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Mean field dynamics of the bounded confidence model in opinion dynamics

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Introducción

Algunos de los modelos sobre dinámicas de opinión ([14], [3], [26], [19], [32], [29]) están basados en valores binarios (discretos) en vez de valores continuos. Esto suele llevar a regímenes atractores que muestran uniformidad de opiniones (todo ceros o todo unos). De hecho, estos modelos no son suficientes para modelar escenarios como la red social de camioneros interesados en la calidad de la comida de un restaurante de carretera o las calificaciones hechas por los críticos de cine sobre las nuevas películas a estrenar. En ambos casos se requiere un espectro de opiniones continuo. Por ejemplo, el modelo continuo se usa ampliamente en política ([11]), donde las personas se sitúan de acuerdo a cuán de izquierda (o de derecha) sus opiniones son, lo cual nos lleva a hacernos preguntas tan naturales como si el resultado de unas elecciones puede ser previsto con antelación.

De entre los diferentes modelos existentes, uno de los más populares es el modelo de confianza acotada introducido por Deffuant et al. ([9]) donde los individuos se escogen al azar e interactúan si sus opiniones difieren en más de un umbral de desviación. Este modelo ha sido estudiado extensamente: desde el punto de vista de la topología abarcando grafos aleatorios ([12]) a topología de rejilla ([33], [34]), desde el punto de vista dimensional puede ser generalizado a opiniones vectoriales multidimensionales ([25], [34]). También hay estudios ([33]) en los que el umbral de desviación es diferente dependiendo del individuo (hay individuos más tolerantes que otros) o en los que la interacción se realiza promediando sobre todas las parejas potenciales de individuos que se toleran unos a otros ([10], [15]). En nuestro caso, consideraremos la red como completa (totalmente conectada) y el valor del umbral de desviación constante en cada individuo. Trabajaremos en dimensión 1, aunque el modelo (y la posterior aproximación a campo medio) puede ser fácilmente generalizado a un espacio multidimensional.

Los sistemas de reputación han emergido últimamente debido a la necesidad de medir la confianza sobre los usuarios mientras se realizan transacciones a través de Internet. Ejemplos populares que utilizan sistemas de reputación son e-Bay ([28]) o Bizrate ([31]). Además, el modelo introducido por Le Boudec et al. ([20], [7]) para representar la evolución de la confianza y los efectos potenciales que un grupo de atacantes puede tener sobre el sistema es una generalización del modelo de confianza acotada, en particular para el caso en el que no hay atacantes ni observaciones directas y la evolución del sistema radica únicamente en la interacción a través de los diferentes individuos.

La aproximación de campo medio es una aproximación determinista para el caso en el que el número de individuos es suficientemente grande. Se ha usado en muy diferentes contextos tales como conexiones TCP ([30],[4]), flujos HTTP ([5]), reparto de ancho de banda en transferencia de archivos ([18]), redes móviles ([8]), enjambres de robots ([21]), sistemas de transporte ([2]) y sistemas de reputación ([20], [24], [23]).

Nuestras contribuciones son las siguientes:

• Demostramos que en el sistema probabilístico, los momentos de orden k de los valores de reputación son decrecientes con el tiempo independientemente del valor de k. La demostración se basa en la desigualdad de Muirhead. Particularizamos este resultado para los momentos de orden 1 y 2.

• Demostramos que en el sistema probabilístico, los momentos de orden k son estacionarios (constantes) con probabilidad 1 si y solo si se alcanza el consenso.

• Demostramos que en el sistema probabilístico, los usuarios tienden a agruparse en clusters. Tras un tiempo finito, el número de clusters permanece constante. Dentro de un cluster, todas las opiniones tienden a la opinión media del cluster.

• Demostramos que el número de clusters a los que se converge para tiempos suficientemente grandes está acotado.

• Generalizamos estos dos últimos resultados a funciones de interacción entre individuos más generales.

• Conjeturamos que cuando el número de individuos participantes en el sistema es grande, el modelo converge a un régimen espacial de campo medio determinista que está caracterizado de manera única por una ecuación integro-diferencial (EID). Demostramos explícitamente algunos subcasos.

• Demostramos que la EID presenta existencia y unicidad de soluciones en el espacio de medidas con signo con la norma de la variación total. La demostración es similar a la clásica del teorema de Picard-Lindelöf.

• Utilizando la EID, demostramos las mismas propiedades que en el modelo probabilístico, esto es, que los momentos de orden k son decrecientes con el tiempo, donde aquí también particularizamos los casos k = 1 y k = 2, y que se converge a una distribución discreta, habiendo convergencia independientemente de la condición inicial. Además, podemos encontrar una condición inicial continua tal que para toda distribución discreta dada, tenemos convergencia a dicha distribución discreta. Estas últimas demostraciones se basan en el teorema de la Convergencia Dominada y en la construcción explícita de la condición inicial.

• Demostramos que partiendo de una condición inicial simétrica, la función es simétrica para todo tiempo.

• Acotamos el crecimiento de la función y damos cotas explícitas para todo tiempo T. La demostración se basa en la descomposición de la región de integración en diversos intervalos y la resolución de una desigualdad donde las variables son las medidas de dichos intervalos.

• Acotamos el crecimiento de las derivadas de cualquier orden de la función. Obtenemos que no hay blow-up en tiempo finito (esto es, que la función y todas sus derivadas son finitas para todo tiempo finito).

• Establecemos una cota sobre el parámetro Δ, para distinguir de forma rápida (sin realizar simulaciones) las situaciones en las que hay consenso (una opinión al final) de las que hay polarización (más de una). Esto tiene aplicación práctica puesto que basta con conocer la condición inicial y el valor de los parámetros para ser capaces garantizar consenso (todas las opiniones coincidirán con el paso del tiempo). La demostración usa técnicas de análisis convexo y la propiedad de que si hubiera polarización, las opiniones han de estar separadas Delta o más. Particularizamos la cota para diversos escenarios que representamos al realizar el análisis numérico.

• Desarrollamos nuestro propio método numérico compuesto para resolver la EID y lo implementamos en C++ y Matlab. Puede ser usado como una herramienta rápida de simulación, en particular cuando la población es muy grande. Debemos tener en cuenta que hoy en día el número de usuarios en redes sociales como Facebook supera los 250 millones ([1]) y que Facebook es el cuarto "país" más poblado del mundo.

• Demostramos el error del método numérico y su coste asintótico en tiempo.

• Utilizando nuestro método numérico, observamos fases de transición mientras variamos el parámetro Δ para una cierta condición inicial. Las fases consisten en el número de valores diferentes de la distribución discreta límite. Realizamos experimentos con diferentes condiciones iniciales: Uniforme y Beta(1,6) y diferentes valores del parámetro w. Si comparamos nuestros resultados para $w = 0.5$ usando el modelo determinista con los resultados en [9] con el modelo probabilístico, los intervalos de Δ en los que se tiene una alta probabilidad de convergencia a n Diracs se corresponden con los mismos intervalos en los que obtenemos convergencia a n Diracs, lo que sugiere que la aproximación de campo medio es suficientemente buena.

• Modelizamos el escenario de una fusión entre dos empresas, clasificando a los trabajadores como indecisos o extremistas de acuerdo con su opinión respecto a la nueva compañía. Obtenemos que, en el peor caso, teniendo un 21% de trabajadores indecisos es suficiente para unir a las dos facciones de extremistas y conseguir consenso y que hay una transición brusca del centro de masas de la distribución en [0,0.5] entre los dos estados (consenso – polarización) finales posibles, pasando esta de 0.1 a 0.5 súbitamente.

La estructura del trabajo es la siguiente. En la sección 2 resumimos toda la notación utilizada en el trabajo. En la sección 3 analizamos el modelo probabilístico para un número finito de individuos. En la sección 5 aplicamos la aproximación de campo medio y repetimos el análisis para el sistema determinista cuando el número de individuos tiende a infinito. Las demostraciones entre los sistemas finito e infinito se encuentran en la sección 4. Presentamos una técnica para acotar el valor crítico de Δ, el cual representa la transición de la distribución límite entre una y dos opiniones en la sección 5.7. El método numérico y las demostraciones de su orden de convergencia y complejidad asintótica se pueden encontrar en la sección 6 y el código fuente en el apéndice A. Finalmente, en la sección 7 presentamos simulaciones que ilustran el comportamiento del sistema: presentamos las diferentes funciones límite a las que la distribución puede tender y también estudiamos diversos diagramas de bifurcación, tanto unidimensionales como bidimensionales, para algunas condiciones iniciales.

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Abstract

In this master thesis we consider a closed social network in which the peers have a reputation value about some subject. We study a probabilistic model for this type of network and prove convergence to a finite set of opinions. Moreover, we give a framework to study the case when the number of peers tends to infinity: a mean-field approach. Under this approach, the model is deterministic. We are able to prove the same convergence results, now to a discrete distribution. We also prove that there is no blow-up in finite time and that there is existence and uniqueness of solution for the equation governing the motion of the deterministic system. We develop a technique for bounding the critical parameter that distinguishes between having one or several different reputation values when time goes to infinity. We also create a numerical method to simulate the deterministic equation and prove its order of convergence and complexity. We show bifurcations for various initial conditions and find the critical value of the parameters for those situations.
Keywords: Social networks, reputation, model, mean field, partial integro-differential equations, numerical methods, nonlinear systems

1 Introduction

Some of the models about opinion dynamics ([14], [3], [26], [19], [32], [29]) are based on binary values rather than continuous. This often leads to attractors that display uniformity of opinions. Indeed, those models are not enough to model other scenarios like the social network of truck drivers interested in the quality of food of a highway restaurant or the critics' ratings about the new opening movies and it is required to have a continuous spectrum of opinions. For example, the continuous model is widely used in politics ([11]), where people are positioned according to how left-(or right-)wing their opinions are.

Among the different models that exist, one of the most popular is the bounded confidence model introduced by Deffuant et al. ([9]) where the peers are selected randomly and they interact between them if their opinions differ by less than a deviation threshold. This model has been extensively studied: from the point of view of the topology ranging from random graphs ([12]) to lattice topology([33], [34]), from the dimensional point of view it can also be generalized to multidimensional vector opinions ([25], [34]). There are also studies ([33]) in which the deviation threshold is different depending on the peer (i.e. there are people more tolerant than others) or in which the interaction is done and averaged among all the potential pairs of peers that tolerate each other ([10], [15]). In our case, we will consider the network as a fully connected one and the deviation test constant throughout every peer. We will work in dimension 1, although the model (and the further mean-field approach) can be easily generalized to a multidimensional space.

Reputation systems have lately emerged due to the necessity to measure trust about users while doing transactions over the internet. Popular examples that use reputation systems are e-Bay ([28]) or Bizrate ([31]). Moreover, the model introduced by Le Boudec et al. ([20], [7]) for representing the evolution of the trust and the potential effects that a group of liars might have while trying to attack the system is a generalization of the bounded confidence model, in particular for the case where there are no liars nor direct observations and the evolution of the system is only carried by interaction throughout the different peers.

The mean-field approach is a deterministic approximation for the case when the number of different peers is big. It has been used in many different contexts such as TCP connections ([30],[4]), HTTP flows ([5]), bandwidth sharing between streaming and file transfers ([18]), mobile networks ([8]), robot swarms ([21]), transportation systems ([2]) and reputation systems ([20], [24], [23]).

We make the following contributions.

- We prove that in the probabilistic system, the users tend to group into clusters. After some finite time, the number of clusters stays constant. Within one cluster, all opinions tend to the mean opinion of the cluster.

- We conjecture that as the number of nodes participating in the system becomes large, this model converges to a deterministic spatial mean-field regime that is uniquely characterized by a partial integro-differential equation (PIDE). We give explicit proofs for some subcases.

- Using the PIDE, we prove the same properties than in the probabilistic model, namely that it converges to a discrete distribution.

• We create a numerical method to solve the PIDE. It may be used as a fast simulation tool, in particular when the number of users is very large. We should keep in mind that nowadays the number of users in social networks such as Facebook exceeds 250 million ([1]). Using our numerical method, we observe transition phases while varying Δ (cf. section 2) for some fixed initial condition. The phases consist on the number of different values of the limit discrete distribution.

• We model the scenario of a company fusion, categorizing the workers into "undecided" and "extremists". We obtain that having 20% of the workers "undecided" is enough to unite the two factions of extremists and achieve consensus.

• We establish a bound on the parameter Δ, in order to determine if there is consensus or not, under the assumption of symmetric initial conditions.

The structure of the thesis is the following. In section 2 we summarize the notation used in the thesis. In section 3 we analyze the probabilistic model for a finite number of peers. In section 5 we apply a mean-field approach and we repeat the analysis for the deterministic system when the number of peers tends to infinity. The proofs between the finite and infinite system are done in section 4. We present a technique for bounding the critical value of Δ which represents the transition of the limit distribution between one Dirac and two Diracs in section 5.7. The numerical method and the proofs for its order of convergence and complexity can be found in section 6 and the code in the appendix. Finally, in section 7 we present simulations that illustrate the behavior of the system: we present some of the different limit functions that the distribution might tend to and we also study the bifurcation diagrams for some initial conditions under certain scenarios.

2 Notation list

Symbol	Meaning
w	confidence factor
Δ	deviation threshold
$x_i^N(t)$	State of the i-th peer after time t in a N peer population
$f(x,t)$	Probability distribution of the reputation values
$\mu_k^N(t)$	k-th moment of the discrete distribution in a N peer population
$x_{(i)}(t)$	Reputation value of the ordered peer i
$C_{i,j}(t)$	Cluster that contains at time t, ordered peers i to j, inclusive
$F(x,t)$	Cumulative distribution of $f(x,t)$
$\mu_k(t)$	k-th moment of the continuous distribution
$f^e(x,t)$	Approximation by the Euler method of $f(x,t)$
$\nu_e^t(x)$	Lebesgue-Stieltjes measure associated to $f^e(x,t)$
$f^r(x,t)$	Piecewise constant approximation of $f^e(x,t)$
$\nu_r^t(x)$	Lebesgue-Stieltjes measure associated to $f^r(x,t)$
$\mu_s^t(x)$	Lebesgue-Steltjes measure associated to the continuation of $f^r(x,s)$ at time t
Δt	Discretization step in the temporal domain
I	Number of intervals of the piecewise constant functions

Table 1: Notation list

3 Finite N model

In this section, we will use the same model as in [9]. In this model, we have a population of N peers, having peer i a reputation value $x_i^N \in [0,1]$. The object of study is the (discrete) probability distribution $f(x,t)$, defined this way:

$$\Pr\{f(x,t)=k\} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{x_i^N(t)=k\}}$$

The dynamics of the model are the following: at any time unit, two peers i and j are selected at random uniformly. Given their reputation values $x_i(t)$ and $x_j(t)$, the system evolves like this:

We first make a test on the deviation between the reputation values of the two peers. On the one hand, if $|x_i^N(t) - x_j^N(t)| > \Delta$, $x_i^N(t+1) = x_i^N(t), x_j^N(t+1) = x_j^N(t)$. This reflects the case where the distance between the peers' beliefs is greater than the *deviation threshold* Δ and therefore the interaction produces no effect on any of them. On the other hand, if $|x_i(t) - x_j(t)| \leq \Delta$, we have that:

$$x_i^N(t+1) = wx_i^N(t) + (1-w)x_j^N(t)$$
$$x_j^N(t+1) = wx_j^N(t) + (1-w)x_i^N(t)$$

In this situation, the reputation values are close enough and each peer influences the other's belief by means of the *confidence factor* $w \in [0,1]$. Large values of w mean that the peers trust very much their own beliefs in comparison to the new information given by the other interacting peer. A special case is when $w = \frac{1}{2}$, in which both peers will have the average value after the transition.

We will prove that all the k-th moments are non-increasing with time and that the system converges with probability 1 to a (discrete) distribution that takes at most $\lceil \frac{1}{\Delta} \rceil$ different values separated by at least Δ. Although the limit function could change, the result always holds regardless of N, w and the initial condition $f(x,0)$. We will refer to this situation as *consensus*, because all the peers have an agreement about their reputation value. Moreover, we will speak about *partial consensus* when there is more than one different value and the values are separated more than Δ apart, and we will speak about *total consensus* when every peer has the same reputation value. From now on, we will drop the superindex N while speaking about $x_i^N(t)$.

It is worth noticeable that Dittmer and Krause ([10], [17]) had similar results with a similar, but deterministic, model. The technique used in both our proofs and theirs might resemble in some cases.

3.1 Moments

Proposition 3.1 *For any time t, let $\mu_k^N(t) = \frac{1}{N^k}\sum_{i=1}^{N} x_i(t)^k$ be the k-th moment. If $t_1 \leq t_2$, then $\mu_k^N(t_1) \geq \mu_k^N(t_2)$*

Proof: We will prove that the sum doesn't increase with any transition independently of the state of the system. Let us suppose that peer i interacts with peer j at some time t. We distinguish two cases:

• If $|x_i(t) - x_j(t)| > \Delta$, then $x_i(t+1) = x_i(t)$ and $x_j(t+1) = x_j(t)$, which implies $\mu_k^N(t+1) = \mu_k^N(t)$ as the reputation values for the rest of the peers remain unchanged.

• If $|x_i(t) - x_j(t)| \leq \Delta$, then:

$$\left\{ \begin{array}{l} x_j(t+1) = w \cdot x_i(t) + (1-w) \cdot x_j(t) \\ x_i(t+1) = w \cdot x_j(t) + (1-w) \cdot x_i(t) \end{array} \right\}$$

Therefore:

$$\mu_k^N(t+1) - \mu_k^N(t) = \frac{1}{N^k}\left(x_i(t)^k + x_j(t)^k - (wx_i(t) + (1-w)x_j(t))^k + (wx_j(t) + (1-w)x_i(t))^k\right)$$

$$= \sum_{p=0}^{k} a_p x_i(t)^p x_j(t)^{k-p} \quad (1)$$

where:

$$a_{k-p} = a_p = \begin{cases} \frac{1}{N^k}\left(\binom{k}{p}(1-w)^p w^{k-p} + \binom{k}{k-p}(1-w)^{k-p}w^p\right) & \text{if } p \neq 0, k \\ \frac{1}{N^k}\left(\binom{k}{0}(1-w)^k + \binom{k}{k}w^k - 1\right) & \text{else} \end{cases}$$

We now have that:

$$\sum_{p=0}^{k} a_p = \frac{1}{N^k}2((1-w)^k + w^k - 1) + \frac{1}{N^k}\sum_{p=1}^{k-1}\left[\binom{k}{i}(1-w)^p w^{k-p} + \binom{k}{k-i}(1-w)^{k-p}w^p\right]$$

$$= -\frac{2}{N^k} + \frac{1}{N^k}((1-w)+w)^k + \frac{1}{N^k}(w+(1-w))^k = 0 \Rightarrow -a_0 - a_k = \sum_{i=1}^{k-1} a_i$$

Applying repeatedly Muirhead's inequality [22] and the fact that $a_0 \leq 0$ and $a_p = a_{k-p}$, we have the following results:

$$a_p x_i(t)^p x_j(t)^{k-p} + a_{k-p}x_i(t)^{k-p}x_j(t)^p \leq a_p x_i(t)^k + a_{k-p}x_j(t)^k \quad \forall\, 0 < p < k$$

Adding all the inequalities we get that:

$$\sum_{p=1}^{k-1}\left[a_p x_i(t)^p x_j(t)^{k-p} + a_{k-p}x_i(t)^{k-p}x_j(t)^p - a_p x_i(t)^k - a_{k-p}x_j(t)^k\right]$$

$$= 2\sum_{p=1}^{k-1} a_p x_i(t)^p x_j(t)^{k-p} + 2a_k x_i(t)^k + 2a_0 x_j(t)^k = 2\sum_{p=0}^{k} a_p x_i(t)^p x_j(t)^{k-p} \leq 0$$

$$\Rightarrow \mu_k^N(t+1) - \mu_k^N(t) \leq 0$$

therefore being $\mu_k^N(t)$ non-increasing. □

We will now show particular cases of this result:

Corollary 3.2 *The mean of $f(x,t)$ is constant throughout time.*

Proof: Particularizing equation (1) to the case $k = 1$, we get that:

$$a_0 = a_1 = 1 - w + w - 1 = 0$$

Therefore, $\mu_k^N(t+1) - \mu_k^N(t) = 0$, regardless of the $x_i(t)$ and $x_j(t)$ chosen in the transition. □

This fact shouldn't be surprising, as there is no extra information introduced in the system. Therefore, the average belief should remain the same.

Corollary 3.3 *The variance of $f(x,t)$ is a non-increasing function of t. Moreover, if peers i and j interact and $x_i(t)$ and $x_j(t)$ pass the deviation test, the drop on the variance is given by $\frac{1}{N^2}2w(1-w)(x_i(t)-x_j(t))^2$. Otherwise the variance is the same.*

Proof: The first part of the statement follows automatically from 3.1. For the second part, we should notice that:

$$\sigma^N(t+1)^2 - \sigma^N(t)^2 = \mu_2^N(t+1) - \mu_2^N(t) - \mu_1^N(t+1)\mu_1^N(t+1) + \mu_1^N(t)\mu_1^N(t) = \mu_2^N(t+1) - \mu_2^N(t)$$

where in the last equality we have used corollary 3.2. Particularizing equation (1) to the case $k = 2$, we get that:

$$a_0 = a_2 = (1-w)^2 + w^2 - 1 = -\frac{1}{N^2}2w(1-w)$$

$$a_1 = \frac{1}{N^2} 4w(1-w)$$

We can easily factor now:

$$\mu_2^N(t+1) - \mu_2^N(t) = \frac{1}{N^2}\left(4w(1-w)x_i(t)x_j(t) - 2w(1-w)x_i(t)^2 - 2w(1-w)x_j(t)^2\right) = -\frac{1}{N^2} 2w(1-w)(x_i(t) - x_j(t))^2$$

Trivially, if the peers don't pass the deviation test, there is no change in the variance. □

Proposition 3.4 *With probability 1, all k-th moments ($k > 1$) are non-increasing until consensus (partial or total) is reached. Moreover, the consensus condition is necessary and sufficient to determine that the moments won't decrease any more.*

Proof: We will first prove that it is a necessary condition. Suppose the contrary: there is no partial nor total consensus and the k-th moments are non-decreasing. By proposition 3.1 they must be constant. As there is no consensus, there exists a pair of peers (i, j) such that $|x_i(0) - x_j(0)| < \Delta$ and $x_i(0) \neq x_j(0)$. If the moments are constant, that means that equation (1) is an equality, which only occurs when the peers that meet don't pass the deviation test or are equal, or when $k = 1$. But if there are only meetings where the meeting peers don't pass the deviation test or are equal, the meeting between peers i and j never occurs as their values have not been modified through time and therefore the meeting would result in a decrement of the moments. These series of events only happen with probability 0. Therefore, the contrary occurs with probability 1, as claimed.

For the sufficient part, if there is partial or total consensus, then every pair of peers either doesn't pass the deviation test or has the same value. This would imply that equation (1) is indeed an equality in case of interaction. Therefore, the moments remain constant regardless of the pair chosen to interact. □

This corollary suggests the idea of convergence, as the moments have always a finite limit (they are bounded, decreasing sequences) and we have established a necessary and sufficient condition for them to be stationary.

3.2 Convergence to Dirac

We define a *cluster* $C_{i,j}(t)$ as the set $\{x_{(i)}(t), \ldots, x_{(j)}(t)\}$ (the set that contains i-th ordered element up to the j-th) such that $|x_k - x_{k+1}| \leq \Delta \; \forall \; i \leq k \leq j-1$ and is maximal (i.e, we can't extend it beyond i or j). In other words, a cluster is a maximal group of peers, such that any peer can pass the deviation test with its neighbors.

Proposition 3.5 *A cluster $C_{i,j}(t)$ can either split into several clusters or remain grouped, but never grow, as t increases. Moreover, the two boundary peers i and j can't decrease or increase respectively.*

Proof: We will proceed by induction on t. We will prove that $C_{i,j}(t)$ is maximal at $t+1$ given that it's maximal at t. Suppose the contrary, that $C_{i,j}(t)$ can grow beyond i. That means that $|x_{(i)}(t+1) - x_{(i-1)}(t+1)| \leq \Delta$. This can happen either because $x_{(i)}(t)$ has decreased, because $x_{(i-1)}(t)$ has increased, or both. We will show that it is impossible that any of these possibilities occurs. $x_{(i)}(t)$ can only decrease if and only if there is an interaction between a peer with order index greater or equal than i and another peer with order index less or equal than $i-1$. Any other interactions don't affect $x_{(i)}(t)$ because after any interaction, both peers will have a rating that lies between the two original ones. However, no two peers such that one has order index $p \geq i$ and the other has order index $q \leq i-1$ can pass the deviation test because $|x_{(p)}(t) - x_{(q)}(t)| \geq |x_{(i)}(t) - x_{(i-1)}(t)| > \Delta$ by hypothesis. Analogously, $x_{(i-1)}(t)$ can only increase if such an interaction takes place and therefore it's impossible that $C_{i,j}(t)$ grows beyond i. For the case that $C_{i,j}$ grows beyond j, the proof is similar. Therefore, the cluster can't grow further. □

Proposition 3.6 *There exists a time T_{cl} after which the number of clusters remains the same for all $t > T_{cl}$.*

Proof: Let $C(t)$ be the number of clusters at time t. We know by proposition 3.5 that $C(t)$ is an increasing sequence. However, $C(t)$ is bounded because there must be a separation of at least Δ between clusters, making the number of clusters bounded by $\left\lceil \frac{1}{\Delta} \right\rceil$. This means that the sequence $C(t)$ is convergent to a limit C, so that there exists some T_{cl} such that for any $t > T_{cl}$, $|C(t) - C| < \frac{1}{2}$. As $C(t)$ only takes integer values, $C(t) = C$ for any $t > T_{cl}$. □

Proposition 3.7 *With probability 1, any initial condition $f(x, 0)$ converges to a distribution x_i of at most $\left\lceil \frac{1}{\Delta} \right\rceil$ different values that satisfy $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} x_i(0)$.*

Proof: Let T_{cl} the time after which the number of clusters remains the same (which exists and is finite by proposition 3.6). Let $C_{i,j}$ be one of the clusters that won't break after T_{cl}. We can assume that $j \neq i$. Otherwise the result holds trivially. With probability 1, we will have an infinite number of iterations such that both peers are in $C_{i,j}$ and every peer in $C_{i,j}$ will appear eventually in those iterations with probability 1. We can restrict ourselves now to the subsequence of times t_k in which both interacting peers are in $C_{i,j}$. Abusing of notation we will call this sequence t. By proposition 3.5, $x_{(j)}(t)$ and $x_{(i)}(t)$ are decreasing and increasing functions of t respectively. Hence, $x_{(j)}(t) - x_{(i)}(t)$ is a decreasing function of t and is clearly bounded by 0. Therefore it converges to some limit $L \geq 0$. We will prove that $L = 0$.

Let us suppose that $L \neq 0$. Then, for any t, we can select an index $i(t)$ such that $\Delta > x_{(i(t)+1)}(t) - x_{(i(t))}(t) \geq \frac{L}{j-i}$. With probability 1, the pair $(i(t), i(t) + 1)$ will meet infinitely often. Therefore, there exists a subsequence t_k such that $\mu_2^N(t_k+1) - \mu_2^N(t_k) \leq -\frac{1}{N^2} 2w(1-w) \left(\frac{L}{j-i}\right)^2$. This would mean that $\mu_2^N(t)$ is not a Cauchy sequence and therefore not convergent. But $\mu_2(t)$ is decreasing and bounded by 0, thus convergent. Contradiction. The contradiction comes from the supposition that $L \neq 0$.

As $L = 0$, $x_{(i)}(t) \to x_{(j)}(t) \to M$. Every element of the cluster is bounded by $x_{(i)}(t)$ and $x_{(j)}(t)$, therefore it also converges to M. Doing this for every different cluster $C_{i,j}$ we prove that every cluster converges to a single value. The number of different values is bounded by the maximum number of clusters, which was proven in proposition 3.6. The condition on the sum can be easily obtained from 3.2. □

We can generalize to a more general class of transition functions to obtain the following result:

Proposition 3.8 *If the model for the transitions is given by:*

$$x_i^N(t + 1) = x_i^N(t) + K(x_j^N(t) - x_i^N(t))$$
$$x_j^N(t + 1) = x_j^N(t) + K(x_i^N(t) - x_j^N(t))$$

where $K(z)$ is a continuous odd function such that $K(z) = wz \;\forall\; z \in [-\Delta, \Delta]$ and its support is $[-\Delta - \varepsilon, \Delta + \varepsilon]$. We will also assume that $|K(z)| > |z| \;\forall z \;\in [-\Delta - \varepsilon, \Delta + \varepsilon] - \{0\}$ and that $K(z)$ is strictly positive in $(0, \Delta + \varepsilon)$. Then, with probability 1, any initial condition $f(x, 0)$ converges to a distribution x_i of at most $\left\lceil \frac{1}{\Delta + \varepsilon} \right\rceil$ different values that satisfy $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} x_i(0)$.

Proof: We can first prove that the variance is non-increasing regardless of which peers are chosen to interact. If the peers are more than $\Delta + \varepsilon$, then they remain with the same reputation value and the variance does not increase. Otherwise, if they are at a distance less or equal than Δ, the variance decreases by $\frac{1}{N^2} 2w(1 - w)(x_i(t) - x_j(t))^2$, as it was proved in proposition 3.3. Finally, if the peers are at a distance which is between Δ and $\Delta + \varepsilon$, the variance increases by:

$$\mu_2(t + 1) - \mu_2(t) = \frac{1}{N^2}[-x_i(t)^2 - x_j(t)^2 + (x_i(t) + K(x_j(t) - x_i(t)))^2 + (x_j(t) + K(x_i(t) - x_j(t)))^2]$$

$$= \frac{1}{N^2}[2x_i(t)K(x_j(t) - x_i(t)) + 2x_j(t)K(x_j(t) - x_i(t)) + K(x_j(t) - x_i(t))^2 + K(x_i(t) - x_j(t))^2]$$

Assuming without loss of generality that $x_i(t) \geq x_j(t)$, we have that:

$$= \frac{2K(x_j(t) - x_i(t))}{N^2}[(x_i(t) - x_j(t) + K(x_j(t) - x_i(t)))]$$

As $x_i(t) \geq x_j(t)$, $K(x_j(t) - x_i(t)) \leq 0$ and:

$$x_i(t) - x_j(t) + K(x_j(t) - x_i(t)) \geq 0 \Leftrightarrow K(x_j(t) - x_i(t)) \geq x_j(t) - x_i(t)$$

which is clearly true by the assumptions. Therefore, $\mu_2(t+1) - \mu_2(t) \leq 0$, regardless of the peers involved in the transition.

We will consider clusters such that the difference between two adjacent peers is strictly less than $\Delta + \varepsilon$. Let T_{cl} the time after which the number of clusters remains the same (which exists and is finite by proposition 3.6). Let $C_{i,j}$ be one of the clusters that won't break after T_{cl}. We can assume that $j \neq i$. Otherwise the result holds trivially. With probability 1, we will have an infinite number of iterations such that both peers are in $C_{i,j}$ and every peer in $C_{i,j}$ will appear eventually in those iterations with probability 1. We can restrict ourselves now to the subsequence of times t_k in which both interacting peers are in $C_{i,j}$. Abusing of notation we will call this sequence t. Let d_i be the distance between the i-th ordered peer and the $i+1$-th one. In other words:

$$d_i = x_{(i+1)} - x_{(i)}$$

The objective is to prove that $x_{(j)}(t) - x_{(i)}(t)$ converges to 0. It is clear that it converges because it is a bounded and decreasing sequence. Let us suppose that it converges to $L > 0$. First, for any time t, let us define the Δ-Distance $S^{\Delta}(t)$ and the $(\Delta + \varepsilon)$-Distance $S^{\Delta+\varepsilon}(t)$ as:

$$S^{\Delta}(t) = \sum_{i=1}^{N-1} d_i 1_{\{0 \leq d_i \leq \Delta\}}$$

$$S^{\Delta+\varepsilon}(t) = \sum_{i=1}^{N-1} d_i 1_{\{\Delta \leq d_i \leq \Delta+\varepsilon\}}$$

We will also need to define the $(\Delta + \varepsilon)$-Number $N^{\Delta+\varepsilon}(t)$ as:

$$N^{\Delta+\varepsilon}(t) = \sum_{i=1}^{N-1} 1_{\{\Delta \leq d_i \leq \Delta+\varepsilon\}}$$

We will now prove that both distances tend to 0 as t goes to infinity by contradiction.

We'll start with $S^{\Delta}(t)$. Pick any subsequence t_k such that $S^{\Delta}(t_k)$ converges to $S^{\Delta} > 0$. We know that there is at least one, because $S^{\Delta}(t)$ is bounded between 0 and 1. The objective is to prove that every convergent subsequence converges to 0. By the pigeonhole principle there exists an index $i(t)$ such that $\Delta \geq d_{i(t)} \geq \frac{S^{\Delta}}{2N}$. With probability 1, the pair $(i(t), i(t)+1)$ will meet infinitely often. Therefore, there exists a subsequence t_{k_q} of t_k such that $\mu_2^N(t_{k_{q+1}}) - \mu_2^N(t_{k_q}) \leq -\frac{1}{N^2}2w(1-w)\left(\frac{S^{\Delta}}{2N^{\Delta}}\right)^2$. This would mean that $\mu_2^N(t)$ is not a Cauchy sequence and therefore not convergent. But $\mu_2(t)$ is decreasing and bounded by 0, thus convergent. Contradiction. The contradiction comes from the supposition that $S^{\Delta} > 0$. Therefore $S^{\Delta}(t_k)$ converges to 0. As this doesn't depend on the t_k chosen, any convergent subsequence converges to 0. As $S^{\Delta}(t)$ is bounded and every convergent subsequence converges to 0, $S^{\Delta}(t)$ converges to 0.

Using the fact that $S^{\Delta}(t)+S^{\Delta+\varepsilon}(t) = x_{(j)}(t)-x_{(i)}(t)$, we know that $S^{\Delta+\varepsilon}(t)$ converges to L. If we prove that it converges to 0 we will get a contradiction and the proof will be finished. Let t_k be a convergent subsequence of $N^{\Delta+\varepsilon}(t)$, which we know it exists because $N^{\Delta+\varepsilon}(t)$ is bounded between 0 and N. As $N^{\Delta+\varepsilon}(t_k)$ is convergent and takes integer values, there exists some t_m such that $N^{\Delta+\varepsilon}(t_k) = N^{\Delta+\varepsilon}\ \forall\ k \geq m$. Let us assume $N^{\Delta+\varepsilon} \neq 0$, otherwise it is trivial that $L = 0$. Then, we have that $\frac{S^{\Delta+\varepsilon}(t_k)}{N^{\Delta+\varepsilon}(t_k)}$ converges to $\frac{L}{N^{\Delta+\varepsilon}}$, which we can bound like this:

$$\Delta \leq \frac{L}{N^{\Delta+\varepsilon}} < \Delta + \varepsilon$$

The second inequality is strict because otherwise that would mean that there is some $d_i \geq \Delta + \varepsilon$, and that is equivalent to the breaking of the cluster, which was impossible by hypothesis. Therefore, for sufficiently large k, we have that there exists some $d_i(t_k)$ such that:

$$\Delta \leq d_i(t_k) \leq \frac{L}{N^{\Delta+\varepsilon}} + m < \Delta + \varepsilon$$

where

$$m = \min\left\{\frac{1}{2}\left(\frac{L}{N^{\Delta+\varepsilon}} - \Delta\right), \frac{1}{2}\left(\Delta + \varepsilon - \frac{L}{N^{\Delta+\varepsilon}}\right)\right\}$$

Using the hypotheses, we know that:

$$\sup_{x\in\left[\Delta, \frac{L}{N^{\Delta+\varepsilon}}+m\right]} \frac{2K(x)}{N^2}(K(x)-x) = \max_{x\in\left[\Delta, \frac{L}{N^{\Delta+\varepsilon}}+m\right]} \frac{2K(x)}{N^2}(K(x)-x) = C < 0$$

Using the same arguments as before, we can choose again a pair $i(t), i(t)+1$ that meets infinitely often. That would prove that $\mu_2(t)$ is not a Cauchy sequence because the sequence decreases by at least C. Therefore it should be non convergent, which is a contradiction. The contradiction comes from the assumption that $N^{\Delta+\varepsilon} \neq 0$, so $N^{\Delta+\varepsilon} = 0$. As we can do this for any subsequence chosen and $N(t)$ is bounded between 0 and N, $N(t)$ is convergent to 0.

To conclude the proof, we have that:

$$\lim_{t\to\infty} x_{(j)}(t) - x_{(i)}(t) = \lim_{t\to\infty} S^{\Delta}(t) + S^{\Delta+\varepsilon}(t) = 0 + 0 = 0$$

The number of different values is bounded by the maximum number of clusters, which was proven in proposition 3.6. The condition on the sum can be easily obtained from 3.2. □

4 From finite to infinite N

In this section we will illustrate how to transform the finite N system into the infinite N system using the mean field approach. Although there isn't yet a full proof for convergence in the general case, we conjecture there is convergence given the results we have observed. We first give an heuristic argument:

4.1 Heuristic argument

We begin taking as time unit $\frac{1}{N-1}$: this will ensure that the rate with which a peer makes a transition per time unit is constant for any value of N. Let $M^N(t)$ be the occupancy measure of the system and $\mathcal{G}^N$ be the generator of M^N, both defined as:

$$M^N(t) = \frac{1}{N}\sum_{n=1}^{N}\delta_{X_n^N(t)}$$

$$\mathcal{G}^N(\varphi)(\nu) = (N-1)\mathbb{E}\left(\varphi\left(M^N\left(t+\frac{1}{N-1}\right)\right) - \varphi(M^N(t))\middle| M^N(t) = \nu\right)$$

Setting $\nu = \frac{1}{N}\sum_{n=1}^{N}\delta_{x_n}$ and $\varphi(\nu) =< h, \nu >$, we can calculate $\mathcal{G}^N(\varphi)(\nu)$ for any bounded test function h:

$$\mathcal{G}^N(\varphi)(\nu) = (N-1)\mathbb{E}\left(\varphi\left(M^N\left(t+\frac{1}{N-1}\right)\right) - \varphi(M^N(t))\middle| M^N(t) = \nu\right)$$

$$= (N-1)\frac{2}{N(N-1)}\frac{1}{N}\sum_{m<n}(h(wx_m + (1-w)x_n) + h(wx_n + (1-w)x_m) - h(x_m) - h(x_n))1_{\{|x_n - x_m| \leq \Delta\}}$$

$$\frac{1}{N^2}\sum_{m<n}(h(wx_m + (1-w)x_n) + h(wx_n + (1-w)x_m) - h(x_m) - h(x_n))1_{\{|x_n - x_m| \leq \Delta\}}$$

$$+\frac{1}{N^2}\sum_{m<n}(h(wx_m + (1-w)x_n) + h(wx_n + (1-w)x_m) - h(x_m) - h(x_n))1_{\{|x_n - x_m| \leq \Delta\}}$$

$$= \frac{1}{N^2}\sum_{m<n}(h(wx_m + (1-w)x_n) + h(wx_n + (1-w)x_m) - h(x_m) - h(x_n))1_{\{|x_n - x_m| \leq \Delta\}}$$

$$+\frac{1}{N^2}\sum_{n<m}(h(wx_m + (1-w)x_n) + h(wx_n + (1-w)x_m) - h(x_m) - h(x_n))1_{\{|x_n - x_m| \leq \Delta\}}$$

$$= \frac{1}{N^2}\sum_{n,m}(h(wx_m + (1-w)x_n) + h(wx_n + (1-w)x_m) - h(x_m) - h(x_n))1_{\{|x_n - x_m| \leq \Delta\}}$$

$$= \int_{[0,1]^2}(h(wy + (1-w)x) + h(wx + (1-w)y) - h(x) - h(y))1_{\{|x-y| \leq \Delta\}}d\nu(x)d\nu(y)$$

$$= 2\int_{[0,1]^2}(h(wx + (1-w)y) - h(x))1_{\{|x-y| \leq \Delta\}}d\nu(x)d\nu(y)$$

Defining ν_t^N as the probability distribution of $X^N(t)$ and assuming that we can use the mean field approximation, which consists in assuming that $X_n^N(t)$ and $X_m^N(t)$ are independent and equally distributed we get that:

$$< \nu^N_{t+\frac{1}{N-1}} - \nu_t^N, h >= \frac{2}{N-1}\int_{[0,1]^2}(h(wx + (1-w)y) - h(x))1_{\{|x-y| \leq \Delta\}}d\nu_t(x)d\nu_t(y)$$

Letting N go to infinity, we have that:

$$< \frac{\partial \nu_t}{\partial t}, h >= 2\int_{[0,1]^2}(h(wx + (1-w)y) - h(x))1_{\{|x-y| \leq \Delta\}}d\nu_t(x)d\nu_t(y)$$

or, in integral form:

$$< h, \nu_t > - < h, \nu_0 >= 2\int_0^t\int_{[0,1]^2}(h(wx + (1-w)y) - h(x))1_{\{|x-y| \leq \Delta\}}d\nu_s(x)d\nu_s(y)ds$$

Taking $h(x) = 1_{\{x \leq z\}}$ and defining $F(z,t) =< h, \nu_t >$ we obtain:

$$F(z,t) - F(z,0) = 2\int_0^t \int_{[0,1]^2} (1_{\{wx+(1-w)y\leq z\}} - h_{\{x\leq z\}})1_{\{|x-y|\leq\Delta\}} d\nu_s(x)d\nu_s(y)ds$$

$$= 2\int_0^t \int_{I_1} dF(x,s)dF(y,s)ds - 2\int_0^t \int_{I_2} dF(x,s)dF(y,s)ds$$

Taking derivatives with respect to t:

$$\frac{\partial F(z,t)}{\partial t} = 2\int_{I_1} dF(x,t)dF(y,t) - 2\int_{I_2} dF(x,t)dF(y,t)dt \tag{2}$$

where I_1 and I_2 are the regions defined by the following restrictions:

$$I_1 = (x,y) \in [0,1]\times[0,1] \; s.t: \begin{cases} y-\Delta \leq x \leq y+\Delta \\ wx+(1-w)y \leq z \end{cases}$$

$$I_2 = (x,y) \in [0,1]\times[0,1] \; s.t: \begin{cases} y-\Delta \leq x \leq y+\Delta \\ x \leq z \end{cases}$$

Note that equation (2) is equivalent to (3).

4.2 Discretized Domain

We can prove convergence for the case in which we work with a discretized domain over [0,1] (such as what the software like Matlab does). Let $\{p_1, p_2, \ldots, p_q\}$ be such discretized domain. To prove this we will use the same scheme as in [6]. We will begin calculating the drift. Let $M_i^N(t)$ be the proportion of opinions that are at the state p_i. Let $v(p_i,p_j)$ be the p_k such that we approximate the new state of p_i after the transition between p_i and p_j by p_k. If we calculate the drift, we have that:

$$f_i^N(\overrightarrow{m}) = \mathbb{E}\left(M_i^N\left(t+\frac{1}{N-1}\right) - M_i^N(t) \middle| M^N(t) = \overrightarrow{m}\right)$$

$$= \frac{1}{N}\sum_{j=1}^{q}\sum_{k=1}^{q}(1_{\{j=i\}} + 1_{\{k=i\}} - 1_{\{v(p_j,p_k)=i\}} - 1_{\{v(p_k,p_j)=i\}})m_j m_k$$

Thus we have a drift of order $\frac{1}{N}$. Taking into account that there are at most 2 objects that change their state after one transition and that the drift is $\mathcal{C}^\infty$ with respect to $\overrightarrow{m}$, the hypotheses of [6] are satisfied. We have that, as $N \to \infty$, $M^N(t)$ converges in probability uniformly over finite time intervals to the solution of:

$$\frac{\partial m_i}{\partial t} = \frac{1}{N}\sum_{j=1}^{q}\sum_{k=1}^{q}(1_{\{j=i\}} + 1_{\{k=i\}} - 1_{\{v(p_j,p_k)=i\}} - 1_{\{v(p_k,p_j)=i\}})m_j m_k = \frac{2}{N}\sum_{j=1}^{q}\sum_{k=1}^{q}(1_{\{j=i\}} - 1_{\{v(p_j,p_k)=i\}})m_j m_k$$

which is the discretized version of equation(2).

5 Infinite N case

In this section we will study the case where the number of peers N goes to infinity. First, we will obtain the partial integro-differential equations that govern the motion of the dynamical system. Next, we will proof similar results to the discrete case. This results include the evolution of the moments or the convergence to a sum of Diracs. We will complete the section stating some other properties of the system: existence and uniqueness of solutions for the PIDE and boundedness of the distribution and its derivatives after finite time, which means that there is no blow-up in finite time. Finally, we describe a technique to get a bound for the cases in which the limit when t goes to infinity is exactly one Dirac, as opposed to more than one.

5.1 Motion of the system

Proposition 5.1 *Assuming 1 interaction per time slot, the probability distribution function satisfies the following partial integro-differential equation:*

$$\frac{\partial F(z,t)}{\partial t} = -2\int_0^z \int_{y-\Delta}^{y+\Delta} f(x,t)f(y,t)dxdy + 2\int_0^{z-w\Delta}\int_{y-\Delta}^{y+\Delta} f(x,t)f(y,t)dxdy + 2\int_{z-w\Delta}^{z+w\Delta}\int_{y-\Delta}^{\frac{z-(1-w)y}{w}} f(x,t)f(y,t)dxdy \quad (3)$$

where $f(x,t) = \dfrac{\partial F(x,t)}{\partial x}$ *is the probability density function associated to* $F(x,t)$.

Proof:

Though we give a proof in section 4, it is possible to heuristically derive the equation by counting all the contributions to the derivative of $F(z,t)$ separately and considering their mean-field limit.

On the one hand, there is a negative contribution for each of the interacting peers that has a reputation value less or equal than z. Therefore the negative contribution will be the integral of $f(x)f(y)$ over all pairs (x,y) that either satisfy $x < z$ and pass the deviation test, which is given by $\int_0^z \int_{x-\Delta}^{x+\Delta} f(x,t)f(y,t)dxdy$ and over all pairs (x,y) that satisfy $y < z$ and pass the deviation test, which is given by $\int_0^z \int_{y-\Delta}^{y+\Delta} f(x,t)f(y,t)dxdy$.

On the other hand, the positive contribution to the derivative corresponds to the pairs (x,y) such that after the interaction, the resulting pair (x',y') satisfies that either x' or y' is less or equal than z. We have contribution for each of the elements of the new pair that is less or equal than z. We'll start calculating the contribution given by all pairs (x,y) such that $x' \le z$. Those points have to satisfy simultaneously:

$$y - \Delta < x < y + \Delta, \quad 0 \le wx + (1-w)y \le z$$

which is equivalent to:

$$y - \Delta < x < \min\left\{y + \Delta, \frac{z-(1-w)y}{w}\right\}$$

To determine the integration limits, we have:

$$y + \Delta \le \frac{z-(1-w)y}{w} \Leftrightarrow wy + w\Delta \le z - y + wy \Leftrightarrow y \le z - w\Delta$$

And also:

$$y - \Delta \le \frac{z-(1-w)y}{w} \Leftrightarrow wy - w\Delta \le z - y + wy \Leftrightarrow y \le z + w\Delta$$

Therefore we can write the contribution as:

$$\int_0^{z-w\Delta}\int_{y-\Delta}^{y+\Delta} f(x,t)f(y,t)dxdy + \int_{z-w\Delta}^{z+w\Delta}\int_{y-\Delta}^{\frac{z-(1-w)y}{w}} f(x,t)f(y,t)dxdy$$

Now we want to calculate the contribution given by all pairs (x,y) such that $y' \le z$ but by symmetry the contribution is the same as the previous one.

Adding all the calculated contributions gives us the result. □

Proposition 5.2 *We have the following expression for the probability density function:*

$$\frac{\partial f(x,t)}{\partial t} = -2f(x,t)\int_{x-\Delta}^{x+\Delta} f(z,t)dz + \frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f(z,t)f\left(\frac{x-(1-w)z}{w},t\right)dz$$

Proof: All we have to do is take derivatives with respect to x from the previous expression. Term by term we get:

$$\frac{\partial}{\partial x}\left\{-2\int_0^x f(y,t)\left(\int_{y-\Delta}^{y+\Delta} f(z,t)dz\right)dy\right\} = -2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(z,t)dz\right)$$

$$\frac{\partial}{\partial x}\left\{2\int_0^{x-w\Delta} f(z,t)\left(\int_{y-\Delta}^{y+\Delta} f(y,t)dy\right)dz\right\} = 2f(x-w\Delta,t)\left(\int_{x-w\Delta-\Delta}^{x-w\Delta+\Delta} f(z,t)dz\right)$$

Defining $h(x,y,t) = \int_{y-\Delta}^{\frac{x-(1-w)y}{w}} f(z,t)dz$ and applying Leibniz integral rule:

$$\frac{\partial}{\partial x}\left\{2\int_{x-w\Delta}^{x+w\Delta} f(y,t)h(x,y,t)dy\right\} = 2f(x+w\Delta,t)h(x,x+w\Delta,t)$$

$$-2f(x-w\Delta,t)h(x,x-w\Delta,t) + 2\int_{x-w\Delta}^{x+w\Delta} f(y,t)\frac{\partial}{\partial x}h(x,y,t)dy$$

$$2f(x+w\Delta,t)h(x,x+w\Delta,t) = 2f(x+w\Delta,t)\int_{x+w\Delta-\Delta}^{x+w\Delta-\Delta} f(z,t)dz = 0$$

$$-2f(x-w\Delta,t)h(x,x-w\Delta,t) = -2f(x-w\Delta,t)\int_{x-w\Delta-\Delta}^{x-w\Delta+\Delta} f(z,t)dz$$

$$\frac{\partial}{\partial x}h(x,y,t) = \frac{1}{w}f\left(\frac{x-(1-w)y}{w},t\right) \Rightarrow 2\int_{x-w\Delta}^{x+w\Delta} f(y,t)\frac{\partial}{\partial x}h(x,y,t)dy = \frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy$$

Adding all the equations we get:

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{\partial F(x,t)}{\partial t}\right\} = -2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(z,t)dz\right) + \frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy \quad (4)$$

□

5.2 Moments

We will prove analogous results as the ones obtained for the discrete N case (see section 3), namely that the k-th moments are decreasing with time. The techniques used are quite similar as the ones used for the discrete case, although there are slight variations.

Throughout this section and the following ones, we will make the hypothesis that $f(x,t)$ is analytical with respect to t.

We will first prove the positiveness of $f(x,t)$, which should be natural thinking that f is a probability distribution:

Proposition 5.3 *If $f(x,0) \geq 0$, then $f(x,t) \geq 0 \quad \forall t$ independently of Δ and w.*

Proof: Let $m(t) = \inf_x f(x,t)$. Note that we have that $m(0) \geq 0$.

We will now follow a contradiction argument. Let us suppose that $m(t) < 0$ for some t. As $m(0) \geq 0$ and $m(t)$ is continuous, there has to exist some T such that $m(T) = 0$ and $m'(T) < 0$. Let T_0 be the minimum of those such T. By the continuity of $f(x,t)$ with respect to t the infimum has to be a minimum. We have, for any x at which the minimum is attained:

$$\begin{aligned}\frac{\partial f(x,T_0)}{\partial t} &= -2f(x,T_0)\int_{x-\Delta}^{x+\Delta} f(z,T_0)dz + \frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f(z,T_0) f\left(\frac{x-(1-w)z}{w}, T_0\right) dz \\ &= \frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f(z,T_0) f\left(\frac{x-(1-w)z}{w}, T_0\right) dz \qquad (5)\end{aligned}$$

which is non-negative by definition of T_0. Therefore we get a contradiction because $m'(T_0) \geq 0$. □

Proposition 5.4 $|\sup\{Supp(f(x,t))\} - \inf\{Supp(f(x,t))\}|$ *does not increase with time.*

Proof: Let $I = [\inf\{\text{Supp}(f(x,0))\}, \sup\{\text{Supp}(f(x,0))\}]$ and let $Z = [0,1] - I$. We want to show that $f(x,t)|_Z = 0$ for all t. Recalling equation (4):

$$\frac{\partial f(x,t)}{\partial t} = -2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(z,t)dz\right) + \frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t) f\left(\frac{x-(1-w)y}{w}, t\right) dy$$

We will prove inductively that $\frac{\partial^n}{\partial t^n} f(x,0) = 0 \quad \forall x \in Z$. For the case $n = 1$, the first term of (4) is zero because $f(x,0) = 0$. Let's suppose the second term doesn't vanish. That cannot happen because $y < x < \frac{x-(1-w)y}{w}$, so y and $\frac{x-(1-w)y}{w}$ can't be both elements of I. Hence, we have proved that for all $x \in Z$, $\left.\frac{\partial}{\partial t} f(x,t)\right|_{t=0} = 0$.

Now suppose that the statement is true for all $n \leq K$. We will prove that is also true for $K+1$. Computing derivatives based on (4) we get:

$$\begin{aligned}\frac{\partial^{K+1} f(x,t)}{\partial t^{K+1}} &= -2\sum_{i=0}^{K}\binom{K}{i}\frac{\partial^i}{\partial t^i} f(x,t)\left(\int_{x-\Delta}^{x+\Delta}\frac{\partial^{K-i}}{\partial t^{K-i}} f(z,t)dz\right) \\ &+ \sum_{i=0}^{K}\binom{K}{i}\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta}\frac{\partial^i}{\partial t^i} f(y,t)\frac{\partial^{K-i}}{\partial t^{K-i}} f\left(\frac{x-(1-w)y}{w}, t\right) dy \qquad (6)\end{aligned}$$

The first sum is zero because by induction $\left.\frac{\partial^i}{\partial t^i} f(x,t)\right|_{t=0} = 0$ if $x \in Z$. For each summand of the second sum, the same argument as above is valid, as the derivatives of $f(x,t)$ of order less or equal than K evaluated at $t = 0$ might only be non-zero at I. Therefore, we prove that every derivative of any order is zero. Making the Taylor expansion of $f(x,t)$ at $f(x,0)$ gives us that $f(x,t) = 0$ for any t if $x \in Z$.

□

This proposition is intuitive: there is no extra information, therefore the lowest belief can't decrease, because there is no other influence than the rest of the peers that have a higher reputation value. Similarly, for the highest belief the intuitive argument is the same.

We now define the (non-centered) moments in the usual way: (note that there is dependence on time)

$$\mu_n(t) = \int_0^1 x^n f(x,t)dx$$

The main objective of the subsection is to study those moments and use their properties to prove stronger results (see subsection 5.3).

Proposition 5.5 *For any n, $\mu_n(t)$, the n-th moment of $f(x,t)$, is a decreasing function of t.*

Proof:

We want to prove that:

$$\frac{\partial}{\partial t}\left\{\mu_n(t)\right\} \leq 0$$

But:

$$\frac{\partial}{\partial t}\mu_n(t) = \frac{\partial}{\partial t}\left(\int_0^1 x^n f(x,t)dx\right) = \int_0^1 x^n \frac{\partial}{\partial t} f(x,t)dx$$

$$= -2\int_0^1 x^n \int_{x-\Delta}^{x+\Delta} f(x,t)f(z,t)dzdx + \int_0^1 x^n \frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f(z,t) f\left(\frac{x-(1-w)z}{w}, t\right) dzdx$$

Let $k = \dfrac{x-(1-w)z}{w} \Rightarrow wdk = dx$. Our goal is to transform the integrals in the plane (x,z) to the plane (k,z). We will repeatedly use Fubini's theorem since all the integrals are finite.

We now distinguish two cases:

(1) Case $1-\Delta w \geq \Delta w$: We define the following regions:

$$J_1 = \left\{(k,z) \middle| -\Delta w \leq z \leq \Delta w,\ \frac{-(1-w)z}{w} \leq k \leq z+\Delta\right\}$$

$$J_2 = \left\{(k,z) \,|\Delta w \leq z \leq 1-\Delta w,\ z-\Delta \leq k \leq z+\Delta\right\}$$

$$J_3 = \left\{(k,z) \middle| 1-\Delta w \leq z \leq 1+\Delta w,\ z-\Delta \leq k \leq z + \frac{1-z}{w}\right\}$$

$$J_4 = \left\{(k,z) \middle| -\Delta w \leq z \leq \Delta w,\ z-\Delta \leq k \leq \frac{-(1-w)z}{w}\right\}$$

$$J_5 = \left\{(k,z) \middle| 1-\Delta w \leq z \leq 1+\Delta w,\ z+\frac{1-z}{w} \leq k \leq z+\Delta\right\}$$

It's easy to see that:

$$\int_{J_4} (wk+(1-w)z)^n f(z,t)f(k,t)dzdk = 0$$

as $k < 0$ for all $z > 0$, which implies $f(k,t) = 0$ by proposition 5.4 and the fact that the initial condition has support contained in $[0,1]$. If $z < 0$, that directly means that $f(z,t) = 0$. Therefore, the integral is 0. In an analogous way we can see that:

$$\int_{J_5} (wk+(1-w)z)^n f(z,t)f(k,t)dzdk = 0$$

because $k \geq z + \dfrac{1-z}{w} > 1$ for all $z < 1$, which implies $f(k,t) = 0$. If $z > 1$, that directly means that $f(z,t) = 0$. Therefore the integral is 0.

We can rewrite the second integral as:

$$\int_0^1 x^n \frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f(z,t)f\left(\frac{x-(1-w)z}{w},t\right)dzdx = \int_{J_1+J_2+J_3} 2(wk+(1-w)z)^n f(z,t)f(k,t)dzdk =$$

$$\int_{J_1+J_2+J_3+J_4+J_5} 2(wk+(1-w)z)^n f(z,t)f(k,t)dzdk = \underbrace{\int_{-\Delta w}^{1+\Delta w}\int_{z-\Delta}^{z+\Delta} 2(wk+(1-w)z)^n f(z,t)f(k,t)dkdz}_{K}$$

(2) Case $1-\Delta w < \Delta w$:

We define the following regions:

$$J_1 = \left\{(k,z)\,\middle|\, -\Delta w \le z \le 1-\Delta w,\ \frac{-(1-w)z}{w} \le k \le z+\Delta\right\}$$

$$J_2 = \left\{(k,z)\,\middle|\, 1-\Delta w \le z \le \Delta w,\ \frac{-(1-w)z}{w} \le k \le z+\frac{1-z}{w}\right\}$$

$$J_3 = \left\{(k,z)\,\middle|\, \Delta w \le z \le 1+\Delta w,\ z-\Delta \le k \le z+\frac{1-z}{w}\right\}$$

$$J_4 = \left\{(k,z)\,\middle|\, -\Delta w \le z \le \Delta w,\ z-\Delta \le k \le \frac{-(1-w)z}{w}\right\}$$

$$J_5 = \left\{(k,z)\,\middle|\, 1-\Delta w \le z \le 1+\Delta w,\ z+\frac{1-z}{w} \le k \le z+\Delta\right\}$$

For the same reasons as above:

$$\int_{J_4} (wk+(1-w)z)^n f(z,t)f(k,t)dzdk = \int_{J_5} (wk+(1-w)z)^n f(z,t)f(k,t)dzdk = 0$$

So, in this case we can also write:

$$\int_0^1 x^n \frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f(z,t) f\left(\frac{x-(1-w)z}{w},t\right) dzdx = \int_{J_1+J_2+J_3+J_4+J_5} 2(wk+(1-w)z)^n f(z,t)f(k,t)dzdk$$

$$= \underbrace{\int_{-\Delta w}^{1+\Delta w}\int_{z-\Delta}^{z+\Delta} 2(wk+(1-w)z)^n f(z,t)f(k,t)dkdz}_{K}$$

Let's study the term (K):

$$K = \int_0^1\int_{z-\Delta}^{z+\Delta} 2(wk+(1-w)z)^n f(z,t)f(k,t)dkdz = \int_0^1\int_{z-\Delta}^{z+\Delta} P(k,z)f(z,t)f(k,t)dkdz$$

We distinguish two cases again:

(1) $\Delta > 1-\Delta$: We define the following regions:

$$R_1 = \{(k,z)\,|{-\Delta} \le k \le 1-\Delta,\ 0 \le z \le k+\Delta\}$$

$$R_2 = \{(k,z)\,|1-\Delta \le k \le \Delta,\ 0 \le z \le 1\}$$

$$R_3 = \{(k,z)\,|\Delta \le k \le 1+\Delta,\ k-\Delta \le z \le 1\}$$

$$R_4 = \{(k,z)\,|{-\Delta} \le k \le \Delta,\ k-\Delta \le z \le 0\}$$

$$R_5 = \{(k,z)\,|1-\Delta \le k \le 1+\Delta,\ 1 \le z \le k+\Delta\}$$

We have that:

$$\int_{R_4} P(k,z)f(z,t)f(k,t)dzdk = 0, \quad \int_{R_5} P(k,z)f(z,t)f(k,t)dzdk = 0$$

We get both equalities because in R_4, $z<0$ and in R_5, $z>1$. Therefore $f(z,t)=0$ in both R_4 and R_5. We can write now:

$$\int_{R_1+R_2+R_3} P(k,z)f(z,t)f(k,t)dzdk = \int_{R_1+R_2+R_3+R_4+R_5} P(k,z)f(z,t)f(k,t)dzdk$$
$$= \int_{-\Delta}^{1+\Delta}\int_{k-\Delta}^{k+\Delta} P(k,z)f(z,t)f(k,t)dzdk = \int_0^1\int_{k-\Delta}^{k+\Delta} P(k,z)f(z,t)f(k,t)dzdk$$

which means that:

$$\int_0^1\int_{k-\Delta}^{k+\Delta} P(k,z)f(z)f(k)dzdk = \int_0^1\int_{k-\Delta}^{k+\Delta} P(z,k)f(z)f(k)dzdk$$

(2) $\Delta \leq 1-\Delta$: We define the following regions:

$$R_1 = \{(k,z)\,|{-\Delta} \leq k \leq \Delta,\ 0 \leq z \leq k+\Delta\}$$
$$R_2 = \{(k,z)\,|\Delta \leq k \leq 1-\Delta,\ k-\Delta \leq z \leq k+\Delta\}$$
$$R_3 = \{(k,z)\,|1-\Delta \leq k \leq 1+\Delta,\ k-\Delta \leq z \leq 1\}$$
$$R_4 = \{(k,z)\,|{-\Delta} \leq k \leq \Delta,\ k-\Delta \leq z \leq 0\}$$
$$R_5 = \{(k,z)\,|1-\Delta \leq k \leq 1+\Delta,\ 1 \leq z \leq k+\Delta\}$$

Again, we have that:

$$\int_{R_4} P(k,z)f(z,t)f(k,t)dzdk = 0, \quad \int_{R_5} P(k,z)f(z,t)f(k,t)dzdk = 0$$

for the same reasons as before. Therefore:

$$\int_{R_1+R_2+R_3} P(k,z)f(z,t)f(k,t)dzdk = \int_{R_1+R_2+R_3+R_4+R_5} P(k,z)f(z,t)f(k,t)dzdk$$
$$= \int_{-\Delta}^{1+\Delta}\int_{k-\Delta}^{k+\Delta} P(k,z)f(z,t)f(k,t)dzdk = \int_0^1\int_{k-\Delta}^{k+\Delta} P(k,z)f(z,t)f(k,t)dzdk$$

From which we conclude that:

$$\int_0^1\int_{k-\Delta}^{k+\Delta} P(k,z)f(z,t)f(k,t)dzdk = \int_0^1\int_{k-\Delta}^{k+\Delta} P(z,k)f(z,t)f(k,t)dzdk$$

independently of the value of Δ and w.

Our objective is to get a symmetric expression on x and z under the integral. Therefore we can do the following:

$$\frac{\partial}{\partial t}\mu_k(t) = \int_0^1\int_{x-\Delta}^{x+\Delta} P(x,z)f(x)f(z)dzdx$$
$$= \int_0^1\int_{x-\Delta}^{x+\Delta}\left(\frac{1}{2}P(x,z)+\frac{1}{2}P(z,x)\right)f(x)f(z)dzdx = \int_0^1\int_{x-\Delta}^{x+\Delta} Q(x,z)f(x)f(z)dzdx$$

where:

$$Q(x,z) = \sum_{i=0}^{n} a_i x^i z^{n-i}, \quad a_{n-i} = a_i = \begin{cases} \binom{n}{i}(1-w)^i w^{n-i} + \binom{n}{n-i}(1-w)^{n-i}w^i & \text{if } i \neq 0,n \\ \binom{n}{n}(1-w)^n + \binom{n}{0}w^n - 1 & \text{else} \end{cases}$$

□

We can now finish the proof as in proposition 3.1 to get that $Q(x,z) \leq 0$. As $f(x)$ is always non-negative (by proposition 5.3), we are integrating a non-positive function. Therefore the result should be always non-positive, which is what we wanted to prove.

Corollary 5.6 *Given $f(x,0)$, for every $k,t>0$, $\mu_k(t)$ is a decreasing function of Δ.*

Proof: Let R_Δ be the region of integration of $\frac{\partial}{\partial t}\mu_k(t)$. First, we notice that $R_\Delta \subset R_{\Delta'}$ if $\Delta < \Delta'$. As we integrate a function which is non-positive for any value of the integration variables, the value of the integral is decreasing with respect to Δ. □

Corollary 5.7 *For any value of Δ, we have:*

$$\frac{\partial}{\partial t}\mu_k(t) \geq \sum_{i=0}^{k} a_i \mu_i(t)\mu_{k-i}(t), \quad a_{k-i} = a_i = \begin{cases} \binom{k}{i}(1-w)^i w^{k-i} + \binom{k}{k-i}(1-w)^{k-i}w^i & \textit{if } i \neq 0,k \\ \binom{k}{k}(1-w)^k + \binom{k}{0}w^k - 1 & \textit{else} \end{cases}$$

We have equality if $\Delta \geq |\sup\{Supp(f(x,0))\} - \inf\{Supp(f(x,0))\}|$.

Proof: Letting $\Delta = 1$ and using corollary 5.6, we integrate now over a square of size 1 and we can calculate each integral separately. Therefore:

$$\frac{\partial}{\partial t}\mu_k(t) \geq \frac{\partial}{\partial t}\mu_k(t)\bigg|_{\Delta=1} = \int_0^1 \int_{x-\Delta}^{x+\Delta} \sum_{i=0}^{k} a_i x^i z^{k-i} f(x,t) f(z,t)\, dz dx \bigg|_{\Delta=1}$$

$$= \int_0^1 \int_0^1 \sum_{i=0}^{k} a_i x^i z^{k-i} f(x,t) f(z,t)\, dz dx = \sum_{i=0}^{k} a_i \mu_i(t)\mu_{k-i}(t)$$

Note that if $\Delta \geq |\sup\{\mathrm{Supp}(f(x,0))\} - \inf\{\mathrm{Supp}(f(x,0))\}| \geq |\sup\{\mathrm{Supp}(f(x,t))\} - \inf\{\mathrm{Supp}(f(x,t))\}|$, we have that:

$$\int_0^1 \int_{x-\Delta}^{x+\Delta} \sum_{i=0}^{k} a_i x^i z^{k-i} f(x,t) f(z,t)\, dz dx = \int_0^1 \int_0^1 \sum_{i=0}^{k} a_i x^i z^{k-i} f(x,t) f(z,t)\, dz dx$$

and therefore equality. □

We will now present, as in section 3, particular results of the previous corollary.

Corollary 5.8 *$\mu_1(t)$ is constant for all t, independently of Δ.*

Proof: On the one hand, particularizing corollary 5.7 to $k=1$, we get that:

$$\frac{\partial}{\partial t}\mu_1(t) \geq 0$$

On the other, by proposition 5.5 we have that:

$$\frac{\partial}{\partial t}\mu_1(t) \leq 0$$

Hence:

$$\frac{\partial}{\partial t}\mu_1(t) = 0 \Rightarrow \mu_1(t) = \mu_1(0)$$

□

Corollary 5.9 *$V_0 \geq \sigma(t)^2 \geq V_0 e^{-4w(1-w)t}$ $\forall t$, where V_0 is the variance of $f(x,0)$.*

Proof: On the one hand, particularizing corollary 5.7 to $k = 2$, we get that:

$$\frac{\partial}{\partial t}\sigma(t)^2 = \frac{\partial}{\partial t}\mu_2(t) \geq 2[(1-w)^2 + w^2 - 1]\mu_2(t) + 4w(1-w)\mu_1(t)^2 = 4w(1-w)\sigma(t)^2$$

On the other, by proposition 5.5 we have that:

$$\frac{\partial}{\partial t}\mu_2(t) \leq 0$$

Hence, by integration:

$$V_0 \geq \sigma(t)^2 \geq V_0 e^{-4w(1-w)t}$$

□

5.3 Existence and uniqueness of the PIDE

Using a similar approach as in Picard's Theorem ([16]), we define the following functional $\mathcal{H}$, which acts over the set of finite signed measures X:

$$\mathcal{H}(\nu)(dx,t) = \nu(dx,0) + \int_0^t \mathcal{L}(\nu(dx,s))ds$$

where $\nu(dx,t)$ is the Lebesgue-Stieltjes measure ([27]) associated to $F(x,t)$ and $\mathcal{L}$ is an operator that acts the following way:

$$< \mathcal{L}(\nu(dx,s)), h >= \int_{[0,1]^2} [h(wx + (1-w)y) + h(wy + (1-w)x) - h(x) - h(y)]\nu(dx,s)\nu(dy,s)$$

The objective is to prove that $\mathcal{H}$ is a contractive application in a Banach space to apply the fixed point theorem. The unique fixed point of $\mathcal{H}$ will be the solution to our partial integro-differential equation. It's very important to notice that if we can find a solution defined in $[0,1]\times[0,T]$ for any positive T we can extend it to $[0,1]\times[0,\infty)$ as the PIDE is autonomous. Therefore, we can prove existence and uniqueness of the solution.

In our case, the set of finite signed measures with the total variation norm is a Banach space.

Proposition 5.10 *$\mathcal{L}$ is $L-$Lipschitz*

Proof: To prove that $\mathcal{L}$ is L-Lipschitz, we need to bound

$$\|\mathcal{L}(\nu) - \mathcal{L}(\nu')\| = \sup_{\|h\|_\infty \leq 1} < \mathcal{L}(\nu) - \mathcal{L}(\nu'), h >$$

$$= \sup_{\|h\|_\infty \leq 1} \int_{[0,1]^2} [h(wx + (1-w)y) + h(wy + (1-w)x) - h(x) - h(y)]|\nu(dx)\nu(dy) - \nu'(dx)\nu'(dy)|$$

We can clearly bound $h(wx + (1-w)y) + h(wy + (1-w)x) - h(x) - h(y)$ by 4, regardless of h, as x and y are points of the square of size 1. Therefore:

$$|\mathcal{L}(\nu) - \mathcal{L}(\nu')| \leq 4\int_{[0,1]^2} |\nu(dx)\nu(dy) - \nu'(dx)\nu'(dy)| \leq 4\int_{[0,1]^2} |\nu(dx)||\nu(dy) - \nu'(dy)| + 4\int_{[0,1]^2} |\nu(dx) - \nu'(dx)||\nu'(dy)|$$

$$= 4\int_{[0,1]} |\nu - \nu'|(dy) + 4\int_{[0,1]} |\nu - \nu'|(dx) = 8\|\nu - \nu'\|$$

We have proved that $\mathcal{L}$ is indeed an 8-Lipschitz operator. □

Proposition 5.11 *$\mathcal{H}$ is contractive*

We will now prove that $\mathcal{H}$ is contractive. We have the following inequalities:

$$|\mathcal{H}(\nu)(dx,t) - \mathcal{H}(\nu')(dx,t)| = \left|\int_0^t \mathcal{L}(\nu(dx,s)) - \mathcal{L}(\nu'(dx,s))ds\right| \leq 8\int_0^t \|\nu(dx,s) - \nu'(dx,s)\|\, ds$$

$$\leq 8\int_0^t \|\nu - \nu'\|\, ds = 8t\,\|\nu - \nu'\| \leq 8T\,\|\nu - \nu'\|$$

Therefore:

$$\|\mathcal{H}(\nu) - \mathcal{H}(\nu')\| \leq 8T\,\|\nu - \nu'\|$$

For $0 < T < \frac{1}{8}$, we know that $\mathcal{H}$ is a contractive mapping and we are done.

Proposition 5.12 *$\mathcal{H}$ maps X into X*

Proof:

We need to prove that $\mathcal{H}(\nu)(dx,t)$ is a measure for every $t < T$. First, let's see the following properties: Let h_1 and h_2 be two continuous functions with compact support. $\mathcal{H}(\nu)(dx,t)$ is linear:

$$< \mathcal{H}(\nu)(dx,t), \alpha h_1 + \beta h_2 > = < \nu(dx,0), \alpha h_1 + \beta h_2 > + \int_0^t < \mathcal{L}(\nu(dx,s)), \alpha h_1 + \beta h_2 > ds$$

$$= \alpha < \nu(dx,0), h_1 > + \alpha\int_0^t < \mathcal{L}(\nu(dx,s)), h_1 > ds + \beta < \nu(dx,0), h_2 > + \beta\int_0^t < \mathcal{L}(\nu(dx,s)), h_2 > ds$$

$$= \alpha < \mathcal{H}(\nu)(dx,t), h_1 > + \beta < \mathcal{H}(\nu)(dx,t), h_2 >$$

$\mathcal{H}(\nu)(dx,t)$ is continuous:

$$\| < \mathcal{H}(\nu)(dx,t), h > \| \leq \| < \nu(dx,0), h > \| + \left\|\int_0^t < \mathcal{L}(\nu(dx,s)), h > ds\right\|$$

$$\leq \|\nu(dx,0)\|\|h\| + \int_0^t \|< \mathcal{L}(\nu(dx,s)), h >\|\, ds \leq \|\nu\|\|h\| + 4T\,\|\nu\|^2\,\|h\| = \|h\|\left(\|\nu\| + 4T\|\nu\|^2\right)$$

By the Riesz representation theorem, there exists a measure μ such that $< \mathcal{H}(\nu), h > = < \mu, h >$ and we are done.

□

5.4 Convergence to Dirac

Built upon the previous proofs, we will now prove the same results as in the discrete N section, namely the convergence to one or several Diracs, or, in other terms, partial or total consensus. Moreover, we also prove that for any Δ and any limit function $g(x)$ that consists of a sum of Diracs which are separated more than Δ apart, there exists an initial condition $f(x,0)$ such that $\lim_{t\to\infty} f(x,t) = g(x)$.

Proposition 5.13 *Let $F_{lim}(x) = \lim_{t\to\infty} F(x,t)$. F_{lim} is a distribution.*

Proof: By proposition 5.5, we know that all the moments $\mu_k(t)$ are decreasing with time and are bounded by 0. Therefore, they are convergent to some quantity μ_k when t goes to infinity. That means that $F(x,t)$ is convergent to some distribution $F_{lim}(x)$. The proof can be found in [13]. □

Knowing that $F_{lim}(x)$ is a distribution, we can now study the induced Lebesgue-Stieltjes to know how the measure is distributed in $[0,1]$:

Proposition 5.14 *Let $F_{lim}(x)$ be $\lim_{t\to\infty} F(x,t)$ and μ_F the Lebesgue-Stieltjes measure associated to $F_{lim}(x)$. Then, the only sets T such that $\mu_F(T) > 0$ are the ones that contain one or more points of the set $S = \{x_0, x_1, \ldots, x_N\}$, where the a_i only depend on the initial condition and Δ and are such that $|a_i - a_j| > \Delta$ for every i, j. That means that $F_{lim}(x)$ is a discrete distribution with support equal to S.*

Proof: By proposition 5.13, we know that F_{lim} exists. Let $k > 1$. We can write therefore:

$$0 = \lim_{t\to\infty} \frac{\partial}{\partial t}\mu_k(t) = \int_{[0,1]} \left(\int_{[x-\Delta,x+\Delta]} P(x,z)d\mu_F(z) \right) d\mu_F(x) = \int_{[0,1]} G(x)d\mu_F(x) \tag{7}$$

First we will justify the interchanging of the limit and the integral between the second and third step. We use twice the dominated convergence theorem. Defining $G(x,t) = \int_{[x-\Delta,x+\Delta]} P(x,z)f(z,t)dz$ and $h(x,t) = G(x,t)f(x,t)$ we have that:

$$|G(x,t)| = \left| \int_{[x-\Delta,x+\Delta]} P(z,x)f(x,t)dx \right| \leq \int_{[x-\Delta,x+\Delta]} |P(z,x)|f(z,t)dz \leq K \int_{[x-\Delta,x+\Delta]} f(z,t)dz \leq K$$

where $K = \max_{[0,1]\times[0,1]} |P(z,x)|$, which exists because we are taking the maximum of a continuous function on a compact set (Weierstrass' Theorem). Therefore $|G(x,t)| \leq K \Rightarrow |h(x,t)| \leq Kf(x,t)$, which is integrable, so we can exchange the limit and the first integral. To prove that we can also exchange it with the second, it's enough to see that $|P(x,z)f(z,t)| \leq Kf(z,t)$, which is also integrable and therefore we are done.

Focusing now on equation (7), we know that $G(x) \leq 0$, therefore there exists a set Ω_0, $\mu_F(\Omega_0) = 0$, such that $\forall x_0 \notin \Omega_0, G(x_0) = 0$. Let $x_1 \in \Omega_1 = [0,1] - \Omega_0$. x_1 exists because the total measure of $[0,1]$ is 1 and therefore $\Omega_1 \neq \emptyset$.

$$G(x_1) = 0 \Rightarrow 0 = \int_{[x_1-\Delta,x_1+\Delta]} P(x_1,z)d\mu_F(z) = \int_{[x_1-\Delta,x_1)\cup(x_1,x_1+\Delta]} P(x_1,z)d\mu_F(z)$$

As $P(x_1,z) < 0$ strictly in $[x_1-\Delta,x_1)\cup(x_1,x_1+\Delta]$, $\mu_F([x_1-\Delta,x_1)\cup(x_1,x_1+\Delta]) = \int_{[x_1-\Delta,x_1)\cup(x_1,x_1+\Delta]} d\mu_F(z) = 0$. We don't know what is the value of $\mu_F(\{x_1\})$. Let $S_1 = \{x_1\}$. Now, if $\Omega_2 = \Omega_1 - [x_1 - \Delta, x_1 + \Delta] \neq \emptyset$, we pick an $x_2 \in \Omega_2$. Again, $\mu_F([x_2 - \Delta, x_2) \cup (x_2, x_2 + \Delta]) = 0$ and the value of $\mu_F(\{x_2\})$ is unknown. Let $S_2 = \{x_1, x_2\}$. We repeat this process until Ω_N is empty, which we will achieve after at most $\lceil \frac{1}{\Delta} \rceil$ steps, since by construction every x_i and x_j are more than Δ apart. We will have then a set $S_N = \{x_1, x_2, \ldots, x_N\}$.This set has clearly measure 1, since we have proved that $\mu_F([0,1] - S_N) = 0$. Taking $S = \{x_i | x_i \in S_N, \mu_F(x_i) > 0\}$ we are done. □

We can now expect what happens when $\Delta \geq \text{Supp}(f(x,0))$, that is, everybody has the chance to interact with anybody: there will only be one opinion in the end, because if there were more than one, they could still interact until there is only one left. Formally:

Corollary 5.15 *If $\Delta \geq Supp(f(x,0))$ there is total consensus, i.e, the set S defined in the previous proposition consists only of a single point.*

Proof: Proceeding as in the previous proposition, we get that $\Omega_2 = \Omega_1 - [x_1 - \Delta, x_1 + \Delta] = \emptyset$. Therefore $S = S_1 = \{x_1\}$. □

Proposition 5.16 *Any combination of Dirac Deltas can be a limit point, provided that they satisfy the conditions of the convergent sets characterized by proposition 5.14.*

Proof: We'll give explicitly an initial condition $f(x,0)$ such that it converges to any given

$$g(x) = \sum_i a_i \delta(x - x_i), \quad |x_i - x_j| > \Delta, \quad \sum_i a_i = 1$$

First we define the intervals $I_i = \left[x_i - \frac{M}{4}, x_i + \frac{M}{4}\right] \cap [0,1]$, where

$$M = \min\{\min_{i,j}\{|x_i - x_j| - \Delta\}, \Delta\}$$

Note that M is strictly positive as all the differences are strictly greater than Δ. With this definition we have that:

$$\mu(I_i) \leq \frac{\Delta}{2} < \Delta \quad \forall i$$

$$d(I_i, I_j) = |x_i - x_j| - \frac{M}{2} \geq \Delta + \frac{M}{2} > \Delta \quad \forall i,j$$

For every i, let $g^i(x)$ be any function such that $\text{Supp}(g^i(x)) \subset I_i$, $\int_{I_i} g^i(x)dx = a_i$, $\int_{I_i} x g^i(x) dx = a_i x_i$. Let $f(x,0) = \sum_i g^i(x)$. We will prove that with this initial condition, the function converges to $g(x)$.

Let $I = \bigcup_i I_i$ and let $Z = [0,1] - I$. We first want to show that $f(x,t)|_Z = 0$ for all t. Recalling equation (4):

$$\frac{\partial f(x,t)}{\partial t} = -2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(z,t)dz\right) + \frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t) f\left(\frac{x-(1-w)y}{w}, t\right) dy$$

Given $f(x,0)$ such that $\text{Supp}(f(x,0)) \subset I$, we will prove inductively that $\frac{\partial^n}{\partial t^n} f(x,0) = 0 \quad \forall x \in Z$. For the case $n = 1$, the first term of (4) is zero because $f(x,0) = 0$. Let's suppose the second term doesn't vanish. If that happens, then there exist some y, i, j such that $y \in I_i$ and $\frac{x-(1-w)y}{w} \in I_j$. $i \neq j$ because $y < x < \frac{x-(1-w)y}{w}$ and $x \in Z$. But if we have such case then $d(I_i, I_j) \leq \left|y - \frac{x-(1-w)y}{w}\right| \leq \Delta$, which is by construction impossible. Hence, we have proved that for all $x \in Z$, $\frac{\partial}{\partial t} f(x,t)\big|_{t=0} = 0$.

Now suppose that the statement is true for all $n \leq K$. We will prove that is also true for $K+1$. Computing derivatives based on (4) we get:

$$\frac{\partial^{K+1} f(x,t)}{\partial t^{K+1}} = -2\sum_{i=0}^{K} \binom{K}{i} \frac{\partial^i}{\partial t^i} f(x,t) \left(\int_{x-\Delta}^{x+\Delta} \frac{\partial^{K-i}}{\partial t^{K-i}} f(z,t) dz\right)$$
$$+ \sum_{i=0}^{K} \binom{K}{i} \frac{2}{w} \int_{x-w\Delta}^{x+w\Delta} \frac{\partial^i}{\partial t^i} f(y,t) \frac{\partial^{K-i}}{\partial t^{K-i}} f\left(\frac{x-(1-w)y}{w}, t\right) dy \tag{8}$$

The first sum is zero because by induction $\frac{\partial^i}{\partial t^i} f(x,t)\Big|_{t=0} = 0$ if $x \in Z$. For each summand of the second sum, the same argument as above is valid, as the derivatives of $f(x,t)$ of order less or equal than K evaluated at $t = 0$ might only be non-zero at I. Therefore, we prove that every derivative of any order is zero. Making the Taylor expansion of $f(x,t)$ at $f(x,0)$ gives us that $f(x,t) = 0$ for any t if $x \in Z$.

Now we will consider the functions $f^i(x,t) = f(x,t)|_{I_i}$. We can write therefore $f(x,t) = \sum_i f^i(x,t)$ because the support of the different f^i will always be contained in I for every t. We claim that $f^i(x,t)$ satisfies (4). If $x \in I_i$ we have for both integrals:

$$f(x,t)\int_{x-\Delta}^{x+\Delta} f(z,t)dz = f^i(x,t)\int_{I_i\cap[x-\Delta,x+\Delta]} f(z,t)dz$$

$$\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy = \int_{I_i\cap[x-w\Delta,x+w\Delta]} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy$$

As $I_i \cap [x-\Delta, x+\Delta] \subset [x-\Delta, x+\Delta]$ and $f(x,t)$ is always non-negative by proposition 5.3, the only possibility for the first integrals to be different is if there is some point (y,t) in $[x-\Delta, x+\Delta]\times[0,\infty)$ outside I_i such that $f(y,t) \neq 0$. y can't be in Z for any t because we have proved that $f = 0$ in Z for every t. y can't be neither in some I_j because that would mean that $d(I_i, I_j) \leq |x-y| \leq \Delta$, which is impossible by definition of the sets I_i, I_j. Using a similar argument we prove the second equality. If, on the contrary, $x \notin I_i$, the derivative should be 0 by definition of f^i. The first term is zero because $f^i(x) = 0$. The second term is zero because $\left\{y, \frac{x-(1-w)y}{w}\right\}$ can't be both located at the same time to the left or to the right of x. Hence at least one of $\left\{f(y,t), f\left(\frac{x-(1-w)y}{w},t\right)\right\}$ is zero and the integral is zero. Therefore f^i satisfies (4).

We can rewrite (4) as:

$$\frac{\partial f^i(x,t)}{\partial t} = -2f(x,t)\left(\int_{I_i\cap[x-\Delta,x+\Delta]} f(z,t)dz\right) + \frac{2}{w}\int_{I_i\cap[x-w\Delta,x+w\Delta]} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy$$

$$= -2f^i(x,t)\left(\int_{x-\Delta}^{x+\Delta} f^i(z,t)dz\right) + \frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f^i(y,t)f^i\left(\frac{x-(1-w)y}{w},t\right)dy$$

Given that f^i satisfies (4) we can apply corollary 5.15 to get that f^i converges to $a_i\delta(x-x_i)$ because $\mathrm{Supp}(f^i(x,0)) = I_i$ and $\mu(I_i) < \Delta$. Doing this for all i, $f(x,t) = \sum_i f^i(x,t)$ converges to $\sum_i a_i\delta(x-x_i)$, which is what we wanted to prove.

□

5.5 Symmetry

Proposition 5.17 *If $f(x,0)$ is symmetric, then $f(x,t)$ is symmetric for all t.*

Proof: Let $h(x,t) = f(x,t) - f(1-x,t)$. We'll find the equation satisfied by $h(x,t)$ and show that $h(x,t) = 0$ is a solution of it.

$$\frac{\partial}{\partial t}h(x,t) = \frac{\partial}{\partial t}(f(x,t) - f(1-x,t)) = -2f(x,t)\int_{x-\Delta}^{x+\Delta} f(z,t)dz + 2f(1-x,t)\int_{1-x-\Delta}^{1-x+\Delta} f(z,t)dz$$

$$+\frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f(z,t)f\left(\frac{x-(1-w)z}{w},t\right)dz - \frac{2}{w}\int_{1-x-\Delta w}^{1-x+\Delta w} f(z,t)f\left(\frac{1-x-(1-w)z}{w},t\right)dz$$

$$= -2f(x,t)\int_{x-\Delta}^{x+\Delta} f(z,t)dz + 2f(1-x,t)\int_{x-\Delta}^{x+\Delta} f(1-z,t)dz$$

$$+\frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f(z,t)f\left(\frac{x-(1-w)z}{w},t\right)dz - \frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f(1-z,t)f\left(1-\frac{x-(1-w)z}{w},t\right)dz$$

Adding and substracting $2f(x,t)\int_{x-\Delta}^{x+\Delta} f(1-z,t)dz$ and $\frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f(1-z,t)f\left(\frac{x-(1-w)z}{w},t\right)dz$:

$$\frac{\partial}{\partial t}h(x,t) = -2f(x,t)\int_{x-\Delta}^{x+\Delta} h(z,t)dz - 2h(x,t)\int_{x-\Delta}^{x+\Delta} f(1-z,t)dz$$

$$+\frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} h(z,t)f\left(\frac{x-(1-w)z}{w},t\right)dz + \frac{2}{w}\int_{x-\Delta w}^{x+\Delta w} f(1-z,t)h\left(\frac{x-(1-w)z}{w},t\right)dz$$

Noting that $\frac{\partial}{\partial t}h(x,T) = 0 \quad \forall x$ if $h(x,T) = 0$ independently of T, we get that $h(x,t) = 0$ is a solution of the PDE and hence proves that if $h(x,0) = 0$ (symmetric initial condition), then $h(x,t) = 0 \quad \forall t \Rightarrow f(x,t) = f(1-x,t) \quad \forall t$. □

5.6 Boundedness after finite time

In this subsection we will prove that $f(x,t)$ remains bounded after a finite time T, given that $f(x,0)$ is bounded. Moreover, the absolute value of the n-th derivative with respect to time $\left|\frac{\partial^n}{\partial t^n}f(x,T)\right|$ is also bounded. The boundedness will also play a crucial role calculating the error of the numerical method implemented in section 6.

Proposition 5.18 *Let $M(t) = \sup|f(x,t)|$. Assume $M(0) < \infty$. Then:*

$$M(T) \leq e^{\left(\frac{2}{w}+\frac{2}{1-w}\right)T}(M(0)+4) - 4 \quad \forall\, T$$

Proof: We have, for all x, due to the non-negativeness of $f(x,t)$:

$$\frac{\partial f(x,t)}{\partial t} = -2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(z,t)dz\right) + \frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy$$

$$\leq \frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy$$

The objective is to bound the integral in terms of $M(t)$. We start fixing some arbitrary t:

Let $A_i = \{x \in \text{Supp}(f(x,t)) | i-1 < f(x,t) \leq i\}$. Note that $A_j = \emptyset\ \forall j > \lceil M(t)\rceil, \forall j \leq 0$ and that the A_i are disjoint.

Let an arbitrary $z \in A_j, \frac{x-(1-w)z}{w} \in A_k$. Then:

$$f(z,t)f\left(\frac{x-(1-w)z}{w},t\right) \leq \max\{j,k\}^2$$

For any x, we have that:

$$\frac{2}{w}\int_{x-w\Delta}^{x} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy \leq \frac{2}{w}\sum_{i,j}\mu\left(\left\{z\,\middle|\, z\in A_i, \frac{x-(1-w)z}{w}\in A_j\right\}\right)\max\{i,j\}^2$$

Decomposing the sum in two, and using the fact that the A_i are disjoint we can get that:

$$\frac{2}{w}\sum_{i,j}\mu\left(\left\{z\,\middle|\, z\in A_i, \frac{x-(1-w)z}{w}\in A_j\right\}\right)\max\{i,j\}^2 = \underbrace{\frac{2}{w}\sum_{i}\mu\left(\left\{z\,\middle|\, z\in A_i, \frac{x-(1-w)z}{w}\in \bigcup_{k\leq i} A_k\right\}\right)i^2}_{I_1}$$

$$+\underbrace{\frac{2}{w}\sum_i \mu\left(\left\{z \,\middle|\, z \in \bigcup_{k<i} A_k, \frac{x-(1-w)z}{w} \in A_i\right\}\right) i^2}_{I_2}$$

We can bound I_1 and I_2 now as:

$$I_1 \leq \frac{2}{w}\sum_i \mu(A_i) i^2$$

$$I_2 \leq \frac{2}{1-w}\sum_i \mu(A_i) i^2$$

We now want to find the worst case, which is the one where we maximize $\sum_i \mu(A_i) i^2$. Note that our variables are the $\mu(A_i)$. However, we have the following restrictions:

$$\sum_i \mu(A_i) \leq 1 \text{ (The support of } f(x,t) \text{ is contained in } [0,1])$$

$$\sum_i (i-1)\mu(A_i) \leq \int_0^1 f(x,t)dx = 1$$

Plugging the second restriction into the objective function, we get that:

$$\sum_{i=1}^{\lceil M(t)\rceil} \mu(A_i) i^2 \leq \sum_{i=1}^{\lceil M(t)\rceil -1} \mu(A_i) i^2 + \frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil -1} - \frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil -1} \sum_{i=1}^{\lceil M(t)\rceil -1} \mu(A_i)(i-1)$$

$$= \frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil -1} + \sum_{i=1}^{\lceil M(t)\rceil -1} \mu(A_i)\left(i^2 - \frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil -1}(i-1)\right)$$

$$= \frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil -1} + \frac{1}{\lceil M(t)\rceil -1} \sum_{i=1}^{\lceil M(t)\rceil -1} \mu(A_i)\left(\lceil M(t)\rceil i - \lceil M(t)\rceil - i)(i - \lceil M(t)\rceil\right)$$

Studying the coefficients:

$$(\lceil M(t)\rceil i - \lceil M(t)\rceil - i)(i - \lceil M(t)\rceil) \leq 0 \quad \text{ if } \lceil M(t)\rceil > i > 1$$

because $i - \lceil M(t)\rceil < 0$ trivially, and:

$$\lceil M(t)\rceil i - \lceil M(t)\rceil - i \geq 0 \Leftrightarrow \lceil M(t)\rceil i - \lceil M(t)\rceil - i + 1 \geq 1 \Leftrightarrow (\lceil M(t)\rceil - 1)(i-1) \geq 1$$

which is clearly true under the constraints on i. However, for $i = 1$:

$$(\lceil M(t)\rceil i - \lceil M(t)\rceil - i)(i - \lceil M(t)\rceil) = \lceil M(t)\rceil - 1\rceil > 0$$

Therefore the maximum of the objective function is attained when $A_i = 0 \quad \forall\ i > 1$ and A_1 is as big as possible. Taking into account the first restriction based on the support of $f(x,t)$, $A_1 = 1$. In that case, we have that:

$$\sum_{i=1}^{\lceil M(t)\rceil} \mu(A_i) i^2 \leq \frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil -1} + 1 \leq \lceil M(t)\rceil + 3 \leq M(t) + 4$$

Therefore:

$$\sup_{A_i}\left\{\sum_i \mu(A_i) i^2\right\} \leq M(t) + 4$$

Finally, for any x we have:

$$\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy \le I_1+I_2 \le \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4)$$

which means that:

$$M'(t) \le \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4)$$

Integrating, we get the following bound:

$$\sup_x |f(x,T)| = M(T) \le e^{\left(\frac{2}{w}+\frac{2}{1-w}\right)T}(M(0)+4)-4$$

which proves the result. □

Proposition 5.19 *Let* $M(t)=\sup_x f(x,t)$*. Then* $\left|\frac{\partial}{\partial t}f(x,t)\right| \le \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4)$

Proof: We have, for all x:

$$\left|\frac{\partial}{\partial t}f(x,t)\right| = \left|-2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(z,t)dz\right)+\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy\right|$$

$$\le \max\left\{\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy,\ 2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(z,t)dz\right)\right\}$$

On the one hand, we have that:

$$2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(z,t)dz\right) \le 2M(t)\int_0^1 f(z,t)dz \le 2M(t)$$

On the other, using proposition 5.18:

$$\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy \le \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4)$$

Therefore:

$$\left|\frac{\partial}{\partial t}f(x,t)\right| \le \max\left\{2M(t),\ \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4)\right\} = \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4)$$

□

Proposition 5.20 *If* $\sup|f(x,0)|<\infty$*, then* $\left|\frac{\partial^n}{\partial t^n}f(x,T)\right| < \infty \quad \forall\, n,T$

Proof: Recalling the expression for the n-th derivative with respect to t:

$$\frac{\partial^{K+1}f(x,t)}{\partial t^{K+1}} = -2\sum_{i=0}^{K}\binom{K}{i}\frac{\partial^i}{\partial t^i}f(x,t)\left(\int_{x-\Delta}^{x+\Delta}\frac{\partial^{K-i}}{\partial t^{K-i}}f(z,t)dz\right)$$
$$+\sum_{i=0}^{K}\binom{K}{i}\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta}\frac{\partial^i}{\partial t^i}f(y,t)\frac{\partial^{K-i}}{\partial t^{K-i}}f\left(\frac{x-(1-w)y}{w},t\right)dy \qquad (9)$$

We will proceed inductively, first fixing T and then bounding the derivatives in increasing order. For $n = 0$, the result follows from proposition 5.19. Let us suppose that

$$\sup_x \left| \frac{\partial^K f(x,T)}{\partial t^K} \right| \leq B(K) \quad \forall\, K < n$$

Then:

$$\left| \frac{\partial^n f(x,T)}{\partial t^n} \right| \leq \left| 2 \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{\partial^i}{\partial t^i} f(x,T) \left(\int_{x-\Delta}^{x+\Delta} \frac{\partial^{n-1-i}}{\partial t^{n-1-i}} f(z,T) dz \right) \right|$$
$$+ \left| \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{2}{w} \int_{x-w\Delta}^{x+w\Delta} \frac{\partial^i}{\partial t^i} f(y,T) \frac{\partial^{n-1-i}}{\partial t^{n-1-i}} f\left(\frac{x-(1-w)y}{w}, T \right) dy \right|$$
$$\leq 4\Delta \sum_{i=0}^{n-1} \binom{n-1}{i} B(i)B(n-1-i) + 4\Delta \sum_{i=0}^{n-1} \binom{n-1}{i} B(i)B(n-1-i) = 8\Delta \sum_{i=0}^{n-1} \binom{n-1}{i} B(i)B(n-1-i) < \infty$$

which bounds the n-th derivative. This proves the inductive step and concludes the proof.

□

5.7 Convexity Approach

In this subsection we describe a technique to find a bound for the critical Δ that distinguishes between having one or two Diracs when t goes to infinity. Although the bound is suboptimal, to the best of our knowledge there have been no proofs in this direction: neither in the deterministic nor in the probabilistic models. The technique is based on the study throughout time of some quantity $Q_K(t)$, namely the scalar product between $f(x,t)$ and some kernel $K(x)$. The problem here is to find a suitable kernel $K(x)$ such that $Q_K(t)$ has nice properties. In our case, we have found that choosing a convex, continuous kernel $K(x)$ we find the desired properties and can proof convergence to one Dirac under some symmetry assumptions on the initial condition.

Proposition 5.21 *Let* $Q_K(t) = \int_0^1 K(x) f(x,t) dx$*, where* $K(x)$ *is continuous and convex and* $f(x,0)$ *is symmetric with respect to* $x = \frac{1}{2}$*. Let* S_K *be the set of* x_0 *that satisfy* $Q_K(0) = \frac{1}{2}[K(x_0) + K(1-x_0)]$ *and* $d = \inf\left\{x \in S_K | x \geq \frac{1}{2}\right\} - \sup\left\{x \in S_K | x \leq \frac{1}{2}\right\}$*. Then one of the* $K(x)$ *such that* d *is positive and minimal is:*

$$K_{opt}(x) = \begin{cases} -x + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{2} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Moreover, in this case d *is given by* $1 - 4\int_0^1 x f(x,0) dx$ *and* K *is not unique.*

Proof: We will start computing the derivative of $Q_K(t)$. Using the same techniques as in proposition 5.5, we get that:

$$\frac{\partial Q_K(t)}{\partial t} = 2 \int_0^1 \int_{x-\Delta}^{x+\Delta} \left[K(wx + (1-w)y) + K(wy + (1-w)x) - K(x) - K(y) \right] f(x,t) f(y,t) dx dy$$

By the convexity of $K(x)$:

$$K(wx + (1-w)y) + K(wy + (1-w)x) \leq wK(x) + (1-w)K(y) + wK(y) + (1-w)K(x) = K(x) + K(y)$$

Therefore $Q_K(t)$ is a decreasing function of t independently of K.

Let $H(x) = \frac{1}{2}\left[K(x) + K(1-x)\right]$. Trivially we have:

$$H(1-x) = \frac{1}{2}\left[K(1-x) + K(x)\right] = H(x)$$

$$H(ax + (1-a)y) = \frac{1}{2}\left[K(ax + (1-a)y) + K(a + (1-a) - ax - (1-a)y)\right]$$

$$\leq \frac{1}{2}\left[aK(x) + (1-a)K(y) + aK(1-x) + (1-a)K(1-y)\right] = aH(x) + (1-a)H(y)$$

which proves that H is also symmetric and convex. Moreover, H is continuous because K is continuous.

On the one hand, we have that $\min_{x\in[0,1]} H(x) = H\left(\frac{1}{2}\right)$. Supposing the contrary, that the minimum is attained at x_0 and it is strictly smaller than $H\left(\frac{1}{2}\right)$. Then:

$$H\left(\frac{1}{2}\right) = H\left(\frac{1}{2}x_0 + \frac{1}{2}(1-x_0)\right) \leq \frac{1}{2}H(x_0) + \frac{1}{2}H(1-x_0) = H(x_0)$$

and we get a contradiction.

On the other hand, we have that $\max_{x\in[0,1]} H(x) = H\left(0\right) = H(1)$. Supposing the contrary, that the maximum is attained at x_0 and it is strictly greater than $H\left(0\right)$. Then:

$$H((1-x_0)\cdot 0 + x_0 \cdot 1) \leq (1-x_0)H(0) + x_0 H(1) = H(0)$$

and again, we get a contradiction.

From these inequalities, the following bound is immediate:

$$Q_K(0) = \int_0^{\frac{1}{2}} K(x)f(x,0)dx + \int_{\frac{1}{2}}^1 K(x)f(x,0)dx = \int_0^{\frac{1}{2}} K(x)f(x,0)dx + \int_0^{\frac{1}{2}} K(1-x)f(x,0)dx = 2\int_0^{\frac{1}{2}} H(x)f(x,0)dx$$

$$H(0) = 2\int_0^{\frac{1}{2}} H(0)f(x,0)dx \geq 2\int_0^{\frac{1}{2}} H(x)f(x,0)dx = Q_K(0) \geq 2\int_0^{\frac{1}{2}} H\left(\frac{1}{2}\right)f(x,0)dx = H\left(\frac{1}{2}\right)$$

By the continuity of H, we can guarantee that the equation $H(x) = Q_K(0)$ has at least one solution in $\left[0, \frac{1}{2}\right]$. We can now assume that $Q_K(0) > H\left(\frac{1}{2}\right)$, because otherwise d would be equal to 0 as $\frac{1}{2}$ is a solution of the equation. As we also explicitly specify the optimal $K_{opt}(x)$, the existence of such a $K(x)$ is proven later by verification.

We will now prove that $H(x) = Q_K(0)$ has exactly one solution in $\left[0, \frac{1}{2}\right)$. Note that the exact value of $Q_K(0)$ is irrelevant in the equation. It suffices that the right hand side is greater than $H\left(\frac{1}{2}\right)$ to have uniqueness of solutions in x. Let us suppose that $H(x_0) = H(x_1) = Q_K(0)$, $x_0 < x_1 < \frac{1}{2}$. We have that:

$$H(x_1) = H\left(\frac{x_1 - x_0}{\frac{1}{2} - x_0}\frac{1}{2} + \frac{\frac{1}{2} - x_1}{\frac{1}{2} - x_0}x_0\right) \leq \frac{\frac{1}{2} - x_1}{\frac{1}{2} - x_0}H(x_0) + \frac{x_1 - x_0}{\frac{1}{2} - x_0}H\left(\frac{1}{2}\right) < H(x_0)$$

Thus, the only solution $x_s < \frac{1}{2}$ that satisfies $H(x_s) = Q_K(0)$ is bounded by the following equation:

$$x_s \leq \frac{1}{2}\frac{Q_K(0) - H(0)}{H\left(\frac{1}{2}\right) - H(0)}$$

because of:

$$Q_K(0) = H(x_s) = H\left((1-2x_s)\cdot 0 + 2x_s \cdot \frac{1}{2}\right) \leq (1-2x_s)H(0) + 2x_s H\left(\frac{1}{2}\right) = H(0) + 2x_s\left[H\left(\frac{1}{2}\right) - H(0)\right]$$

Indeed, we can select a $K_{opt}(x)$ such that the bound is attained. This $K_{opt}(x)$ is given by:

$$K_{opt}(x) = \begin{cases} 2\left[H\left(\frac{1}{2}\right) - H(0)\right]x + H(0) & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2\left[H\left(\frac{1}{2}\right) - H(0)\right](1-x) + H(0) & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Being K symmetric, $H(x) = K(x)$. Finally, we will prove that the calculation of x_s is independent of $H(0)$ and $H\left(\frac{1}{2}\right)$ because:

$$Q_K(0) = 4\left[H\left(\frac{1}{2}\right) - H(0)\right]\int_0^{\frac{1}{2}} xf(x,0)dx + H(0) = H(x_s)$$

Solving the equation, we get that $x_s = 2\int_0^{\frac{1}{2}} xf(x,0)dx$, which is independent of $H(0)$ and $H\left(\frac{1}{2}\right)$. Therefore we can pick any pair of values for $H(0)$ and $H\left(\frac{1}{2}\right)$, provided that they are different. In our case, for simplicity, we pick $H(0) = \frac{1}{2}, H\left(\frac{1}{2}\right) = 0$.

Knowing that the only solution is x_s and, by symmetry $1 - x_s$, the distance between the two solutions is given by:

$$d = 1 - x_s - x_s = 1 - 4\int_0^{\frac{1}{2}} xf(x,0)dx$$

which is what we wanted to prove.

□

Theorem 5.22 *If $f(x,0)$ is a symmetric initial condition, then $\lim_{t\to\infty} f(x,t) = \delta\left(x - \frac{1}{2}\right)$ for any*

$$\Delta \geq \Delta_c = \max\left(1 - 4\int_0^{\frac{1}{2}} xf(x,0)dx, \frac{1}{2}\right)$$

Proof: Let $\Delta \geq \Delta_c$. As $\Delta \geq \frac{1}{2}$ by proposition 5.14 we know that the limit is either 1 or 2 Diracs. Suppose that the limit is $g(x) = \frac{1}{2}[\delta(x - x_0) + \delta(x - (1 - x_0)]$, $1 - 2x_0 > \Delta$, where we also have used the symmetry of the initial condition in the expression of the limit. Let $K(x)$ and $H(x)$ be defined as in proposition 5.21, and let $K_{opt}(x)$ be the optimal kernel. Then, $H(x_0) = \lim_{t\to\infty} Q_K(t) \leq Q_K(0)$. Let S_{inf} be the set of solutions to $H(x) = \lim_{t\to\infty} Q_K(t)$ and S_0 the set of solutions to $H(x) = Q_K(0)$. We have proved in 5.21 that both sets consist of at most two points. If we assume that $H(x)$ is decreasing on $\left[0, \frac{1}{2}\right]$, then:

$$d_{inf} = \inf\left\{x \in S_{inf} | x \geq \frac{1}{2}\right\} - \sup\left\{x \in S_{inf} | x \leq \frac{1}{2}\right\} \leq \inf\left\{x \in S_0 | x \geq \frac{1}{2}\right\} - \sup\left\{x \in S_0 | x \leq \frac{1}{2}\right\} = d_0$$
$$= 1 - 4\int_0^{\frac{1}{2}} xf(x,0)dx \leq \Delta$$

We get a contradiction with the fact that the two solutions of the equation $H(x_0) = \lim_{t\to\infty} Q_K(t)$ have to be separated by more than Δ. The contradiction comes from the assumption that $\lim_{t\to\infty} f(x,t) = \frac{1}{2}[\delta(x - x_0) + \delta(x - (1 - x_0)]$.

Finally, we prove that $H(x)$ is decreasing on $\left[0, \frac{1}{2}\right]$ (and increasing on $\left[\frac{1}{2}, 1\right]$ by symmetry). Let us suppose that for some $x_0 < x_1 < \frac{1}{2}$, $H(x_0) < H(x_1)$. Then, by convexity:

$$H(x_1) = H\left(\frac{x_1 - x_0}{\frac{1}{2} - x_0}\frac{1}{2} + \frac{\frac{1}{2} - x_1}{\frac{1}{2} - x_0}x_0\right) \leq \frac{x_1 - x_0}{\frac{1}{2} - x_0}H\left(\frac{1}{2}\right) + \frac{\frac{1}{2} - x_1}{\frac{1}{2} - x_0}H(x_0) \leq H(x_0)$$

which is a contradiction, and we are done.

□

Using the theorem, we can get bounds for interesting cases. For example:

Corollary 5.23 *Let* $f(x,0) = 1_{[0,1]}$ *(uniform distribution in* $[0,1]$*). Then, for any* $\Delta \geq \frac{1}{2}$ *we have (total) consensus, i.e:*

$$\lim_{t\to\infty} f(x,t) = \delta\left(x - \frac{1}{2}\right)$$

Proof:

$$1 - 4\int_0^{\frac{1}{2}} x dx = \frac{1}{2}$$

By applying theorem 5.22 we get the desired result. □

In other words, if we have a uniform distribution regarding the opinions about a certain topic, the population will end up having the same opinion (center) if every person is at least as tolerant such that they accept opinions from the rest that are half of the spectrum away or less.

Another interesting distribution is one where we have three types of people: extremists (in both sides) with density $\frac{1-\alpha}{2}$ each and totally undecided people (at the middle) with density α. We can get the following result:

Corollary 5.24 *Let* $f(x,0) = \left(\frac{1-\alpha}{2}\right)\delta(x) + \alpha\delta\left(x - \frac{1}{2}\right) + \left(\frac{1-\alpha}{2}\right)\delta(x-1)$*. We have consensus in the following cases:*

$$\begin{cases} \Delta \geq 1 - 2\alpha & \text{if } \alpha \leq \frac{1}{4} \\ \Delta \geq \frac{1}{2} & \text{if } \alpha \geq \frac{1}{4} \end{cases}$$

Proof:

$$1 - 4\int_0^{\frac{1}{2}} x f(x,0) dx = 1 - 2\alpha$$

By applying theorem 5.22 we get the desired result. □

6 Numerical Approach

In order to watch the dynamics of the system, and after the impossibility to find a closed solution of the equation, we have developed a numerical method and a simulator to test the system under given initial conditions. The simulator was programmed in 600 lines of C++ code, and the parsing and plotting of the result in Matlab. In this section, we present the algorithm used, and analyze its error and complexity. It is important to remark that this algorithm improves the running time of the probabilistic methods used in [25] for a large number of users.

6.1 Algorithm

The algorithm used takes as input an initial condition $f^r(x,0)$, which is a piecewise constant function of I intervals, a time T after which we want to calculate an approximate solution and a maximum error ε and outputs an approximation of the solution $f^r(x,T)$. It works as follows:

First, we perform a discretization in t. In steps of Δt we approximate $f^r(x,t+\Delta t)$ by using a forward Euler method. In other words, we say that:

$$f^r(x,t+\Delta t) \approx f^r(x,t) + \Delta t \partial_t f^r(x,t) = f^e(x,t+\Delta t)$$

Here we exploit the fact that $f^r(x,t)$ is a piecewise constant function, so that we can calculate analytically the derivative which is a piecewise linear function. The deduction of the formula for the derivative is explained later. Hence, $f^e(x,t+\Delta t)$ is also piecewise linear, as it is the sum of a piecewise linear and a piecewise constant function. Then, we approximate $f^e(x, t + \Delta t)$ with another piecewise constant function (which we will call $f^r(x, t + \Delta t)$ for simplicity) of $I_{t+\Delta t}$ intervals, so that we can reuse the same scheme and we can compute explicitly the expression for the derivative. The error is chosen in such a way that the distance between the associated Lebesgue-Stieltjes measures is minimized (see section 5 for the distance used). We perform this loop until we calculate $f^r(x, T)$ in steps of Δt.

Knowing beforehand the complexity, we can choose the parameters Δt and I_t so that the total error is less than the specified. We have two ways of selecting them, either in a fixed or in an adaptative way:

The first way consists on having a constant number of intervals throughout the algorithm. Although the internal loop is executed faster (only once), we might overestimate the number of intervals at some time, where the equation is not stiff enough or Δt is very small. In contrast, if we decide to adapt the number of intervals at each step so that we bound the maximum error per iteration, we are sure that we won't have more than the necessary intervals, but at the cost of possibly having to recalculate $f^r(x, t)$ several times, when errors are big. In any case, the asymptotic cost of both algorithms is the same, as the calculation of $f^r(x, t)$ is not the bottleneck, which is the calculation of $f^e(x, t)$.

Both algorithms can be seen in the following figures:

Input: $f^r(x.0), T, \varepsilon_{max}$
Output: $f^r(x, T)$

Pick Δt and I according to ε_{max};
for $t \leftarrow 0$ **to** T **step** Δt **do**
  $f^e(x, t + \Delta t) \leftarrow f^r(x, t) + \Delta t \partial_t f^r(x, t)$;
  $f^r(x, t + \Delta t) \leftarrow$ `PiecewiseConstantApproximation(`$f^e(x, t + \Delta t), I$`)`;
end

Algorithm 1: Fixed I_t

Input: $f^r(x.0), T, \varepsilon_{max}, \Delta t$
Output: $f^r(x, T)$

for $t \leftarrow 0$ **to** T **step** Δt **do**
  $f^e(x, t + \Delta t) \leftarrow f^r(x, t) + \Delta t \partial_t f^r(x, t)$;
  $I \leftarrow 1$;
  repeat
    $f^r(x, t + \Delta t) \leftarrow$ `PiecewiseConstantApproximation(`$f^e(x, t + \Delta t), I$`)`;
    $\varepsilon_{curr} \leftarrow$ `GetError(`$f^r(x, t + \Delta t), f^e(x, t + \Delta t)$`)`;
    $I \leftarrow 2I$;
  until $\varepsilon_{curr} < \varepsilon_{max}$;
end

Algorithm 2: Adaptative I_t

6.2 Calculus of the optimal $f^r(x, t)$

The objective in this subsection is to determine which is the best approximation $f^r(x, t)$ (piecewise constant function) to $f^e(x, t)$ (piecewise linear function) such that the distance between the associated Lebesgue-Stieltjes measures with the total variation norm. Let $\nu_r(x)$ and $\nu_e(x)$ be the associated measures. Note that we can minimize the error separately for each interval independently. Therefore, given a $\nu_e(x)$ associated to $f^e(x) = ax + b$ and an interval $X = [x_s, x_e]$ we want to find:

$$\min_{\nu_r} \int_X |d\nu_e(x) - d\nu_r(x)| = \min_M \int_{x_s}^{x_e} |ax + b - M| dx$$

If $a = 0$, then $M = b$ clearly minimizes the expression and the error is 0. Let's suppose $a \neq 0$. If M lies between $ax_s + b$ and $ax_e + b$, then:

$$\min_M \int_{x_s}^{x_e} |ax + b - M|dx = \min_M \frac{1}{2a}[(ax_e + b - M)^2 + (ax_s + b - M)^2] = \frac{a}{4}(x_e - x_s)^2$$

The minimum is attained for

$$M_{min} = \frac{\int_{x_s}^{x_e} (ax + b)dx}{x_e - x_s} = \frac{a}{2}(x_s + x_e) + b$$

If M lies outside ax_s+b and ax_e+b, then $\int_{x_s}^{x_e} |ax+b-M|dx = \frac{a}{2}(x_e-x_s)^2+(x_e-x_s)(\min\{ax_s, ax_e\}+b-M)$ if M lies below the minimum of $ax_s + b$ and $ax_e + b$ or $\int_{x_s}^{x_e} |ax + b - M|dx = \frac{a}{2}(x_e - x_s)^2 + (x_e - x_s)(M - \max\{ax_s, ax_e\} - b)$ if M lies above the maximum. Therefore the overall minimum is $\frac{a}{4}(x_e - x_s)^2$. It is noticeable that the x_{min} such that $ax_{min} + b = M_{min}$ is precisely the midpoint between x_s and x_e.

Let f^r be defined piecewise in the intervals $X_i = [x_i, x_{i+1}]$ and let $M_{min,i}$ be the value of M that minimizes the distance between the measures for the interval X_i. We should notice that in this case, the integral is preserved, as:

$$\int_0^1 f^r(x)dx = \int_0^1 \sum_{i=1}^{I} M_{min,i} 1_{X_i} dx = \sum_{i=1}^{I} \int_{X_i} M_{min,i} dx = \sum_{i=1}^{I} \int_{X_i} \frac{\int_{x_i}^{x_{i+1}} f^e(y)dy}{x_{i+1} - x_i} = \sum_{i=1}^{I} \int_{x_i}^{x_{i+1}} f^e(y)dy = \int_0^1 f^e(y)dy$$

Therefore, both $f^r(x, t)$ and $f^e(x, t)$ have integral 1 for all t independently of the rest of the parameters. This is important because it is used when we bound quantities related to $f^r(x, t)$ by applying proposition 5.19.

6.3 Analytical expression of $\partial_t f^r(x, t)$

Now we will give an exact expression for the derivative, given that $f^r(x, t)$ is piecewise constant. Let us suppose that for a given t:

$$f^r(x, t) = \sum_{i=1}^{I} a_i[H(x - x_{i+1}) - H(x - x_i)]$$

where $H(x)$ is the Heaviside step function. Looking at (4), we will calculate the contribution from each integral separately.

It's important to notice that it's enough to calculate for any x_i and x_j:

$$I_1^{i,j}(x) = \int_{x-\Delta}^{x+\Delta} H(x - x_i)H(z - x_j)dz = \int_{-\Delta}^{\Delta} H(x - x_i)H(x + u - x_j)du$$

$$I_2^{i,j}(x) = \frac{1}{w} \int_{x-w\Delta}^{x+w\Delta} H(z - x_i)H\left(\frac{x - (1 - w)z - wx_j}{w}\right) dz = \int_{-\Delta}^{\Delta} H(x + wu - x_i)H(x - (1 - w)u - x_j)du$$

The result of $I_1^{i,j}(x)$ depends on the relative order between x_i and x_j. It is summarized in table 2:

The result of $I_2^{i,j}(x)$ also depends on the relative order between x_i and x_j. Let

$$m = \max\{(1 - w)x_i + wx_j, x_i - w\Delta\}$$

We classify the different cases and the result of $I_2^{i,j}(x)$ in table 3:

Case	$I_1^{i,j}(x)$
$x_i \leq x_j - \Delta \leq x_j + \Delta$	$\begin{cases} 0 & \text{if } x \leq x_j - \Delta \\ x-(x_j-\Delta) & \text{if } x_j - \Delta \leq x \leq x_j + \Delta \\ 2\Delta & \text{if } x_j + \Delta \leq x \end{cases}$
$x_j - \Delta \leq x_i \leq x_j + \Delta$	$\begin{cases} 0 & \text{if } x \leq x_i \\ x-(x_j-\Delta) & \text{if } x_i \leq x \leq x_j + \Delta \\ 2\Delta & \text{if } x_j + \Delta \leq x \end{cases}$
$x_j - \Delta \leq x_j + \Delta \leq x_i$	$\begin{cases} 0 & \text{if } x \leq x_i \\ 2\Delta & \text{if } x_i \leq x \end{cases}$

Table 2: $I_1^{i,j}(x)$

Finally, we can calculate $\partial_t f^r(x,t)$ as:

$$\begin{aligned}\partial_t f^r(x,t) = -2\sum_{i,j} a_i a_j (I_1^{i,j}(x) + I_1^{i+1,j+1}(x) - I_1^{i,j+1}(x) - I_1^{i+1,j}(x)) \\ +2\sum_{i,j} a_i a_j (I_2^{i,j}(x) + I_2^{i+1,j+1}(x) - I_2^{i,j+1}(x) - I_2^{i+1,j}(x))\end{aligned}$$

6.4 Analysis of the error

We are interested in estimating the error that we are making while approximating $f(x,T)$ by $f^r(x,T)$. Again, we will use as metric the distance between the associated Lebesgue-Stieltjes measures. Let $\nu_e^t(x)$ and $\nu_r^t(x)$ be the Lebesgue-Stieltjes measures associated to $f^e(x,t)$ and $f^r(x,t)$ respectively. We also need to define:

$$g^s(x,t) = f(x,t) \text{ s.t. } f(x,s) = f^r(x,s)$$

that is, the prolongation of $f^r(x,s)$. Let also $\mu_s^t(x)$ be the associated measure to $g^s(x,t)$. Note that $\nu_r^t(x) = \mu_t^t(x)$. Thus, we want to calculate:

$$\varepsilon_{tot} = \|\mu_0^T(x) - \nu_r^T(x)\| = \left\| \sum_{k=1}^{T/(\Delta t)} \mu_{(k-1)\Delta t}^T(x) - \mu_{k\Delta t}^T(x) \right\| \leq \sum_{k=1}^{T/(\Delta t)} \left\| \mu_{(k-1)\Delta t}^T(x) - \mu_{k\Delta t}^T(x) \right\|$$

We will start calculating the error done in one iteration of the loop, which is:

$$\|\mu_{k\Delta t}^{k\Delta t}(x) - \mu_{(k-1)\Delta t}^{k\Delta t}(x)\| \leq \|\nu_r^{k\Delta t}(x) - \nu_e^{k\Delta t}(x)\| + \|\nu_e^{k\Delta t}(x) - \mu_{(k-1)\Delta t}^{k\Delta t}(x)\|$$

We begin calculating the error done by the approximation to constant splines. Let I_0 be the smallest I such that $w\Delta$, $(1-w)\Delta$ and Δ are multiples of $\frac{1}{I}$. We assume that I is a multiple of I_0. We first calculate the error when $I = I_0$ and then we will prove that the error is proportional to $\frac{1}{I}$ for large enough I. We multiply the error for each interval by the number of intervals I_0. Keeping in mind that for any interval, the slope of $f^e(x,k\Delta t)$ is bounded by $\dfrac{\Delta y}{\Delta x} = \dfrac{2\Delta t \sup \|\partial_t f^r(x,(k-1)\Delta t)\|}{1/I_0}$:

$$\varepsilon_1(I_0) \leq I_0 \frac{\text{Max. Slope}}{4}\left(\frac{1}{I_0}\right)^2 = \frac{\Delta t}{2} \sup \|\partial_t f^r(x,(k-1)\Delta t)\| \tag{10}$$

However, if we divide each interval in two, the error is halved, because the error with two intervals equals $2\dfrac{\text{Slope}}{4}\left(\dfrac{1}{1/2I_0}\right)^2$, where with one is equal to $\dfrac{\text{Slope}}{4}\left(\dfrac{1}{1/I_0}\right)^2$. Therefore, for sufficiently large I we can write:

$$\varepsilon_1(I) = \varepsilon_1(I_0)\frac{I_0}{I} \tag{11}$$

Case	$I_2^{i,j}(x)$
$m \leq x_i + w\Delta \leq x_j - (1-w)\Delta \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_j - (1-w)\Delta \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$x_i + w\Delta \leq m \leq x_j - (1-w)\Delta \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_j - (1-w)\Delta \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$m \leq x_j - (1-w)\Delta \leq x_i + w\Delta \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} - \frac{x_i-x}{w} & \text{if } x_j - (1-w)\Delta \leq x \leq x_i + w\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_i + w\Delta \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$x_i + w\Delta \leq x_j - (1-w)\Delta \leq m \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq m \\ \frac{x-x_j}{1-w} + \Delta & \text{if } m \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$m \leq x_j - (1-w)\Delta \leq x_j + (1-w)\Delta \leq x_i + w\Delta$	$\begin{cases} 0 & \text{if } x \leq x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} - \frac{x_i-x}{w} & \text{if } x_j - (1-w)\Delta \leq x \leq x_j + (1-w)\Delta \\ \Delta - \frac{x_i-x}{w} & \text{if } x_j + (1-w)\Delta \leq x \leq x_i + w\Delta \\ 2\Delta & \text{if } x_i + w\Delta \leq x \end{cases}$
$x_i + w\Delta \leq x_j - (1-w)\Delta \leq x_j + (1-w)\Delta \leq m$	$\begin{cases} 0 & \text{if } x \leq m \\ 2\Delta & \text{if } m \leq x \end{cases}$
$x_j - (1-w)\Delta \leq m \leq x_i + w\Delta \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq m \\ \frac{x-x_j}{1-w} - \frac{x_i-x}{w} & \text{if } m \leq x \leq x_i + w\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_i + w\Delta \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$x_j - (1-w)\Delta \leq x_i + w\Delta \leq m \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq m \\ \frac{x-x_j}{1-w} + \Delta & \text{if } m \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$x_j - (1-w)\Delta \leq m \leq x_j + (1-w)\Delta \leq x_i + w\Delta$	$\begin{cases} 0 & \text{if } x \leq m \\ \frac{x-x_j}{1-w} + \frac{x_i-x}{w} & \text{if } m \leq x \leq x_j + (1-w)\Delta \\ \Delta - \frac{x_i-x}{w} & \text{if } x_j + (1-w)\Delta \leq x \leq x_i + w\Delta \\ 2\Delta & \text{if } x_i + w\Delta \leq x \end{cases}$
$x_j - (1-w)\Delta \leq x_i + w\Delta \leq x_j + (1-w)\Delta \leq m$	$\begin{cases} 0 & \text{if } x \leq m \\ 2\Delta & \text{if } m \leq x \end{cases}$
$x_j - (1-w)\Delta \leq x_j + (1-w)\Delta \leq m \leq x_i + w\Delta$	$\begin{cases} 0 & \text{if } x \leq m \\ \Delta - \frac{x_i-x}{w} & \text{if } m \leq x \leq x_i + w\Delta \\ 2\Delta & \text{if } x_i + w\Delta \leq x \end{cases}$
$x_j - (1-w)\Delta \leq x_j + (1-w)\Delta \leq x_i + w\Delta \leq m$	$\begin{cases} 0 & \text{if } x \leq m \\ 2\Delta & \text{if } m \leq x \end{cases}$

Table 3: $I_2^{i,j}(x)$

We will now bound $\sup \|\partial_t f^r(x,(k-1)\Delta t)\|$. We will proceed successively by k to bound $\sup |f^r(x, k\Delta t)|$. We define $M(t) = \sup |f^r(x,t)|$ and we also suppose that $\sup |f^r(x,0)| = M(0) = M < \infty$. Now:

$$
\begin{aligned}
M(\Delta t) = \sup |f^r(x,\Delta t)| \leq \sup |f^e(x,\Delta t)| \leq \sup |f^r(x,0)| + \Delta t \sup |\partial_t f^r(x,0)| \\
\leq M + \Delta t K_1 M + \Delta t K_2 = (1 + \Delta t K_1)M + \Delta t K_2
\end{aligned}
$$

where $K_1 = \frac{2}{w} + \frac{2}{1-w}, K_2 = \frac{8}{w} + \frac{8}{1-w}$. The first inequality is true because when we approximate by piecewise constant splines, the maximum of the function decreases. The second equality is trivial and the third is true by proposition 5.18. Continuing with this process:

$$M(2\Delta t) \leq (1+\Delta t K_1)M(\Delta t) + \Delta t K_2 = (1+\Delta t K_1)^2 M + \Delta t K_2(1+\Delta t K_1 + 1)$$

$$M(3\Delta t) \leq (1+\Delta t K_1)M(2\Delta t) + \Delta t K_2 = (1+\Delta t K_1)^3 M + \Delta t K_2 \sum_{i=0}^{2}(1+\Delta t K_1)^i$$

$$\vdots$$

$$M(k\Delta t) \leq (1+\Delta t K_1)M((k-1)\Delta t) + \Delta t K_2 = (1+\Delta t K_1)^k M + \Delta t K_2 \sum_{i=0}^{k-1}(1+\Delta t K_1)^i$$

$$\vdots$$

$$M\left(\frac{T}{\Delta t}\Delta t\right) \leq (1+\Delta t K_1)^{\frac{T}{\Delta t}} M + \Delta t K_2 \sum_{i=0}^{T/\Delta t - 1}(1+\Delta t K_1)^i$$

We can now bound $M(k\Delta t)$ the following way. As K_1 and K_2 are positive:

$$(1+\Delta t K_1)^k M \leq (1+\Delta t K_1)^{\frac{T}{\Delta t}} M$$

$$\Delta t K_2 \sum_{i=0}^{k-1}(1+\Delta t K_1)^i = \Delta t K_2 \frac{(1+\Delta t K_1)^k - 1}{\Delta t K_1} \leq \frac{K_2}{K_1}((1+\Delta t K_1)^{\frac{T}{\Delta t}} - 1) \leq \frac{K_2}{K_1}(1+\Delta t K_1)^{\frac{T}{\Delta t}}$$

We now consider $h(\Delta t) = (1+K_1\Delta t)^{\frac{T}{\Delta t}}$. We have that:

$$h'(\Delta t) = (1+K_1\Delta t)^{\frac{T}{\Delta t}} \left(-\frac{T\ln(1+K_1\Delta t)}{(\Delta t)^2} + \frac{TK_1}{\Delta t(1+K_1\Delta t)}\right)$$

We will prove that $h'(\Delta t)$ is negative so that we can bound $h(\Delta t)$ by $h(0)$. It is enough to see that:

$$\frac{TK_1}{\Delta t(1+K_1\Delta t)} \leq \frac{T\ln(1+K_1\Delta t)}{(\Delta t)^2} \Leftrightarrow \frac{K_1\Delta t}{(1+K_1\Delta t)} \leq \ln(1+K_1\Delta t)$$

For $\Delta t = 0$, both sides are equal. If we compare derivatives, the derivative of the left-hand side is equal to $\frac{K_1}{1+K_1\Delta t} - \frac{K_1^2\Delta t}{(1+K_1\Delta t)^2}$ whereas the derivative of the right-hand side is equal to $\frac{K_1}{1+K_1\Delta t}$, which is clearly greater. Therefore, $h'(\Delta t)$ is negative and we can make the following bound:

$$h(\Delta t) \leq h(0) = \lim_{\Delta t\to 0}(1+K_1\Delta t)^{\frac{T}{\Delta t}} = e^{K_1 T}$$

We can therefore bound, for any k:

$$M(k\Delta t) \leq (1+\Delta t K_1)^{\frac{T}{\Delta t}} M + \frac{K_2}{K_1}(1+\Delta t K_1)^{\frac{T}{\Delta t}} \leq e^{K_1 T}\left(M + \frac{K_2}{K_1}\right)$$

Using proposition 5.19, for any k:

$$\sup \|\partial_t f^r(x,(k-1)\Delta t)\| \leq K_1 M((k-1)\Delta t) + K_2 \leq K_1 e^{K_1 T}\left(M + \frac{K_2}{K_1}\right) + K_2 = C_1$$

Substituting in equation (11), we get that:

$$\varepsilon_1 \leq \frac{C_1 I_0}{2}\frac{\Delta t}{I} = O\left(\frac{\Delta t}{I}\right) \tag{12}$$

We now calculate the error of the Euler forward method. We have that:

$$\varepsilon_2 = \|\nu_e^{k\Delta t}(x) - \mu_{(k-1)\Delta t}^{k\Delta t}(x)\| = \int_0^1 |g^{(k-1)\Delta t}(x, k\Delta t) - g^{(k-1)\Delta t}(x,(k-1)\Delta t) - \Delta t \partial_t g^{(k-1)\Delta t}(x,(k-1)\Delta t)|dx$$

$$\leq \frac{1}{2}(\Delta t)^2 \sup \|\partial_{tt}^2 g^{(k-1)\Delta t}(x,(k-1)\Delta t)\| + O\left((\Delta t)^3\right)$$

Using proposition 5.20, we can bound, for any k:

$$\sup \|\partial_{tt}^2 g^{(k-1)\Delta t}(x,(k-1)\Delta t)\| \leq 16\Delta \sup \|\partial_t g^{(k-1)\Delta t}(x,(k-1)\Delta t)\| \sup \|g^{(k-1)\Delta t}(x,(k-1)\Delta t)\|$$

$$\leq 16\Delta\left(K_1 e^{K_1 T}\left(M + \frac{K_2}{K_1}\right) + K_2\right) e^{K_1 T}\left(M + \frac{K_2}{K_1}\right) = C_2$$

Therefore:

$$\varepsilon_2 \leq \frac{C_2}{2}(\Delta t)^2 + O((\Delta t)^3) = O((\Delta t)^2) \tag{13}$$

Adding equations (12) and (13) we get that:

$$\|\mu_{k\Delta t}^{k\Delta t}(x) - \mu_{(k-1)\Delta t}^{k\Delta t}(x)\| \leq \varepsilon_1 + \varepsilon_2 = O\left((\Delta t)^2 + \frac{\Delta t}{I}\right)$$

Finally, we will bound $\|\mu_{(k-1)\Delta t}^{T}(x) - \mu_{k\Delta t}^{T}(x)\|$ in terms of $\|\mu_{k\Delta t}^{k\Delta t}(x) - \mu_{(k-1)\Delta t}^{k\Delta t}(x)\|$:

$$\frac{\partial}{\partial t}\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx \leq \int_0^1 |\partial_t g^{k\Delta t}(x,t) - \partial_t g^{(k-1)\Delta t}(x,t)|$$

$$\leq \underbrace{\int_0^1 2\left|-g^{k\Delta t}(x,t)\int_{x-\Delta}^{x+\Delta} g^{k\Delta t}(z,t)dz + g^{(k-1)\Delta t}(x,t)\int_{x-\Delta}^{x+\Delta} g^{(k-1)\Delta t}(z,t)dz\right|}_{I}$$

$$+ \underbrace{\int_0^1 \frac{2}{w}\left|\int_{x-w\Delta}^{x+w\Delta} g^{k\Delta t}(z,t) g^{k\Delta t}\left(\frac{x-(1-w)z}{w}, t\right)dz - \int_{x-w\Delta}^{x+w\Delta} g^{(k-1)\Delta t}(z,t) g^{(k-1)\Delta t}\left(\frac{x-(1-w)z}{w}, t\right)dz\right|}_{J}$$

We will first bound I. We have that:

$$I \leq \underbrace{2\int_0^1 |g^{(k-1)\Delta t}(x,t) - g^{k\Delta t}(x,t)| \int_{x-\Delta}^{x+\Delta} g^{(k-1)\Delta t}(z,t)dzdx}_{I_1} + \underbrace{2\int_0^1 g^{k\Delta t}(x,t)\int_{x-\Delta}^{x+\Delta} |g^{(k-1)\Delta t}(z,t) - g^{k\Delta t}(z,t)|dzdx}_{I_2}$$

On the one hand:

$$I_1 \le 2\int_0^1 |g^{(k-1)\Delta t}(x,t) - g^{k\Delta t}(x,t)|dx$$

On the other:

$$I_2 \le 2\int_0^1 g^{k\Delta t}(x,t)\int_0^1 |g^{(k-1)\Delta t}(z,t) - g^{k\Delta t}(z,t)|dzdx \le 2\int_0^1 |g^{(k-1)\Delta t}(x,t) - g^{k\Delta t}(x,t)|dx$$

Now we will bound J:

$$J \le \underbrace{\frac{2}{w}\int_0^1\int_{x-w\Delta}^{x+w\Delta} g^{k\Delta t}(z,t)\left|g^{k\Delta t}\left(\frac{x-(1-w)z}{w},t\right) - g^{(k-1)\Delta t}\left(\frac{x-(1-w)z}{w},t\right)\right| dzdx}_{J_1}$$
$$+\underbrace{\frac{2}{w}\int_0^1\int_{x-w\Delta}^{x+w\Delta}\left|g^{k\Delta t}(z,t) - g^{(k-1)\Delta t}(z,t)\right| g^{(k-1)\Delta t}\left(\frac{x-(1-w)z}{w},t\right) dzdx}_{J_2}$$

$$J_1 = 2\int_0^1\int_{x-\Delta}^{x+\Delta} g^{k\Delta t}(x,t)\left|g^{k\Delta t}(z,t) - g^{(k-1)\Delta t}(z,t)\right| dzdx \le 2\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx$$

$$J_2 = 2\int_0^1\int_{x-\Delta}^{x+\Delta}\left|g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)\right| g^{(k-1)\Delta t}(z,t)\, dzdx \le 2\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx$$

Adding all the equations together we get that:

$$\frac{\partial}{\partial t}\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx \le I + J \le I_1 + I_2 + J_1 + J_2 \le 8\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx$$

Integrating:

$$\left\|\mu^t_{k\Delta t}(x) - \mu^t_{(k-1)\Delta t}(x)\right\| = \int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)|dx$$
$$\le e^{8(t-(k-1)\Delta t)}\int_0^1 |g^{k\Delta t}(x,(k-1)\Delta t) - g^{(k-1)\Delta t}(x,(k-1)\Delta t)|dx = e^{8(t-(k-1)\Delta t)}\|\mu^{k\Delta t}_{k\Delta t}(x) - \mu^{k\Delta t}_{(k-1)\Delta t}(x)\|$$

Therefore:

$$\varepsilon_{tot} \le \sum_{k=1}^{T/(\Delta t)}\left\|\mu^T_{(k-1)\Delta t}(x) - \mu^T_{k\Delta t}(x)\right\| \le e^{8T}\sum_{k=1}^{T/(\Delta t)}\left\|\mu^{k\Delta t}_{(k-1)\Delta t}(x) - \mu^{k\Delta t}_{k\Delta t}(x)\right\| = e^{8T}\frac{T}{\Delta t}O\left((\Delta t)^2 + \frac{\Delta t}{I}\right) = O\left(\Delta t + \frac{1}{I}\right)$$

6.5 Analysis of the complexity

We will now give the complexity analysis of both algorithms. For simplicity of the analysis, we will assume that I is large enough so that $w\Delta$, $(1-w)\Delta$ and Δ are multiples of $\frac{1}{I}$.

For the first algorithm we have that the computation of the derivative takes $O(I^2)$, since we have a double sum over I intervals. Also, this produces $O(I^2)$ splines since every $I_k^{i,j}(x), k = 1, 2$ is composed of at most 4 splines. Since the splines are not produced in increasing order of x, we need to sort them, which takes $O(I^2 \log I)$ time. Taking into account the expression of the derivative and the assumption on I, the support of every spline is the union of some of the intervals, i.e, there isn't any spline such that its support doesn't fully cover some interval. Therefore, we can compress our $O(I^2)$ splines into $O(I)$ splines in one pass ($O(I^2)$ time). Finally, we only need one pass to make the piecewise constant spline approximation since now everything is sorted and compressed. This takes $O(I)$ time.

Since all this loop is executed $\frac{T}{\Delta t}$ times, the running time has complexity $O\left(\frac{1}{\Delta t} I^2 \log I\right)$.

For the second algorithm, the procedure (and the cost) is the same until the piecewise constant approximation. In this case, we double the number of intervals until we are below some error ε_{max}. Therefore, the total cost is $O\left(\sum_{i=0}^{k} 2^i\right) = O(2^{k+1})$ for some k because both the error calculation and the piecewise constant approximation are linear in the number of intervals. As we know from the previous subsection that the error per iteration is $O\left(\frac{1}{I}\right)$ once fixed Δt, k is $O(-\log(\varepsilon_{max}))$ and therefore the complexity is $O\left(\frac{1}{\varepsilon_{max}}\right)$. Adding this for the $\frac{T}{\Delta t}$ executions of the loop, we get that the total running time is $O\left(\frac{1}{\Delta t}\frac{1}{\varepsilon_{max}^2} \log\left(\frac{1}{\varepsilon_{max}}\right)\right)$.

7 Simulations

In this section, we present the results got by simulating using the above described algorithm. We study different scenarios for the initial distribution: uniform, extremist and undecided. We plot different bifurcations (in terms of how many Diracs we have at the end) depending on Δ. Moreover, we compare the experimental results with the bounds obtained in section 5 and the probabilistic Monte Carlo simulations presented in [9].

7.1 Evolution of the system: different settings

In order to illustrate the behavior of the system as time passes, we show how the system evolves from a uniform distribution to one (or more) Diracs, depending on the deviation threshold Δ. We run those sets of experiments for 3 different values of w : $0.5, 0.75$ and 0.9 and we plot the probability function at times $t = 0$, $t = 20$ and $t = 100$. The simulations have been done with the parameters $I = 200, \Delta t = 0.1, T = 100$. Although the set of parameters might theoretically yield a big error, in practice this error is much smaller.

From the images, we can't appreciate any effect on the choice of w but the speed of convergence. The functions converge to the same number of Diracs centered at the same places.

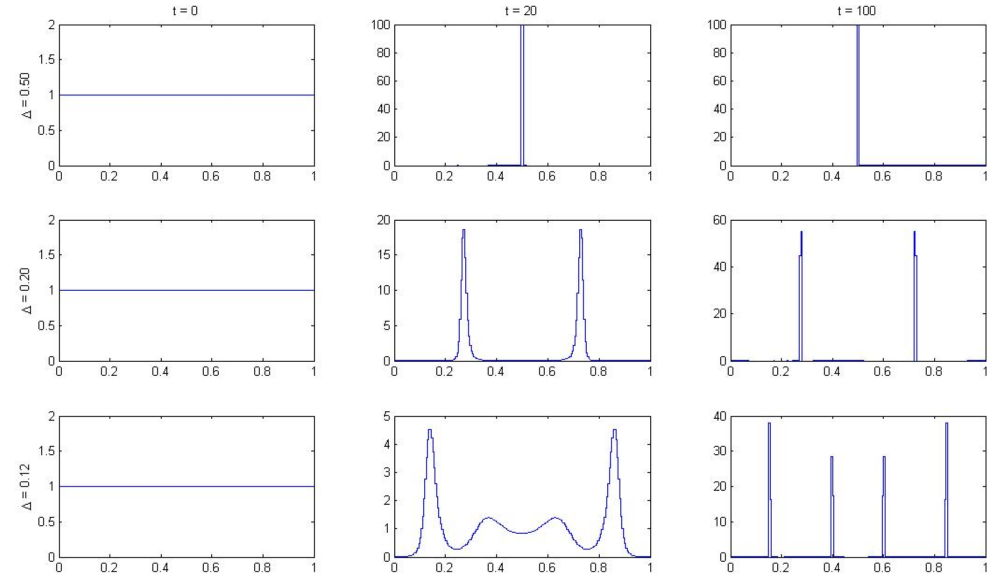

Figure 1: $w = 0.5$. Evolution

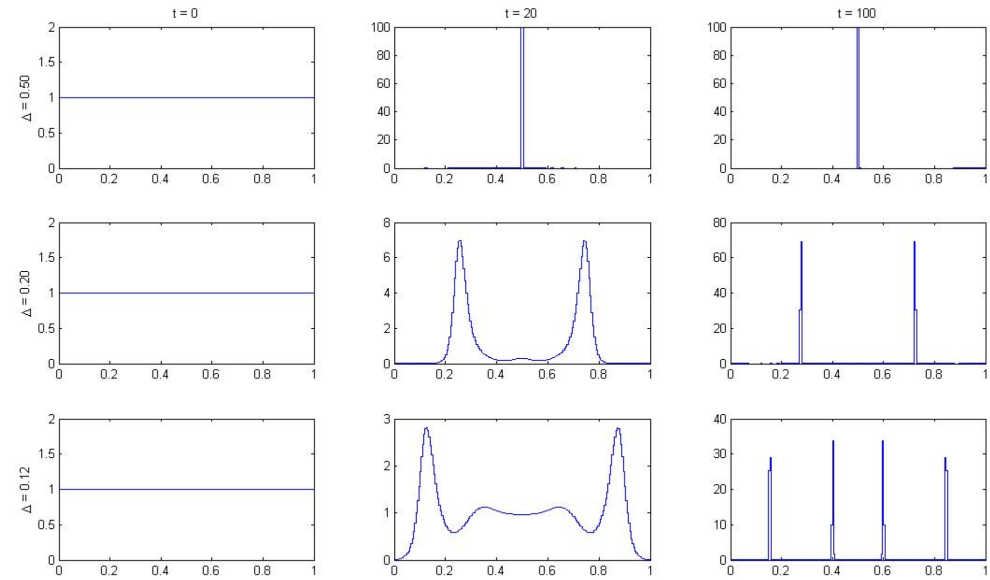

Figure 2: $w = 0.75$. Evolution

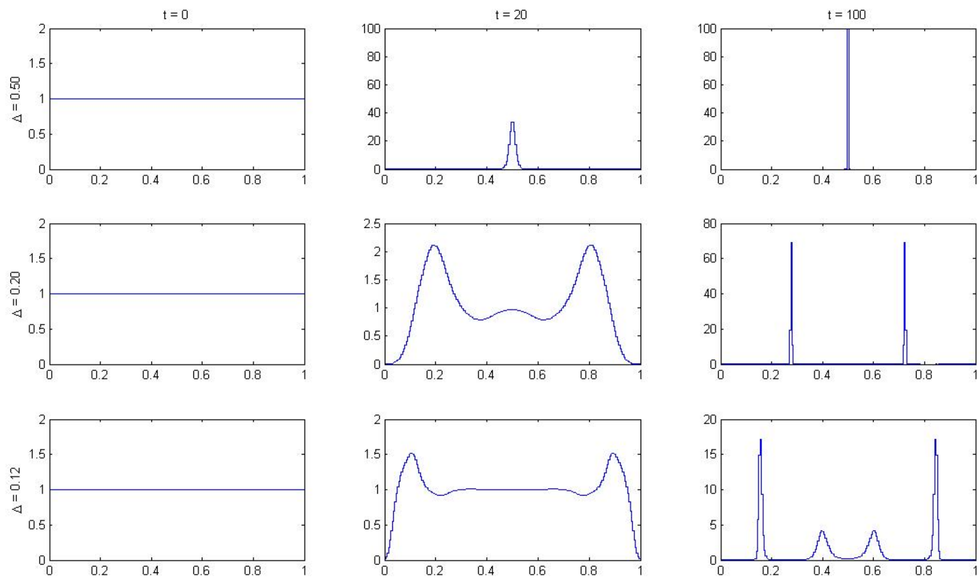

Figure 3: $w = 0.9$. Evolution

7.2 Extremists and Undecided

We now present some common scenarios: imagine a company fusion and the opinion of the employees about the new company, or a rough categorization of voters in an election. We can characterize these opinions as extremists (either 0 or 1) or undecided (0.5). The density of the opinions is α for the undecided and $\frac{1-\alpha}{2}$ for each of the extremist classes. We plot the result (1 Dirac or 2 Diracs) for each pair (α, Δ) in $[0,1] \times \left[\frac{1}{2}, 1\right]$. Notice that values of Δ smaller than $\frac{1}{2}$ would result in no motion at all. We do this for the previous set of values for w and find that in every case, the fraction of undecided people necessary to achieve consensus is much smaller from what one would expect (see figures 4 to 6). We also plot the center of masses of the first half of the distribution to show that it is not a smooth function of α and that close to the critical value $\Delta_c(\alpha)$ there is a jump. Again, we do this for the previous 3 values of w. All the simulations have been done with $I = 200, \Delta t = 0.1, T = 100$.

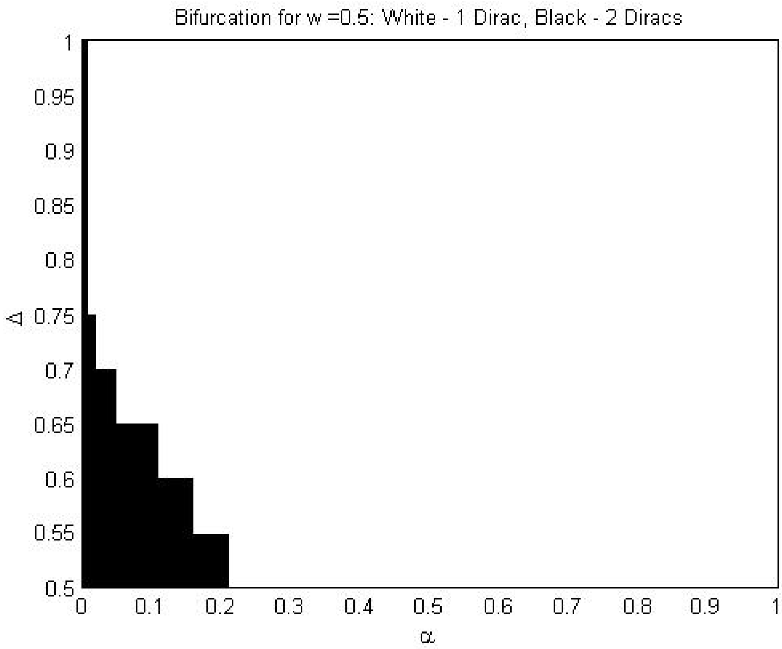

Figure 4: $w = 0.5$. Bifurcation diagram: 1 vs 2 Diracs

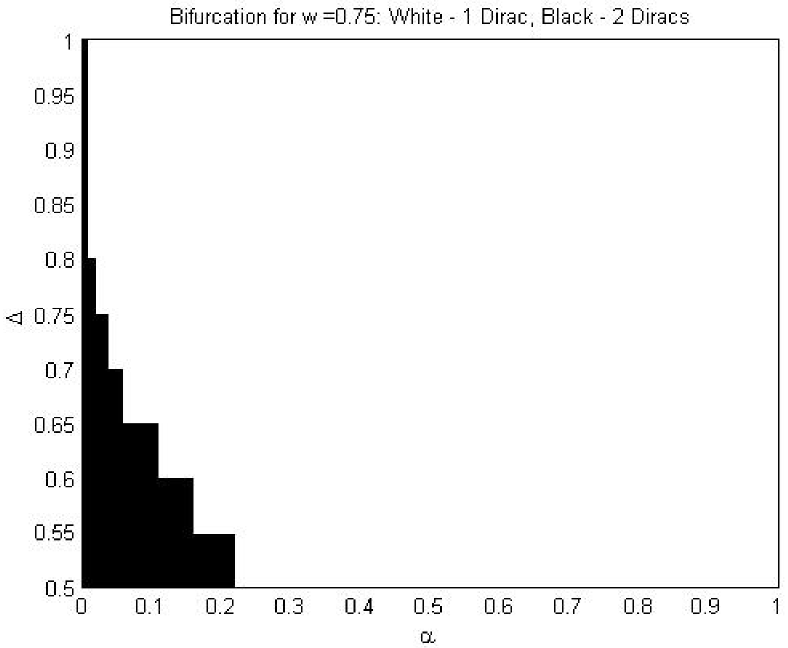

Figure 5: $w = 0.75$. Bifurcation diagram: 1 vs 2 Diracs

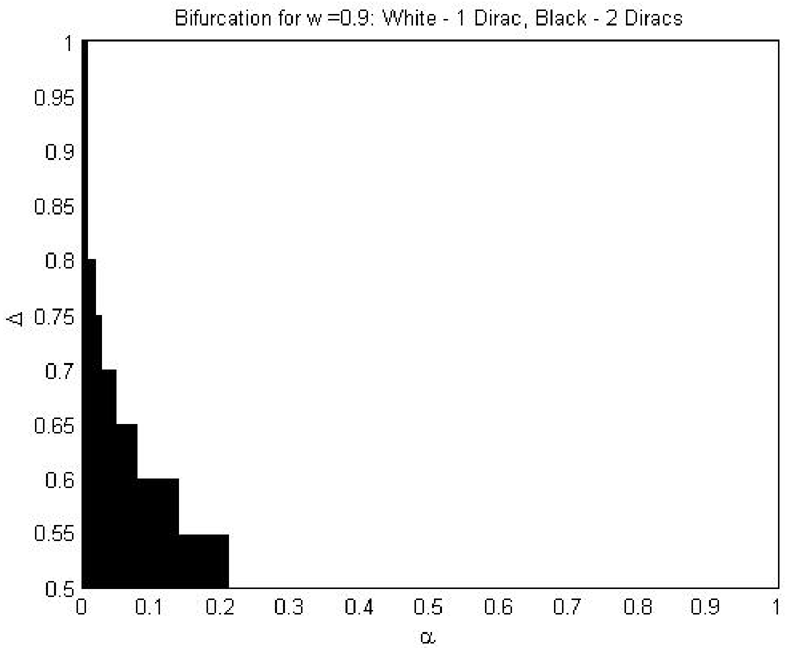

Figure 6: $w = 0.9$. Bifurcation diagram: 1 vs 2 Diracs

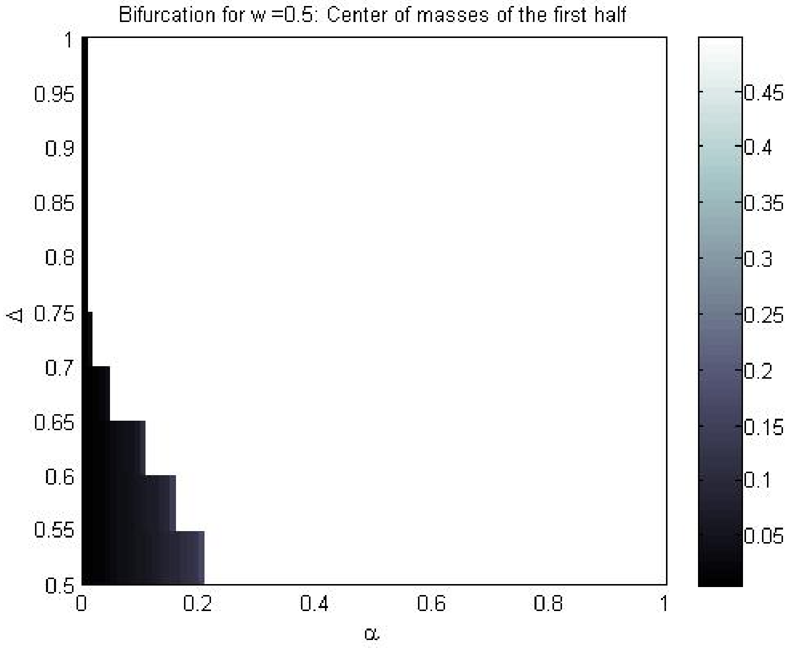

Figure 7: $w = 0.5$. Center of masses of the first half

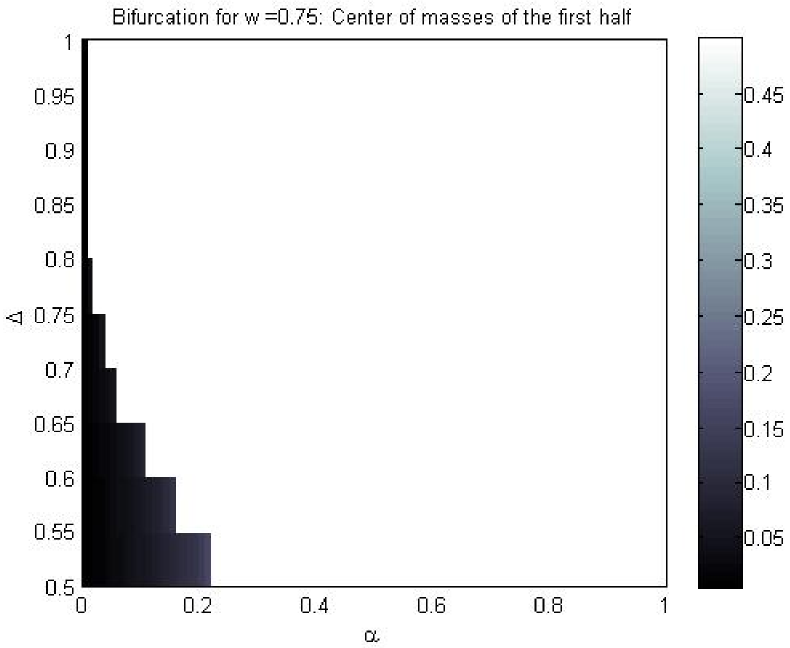

Figure 8: $w = 0.75$. Center of masses of the first half

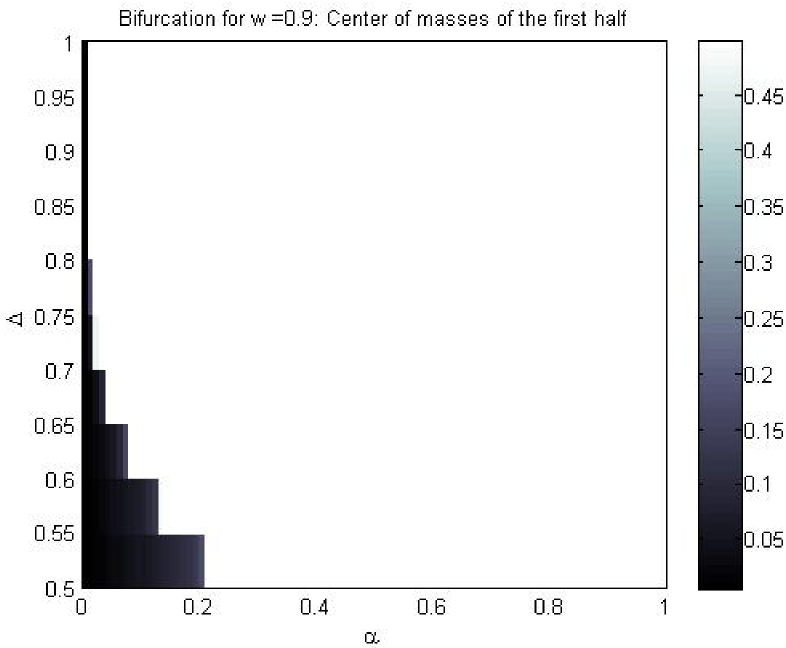

Figure 9: $w = 0.9$. Center of masses of the first half

Zooming in the critical region:

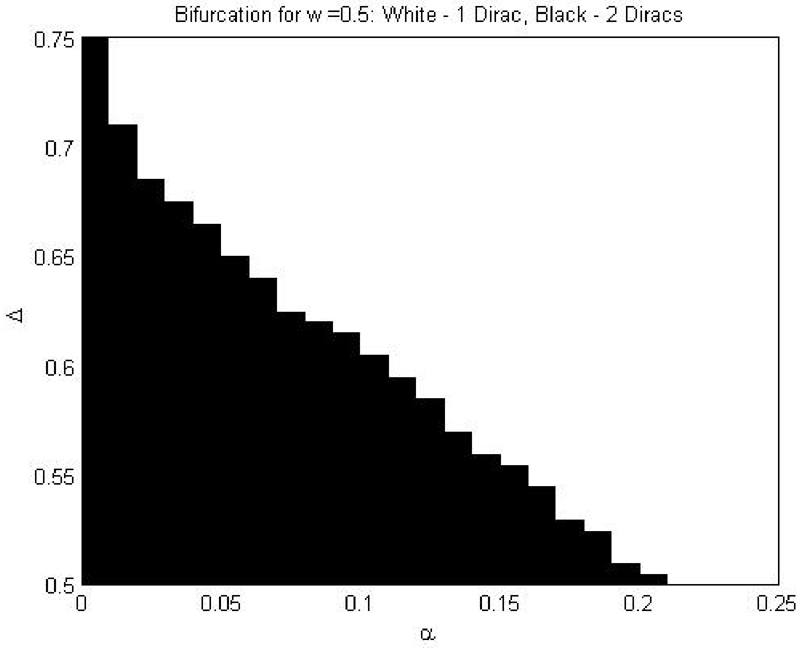

Figure 10: $w = 0.5$. Bifurcation diagram: 1 vs 2 Diracs. Zoomed

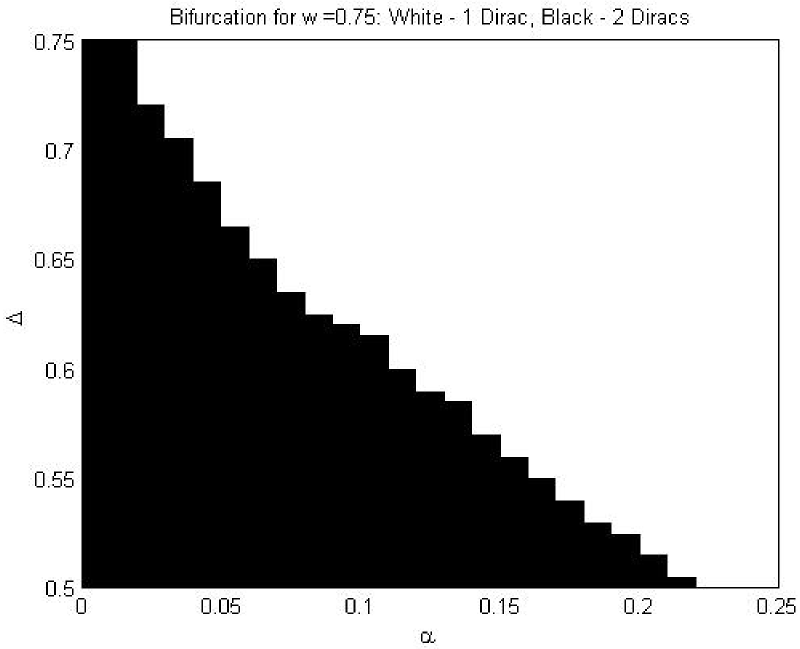

Figure 11: $w = 0.75$. Bifurcation diagram: 1 vs 2 Diracs. Zoomed

We can see that the curve described by the interphase between the 1-Dirac and 2-Diracs regions is clearly nonlinear, due to the nonlinearity of the equation.

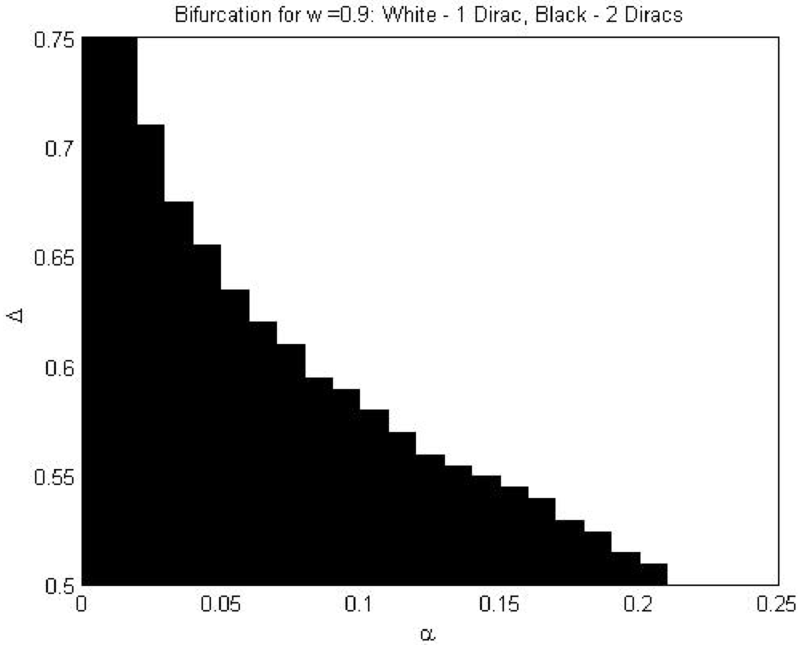

Figure 12: $w = 0.9$. Bifurcation diagram: 1 vs 2 Diracs. Zoomed

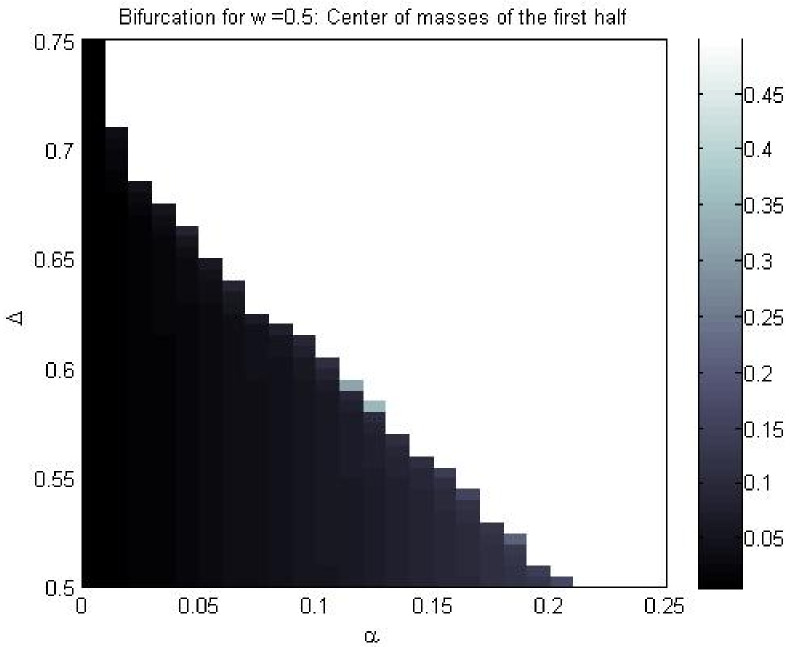

Figure 13: $w = 0.5$. Center of masses of the first half. Zoomed

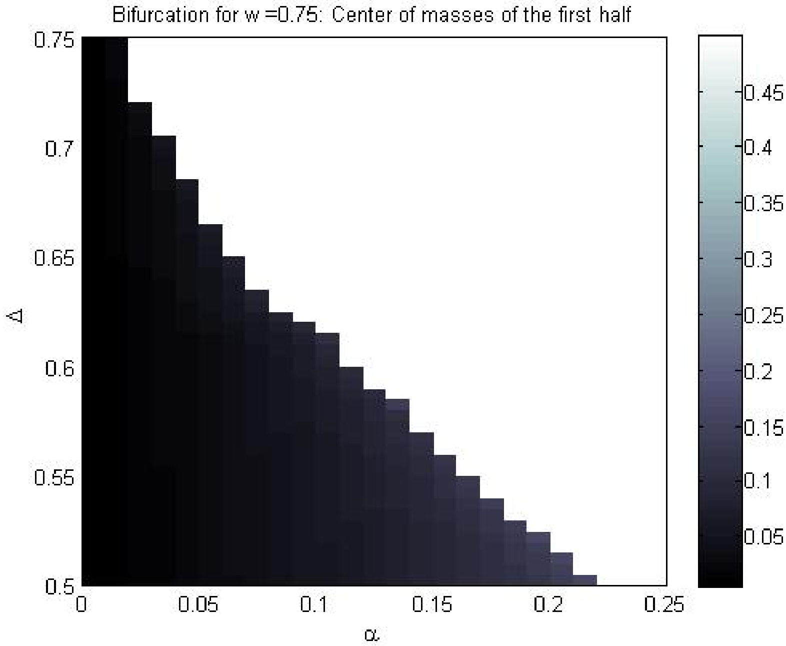

Figure 14: $w = 0.75$. Center of masses of the first half. Zoomed

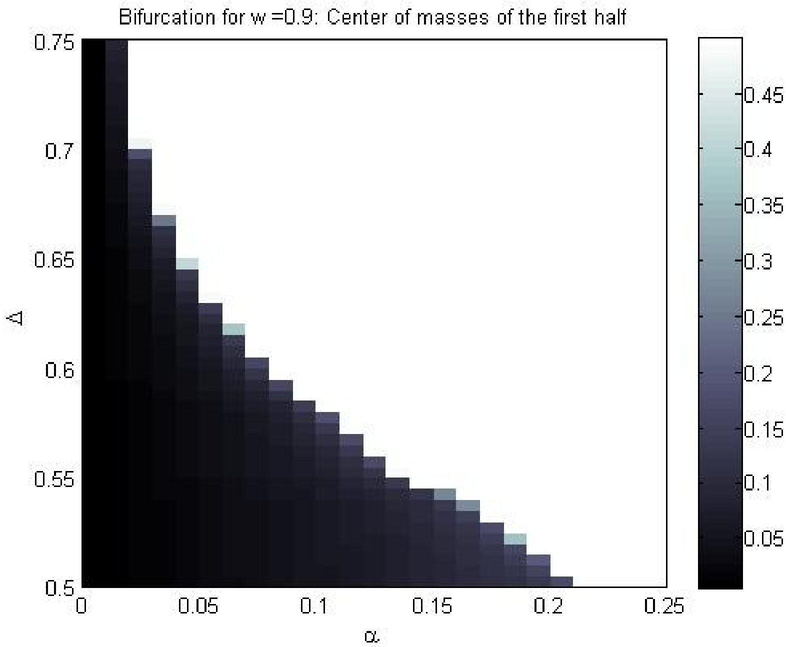

Figure 15: $w = 0.9$. Center of masses of the first half. Zoomed

7.3 Initial uniform conditions in terms of delta

We present here the evolution of the number of Diracs with respect to Δ, using as initial condition a uniform distribution. Note that we have capped the situations with more than 7 Diracs into the category "7 or more", which are represented by 7 in the graph. For a Dirac to be considered as such, we require that it has at least 1% of the total mass. Otherwise we consider it as a zero. Again, the results are plotted for the 3 different values of w.

We observe that the results are almost independent of w, as there is almost no difference between the 3 pictures (see 19 for the combined plot of all 3 functions). Another interesting thing to remark is that if we compare our results for $w = 0.5$ with the deterministic model with the ones in [9] with the probabilistic model, the intervals of Δ in which they have a high probability of convergence to n Diracs correspond to the same intervals in which we have convergence to n Diracs. This suggests that the approximation for $N = \infty$ is good enough to preserve properties such as the final state.

All simulations have been done with the following parameters: $I = 200, \Delta t = 0.1, T = 100$.

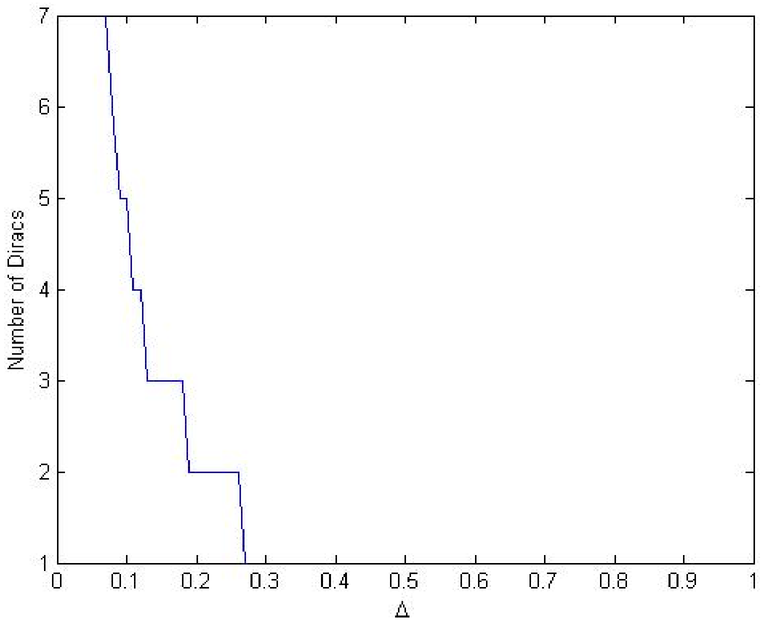

Figure 16: $w = 0.5$. Δ vs Number of Diracs. Uniform initial conditions

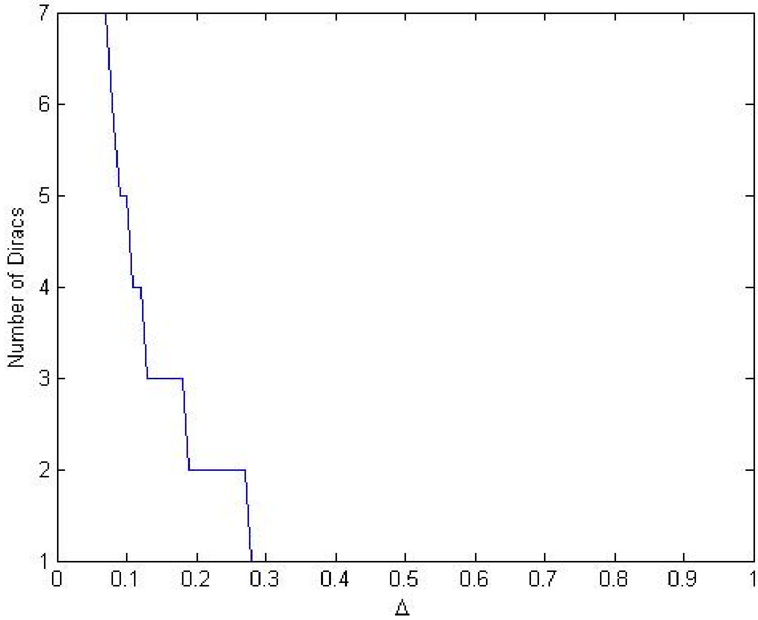

Figure 17: $w = 0.75$. Δ vs Number of Diracs. Uniform initial conditions

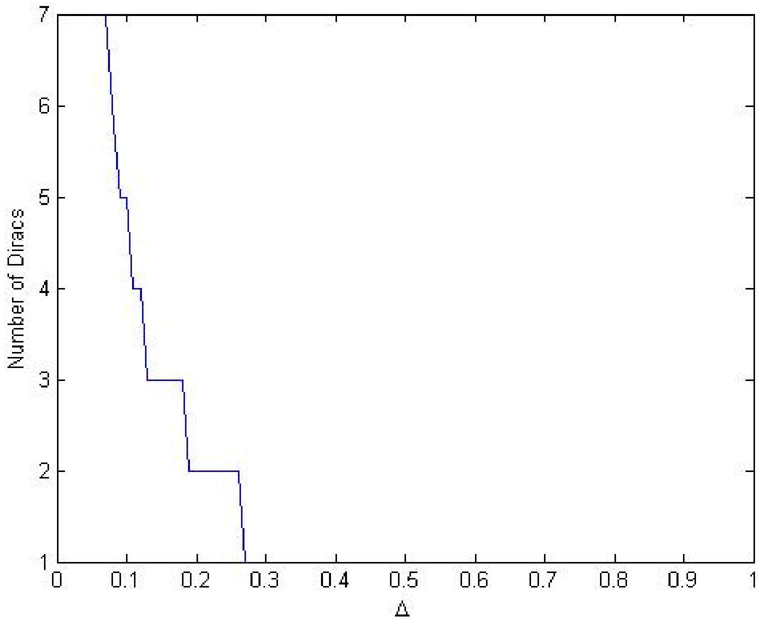

Figure 18: $w = 0.9$. Δ vs Number of Diracs. Uniform initial conditions

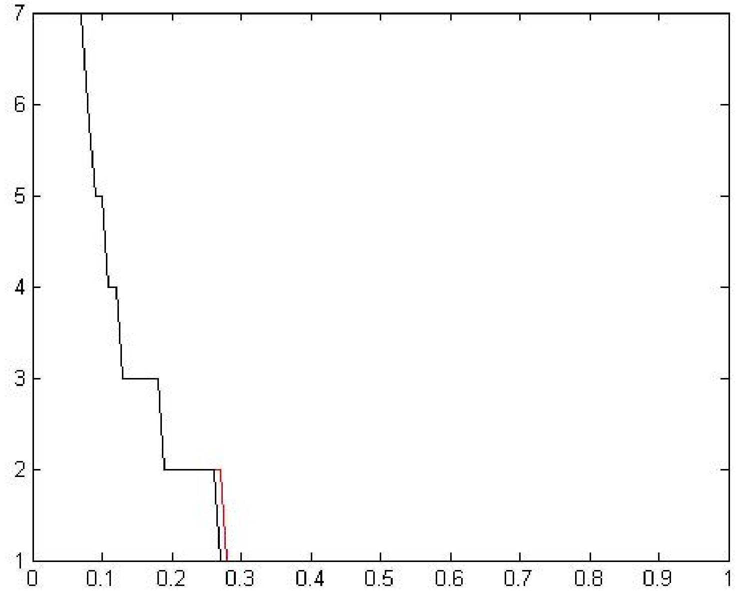

Figure 19: Δ vs Number of Diracs. Uniform initial conditions. Blue - $w = 0.5$ (below black), Red - $w = 0.75$, Black - $w = 0.9$

7.4 Comparison with the bound of the critical value (section 5)

In this subsection, we compare the critical values of Δ obtained by means of the numerical simulation, and the sufficient bounds proved in section 5. We find that the simulations verify the correctness of our bound, but there is still some uncovered space where we can't prove that there is convergence to one Dirac.

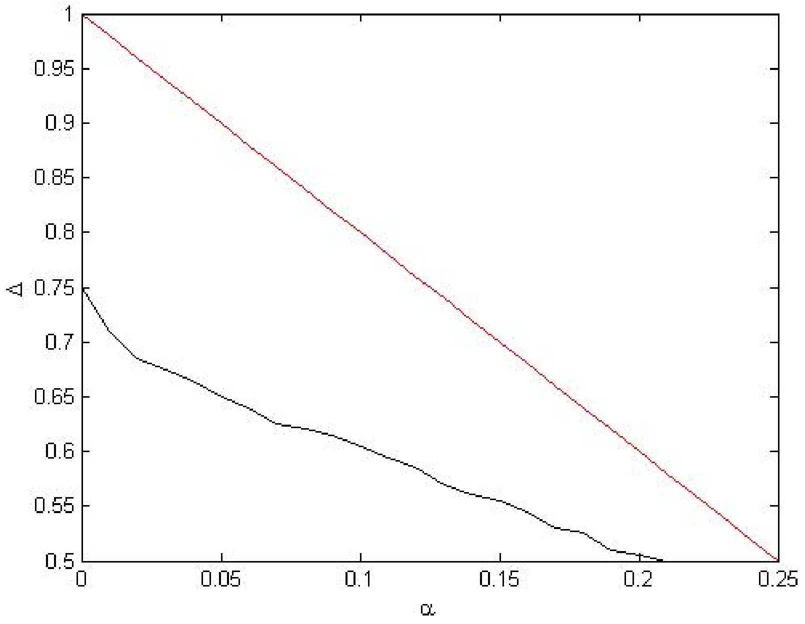

Figure 20: $w = 0.5$. Comparison between the numerical bound (black) and the sufficient condition on it (red) for the critical value of Δ

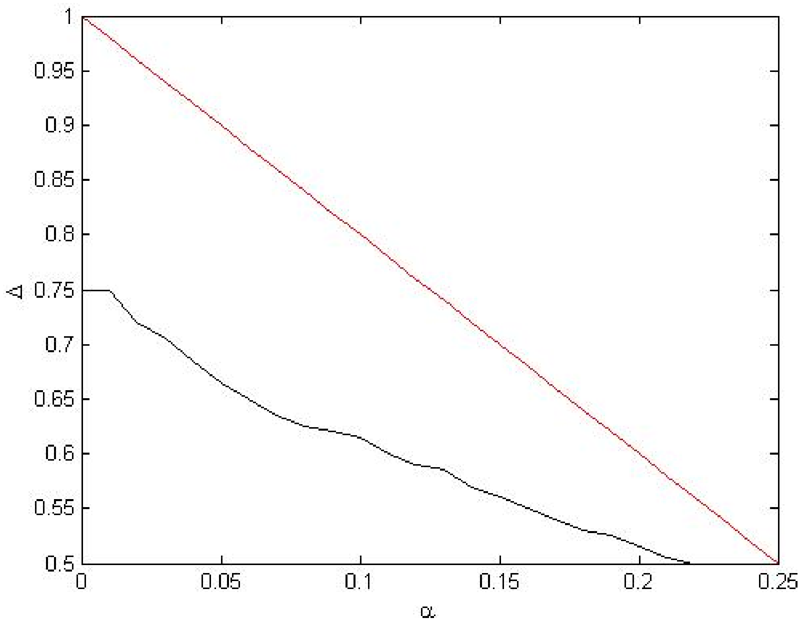

Figure 21: $w = 0.75$. Comparison between the numerical bound (black) and the sufficient condition on it (red) for the critical value of Δ

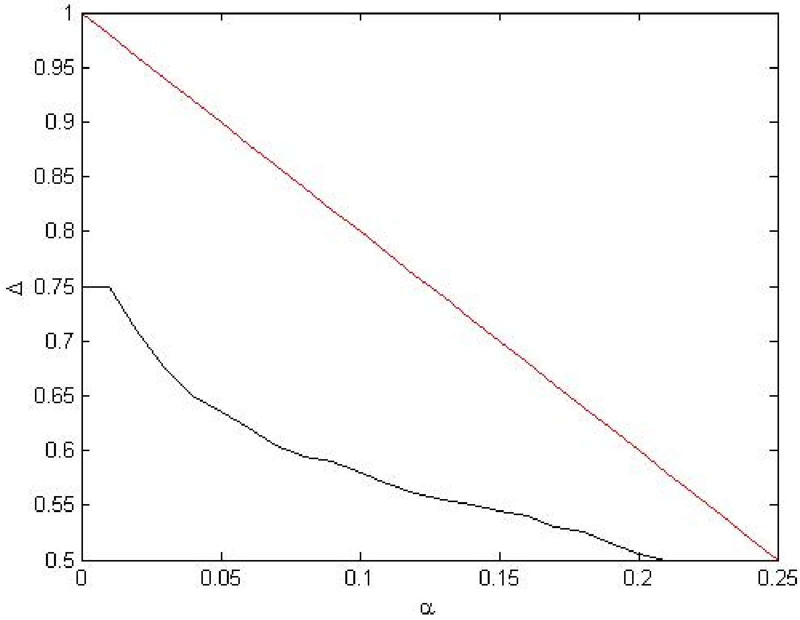

Figure 22: $w = 0.9$. Comparison between the numerical bound (black) and the sufficient condition on it (red) for the critical value of Δ

7.5 Beta distribution as initial condition

Here we study the evolution of the number of Diracs with respect to Δ, using as initial condition a Beta(1,6) distribution. The functions that have 5 or more Diracs have been put into the category represented with a 5. Again, we consider a Dirac if it has 1% of the total mass or more. We present the results for the 3 different values of w.

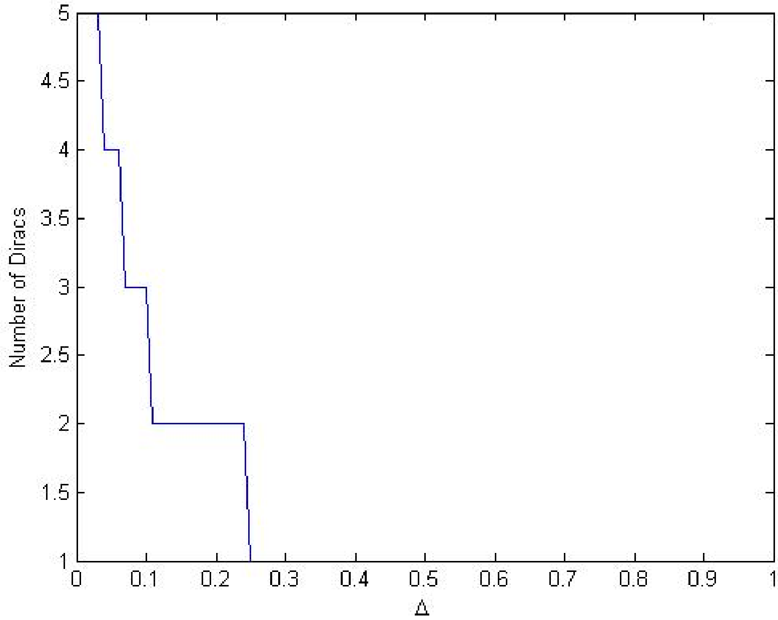

Figure 23: $w = 0.5$. Δ vs Number of Diracs. Beta(1,6) initial conditions

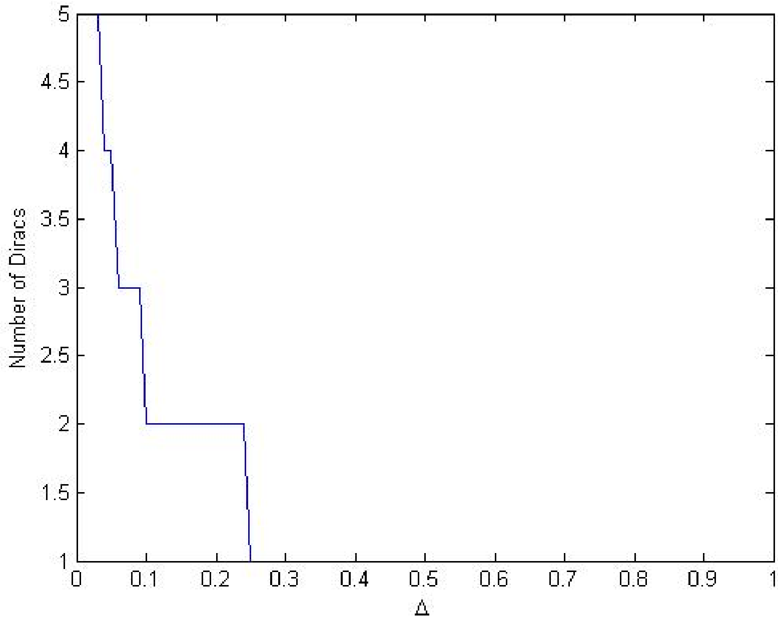

Figure 24: $w = 0.75$. Δ vs Number of Diracs. Beta(1,6) initial conditions

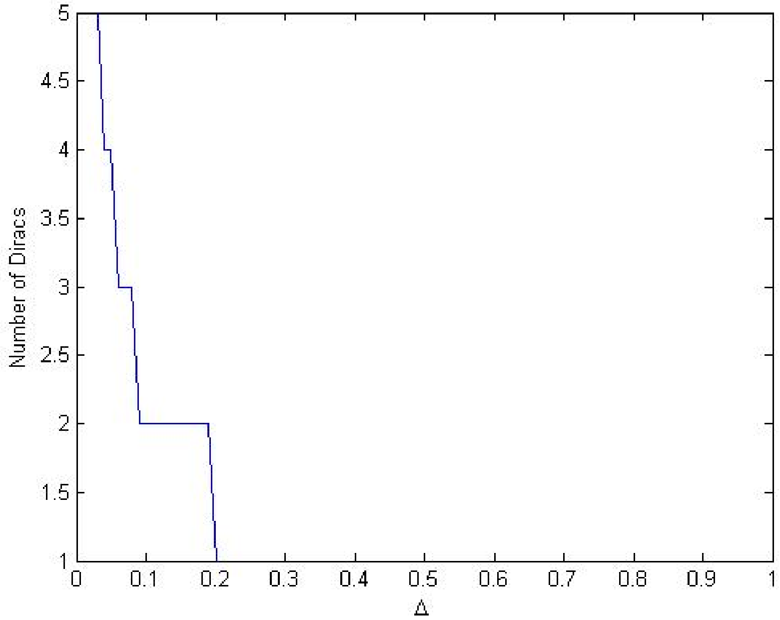

Figure 25: $w = 0.9$. Δ vs Number of Diracs. Beta(1,6) initial conditions

We can observe again the same phenomenon as in the uniform case, namely that the influence of w is negligible. If we compare the results from the ones in subsection 7.3, we can conclude that the final result depends on the initial condition, even for the same parameters w and Δ. Moreover, we can see that for a fixed (w, Δ), if we start with a Beta distribution the number of Diracs will be smaller or equal than if we start with a uniform one. This is explained by the fact that at the Beta distribution the mass is more concentrated than at the Uniform (in our case: to the left) and therefore it should be harder (i.e, Δ should be smaller) to split in the same number of Diracs.

8 Conclusions and future work

Summarizing the conclusions, we can observe that the mean-field approach gives the same results as the deterministic model when t goes to infinity. This suggests the idea that the following diagram commutes:

$$\begin{array}{ccc} f^N(x,t) & \xrightarrow{t \to \infty} & f^N(x,\infty) \\ {\scriptstyle N \to \infty}\big\downarrow & & \big\downarrow \\ f(x,t) & \xrightarrow{t \to \infty} & f(x,\infty) \end{array}$$

Figure 26: Commutative Diagram between Finite and Infinite N and finite and infinite t

We have proved that the horizontal arrows are well defined (i.e the limits exist) and we conjecture that the left arrow is true (there is convergence to mean field). From the simulations we have seen that if we sample N ratings from a distribution $\mathcal{D}$, with high probability, the final state will be the same as if we start with the distribution $\mathcal{D}$ and we run the numerical method for a large time.

Moreover, in the extremist / undecided scenario, we find that with only 21% of the peers being undecided, there is convergence regardless of the deviation threshold set (assuming it is always larger than 0.5). In the worst case, only 21 people are enough to achieve consensus between 79 other people with completely opposite beliefs. This is much lower from what one would estimate.

Regarding the future work, there are several lines of research that could be done afterwards. First, the proof for the mean field convergence should be completed. The next priority should be the focus on generalizing the results to fit the model from Le Boudec et al, which also takes liars and direct observations into account. Another feature that could be incorporated is the possibility of having different values of Δ depending on the population. This could model for example, the fact that the extremists are usually less tolerant than the undecided people. One should also try to sharpen the bounds found in theorem 5.22, which only finds a critical Δ over $\frac{1}{2}$, in order to distinguish the cases where we have consensus from the ones where we haven't. Other interesting things to do could be collecting real-world traces from surveys over time to evaluate the performance of the model.

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A Code

A.1 Line.h

```
class Line { // y = m*x + c
      public:
             Line(double new_m, double new_c);
             Line();
             double getM() const;
             double getC() const;
             pair<double,double> getMC();
             void setM(double m);
             void setC(double c);
             void setMC(const pair<double,double>& mc);
             const Line& operator+=(const Line& next_line);
             Line operator-(void) const;
      private:
              double m;
              double c;
};
```

A.2 Halfline.h

```
class HalfLine {
      public:
               HalfLine(const Line& new_L, const double new_start);
               HalfLine();
               const Line getL() const;
               double getStart() const;
               friend inline bool operator<(const HalfLine& lhs, const HalfLine& rhs){
                   return (lhs.getStart() < rhs.getStart());
               }

      private:
              Line L;
              double start;
};
```

A.3 Spline.h

```
class Spline {
      public:
               Spline(const double new_x_start, const double new_x_end, const double new_y_start,
                      const double new_y_end);
               Spline(const double new_x_start, const double new_x_end, const Line& new_L);
               double getX_Start() const;
               double getX_End() const;
               double getY_Start() const;
               double getY_End() const;
               Line getL() const;
               void PrintSpline();
               void PrintSplineMatlab();
      private:
              double x_start;
              double x_end;
              double y_start;
```

```
            double y_end;
            Line L;
};
```

A.4 SplineUtilities.cpp

```
// Computes the integral of the spline s between x_start and x_end
// We need to provide x_start and x_end such that they are contained in the
// domain of definition of the spline.
// The computation of the integral is done as the area of a trapezoid
double CalculateIntegral(Spline& s, double x_start, double x_end){
       if (x_start < s.getX_Start() or x_end > s.getX_End()){
          cerr << "Error: Bounds out of the Spline domain" << endl;
          assert(false);
       }
       // value of the spline at the starting and end points of the interval
       double integral_start, integral_end;
       Line L = s.getL();
       integral_start = L.getM()*x_start + L.getC();
       integral_end = L.getM()*x_end + L.getC();
       return (integral_start + integral_end) * (x_end - x_start) / 2.0;
}

// Computes the integral of |s - threshold| over all the interval spanned by s.
// We use the fact that the splines are linear to know that there will be at most
// one cut with the x-axis.
double CalculateError(Spline& s, double threshold){
       Line L = s.getL();
       double length = s.getX_End() - s.getX_Start();
       // Constant spline. No cuts with the x-axis.
       if (L.getM() == 0.0) return fabs(threshold - L.getC())*length;
       else {
            // Look for the cut.
            double x_cut = (threshold - L.getC()) / (L.getM());
            // Cut inside the interval
            if (x_cut >= s.getX_Start() && x_cut <= s.getX_End()){
               return (1.0/2.0 * (x_cut - s.getX_Start())*fabs(s.getY_Start() - threshold)
                    + 1.0/2.0 * (s.getX_End() - x_cut)*fabs(s.getY_End() - threshold));
            }
            // Outside
            else {
                 return fabs((s.getY_Start() + s.getY_End() - 2*threshold) * length / 2.0);
            }
       }
}

// Merges the two vectors of Splines v1 and v2 into v3 as if all the splines
// were added together. v1 and v2 don't need to be sorted. v3 needs to be empty.
// v3 will be a sorted vector of splines.
void MergeSplines(vector<Spline>& v1, vector<Spline>& v2, vector<Spline>& v3){
     vector<HalfLine> vH;
     for (int i = 0; i < v1.size(); i++){
         Line L = v1[i].getL();
         vH.push_back(HalfLine(L,v1[i].getX_Start()));
         vH.push_back(HalfLine(-L,v1[i].getX_End()));
     }
     for (int i = 0; i < v2.size(); i++){
```

```
        Line L = v2[i].getL();
        vH.push_back(HalfLine(L,v2[i].getX_Start()));
        vH.push_back(HalfLine(-L,v2[i].getX_End()));
    }
    sort(vH.begin(),vH.end());
    Line curr_line = vH[0].getL();
    double curr_x = vH[0].getStart();
    for (int i=1;i<vH.size();i++){
        // We discard the intervals of length less than 1e-8. This is done to avoid
        // intervals that start or end at the same place
        if (vH[i].getStart() - curr_x > 1e-8)
           v3.push_back(Spline(curr_x,vH[i].getStart(),curr_line));
        curr_x = vH[i].getStart();
        curr_line += vH[i].getL();
    }
}

// Computes the integral \int_{-\Delta}^{\Delta} a1*H(x-x1)*a2*H(x+u-x2) du
// where H(x) is the Heaviside step function and appends it to v_sp in Spline form.
void IntegralHeaviside_1(double a1, double a2, double x1, double x2, vector<Spline>& v_sp){
       if (x1 < x2 - Delta){
          v_sp.push_back(Spline(x2 - Delta, x2 + Delta, Line(a1*a2,-a1*a2*(x2 - Delta))));
          v_sp.push_back(Spline(x2 + Delta, 2, 2*a1*a2*Delta, 2*a1*a2*Delta));
       }
       else if (x1 <= x2 + Delta){
          v_sp.push_back(Spline(x1, x2 + Delta, Line(a1*a2,-a1*a2*(x2 - Delta))));
          v_sp.push_back(Spline(x2 + Delta, 2, 2*a1*a2*Delta, 2*a1*a2*Delta));
       }
       else {
          v_sp.push_back(Spline(x1, 2, 2*a1*a2*Delta, 2*a1*a2*Delta));
       }
}

// Computes the integral \int_{-\Delta}^{\Delta} a1*H(x+wu-x1)*a2*H(x+wu-u-x2) du
// and appends it to v_sp in Spline form.
void IntegralHeaviside_2(double a1, double a2, double x1, double x2, vector<Spline>& v_sp){
     double c1 = max(x1*(1-w) + x2*w,x1 - w*Delta);
     double c2 = x1 + w*Delta;
     double b1 = x2 - (1-w)*Delta;
     double b2 = x2 + (1-w)*Delta;
     // Odd scenario:
     if (c1 <= c2){
        // Case 1:
        if (c1 <= b1 && c2 <= b1){
           v_sp.push_back(Spline(b1, b2, Line(a1*a2/(1-w), -a1*a2*x2/(1-w)+a1*a2*Delta)));
           v_sp.push_back(Spline(b2, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
        }
        // Case 3:
        else if (c1 <= b1 && c2 <= b2){
           v_sp.push_back(Spline(b1, c2, Line(a1*a2/(1-w) + a1*a2/w, -a1*a2*x2/(1-w)-a1*a2*x1/w)));
           v_sp.push_back(Spline(c2, b2, Line(a1*a2/(1-w), -a1*a2*x2/(1-w)+a1*a2*Delta)));
           v_sp.push_back(Spline(b2, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
        }
        // Case 5:
        else if (c1 <= b1 && c2 >= b2){
           v_sp.push_back(Spline(b1, b2, Line(a1*a2/(1-w) + a1*a2/w, -a1*a2*x2/(1-w)-a1*a2*x1/w)));
           v_sp.push_back(Spline(b2, c2, Line(a1*a2/w, -a1*a2*x1/w+a1*a2*Delta)));
```

```
            v_sp.push_back(Spline(c2, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
         }
         // Case 7:
         else if (c1 <= b2 && c2 <= b2){
            v_sp.push_back(Spline(c1, c2, Line(a1*a2/(1-w) + a1*a2/w, -a1*a2*x2/(1-w)-a1*a2*x1/w)));
            v_sp.push_back(Spline(c2, b2, Line(a1*a2/(1-w), -a1*a2*x2/(1-w)+a1*a2*Delta)));
            v_sp.push_back(Spline(b2, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
         }
         // Case 9:
         else if (c1 <= b2 && c2 >= b2){
            v_sp.push_back(Spline(c1, b2, Line(a1*a2/(1-w) + a1*a2/w, -a1*a2*x2/(1-w)-a1*a2*x1/w)));
            v_sp.push_back(Spline(b2, c2, Line(a1*a2/w, -a1*a2*x1/w+a1*a2*Delta)));
            v_sp.push_back(Spline(c2, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
         }
         // Case 11:
         else if (c1 >= b2 && c2 >= b2){
            v_sp.push_back(Spline(c1, c2, Line(a1*a2/w, -a1*a2*x1/w+a1*a2*Delta)));
            v_sp.push_back(Spline(c2, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
         }
         else assert(false);
      }
      // Even scenario:
      else {
         // Case 2:
         if (c2 <= b1 && c1 <= b1){
            v_sp.push_back(Spline(b1, b2, Line(a1*a2/(1-w), -a1*a2*x2/(1-w)+a1*a2*Delta)));
            v_sp.push_back(Spline(b2, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
         }
         // Case 4:
         else if (c2 <= b1 && c1 <= b2){
            v_sp.push_back(Spline(c1, b2, Line(a1*a2/(1-w), -a1*a2*x2/(1-w)+a1*a2*Delta)));
            v_sp.push_back(Spline(b2, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
         }
         // Case 6:
         else if (c2 <= b1 && c1 >= b2){
            v_sp.push_back(Spline(c1, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
         }
         // Case 8:
         else if (c2 <= b2 && c1 <= b2){
            v_sp.push_back(Spline(c1, b2, Line(a1*a2/(1-w), -a1*a2*x2/(1-w)+a1*a2*Delta)));
            v_sp.push_back(Spline(b2, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
         }
         // Case 10:
         else if (c2 <= b2 && c1 >= b2){
            v_sp.push_back(Spline(c1, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
         }
         // Case 12:
         else if (c2 >= b2 && c1 >= b2){
            v_sp.push_back(Spline(c1, 2.0, 2*a1*a2*Delta, 2*a1*a2*Delta));
         }
         else assert(false);
      }
}


// Computes the first integral between two constant splines s1 and s2 and appends the
// result in v_sp. The integral computed is \int_{-\Delta}^{\Delta} s_1(x)s_2(x+u) du
```

```
// and the resultant spline is a function of x. The resultant spline is rescaled by mul.
void IntegralSplines_1(Spline &s1, Spline &s2, vector<Spline>& v_sp, double mul){
     IntegralHeaviside_1(mul*s1.getY_Start(),s2.getY_Start(),s1.getX_Start(),s2.getX_Start(),v_sp);
     IntegralHeaviside_1(-mul*s1.getY_Start(),s2.getY_Start(),s1.getX_Start(),s2.getX_End(),v_sp);
     IntegralHeaviside_1(-mul*s1.getY_Start(),s2.getY_Start(),s1.getX_End(),s2.getX_Start(),v_sp);
     IntegralHeaviside_1(mul*s1.getY_Start(),s2.getY_Start(),s1.getX_End(),s2.getX_End(),v_sp);
}

// Computes the second integral between two constant splines s1 and s2 and appends the
// result in v_sp. The integral computed is \int_{-\Delta}^{\Delta} s_1(x+wu)s_2(x+wu-u) du
// and the resultant spline is a function of x. The resultant spline is rescaled by mul.
void IntegralSplines_2(Spline &s1, Spline &s2, vector<Spline>& v_sp, double mul){
     IntegralHeaviside_2(mul*s1.getY_Start(),s2.getY_Start(),s1.getX_Start(),s2.getX_Start(),v_sp);
     IntegralHeaviside_2(-mul*s1.getY_Start(),s2.getY_Start(),s1.getX_Start(),s2.getX_End(),v_sp);
     IntegralHeaviside_2(-mul*s1.getY_Start(),s2.getY_Start(),s1.getX_End(),s2.getX_Start(),v_sp);
     IntegralHeaviside_2(mul*s1.getY_Start(),s2.getY_Start(),s1.getX_End(),s2.getX_End(),v_sp);
}

// Computes the derivative of the function given by the Splines stored in 'input'.
// Input should be a piecewise constant function with intervals of equal length.
// The derivative will be stored as an unsorted sum of splines in 'derivative'
// and scaled by dt in order to compute the Taylor approximation of f(x,dt) as
// f(x,0) + dt*f'(x,0).
void DerivativeComputation(vector<Spline>& input, vector<Spline>& derivative,
                          double dt){
     for (int i=0;i<input.size();i++){
         for (int j=0;j<input.size();j++){
             IntegralSplines_1(input[i],input[j],derivative,-2.0*dt);
             IntegralSplines_2(input[i],input[j],derivative,2.0*dt);
         }
     }
}

// Computes a suitable NBuckets-Partition of the function given by sorted_input
// in sorted-spline form and stores the result in Partition. The function returns
// true if the error made by the approximation as a Partition is less than
// threshold, otherwise returns false. It returns the error in 'error'.
bool ComputePartitionAndError(vector<Spline> &sorted_input, vector<Spline>& Partition,
                             double tolerance, int NBuckets, double& error){
     Partition.clear();
     error = 0.0;
     Spline curr_spline = sorted_input[0];
     int curr_index = 0;
     for (int i = 1; i <= NBuckets; i++){
         vector<Spline> curr_interval;
         double curr_integral = 0.0;
         while (curr_index < sorted_input.size() && curr_spline.getX_End() <= (double)i / (double)(NBuckets)) {
              Spline rectified_spline(max(curr_spline.getX_Start(),(double)(i-1) / (double)(NBuckets)),
                                    curr_spline.getX_End(), curr_spline.getL());
               curr_integral += CalculateIntegral(rectified_spline,
                                max(curr_spline.getX_Start(),(double)(i-1) / (double)(NBuckets)),
                                curr_spline.getX_End());
               curr_spline = sorted_input[curr_index+1];
               curr_index++;
         }
         if (curr_index < sorted_input.size()){ // Spline cut by the Partition
            Spline rectified_spline(max(curr_spline.getX_Start(),(double)(i-1) / (double)(NBuckets)),
```

```
                                    min(curr_spline.getX_End(),(double)(i) / (double)(NBuckets)),
                                    curr_spline.getL());
            curr_interval.push_back(rectified_spline);
            curr_integral += CalculateIntegral(curr_spline,
                                max(curr_spline.getX_Start(),(double)(i-1) / (double)(NBuckets)),
                                min(curr_spline.getX_End(),(double)(i) / (double)(NBuckets)));
         }
         Partition.push_back(Spline((double)(i-1) / (double)(NBuckets),
                                    (double)(i) / (double)(NBuckets),
                                    curr_integral*NBuckets, curr_integral*NBuckets));
         for (int j=0; j < curr_interval.size(); j++){
             error += CalculateError(curr_interval[j],curr_integral*NBuckets);
         }
     }
     return (error < tolerance);
}

// Simulates the system which starts from an initial condition given by
// initial_condition after Tmax seconds, in steps of dt. tolerance is the threshold
// for a Partition to be considered suitable. Stores the final result in 'result'.
void SimulateFunction(vector<Spline>& initial_condition, double tolerance, double Tmax,
                      double dt, vector<Spline>& result, int NBuckets){
     result.clear();
     vector<Spline> Partition = initial_condition;
     for (double i = 0; i <= Tmax; i+=dt){
         cerr << "time = " << i << endl;
         vector<Spline> derivative;
         DerivativeComputation(Partition,derivative,min(Tmax - i, dt));
         vector<Spline> exact_value;
         MergeSplines(Partition,derivative,exact_value);
         double error;
         bool result_trial = ComputePartitionAndError(exact_value,Partition,
                                 tolerance,NBuckets,error);
         cerr << "Truncation Error = " << error << endl;
         for (int j = 0; j < Partition.size(); j++){
             Partition[j].PrintSplineMatlab(); cout << " ";
         }
         cout << endl;
     }
     result = Partition;
}
```