Λ-adic form attached to f_E, which is the specialization of $\mathscr{F}_{\mathbb{I}}$ at $X = 0$. The proof of (26) consists in using the Galois representation attached to $\mathscr{F}_{\mathbb{I}}$ by Theorem 1.5 to explicitly compute an annhilator under the local Tate pairing of the extension class in $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ defined by the Tate module of E. We next discuss this point in further detail.

Let $z \in E[p^n]$, so that $z^{p^n} = q_E^{d(z)}$ for certain integer $d(z)$ due to the uniformization (25). Conversely, given $d \in \mathbb{Z}/p^n\mathbb{Z}$, one can solve the previous equation for some $z \in \overline{\mathbb{Q}}_p^\times$ which will be well defined modulo $\mu_{p^n} = \mu_{p^n}(\overline{\mathbb{Q}}_p)$. Thus the association $z \mapsto d(z)$ yields an exact sequence of D_p-modules

$$0 \to \mu_{p^n} \to E[p^n] \to \mathbb{Z}/p^n\mathbb{Z} \to 0,$$

with D_p acting trivially on $\mathbb{Z}/p^n\mathbb{Z}$ and via the mod p^n cyclotomic character on μ_{p^n}. Taking the projective limit along n and tensoring with $\mathbb{Q}_p$, the above sequence becomes

$$0 \to \mathrm{Ta}_p(\mu_{p^\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to V_p(E) \to \mathbb{Q}_p \to 0, \tag{27}$$

where $\mathrm{Ta}_p(\mu_{p^\infty}) \otimes \mathbb{Q}_p$ is identified with $\mathbb{Q}_p(1)$ after a (non-canonical) choice of $\mathbb{Z}_p$-basis of $\mathrm{Ta}_p(\mu_{p^\infty})$. Thus we see that $V_p(E)$ is a $\mathbb{Q}_p[D_p]$-module extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$, and it therefore yields an extension class

$$\kappa(V_p(E)) \in \mathrm{Ext}^1_{\mathbb{Q}_p[D_p]}(\mathbb{Q}_p, \mathbb{Q}_p(1)) = \mathrm{H}^1(D_p, \mathbb{Q}_p(1))$$

given by the image of 1 under the coboundary map $d_0 : \mathrm{H}^0(D_p, \mathbb{Q}_p) = \mathbb{Q}_p \to \mathrm{H}^1(D_p, \mathbb{Q}_p(1))$.

On the other hand, it follows from Kummer theory that one has a canonical isomorphism

$$\mathrm{H}^1(D_p, \mathbb{Q}_p(1)) \cong \varprojlim_n \left(\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{p^n}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Therefore, each element $q \in \mathbb{Q}_p^\times$ determines a *Kummer cohomology class*

$$\kappa_{\mathrm{Kum}}(q) \in \mathrm{H}^1(D_p, \mathbb{Q}_p(1)),$$

explicitly given by the natural image under $\mathrm{H}^1(D_p, \mathbb{Z}_p(1)) \to \mathrm{H}^1(D_p, \mathbb{Q}_p(1))$ of the cocycle $\sigma \mapsto \{(q^{1/p^n})^{\sigma-1}\}_n$ for the choice of a compatible sequence $\{q^{1/p^n}\}_n$ of p-power roots of q. In particular, one can associate to the Tate period q_E its Kummer class $\kappa_{\mathrm{Kum}}(q_E)$.

Lemma 2.2. *The sequence* (27) *is non-split, and its extension class* $\kappa(V_p(E))$ *coincides with the Kummer class* $\kappa_{\mathrm{Kum}}(q_E)$.

Proof. The equality $\kappa(V_p(E)) = \kappa_{\mathrm{Kum}}(q_E)$ follows immediately from the previous description of $\kappa_{\mathrm{Kum}}(q)$, since $\delta_{0,n}(1) \in \mathrm{H}^1(D_p, \mathbb{Z}/p^n\mathbb{Z}(1))$ for the degree zero coboundary map $\delta_{0,n}$ of the exact sequence $0 \to \mu_{p^n} \to E[p^n] \to \mathbb{Z}/p^n\mathbb{Z} \to 0$ is given by $\xi_n = (\sigma \mapsto (q_E^{1/p^n})^{\sigma-1})$. For the first assertion, note that we may identify $\xi = \{\xi_n\} \in \varprojlim_n \mathrm{H}^1(D_p, \mathbb{Z}/p^n\mathbb{Z}(1))$ with its (pre-)image $\{q_E \bmod (\mathbb{Q}_p^\times)^n\}_n$ under the Kummer isomorphism. Then we note that the valuation homomorphism $v_p : \mathbb{Q}_p^\times \to \mathbb{Z}$ yields a well-defined mapping $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{p^n} \to \mathbb{Z}/p^n\mathbb{Z}$, and thus also a map

$$v : \varprojlim_n \mathbb{Q}_p/(\mathbb{Q}_p^\times)^{p^n} \to \mathbb{Z}_p.$$

Under the previous identification, $v(\xi) = v_p(q_E)$, and therefore ξ must be of infinite order in $\mathbb{Z}_p$, since $q_E \in p\mathbb{Z}_p$. Suppose now that the sequence (27) splits. Then there exists a D_p-stable subspace X of $V_p(E)$ which is mapped isomorphically onto $\mathbb{Q}_p$ under the projection in (27); the image of $\mathrm{Ta}_p(E) \cap X$ in $\mathbb{Z}_p$ under this mapping is of the form $p^N\mathbb{Z}_p$ for some $N \geq 0$, and it follows that $p^N\xi = 0$, contradicting the fact that ξ is of infinite order. $\square$

Let $\rho_{\mathbb{I}} : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathbb{I}_{\mathscr{L}})$ be the Galois representation attached by Theorem 1.5 to the Λ-adic form $\mathscr{F}_{\mathbb{I}} = \sum a_{\mathbb{I}}(n;X)q^n \in \mathbb{I}[[q]]$, and denote by $V = \mathbb{I}_{\mathscr{L}}^2$ a representation space for $\rho_{\mathbb{I}}$. Recall that Theorem 1.7 asserts the existence of an exact sequence of D_p-modules

$$0 \to \mathbb{I}_{\mathscr{L}}(\kappa\varepsilon_{\mathrm{cyc}}^{-1}\nu^{-1}) \to {\mathbb{I}_{\mathscr{L}}}^2 \to \mathbb{I}_{\mathscr{L}}(\nu) \to 0, \tag{28}$$

where $\nu : D_p \to \mathbb{I}_{\mathscr{L}}^\times$ is the unramified character mapping an arithmetic Frobenius Frob_p to $a_{\mathbb{I}}(p;X)$, and $\kappa : D_p \to {\mathbb{I}_{\mathscr{L}}}^\times$ is such that $\kappa \bmod P_{k,\varepsilon_r} = \varepsilon_r\langle\varepsilon_{\mathrm{cyc}}\rangle^k$. The representation $\rho_{\mathbb{I}}$ can be descended to the integral closure $\widetilde{\mathbb{I}}$ of $\mathbb{I}$ in $\mathbb{I}_{\mathscr{L}}$, and if P is a height one prime in $\widetilde{\mathbb{I}}$, we have a canonical isomorphism $\widetilde{\mathbb{I}}_P \cong \mathbb{I}_P$. Thus if P denotes the arithmetic prime in $\widetilde{\mathbb{I}}$ above P_2 corresponding to f_E, we see that the exact sequence (28) gives rise to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{I}_P(\kappa\varepsilon_{\mathrm{cyc}}^{-1}\nu^{-1}) & \longrightarrow & \mathbb{I}_P^2 & \longrightarrow & \mathbb{I}_P(\nu) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & V_p(E) & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0, \end{array} \tag{29}$$

where the vertical arrows are given by specialization at P. After twisting the first and second arrows in (29) by the Λ-adic character $\nu\varepsilon_{\mathrm{cyc}}^2\kappa^{-1}$ and its (trivial) specialization at P_2 respectively, we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{I}_P(\varepsilon_{\mathrm{cyc}}) & \longrightarrow & \mathbb{I}_P^2(\nu\varepsilon_{\mathrm{cyc}}^2\kappa^{-1}) & \longrightarrow & \mathbb{I}_P(\nu^2\varepsilon_{\mathrm{cyc}}^2\kappa^{-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & V_p(E) & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0. \end{array} \tag{30}$$

Finally, letting $t := \kappa - 2 \in \Lambda$, we see that $\Lambda/(t^2) \cong \mathbb{Q}_p[\varepsilon]$, where $\varepsilon = t \bmod (t^2)$, and the reduction mod t^2 of (30) becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\mathbb{Q}}_p(1) & \longrightarrow & \tilde{V} & \longrightarrow & \tilde{\mathbb{Q}}_p(\psi) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & V_p(E) & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0, \end{array} \tag{31}$$

where ψ denotes the character $\nu_k^2 \langle \varepsilon \rangle^{2-k} \bmod (t^2)$. We have an exact sequence of $\mathbb{Q}_p[D_p]$-modules

$$0 \to \mathbb{Q}_p \xrightarrow{\varepsilon} \tilde{\mathbb{Q}}_p(\psi) \to \operatorname{coker}(\varepsilon) = \mathbb{Q}_p \to 0 \tag{32}$$

induced by the mapping $\varepsilon : q \mapsto \varepsilon . q$. The degree zero coboundary $\mathbb{Q}_p$-linear map in the long exact sequence associated to (32) is explicitly given by

$$\begin{array}{rcl} \delta_\psi : \mathbb{Q}_p^{D_p} = \mathbb{Q}_p & \to & \mathrm{H}^1(D_p, \mathbb{Q}_p) \\ 1 & \mapsto & \left(\sigma \mapsto \dfrac{\mathrm{d}\psi}{\mathrm{d}t}(\sigma) \right), \end{array}$$

where $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ is the composite map

$$D_p \xrightarrow{\psi} 1 + \varepsilon \cdot \mathbb{Q}_p \subset \tilde{\mathbb{Q}}_p \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \mathbb{Q}_p,$$

seen as an element in $\mathrm{Hom}(D_p, \mathbb{Q}_p) = \mathrm{H}^1(D_p, \mathbb{Q}_p)$. Note that $\psi(\sigma) = \psi_0(\sigma) + \varepsilon \cdot \frac{\mathrm{d}\psi}{\mathrm{d}t}(\sigma)$, where $\psi_0(\sigma) = \psi(\sigma)$ modulo ε, and that $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ is nonzero, since ψ is nontrivial. Write $\kappa(\psi)$ for cocycle $\delta_\psi(1)$. The expression of $\kappa(\psi) \in \mathrm{H}^1(D_p^{\mathrm{ab}}, \mathbb{Q}_p)$ in the basis κ_{nr}, κ_{cyc} is obviously

$$\kappa(\psi) = \frac{\mathrm{d}\psi}{\mathrm{d}t}([p, \mathbb{Q}_p]) \cdot \kappa_{\mathrm{nr}} + \frac{\mathrm{d}\psi}{\mathrm{d}t}([\gamma, \mathbb{Q}_p]) \cdot \kappa_{\mathrm{cyc}}, \tag{33}$$

and we see that the ratio $(\frac{\mathrm{d}\psi}{\mathrm{d}t}([p, \mathbb{Q}_p]) : \frac{\mathrm{d}\psi}{\mathrm{d}t}([\gamma, \mathbb{Q}_p]))$ is uniquely determined by the infinitesimal deformation $\tilde{\mathbb{Q}}_p(\psi)$ of $\mathbb{Q}_p$.

The cohomology groups $\mathrm{H}^1(D_p, \mathbb{Q}_p)$ and $\mathrm{H}^1(D_p, \mathbb{Q}_p(1))$ are both 2-dimensional $\mathbb{Q}_p$-vector spaces, and the local Tate pairing

$$\langle \ , \ \rangle : \mathrm{H}^1(D_p, \mathbb{Q}_p(1)) \times \mathrm{H}^1(D_p, \mathbb{Q}_p) \to \mathrm{H}^2(D_p, \mathbb{Q}_p(1)) = \mathbb{Q}_p$$

furnishes a perfect duality between them.

Lemma 2.3. *The cohomology class $\kappa(\psi)$ coming from the connecting homomorphism associated to the exact sequence* (32) *is an annihilator under the local Tate pairing of the cohomology class $\kappa_{\mathrm{Kum}}(q_E)$.*

Proof. Consider the commutative diagram in cohomology induced by the diagram (31):

$$\begin{array}{ccccccc}
\mathrm{H}^1(D_p,\tilde{V}) & \longrightarrow & \mathrm{H}^1(D_p,\tilde{\mathbb{Q}}_p) & \longrightarrow & \mathrm{H}^2(D_p,\tilde{\mathbb{Q}}_p(1)) & \longrightarrow & 0 \\
\uparrow & & {\scriptstyle d_1}\uparrow & & {\scriptstyle d_2}\uparrow & & \\
\mathrm{H}^1(D_p,V(E)) & \longrightarrow & \mathrm{H}^1(D_p,\mathbb{Q}_p) & \xrightarrow{\delta_1} & \mathrm{H}^2(D_p,\mathbb{Q}_p(1)) = \mathbb{Q}_p & \longrightarrow & 0 \\
 & & {\scriptstyle \delta_\psi}\uparrow & & {\scriptstyle d}\uparrow & & \\
 & & \mathrm{H}^0(D_p,\mathbb{Q}_p) = \mathbb{Q}_p & \xrightarrow{\delta_0} & \mathrm{H}^1(D_p,\mathbb{Q}_p(1)) & &
\end{array}$$

The connecting homomorphism d vanishes, and therefore the cohomology class $\kappa(\psi)$ is in the kernel of δ_1, since it is in the image of δ_ψ. On the other hand, $\kappa_{\mathrm{Kum}}(q_E)$ is in the image of δ_0. Thus it suffices to show that the kernel of δ_1 is orthogonal under the local Tate pairing to the image of δ_0. Indeed, the connecting homomorphisms $\delta_0 : \mathbb{Q}_p \to \mathrm{H}^1(D_p, \mathbb{Q}_p(1))$ and $\delta_1 : \mathrm{H}^1(D_p, \mathbb{Q}_p) \to \mathbb{Q}_p$ are transposes of one another, since the exact sequence

$$0 \to \mathbb{Q}_p(1) \to V(E) \to \mathbb{Q}_p \to 0 \tag{34}$$

is self dual with respect to the multiplication pairing $\mathbb{Q}_p(1) \times \mathbb{Q}_p \to \mathbb{Q}_p(1)$ and the Weil pairing $V(E) \times V(E) \to \mathbb{Q}_p(1)$. Since the extension class in $\mathrm{H}^1(D_p, \mathbb{Q}_p(1))$ corresponding to (34) lies in the line spanned by $\kappa_{\mathrm{Kum}}(q_E)$, the conclusion follows. □

Corollary 2.4.

$$\mathcal{L}(E) = -2 \cdot \left.\frac{\mathrm{d}a_p(\kappa)}{\mathrm{d}x}\right|_{\kappa=2}.$$

Indeed, the local Tate pairing of the cohomology classes $\kappa_{\mathrm{Kum}}(q_E)$ and $\kappa(\psi) = \frac{\mathrm{d}\psi}{\mathrm{d}t}$ can be computed as

$$\langle \kappa_{\mathrm{Kum}}(q_E), \kappa(\psi)\rangle = \frac{\mathrm{d}\psi}{\mathrm{d}t}([q_E, \mathbb{Q}_p]).$$

Writting $q_E = p^{\mathrm{ord}_p(q_E)}u$ for $u \in \mathbb{Z}_p^\times$, we have $\psi([q_E, \mathbb{Q}_p]) = a_p(k)^{-2n}\langle u\rangle^{2-k}$. Since $\langle u\rangle^{2-k} = \exp((2-k)\log_p(u))$, we see that

$$\begin{aligned}
\left.\frac{\mathrm{d}\langle u\rangle^{2-k}}{\mathrm{d}k}\right|_{k=2} &= -\log_p(u) \cdot \exp((2-k)\log_p(u))|_{k=2} \\
&= -\log_p(q_E),
\end{aligned}$$

since $\log_p(u) = \log_p(q_E)$ by our choice of p-adic logarithm (i.e. such that $\log_p(p) = 0$). Since $\psi([q_E, \mathbb{Q}_p]) = a_p(k)^{-2n}\langle u\rangle^{2-k}$ for $n = \mathrm{ord}_p(q_E)$, we have

$$\begin{aligned}
0 &= \left.\frac{\mathrm{d}\psi([q_E, \mathbb{Q}_p])}{\mathrm{d}k}\right|_{k=2} \\
&= -2 \cdot \mathrm{ord}_p(q_E) \cdot a_p'(k)\langle u\rangle^{2-k} + a_p(k)^{-2\mathrm{ord}_p(q_E)} \cdot \log_p(\langle u\rangle)\langle u\rangle^{2-k}|_{k=2} \\
&= -2 \cdot \mathrm{ord}_p(q_E) \cdot a_p'(k)|_{k=2} + \log_p(q_E),
\end{aligned}$$

and this concludes the proof of the corollary.

2.3 The adjoint square Selmer group

Given an ordinary modular Galois representation ρ_f, its adjoint square representation $\mathrm{Ad}(\rho_f)$ can be defined, and the ordinarity assumption implies that $\mathrm{Ad}(\rho_f)$ presents an exceptional zero phenomenon in the sense of Greenberg. In this section we review the definition of a panoply of so-called Selmer groups attached to $\mathrm{Ad}(\rho_f)$, which are certain cohomology groups obtained by imposing local conditions on cocycles in the Galois cohomology of $\mathrm{Ad}(\rho_f)$. The usefulness of these groups becomes apparent when one interprets them in terms of more explicit objects. In particular, we show how the various Selmer groups attached to $\mathrm{Ad}(\rho_f)$ relate to various deformation problems defined by $\overline{\rho}_f$.

Let f be an ordinary modular cusp form, K a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ containing all the Hecke eigenvalues of f. Let $\rho_f : G_{\mathbb{Q}} \to \mathbf{GL}_2(K)$ be the Galois representation attached to f and U the representation K-space of ρ_f. The adjoint square representation $\mathrm{Ad}(\rho_f)$ is the Galois representation given by the adjoint action of $G_{\mathbb{Q}}$ on the four-dimensional K-space $V = \mathrm{End}_K(U)$. Thus for $\sigma \in G_{\mathbb{Q}}$ and $\varphi \in V$, $\sigma.\varphi$ is the K-endomorphism of U given by $u \mapsto \sigma.\varphi(\sigma^{-1}.u)$. Equivalently, in terms of the matrix representation ρ_f, $\mathrm{Ad}(\rho_f)$ is the $G_{\mathbb{Q}}$-representation on $\mathbf{GL}_4(K) \cong \mathrm{Aut}_K(V)$ given by conjugation via ρ_f, i.e. $\sigma \in G_{\mathbb{Q}}$ acting on $A \in \mathbf{GL}_4(K)$ by $\rho_f(\sigma) \cdot A \cdot \rho_f(\sigma)^{-1}$.

Since f is ordinary, there exists a D_p-stable K-line U^0 in U such that the quotient U/U^0 is unramified. We fix such a line U_0, thus obtaining a two-step decreasing filtration of D_p-modules

$$U \supset U^0 \supset \{0\}. \tag{35}$$

Taking a K-basis of U with the first vector in U, the resulting matrix representation of $\rho_f|_{D_p}$ is upper-triangular:

$$\rho_f|_{D_p} = \begin{pmatrix} \varepsilon_f & a_f \\ & \delta_f \end{pmatrix}, \tag{36}$$

where $\varepsilon_f, \delta_f : D_p \to K^\times$ are continuous character, δ_f being such that $\delta_f|_{I_p} = 1$, and $a_f : D_p \to K$ is a continuous function. Let $W = \mathrm{End}^0_K(U) \subset V$ be the representation space of the sub-representation $\mathrm{Ad}^0(\rho_f)$ of $\mathrm{Ad}(\rho_f)$ given by the trace-zero K-endomorphisms of U. The filtration (35) induces decreasing D_p-stable filtrations on V and W:

$$\begin{aligned} V &\supset V^0 \supset V^{00} \supset \{0\}; \\ W &\supset W^0 \supset W^{00} \supset \{0\}, \end{aligned}$$

where

$$\begin{aligned}
V^0 &= \{\varphi \in V \mid \varphi(U^0) \subset U^0\},\\
W^0 &= V^0 \cap W;\\
V^{00} &= \{\varphi \in V \mid \varphi(U^0) = \{0\}\},\\
W^{00} &= V^{00} \cap W = V^{00}.
\end{aligned}$$

With the above choice of K-basis of U, we see that V^0 (resp. W^0) is identified with the subspace of $\mathrm{M}_2(K)$ consisting of upper triangular matrices (resp. trace zero upper triangular matrices), and $V^{00} = W^{00}$ with the upper nilpotent ones.

We will also need to consider the following subspaces:

$$\begin{aligned}
V^{\mathrm{ord}} &= \{\varphi \in V \mid \varphi(U) \subset U^0\} = \left\{\begin{pmatrix} * & * \\ & \end{pmatrix}\right\} \subset V^0;\\
V^{\mathrm{split}} &= V^{\mathrm{ord}}/V^{00} = \left\{\begin{pmatrix} * & \\ & \end{pmatrix}\right\}.
\end{aligned}$$

Note that $V^{\mathrm{ord}} \cap W^0 = W^{00}$.

Remark 5. As can be checked with a straightforward calculation, the adjoint action of D_p on V^0/V^{00} is trivial. Thus in particular the action of D_p on $V^{\mathrm{split}} \cong W^0/W^{00}$ is trivial. One then expects after R. Greenberg that the p-adic L-function $L_p(\mathrm{Ad}(\rho_f), s)$ presents an exceptional zero phenomenon.

Fix a $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice $T \subset U$, and define the representations

$$\begin{aligned}
\mathrm{Ad}(\rho_f)_T : G_{\mathbb{Q}} &\to \mathbf{GL}_4(\mathcal{O}),\\
\mathrm{Ad}(\rho_f)_{U/T} : G_{\mathbb{Q}} &\to \mathbf{GL}_4(K/\mathcal{O}),
\end{aligned} \tag{37}$$

given by the adjoint action of $G_{\mathbb{Q}}$ on $V_T = V \cap \mathrm{End}_{\mathcal{O}}(T)$ and on $V_{U/T} = V/V_T = V_T \otimes_{\mathcal{O}} K/\mathcal{O}$ respectively. We put $W_T = W \cap V_T$, and intersecting with T the filtration (35) of U we obtain a two-step D_p-stable decreasing filtration of T:

$$T \supset T^0 \supset \{0\}. \tag{38}$$

Replacing V and W by V_T and W_T respectively in the above definitions, we define V_T^0, V_T^{00}, W_T^0, V_T^{ord} and V_T^{split} and identifying $V_T^?$ with its natural image in $V^?$, we define $V_{U/T}^? = V^?/V_T^? = V_T^? \otimes_{\mathcal{O}} K/\mathcal{O}$ just as before, and $W_{U/T}^? = W^?/W_T^? = W_T^? \otimes_{\mathcal{O}} K/\mathcal{O}$ for $? = 0$, 00, ord or split.

For every a prime number q consider the inflation-restriction exact sequence

$$0 \to \mathrm{H}^1(\mathbb{Q}_q^{\mathrm{unr}}/\mathbb{Q}_p, V^{I_q}) \xrightarrow{\mathrm{inf}_q} \mathrm{H}^1(\mathbb{Q}_q, V) \xrightarrow{\mathrm{res}_q} \mathrm{H}^1(\mathbb{Q}_q^{\mathrm{unr}}, V).$$

Denote by L_q^{unr} the subspace of the local cohomology group $\mathrm{H}^1(\mathbb{Q}_q, V)$ given by $\mathrm{Ker}(\mathrm{res}_q)$. A *Selmer structure* $\mathcal{L}$ on V is the data of a subspace $L_q \subset \mathrm{H}^1(\mathbb{Q}_q, V)$ for every rational prime q such that $L_q = L_q^{\mathrm{unr}}$ for every q outside a finite set of primes S containing p.

Definition 2.5. Let $\mathcal{L} = \{L_q\}$ be a Selmer structure on $V = \mathrm{End}_K(U)$. The *Selmer group* of the adjoint square of f associated to $\mathcal{L}$ is the subspace of $\mathrm{H}^1(\mathbb{Q}, V)$ cut out by the local conditions $\{L_q\}$:

$$\begin{aligned}\mathrm{Sel}^{\mathcal{L}}(\mathbb{Q}, \mathrm{Ad}(\rho_f)) &:= \bigcap_q \mathrm{Ker}\left(\mathrm{H}^1(\mathbb{Q}, V) \xrightarrow{\mathrm{res}_q} \mathrm{H}^1(\mathbb{Q}_q, V)/L_q\right) \\ &= \{c \in \mathrm{H}^1(\mathbb{Q}, V) \mid \ c_q := \mathrm{res}_q c \in L_q \quad \text{for all } q\}.\end{aligned}$$

We will only deal with Selmer structures $\mathcal{L}$ for which one can take $S = \{p\}$ in the definition of $\mathcal{L}$. These will define the Selmer groups:

$$\mathrm{Sel}^s(\mathbb{Q}, \mathrm{Ad}(\rho_f)) \ \subset \ \mathrm{Sel}^{00}(\mathbb{Q}, \mathrm{Ad}(\rho_f)) \ \subset \ \mathrm{Sel}^0(\mathbb{Q}, \mathrm{Ad}(\rho_f)),$$

and

$$\mathrm{Sel}^{\mathrm{split}}(\mathbb{Q}, \mathrm{Ad}(\rho_f)) \ \subset \ \mathrm{Sel}^{\mathrm{ord}}(\mathbb{Q}, \mathrm{Ad}(\rho_f)),$$

after we specify that local condition L_p in each of these five cases. We set

$$\begin{aligned}L_p^s &:= \mathrm{Im}\left(\mathrm{H}^1(\mathbb{Q}_p, W^{00}) \to \mathrm{H}^1(\mathbb{Q}_p, V)\right) \\ L_p^{00} &:= \mathrm{Ker}\left(\mathrm{H}^1(\mathbb{Q}_p, V) \to \mathrm{H}^1(\mathbb{Q}_p^{\mathrm{unr}}, V/W^{00})\right),\end{aligned} \tag{39}$$

and define L_p^0, L_p^{ord} and L_p^{split} replacing W^{00} in (39) by W^0, V^{ord} and V^{split} respectively.

We will also need to consider certain $\mathcal{O}$-torsion Selmer groups. They are denoted by $\mathrm{Sel}_R^{\mathcal{L}}(\mathbb{Q}, \mathrm{Ad}(\rho_f)_{U/T})$, and are defined by taking cohomology with values in $V_{U/T}$ and replacing in the above definitions $V^?$ and $W^?$ by $V_{U/T}^?$ and $W_{U/T}^?$, respectively.

We now consider the duals of the above constructions. Let $\mathrm{Ad}(\rho_f)^*(1)$ be the $G_{\mathbb{Q}}$-representation given by the linear dual $V^* = \mathrm{Hom}_K(V, K)$ of V, twisted with the cyclotomic character $\varepsilon_{\mathrm{cyc}}$. Here, $\sigma \in G_{\mathbb{Q}}$ acts on $f \in V^*$ by $\sigma.f = (v \mapsto f(\sigma^{-1}.v))$, so that the usual non-canonical isomorphism $V \cong V^*$ defined with the choice of a K-basis becomes Galois-equivariant. We define $\mathrm{Ad}(\rho_f)_T^*(1)$ and $\mathrm{Ad}(\rho_f)_{U/T}^*(1)$ by letting $G_{\mathbb{Q}}$ act similarly on $V_T^* := \mathrm{Hom}_{\mathcal{O}}(V_T, \mathcal{O})$ and $V_{U/T}^* = \mathrm{Hom}(V_{U/T}, K/\mathcal{O})$, respectively. The same for V replaced by W. The D_p-stable filtrations on V, W and T induce filtrations on their duals by taking the orthogonal complement of each filter under the corresponding local Tate pairings. With these one defines in a manner completely analogous as before the meaning of the dual Selmer groups $\mathrm{Sel}^?(\mathbb{Q}, \mathrm{Ad}(\rho_f)_!^*(1))$ for $? = s$, 00, 0, ord, split, and $! =$, T or U/T.

Deformation theoretic interpretation of classical Iwasawa theory

Since it is quite illuminating, we pause here to give an interpretation of the Selmer group of Greenberg in the cyclotomic setting of classical Iwasawa theory, following [Gr94b].

Let p be an odd prime. The cyclotomic character $\chi : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$, giving the Galois action on the group μ_{p^∞} of p-power roots of unity, factors through $G_\infty = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$, inducing an isomorphism $G_\infty \cong \mathbb{Z}_p^\times$. The canonical decomposition

$$\mathbb{Z}_p^\times = \mu_{p-1} \times 1 + p\mathbb{Z}_p$$

is thus identified with the canonical decomposition $G_\infty = \Delta \times \Gamma$ of G_∞ into its maximal torsion subgroup Δ and its pro-p-free quotient $\Gamma = G_\infty/\Delta$. The restriction homomorphism allows us to write $\Delta = \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ and $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, where $\mathbb{Q}_\infty = \bigcup_{r \geq 0} \mathbb{Q}_r$ is the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$, its rth piece $\mathbb{Q}_r$ being uniquely determined by the requirement that $\mathrm{Gal}(\mathbb{Q}_r/\mathbb{Q})$ is identified via restriction with the unique closed subgroup $\mathrm{Gal}(\mathbb{Q}(\mu_{p^{r+1}})/\mathbb{Q}(\mu_p))$ of $\Gamma \cong \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$ of order p^r.

It is a consequence of Kummer congruences that for an *odd* integer i, $0 \leq i < p-1$, $i \neq 1$, there exists a unique continuous $\mathbb{Z}_p$-valued function $L_{p,i}(s)$ of a p-adic variable $s \in \mathbb{Z}_p$ characterized by the interpolation property

$$L_{p,i}(n) = (1 - p^{-n})\zeta(n), \tag{40}$$

for all odd integers $n \leq -1$ with $n \equiv i \pmod{p-1}$. This is Kubota-Leopoldt's p-adic L-function, and Iwasawa showed that it is given by a so-called Iwasawa function, i.e. $L_{p,i}(s) = \kappa^s(g_i)$, for a certain $g_i \in \Lambda$.

Let L_∞ be the maximal abelian pro-p-extension of K_∞ everywhere unramified, and put $Y_\infty = \mathrm{Gal}(L_\infty/K_\infty)$. Since Y_∞ is abelian, the Galois group $G_\infty = \Delta \times \Gamma$ naturally acts on Y_∞ via $\gamma.y = \tilde{\gamma} y \tilde{\gamma}^{-1}$ for $\gamma \in G_\infty$ and $y \in Y_\infty$, where $\tilde{\gamma}$ denote any lifting of γ to $\mathrm{Gal}(L_\infty/\mathbb{Q})$. In particular Y_∞ thus becomes a Γ-module, and since it is a pro-p-group, a $\Lambda = \mathbb{Z}_p[[\Gamma]]$-modules. Since $\#\Delta$ is prime to p, one has a natural decomposition of Λ-modules:

$$Y_\infty = \bigoplus_{i=0}^{p-2} Y_\infty^{(i)}, \tag{41}$$

where $Y_\infty^{(i)}$ denotes the ω^i-isotypic component of Y_∞, and ω is the Teichmüller character $\omega : \Delta \to \mu_{p-1} \subset \mathbb{Z}_p^\times$ which generated the cyclic character group $\tilde{\Delta}$. It follows from Class Field Theory that Y_∞ is a finitely generated Λ-module, so the same is true for each $Y_\infty^{(i)}$; since they are obviously torsion Λ-modules, the characteristic ideal $\mathrm{char}_\Lambda(Y_\infty^{(i)})$ of each $Y_\infty^{(i)}$ is defined, which will be a principal ideal in Λ. Then the Main Conjecture of Iwasawa, as proved by Mazur and Wiles in [MW84], is the assertion that one may construct a generator of $\mathrm{char}_\Lambda(Y_\infty^{(i)})$ via interpolation of special values of L-functions: more precisely, that

$$\mathrm{char}_\Lambda(Y_\infty^{(i)}) = g_i\Lambda.$$

Again by the congruences of Kummer, it follows that for *even* integers j, $0 \leq j < p-1$, $j \neq 0$, there exists a unique p-adic L-function $L_{p,j}(s) = \kappa^s(\theta_j) : \mathbb{Z}_p \to \mathbb{Z}_p$ characterized by the interpolation formulae

$$L_{p,j}(n) = (1-p^{n-1})\frac{\zeta(n)}{\Omega_n}, \tag{42}$$

for all even integers $n \geq 2$ with $n \equiv j \pmod{p-1}$, where $\Omega_n \in \mathbb{C}^\times$ is a nonzero constant such that the functional equation safisfied by $\zeta(s)$ yields the identity

$$\frac{\zeta(n)}{\Omega_n} = \zeta(1-n). \tag{43}$$

If follows from (43) that one actually has $L_{p,j} = \kappa^{1-s}(\theta_i)$ for i with $i \equiv 1-j \pmod{p-1}$.

Let M_∞ denote the maximal abelian pro-p-extension of K_∞ everywhere unramified except the places in K_∞ above p, and put $X_\infty = \mathrm{Gal}(M_\infty/K_\infty)$. In complete analogy with Y_∞, the $\mathbb{Z}_p$-module X_∞ can be naturally endowed with a Λ-modules structure, and one has a decomposition $X_\infty = \bigoplus_{j=0}^{p-2} X_\infty^{(j)}$ of finite torsion Λ-modules. Then a reformulation of the theorem of Mazur and Wiles is the assertion that the following identity holds:

$$\mathrm{char}_\Lambda(X_\infty^{(j)}) = f_j\Lambda.$$

2.4 The locally split universal deformation ring

Suppose now that the modular representation ρ_f is *residually CM*, i.e. there exists a quadratic imaginary field M and a Hecke character $\psi : G_M \to \mathcal{O}^\times$ such that

$$\overline{\rho}_f = \mathrm{Ind}_M^{\mathbb{Q}}\overline{\psi}.$$

Then the p-adic representation attached to the character ψ:

$$\rho_\psi := \mathrm{Ind}_M^{\mathbb{Q}}\psi : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathcal{O})$$

is obviously a deformation of $\overline{\rho}_f$; since it also split at p, there exists a unique local morphism of $\mathcal{O}$-algebras

$$\varphi_\psi : R^{\mathrm{split}} \to \mathcal{O}$$

bringing the universal deformation ρ^{split} into ρ_ψ. Let I_ψ be the kernel of φ_ψ. The following proposition is a direct adaptation of the arguments in [Wil95] Prop.1.2:

Proposition 2.6. *There is a canonical isomorphism of $\mathcal{O}$-modules:*

$$\mathrm{Hom}_{\mathcal{O}}(I_\psi/I_\psi^2, K/\mathcal{O}) \cong \mathrm{Sel}^{\mathrm{split}}(\mathbb{Q}, \mathrm{Ad}(\rho_f)_{U/T}). \tag{44}$$

Proof. We first construct for each integer $n > 0$ a morphism

$$\varphi : \mathrm{Sel}^{\mathrm{split}}(\mathbb{Q}, \mathrm{Ad}(\rho_f)_{U/T}[\pi^n]) \to \mathrm{Hom}_{\mathcal{O}}(I_\psi/I_\psi^2, \pi^{-n}\mathcal{O}/\mathcal{O}), \tag{45}$$

which will be compatible with the natural maps induced by the inclusions $\mathrm{Ker}(\pi^n) \hookrightarrow \mathrm{Ker}(\pi^m)$ for $m \geq n > 0$ thus yielding (44). Let $\mathcal{O}_n[\varepsilon]$ be the ring of dual numbers $\mathcal{O}/\pi^n[X]/(X^2) \cong \mathcal{O}/\pi^n \oplus \varepsilon.\mathcal{O}/\pi^n$, where $\varepsilon = X \pmod{X^2}$. Let $[\xi]$ be a cocycle in $\mathrm{Sel}^{\mathrm{split}}(\mathbb{Q}, \mathrm{Ad}(\rho_f)_{U/T}[\pi^n])$, and consider the lifting of ρ_ψ to $\mathcal{O}_n[\varepsilon]$ given by the infinitessimal deformation

$$\begin{array}{rcl} \rho_\xi : G_{\mathbb{Q}} & \to & \mathbf{GL}_2(\mathcal{O}_n[\varepsilon]) \\ \sigma & \mapsto & \rho_\psi(\sigma) + \xi(\sigma)\rho_\psi(\sigma).\varepsilon, \end{array}$$

where ρ_ψ is seen as taking values in $\mathbf{GL}_2(\mathcal{O}_n[\varepsilon])$ by composition with the projection map $\mathcal{O} \to \mathcal{O}/\pi^n \subset \mathcal{O}/\pi^n \oplus \varepsilon.\mathcal{O}/\pi^n$, the inclusion being along the first summand in $\mathcal{O}_n[\varepsilon]$, and one uses the natural inclusion of $\mathcal{O}/\pi^n$-algebras (since $\varepsilon^2 = 0$)

$$\mathrm{M}_2(\mathcal{O}/\pi^n) \hookrightarrow 1_2 + \varepsilon.\mathrm{M}_2(\mathcal{O}/\pi^n) \subset \mathbf{GL}_2(\mathcal{O}_n[\varepsilon])$$

to see $\xi(\sigma) \in \mathbf{GL}_2(\mathcal{O}_n[\varepsilon])$. The correspondence is well-defined, since ρ_ξ is easily checked to be a continuous homomorphism, and affecting ξ by a coboundary does not change the equivalence class of the resulting lifting to $\mathcal{O}_n[\varepsilon]$. The local restrictions imposed on ξ imply that ρ_ξ is a split lifting of ρ_0. Indeed, we only need to check the local condition at p. Consider the exact sequence

$$\begin{array}{rl} 0 \to & \mathrm{H}^1(D_p/I_p, (V_{U/T}[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{I_p}) \to \mathrm{H}^1(D_p, V_{U/T}[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n]) \to \\ \to & \mathrm{H}^1(I_p, V_{U/T}[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{D_p/I_p} \end{array}$$

coming from the inflation-restriction exact sequence. By hypothesis, the cocycle ξ is in the image of the first term in the above sequence. Let X be the subspace of $(V_{U/T}[\pi^n]/V_{U/T}^0[\pi^n])^{I_p}$ such that the natural inclusion map $V_{U/T}^0 \subset V_{U/T}$ yields the exact sequence

$$0 \to (V_{U/T}^0[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{I_p} \to (V_{U/T}[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{I_p} \to X \to 0. \tag{46}$$

Since the action of D_p on $V_{U/T}[\pi^n]/V_{U/T}^0[\pi^n]$ is given by η/η' and we have $(\eta/\eta')|_{D_p} \neq 1 \pmod{\pi}$ due to the p-distinguishedness hypothesis, we see that $X^{D_p/I_p} = X^{\mathrm{Frob}_p} = 0$ and $\mathrm{H}^1(D_p/I_p, X) = X/(\mathrm{Frob}_p - 1)X = 0$ so that the long exact sequence of D_p/I_p-modules induced by (46) gives rise to an isomorphism

$$\mathrm{H}^1(D_p/I_p, (V_{U/T}^0[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{I_p}) \cong \mathrm{H}^1(D_p/I_p, (V_{U/T}[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{I_p}).$$

Thus ξ admits a representative mapping D_p into $V^0_{U/T}[\pi^n]$ and I_p into $W^{\mathrm{split}}_{U/T}[\pi^n]$, and now we see that ρ_ξ is indeed a split lifting of ρ_0 from the formula

$$\begin{pmatrix} * & \\ & \end{pmatrix}\begin{pmatrix} \eta & \\ & \eta' \end{pmatrix} = \begin{pmatrix} *\cdot\eta & \\ & \end{pmatrix}.$$

By the universal property of R^{split}, the split lifting ρ_ξ induces a unique morphism of local $\mathcal{O}$-algebras

$$\varphi_\xi : R^{\mathrm{split}} \to \mathcal{O}_n[\varepsilon]$$

such that $\rho_\xi = \varphi_\xi \circ \rho^{\mathrm{split}}$ for the universal split deformation ρ^{split} of ρ_0. The image of φ_ξ is entirely contained in $\varepsilon.\mathcal{O}/\pi^n$ when restricted to I_ψ, since $\rho_\xi \equiv \rho_\psi \pmod{\varepsilon}$, and we have $\varphi_\xi(I_\psi^2) = \{0\}$. Thus the correspondence $\xi \mapsto \rho_\xi$ induces a map $\varphi : \xi \mapsto \varphi_\xi$ as in (45), which is easily checked to be an $\mathcal{O}$-linear map.

Suppose that $\varphi_\xi(I_\psi) = 0$. Then φ_ψ factor through $R^{\mathrm{split}}/I_\psi \cong \mathcal{O}_\psi$, and thus $[\rho_\psi] = [\rho_\xi]$, and it follows from Schur's lemma that ξ must be a coboundary. So φ is injective.

We check the surjectivity of φ. Take $\phi \in \mathrm{Hom}_{\mathcal{O}}(I_\psi/I_\psi^2, \mathcal{O}/\pi^n)$, which obviously induces (by abuse of notation) $\phi \in \mathrm{Hom}_{\mathcal{O}}(I_\psi/(I_\psi^2, \mathrm{Ker}\phi), \mathcal{O}/\pi^n)$. Take a representative ρ of $\rho^{\mathrm{split}} : G_{\mathbb{Q}} \to \mathbf{GL}_2(R^{\mathrm{split}})$ such that

$$\rho_\psi = (\rho^{\mathrm{split}} \bmod I_\psi). \tag{47}$$

Then ϕ induces a representation $\rho_\phi : G_{\mathbb{Q}} \to \mathbf{GL}_2(R^{\mathrm{split}}/(I_\psi^2, \mathrm{Ker}\phi))$ and we define

$$\xi_\phi : G_{\mathbb{Q}} \to \mathbf{GL}_2(R^{\mathrm{split}}/(I_\psi^2, \mathrm{Ker}\phi)) : g \mapsto \xi_\phi(g) = \rho_\phi(g)\rho_\psi(g)^{-1}.$$

By (47), ξ_ϕ actually take values in $1_2 + \mathrm{M}_2(I_\psi/(I_\psi^2, \mathrm{Ker}\phi))$, and we see that its image contained in $V_{U/T}[\pi^n]$ via the composite $\phi : I_\psi/(I_\psi^2, \mathrm{Ker}\phi) \hookrightarrow \mathcal{O}/\pi^n \subset \mathcal{O}_n[\varepsilon]$, where the inclusion is via the projection $1 \mapsto \varepsilon$ to the second direct summand in $\mathcal{O}_n[\varepsilon] = \mathcal{O}/\pi^n \oplus \varepsilon.\mathcal{O}/\pi^n$. It is easily checked that ξ_ϕ is actually a cocycle. By construction, we have $\varphi(\xi_\phi) = \rho_{\xi_\phi} = \phi$, which proves the surjectivity of φ. □

2.5 Greenberg's adjoint square $\mathcal{L}$-invariant

In this section we recall the cohomological definition of the $\mathcal{L}$-invariant of Greenberg particularized to the case of the adjoint square Galois representation attached to an ordinary modular form. When the modular representation ρ_f comes from the Tate module of a rational elliptic curve $E_{/\mathbb{Q}}$ with split multiplicative reduction at p, this invariant coincides with the $\mathcal{L}$-invariant of E first introduced by Mazur, Tate and Teitelbaum in terms of the Tate period q_E.

Lemma 2.7. *The torsion group* $\mathrm{Sel}^{00}(\mathbb{Q}, \mathrm{Ad}(\rho_f)_{U/T})$ *is finite.*

Proof. Let $\overline{\rho} = \rho_f \pmod{\lambda}$. Since the universal deformation ring $R_{\overline{\rho}}$ is isomorphic to a universal ordinary Hecke algebra R^o, we see that $R_k := R^o \otimes W/PW$, for $P = P_f \cap W[[\Gamma]]$, is isomorphic to a suitable Hecke algebra of weight k and finite level. Thus R_k is a finite W-module, and its W-free quotient is reduced. It follows that $\mathrm{Sel}^{00}(\mathbb{Q}, V^*_{U/T}(1))$ is isomorphic to the finite module $\Omega_{R_k/W} \otimes_{R_k} W$, where the tensor product is taken along the algebra homomorphism $R_k \to W$ associated to ρ_f. □

Corollary 2.8. *The following K-vector spaces are all null*

$$\mathrm{Sel}^s(\mathbb{Q}, \mathrm{Ad}(\rho_f)) = \mathrm{Sel}^s(\mathbb{Q}, \mathrm{Ad}(\rho_f)^*(1)) = 0. \tag{48}$$

Proof. Indeed, it follows from Lemma 2.7 that the Selmer space $\mathrm{Sel}^{00}(\mathbb{Q}, \mathrm{Ad}(\rho_f))$ vanishes, and Greenberg shows in [Gr94a] that $\dim_K \mathrm{Sel}^s(\mathbb{Q}, \mathrm{Ad}(\rho_f)) = \dim_K \mathrm{Sel}^s(\mathbb{Q}, \mathrm{Ad}(\rho_f)^*(1))$. The assertion follows from the inclusion $\mathrm{Sel}^s(\mathbb{Q}, \mathrm{Ad}(\rho_f)) \subset \mathrm{Sel}^{00}(\mathbb{Q}, \mathrm{Ad}(\rho_f))$. □

Denote by S the set of ramified primes for ρ_f, including p. The exact sequence of Poitou and Tate yields the exact sequence

$$0 \to \mathrm{Sel}^s(\mathbb{Q}, \mathrm{Ad}(\rho_f)) \to \mathrm{H}^1(G_{\mathbb{Q},S}, V) \to \prod_{q \in S} \mathrm{H}^1(\mathbb{Q}_q, V)/L^s_q \to \mathrm{Sel}^s(\mathbb{Q}, \mathrm{Ad}(\rho_f)^*(1))^*. \tag{49}$$

Thus by (48) the restriction map $\bigoplus_{q \in S} \mathrm{res}_q$ gives rise to the isomorphism

$$\mathrm{H}^1(G_{\mathbb{Q},S}, V) \cong \prod_{q \in S} \mathrm{H}^1(\mathbb{Q}_q, V)/L^s_q. \tag{50}$$

Consider the subspace H of the right hand term of (50) determined by the image of the composite map $\mathrm{H}^1(\mathbb{Q}_p, W^0) \to \mathrm{H}^1(\mathbb{Q}_p, V) \to \mathrm{H}^1(\mathbb{Q}_p, V)/L^s_p$ under the natural embedding $\mathrm{H}^1(\mathbb{Q}_p, V)/L^s_p \hookrightarrow \prod_q \mathrm{H}^1(\mathbb{Q}_q, V)/L^s_q$. This subspace is isomorphic to $W^0/W^{00} \cong K$ (cf.[Gr94a]), and we denote by $\mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}}$ the subspace of $\mathrm{H}^1(G_{\mathbb{Q},S}, V)$ given by the inverse image of H under the restriction map res_p. Thus $\mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}}$ is mapped isomorphically via res_p onto a one-dimensional subspace of $\mathrm{H}^1(\mathbb{Q}_p, W^0/W^{00}) = \mathrm{Hom}(D_p^{\mathrm{ab}}, K) \cong K^2$, the last isomorphism being induced by the map on cocycles

$$\xi \mapsto \left(\frac{\xi([u, \mathbb{Q}_p])}{\log_p(u)}, \xi([p, \mathbb{Q}_p]) \right) = (x(\xi), y(\xi))$$

for the local Artin Symbol $[\cdot, \mathbb{Q}_p] : \mathbb{Q}_p^\times \to D_p^{\mathrm{ab}}$ and any $u \in \mathbb{Z}_p^\times$ of infinite order. Consider the unramified cocycle $\kappa_{\mathrm{nr}} \in \mathrm{Hom}(D_p/I_p, K)$ mapping Frob_p to 1, the ramified cocycle κ_{cyc} induced by the restriction to D_p of $\log_p(\varepsilon_{\mathrm{cyc}}) \in \mathrm{Hom}(G_{\mathbb{Q}}, K) = \mathrm{H}^1(\mathbb{Q}, K)$. The cocycles κ_{cyc} and κ_{nr} gives rise to a basis of $\mathrm{H}^1(\mathbb{Q}_p, K)$, and we see that $(x(\xi), y(\xi))$ are the coordinates of $[\xi]$ in this basis. Let $[\xi]$ be a nonzero cocycle in $\mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}}$. We can uniquely write

$$\mathrm{res}_p([\xi]) = x(\xi) \cdot \kappa_{\mathrm{cyc}} + y(\xi) \cdot \kappa_{\mathrm{nr}},$$

and the ratio $x(\xi) : y(\xi)$ is independent of the choice of $[\xi]$. If ξ is unramified at p (i.e. $x(\xi) = 0$), it gives rise to a nonzero element in $\mathrm{Sel}^s(\mathbb{Q}, \mathrm{Ad}(\rho_f))$, contradicting Lemma 2.8. Thus we see that $\mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}}$ sits inside $\mathrm{Hom}(\mathbb{Q}_p, K) \cong K^2$ as the image of a K-linear map given by multiplication by a scalar $\mathcal{L}(\mathrm{Ad}(\rho_f)) \in K$. This scalar is the so-called $\mathcal{L}$-*invariant of Greenberg* attached to the adjoint square representation of f, and we see that it can be explicitly given by the formula

$$\mathcal{L}(\mathrm{Ad}(\rho_f)) = c([p, \mathbb{Q}_p]) \cdot \frac{\log_p(\gamma_o)}{c([\gamma_o, \mathbb{Q}_p])} \tag{51}$$

for $\gamma_o = 1 + p \in 1 + p\mathbb{Z}_p$ and any nonzero $c \in \mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}}$.

To compute the $\mathcal{L}$-invariant of $\mathrm{Ad}(\rho_f)$ one must thus construct a global cohomology class $c \in \mathrm{H}^1(G_{\mathbb{Q},S}, V)$ which is unramified at each prime $q \in S \smallsetminus \{p\}$ and such that its restriction to D_p has image contained in the upper triangular trace zero matrices, and such that one can evaluate the expression in (51). The Λ-adic Galois representation $\rho_{\mathscr{F}}$ attached to the Hida family $\mathscr{F}$ containing f, and its relation to deformation problems defined by $\overline{\rho}_f$ often allows one to explicitly produce such a cocycle. This is the approach taken by Hida to obtain a formula for $\mathcal{L}(\mathrm{Ad}(\rho_f))$ in terms of $a_p(\mathscr{F})$.

2.6 The exceptional zero conjecture: the proof by Hida

Hida obtained a generalization of a result by Greenberg and Stevens recalled in § 2.2 which applies to the adjoint square representation attached to an ordinary eigenform f. When f corresponds to a rational elliptic curve with split multiplicative reduction at p, one recovers their result. A crucial point in the arguments by Hida is made by the universality of the ordinary Hecke algebra under certain conditions, which enlightens a relationship between adjoint square Selmer groups and deformation problems attached to f.

Proposition 2.9 (Hida [H04]). *Let f be an ordinary eigenform with trivial Nebentypus. Suppose that its associated Galois representation ρ_f is p-distinguished and absolutely irreducible when restricted to $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$. Denote by $\mathscr{F} = a_p(\mathscr{F})q^n \in \mathbb{I}[[q]]$ its associated Λ-adic form. Then*

$$\mathcal{L}(\mathrm{Ad}(\rho_f)) = -2\log_p(\gamma)a_p(f)^{-1}\frac{\mathrm{d}}{\mathrm{d}X}a_p(\mathscr{F})|_{X=0}.$$

In the above statement, recall that $\mathbb{I}$ is a finite flat extension of $\mathcal{O}[[X]]$, where $\mathcal{O}$ denotes de ring of integers of a finite extension $K/\mathbb{Q}_p$ containing all the Hecke eigenvalues of f. Thus the derivative appearing in the right hand term makes sense.

Lemma 2.10. *As subspaces of $\mathrm{H}^1(G_{\mathbb{Q},S}, V)$, the following equality holds:*

$$\mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}} = \mathrm{Sel}^0(\mathbb{Q}, \mathrm{Ad}(\rho_f)).$$

Proof. Denote by $U_p^{s,0}(V)$ (respectively, $U_p^{s,00}(V)$) be the natural image of $\mathrm{H}^1(\mathbb{Q}_p, W^0)$ (respectively, $\mathrm{H}^1(\mathbb{Q}_p, W^{00})$) in $\mathrm{H}^1(\mathbb{Q}_p, V)$. Then we have the exact sequence

$$0 \to U_p^{s,0}(V)/U_p^{s,00}(V) \xrightarrow{\alpha} \mathrm{H}^1(\mathbb{Q}_p, \mathrm{Ad}(\rho_f))/U_p^{00}(V) \to \mathrm{H}^1(\mathbb{Q}_p, \mathrm{Ad}(\rho_f))/U_p^0(V) \to 0,$$

where α is the natural map, and the last term is defined so that the sequence becomes exact. Let ξ be a cocycle in $\mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}}$. By definition, $\mathrm{res}_p[\xi]$ lies in $U_p^{s,0}(V)/U_p^{s,00}(V)$, and via the embedding α we may regard it uniquely as an element in $\mathrm{Sel}^0(\mathbb{Q}, \mathrm{Ad}(\rho_f))$, since $\mathrm{Im}(\alpha) = U_p^0(V)/U_p^{00}(V)$ and $\mathrm{Sel}^{00}(\mathbb{Q}, \mathrm{Ad}(\rho_f)) = 0$ by Lemma 2.7. The lemma follows immediately from this. □

We can now proof Proposition 2.9:

Proof. Let U be the representation space of ρ_f and choose a D_p-stable $\mathcal{O}$-lattice $T \subset U$ such that the resulting matrix representation, which with a slight abuse of notation we write $\rho_f : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathcal{O})$ has the local shape

$$\rho_f|_{D_p} = \begin{pmatrix} \varepsilon_f & \beta_f \\ & \delta_f \end{pmatrix}.$$

By the universality of the pair $(R^{\mathrm{univ}}, \rho^{\mathrm{univ}})$, there is a point $P_T \in \mathrm{Spf}(R^{\mathrm{univ}})(\mathcal{O})$ corresponding to ρ_f such that $\rho_f = \rho^{\mathrm{univ}} \pmod{P_T}$.

Since $\mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}} = \mathrm{Sel}^0(\mathbb{Q}, \mathrm{Ad}(\rho_f))$ by Lemma 2.10, it follows from Proposition 2.6 that the one-dimensional subspace $\mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}} \subset \mathrm{H}^1(G_{\mathbb{Q},S}, V)$ is isomorphic to the tangent space of $R^{\mathrm{univ}}_{\overline{\rho}_f}$ at P_T:

$$\mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}} \cong \mathrm{Hom}_K(P_T/P_T^2, \mathrm{Ad}(\rho_f)).$$

Recall that the isomorphism is explicitly given by the following correspondence. A cocycle ξ naturally defines an infinitessimal deformation $\rho_\xi : G_{\mathbb{Q}} \to \mathbf{GL}_2(K[\varepsilon])$ of ρ_f by the formula

$$\rho_\xi(\sigma) = \rho_f(\sigma) + \xi(\sigma)\rho_f(\sigma) \cdot \varepsilon;$$

the cohomology relation for ξ corresponds to the strict equivalence relation for ρ_ξ, and the set of strict equivalence classes of such infinitessimal deformations naturally corresponds to the tangent space of $\mathrm{Spf}(R_{P_T})$ at P_T.

Let κ be a generator of $\mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}}$ and write

$$\kappa(\sigma) = \begin{pmatrix} -a(\sigma) & b(\sigma) \\ & a(\sigma) \end{pmatrix} \qquad \text{for } \sigma \in D_p.$$

The $\mathcal{L}$-invariant of $\mathrm{Ad}(\rho_f)$ is then given by the formula

$$\mathcal{L}(\mathrm{Ad}(\rho_f)) = a([p,\mathbb{Q}_p]) \cdot \frac{\log_p(\gamma_o)}{a([\gamma_o,\mathbb{Q}_p])} \tag{52}$$

for the topological generator $\gamma_o \in \Gamma = 1 + p\mathbb{Z}_p$. Note that a can not be identically on I_p, since if $\kappa|_{D_p}$ where unramified modulo nilpotent matrices, then it would give rise to a nonzero element in $\mathrm{Sel}^s(\mathbb{Q}, \mathrm{Ad}(\rho_f))$, contradicting Corollary 2.8.

Since $(R^{\mathrm{univ}}_{P_T}, \rho^{\mathrm{univ}}) \cong (\Lambda_{P_f}, \rho_{\mathscr{F}})$ by hypothesis, we see that $\frac{\mathrm{d}}{\mathrm{d}X}\rho_{\mathscr{F}}$ gives rise to a generator of $\mathrm{H}^1_p(G_{\mathbb{Q},S}, V)^{\mathrm{cyc}}$. Thus for all $\sigma \in D_p$ the identity

$$\kappa(\sigma)\rho_f(\sigma) = C \cdot \frac{\mathrm{d}}{\mathrm{d}X}\rho_{\mathscr{F}}(\sigma) \tag{53}$$

holds for a certain constant $C \in K^\times$. Writing

$$\rho_{\mathscr{F}}(\sigma) = \begin{pmatrix} \varepsilon_{\mathscr{F}}(\sigma) & \beta_{\mathscr{F}}(\sigma) \\ & \delta_{\mathscr{F}}(\sigma) \end{pmatrix}$$

for $\sigma \in D_p$, a comparison of the lower right terms in both sides of (53) after expanding the product in the left hand side yields the formulae:

$$a([p,\mathbb{Q}_p]) \cdot \delta_f([p,\mathbb{Q}_p]) = C \cdot \frac{\mathrm{d}}{\mathrm{d}X}\delta_{\mathscr{F}}([p,\mathbb{Q}_p])\Big|_{X=0}; \tag{54}$$

$$a([\gamma_o,\mathbb{Q}_p]) \cdot \delta_f([\gamma_o,\mathbb{Q}_p]) = C \cdot \frac{\mathrm{d}}{\mathrm{d}X}\delta_{\mathscr{F}}([\gamma_o,\mathbb{Q}_p])\Big|_{X=0}. \tag{55}$$

We have $\delta^o_{\mathscr{F}}([p,\mathbb{Q}_p]) = a_p(\mathscr{F})$. Since the character $\gamma^s \mapsto (1+X)^{s/2}$ interpolates the characters $\varepsilon_{\mathrm{cyc}}^{m(p-1)}$ for all integers m, and since $\varepsilon_{\mathrm{cyc}}([p,\mathbb{Q}_p]) = 1$, we have $\delta_{\mathscr{F}}([p,\mathbb{Q}_p]) = \delta^o_{\mathscr{F}}([p,\mathbb{Q}_p]) \cdot (1+X)^{s/2}([p,\mathbb{Q}_p]) = a_p(\mathscr{F})$. Thus we see from (54) that

$$a([p,\mathbb{Q}_p]) = C \cdot a_p(f)^{-1} \cdot \frac{\mathrm{d}a_p(\mathscr{F})}{\mathrm{d}X}\Big|_{X=0}, \tag{56}$$

given that $\delta_f([p,\mathbb{Q}_p]) = \delta^o_{\mathscr{F}}([p,\mathbb{Q}_p])|_{X=0} = a_p(\mathscr{F})|_{X=0} = a_p(f)$.

Since $\det(\rho^o_{\mathscr{F}})$ is the universal deformation of $(\det(\rho_f) \bmod \mathfrak{m}) : G_{\mathbb{Q},S} \to (\mathcal{O}/\mathfrak{m})^\times$, we see that

$$\rho_{\mathscr{F}} = \rho^o_{\mathscr{F}} \otimes (1+X)^{-s/2} : G_{\mathbb{Q},S} \to \mathbf{GL}_2(\Lambda_P),$$

where $\gamma_o^s \mapsto (1+X)^s : G_{\mathbb{Q},\{p\}} \to \Lambda^\times$ is the universal deformation of the trivial character, sending σ to $(1+X)^s$ if $\varepsilon_{\mathrm{cyc}}(\sigma) = \gamma_o^s$. It follows that $\delta_{\mathscr{F}}([\gamma_o^s,\mathbb{Q}_p]) = (1+X)^{-s/2}$ since $\delta^o_{\mathscr{F}}$ is unramified at p. Therefore $\frac{\mathrm{d}}{\mathrm{d}X}\delta_{\mathscr{F}}([\gamma_o^s,\mathbb{Q}_p]) = -s/2$ so that $\frac{\mathrm{d}}{\mathrm{d}X}\delta_{\mathscr{F}}([\gamma_o,\mathbb{Q}_p]) = -1/2$. Thus we see from (55) that

$$a([\gamma_o,\mathbb{Q}_p]) = C \cdot -\frac{1}{2}, \tag{57}$$

given that due to the unramifiedness of $\delta^o_{\mathscr{F}}$ we have

$$\delta([\gamma_o,\mathbb{Q}_p]) = \delta_{\mathscr{F}}([\gamma_o^s,\mathbb{Q}_p])|_{X=0} = (\delta^o_{\mathscr{F}} \otimes (1+X)^{-s/2})([\gamma_o,\mathbb{Q}_p])|_{X=0} = 1.$$

Now the formula for $\mathcal{L}(\mathrm{Ad}(\rho_f))$ is obtained from (56) and (57) using the definition (52). □

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