Title: Continuous-discontinuous models of failure based on non-local displacements

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1 Introduction

In engineering, it is important to understand the entire failure processes of materials. When modelling failure phenomena, the inception and the propagation of damage must be taken into account. Nevertheless, if a realistic description of the entire failure process is desired, the numerical strategy should consider also the possible evolution of microcracks into macrocracks and the correct macrocrack-microcrack interaction.

To simulate numerically a failure process, either a continuous or a discontinuous approach may be employed. Continuous models are used to model the first stages of failure of quasi-brittle materials, such as concrete. If a local continuum model is employed, the numerical simulations present a pathological mesh sensitivity and physically unrealistic results are obtained. To solve the pathological mesh dependence, a regularisation technique must be used to incorporate non-locality into the model. This non-locality can be incorporated in two different ways. On the one hand, in integral-type models, a non-local state variable is defined as the weighted average of the local state variable in a neighbourhood of the point under consideration. On the other hand, in gradient-type models, higher-order derivatives are added to the partial differential equation that describes the evolution of the non-local variable. These two approaches exhibit similar results.

Nevertheless, these non-local models, in either integral or differential format, cannot be used in the final stage of failure, when the body is physically separated in two or more parts. Since in these models the body is treated as a continuum body, numerical interaction between the separated parts of the body persists and unrealistic results may be obtained. Therefore, if a realistic analysis is desired, discontinuities must be introduced. Compared to a continuous failure analysis, a continuous-discontinuous failure analysis may lead to a more realistic description of the entire process.

In this work, a new continuous-discontinuous approach is presented. This new technique is based on a non-local model (such as continuum damage or plasticity) based on non-local displacements [1]. On the one hand, this work deals with the continuous analysis. Although the aim is to develop a technique useful with any underlying continuous model, only the damage model has been used until now. The goal is to analyse the regularisation capabilities of the non-local damage model with 2D elements. Therefore, the implementation of this model is carried out. On the other hand, the discontinuous technique is presented.
An outline of this work follows. Section 2 deals with the continuous model of failure based on non-local displacements. The gradient version of this model is presented in Section 2.1. Special emphasis is placed on the definition of the boundary conditions. The regularisation capabilities and the size effect capture are illustrated by means of some numerical examples in Section 2.2. Sections 3 and 4 deal with the discontinuous strategy. On the one hand, in Section 3, the continuous-discontinuous technique in standard media is analysed. Three one-dimensional numerical examples illustrate this strategy in Section 3.7. On the other hand, Section 4 deals with the continuous-discontinuous technique based on non-local displacements. A numerical example is presented in Section 4.6 to validate this strategy. The concluding remarks and the future work are explained in Section 5.

Note. In this work, continuum damage models are used. The basic features of these models are briefly described in Appendix A. In Appendix B, some aspects referred to the definition of the Heaviside function are taken into account.
2 Continuous models of failure based on non-local displacements

2.1 Gradient version: boundary conditions

In this section the gradient version of the damage model based on non-local displacements is presented. This model computes the non-local displacements $\tilde{u}$ from the local displacements $u$ as the solution of the second-order PDE

$$\tilde{u}(x,t) - l_c^2 \nabla^2 \tilde{u}(x,t) = u(x,t)$$

(1)

where $l_c$ is the characteristic length of the non-local damage model.

The model is summarised in Table 1.

<table>
<thead>
<tr>
<th>Constitutive equation</th>
<th>$\sigma(x,t) = (1 - D(x,t)) C : \varepsilon(x,t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local strains</td>
<td>$\varepsilon(x,t) = \nabla^s u(x,t)$</td>
</tr>
<tr>
<td>Non-local displacements</td>
<td>$\tilde{u}(x,t) - l_c^2 \nabla^2 \tilde{u}(x,t) = u(x,t) + \text{B.C.}$</td>
</tr>
<tr>
<td>Non-local strains</td>
<td>$\varepsilon_{NL}(x,t) = \nabla^s \tilde{u}(x,t)$</td>
</tr>
<tr>
<td>Non-local state variable</td>
<td>$Y_{NL}(x,t) = Y(\varepsilon_{NL}(x,t))$</td>
</tr>
<tr>
<td>Damage evolution</td>
<td>$D(x,t) = D(Y_{NL})$</td>
</tr>
</tbody>
</table>

where $\sigma$ is the Cauchy stress tensor, $\varepsilon$ the small strain tensor, $C$ the tensor of elastic moduli, $D$ the damage parameter and $\nabla^s$ the symmetrised gradient.

In gradient non-local damage models, homogeneous Neumann boundary conditions are typically prescribed [1].

$$n \nabla \tilde{Y} = 0 \quad \text{on} \quad \partial \Omega$$

(2)

The main reason for choosing this type of condition is the difficulty to motivate Dirichlet boundary conditions (i.e. prescribing $\tilde{Y}$).
In the damage model based on non-local displacements, the boundary conditions seem to be easier to interpret. In this model, Dirichlet boundary conditions may be prescribed for \( \tilde{u} \) [1].

\[
\tilde{u} = u \quad \text{on } \partial \Omega \tag{3}
\]

These boundary conditions have a clear physical interpretation: non-local displacements must coincide with local displacements in all the domain boundary (that is, for both the Dirichlet and Neumann boundaries of the mechanical problem).

As discussed in [2], Eq. (1) can be also combined with Neumann boundary condition

\[
\frac{\partial \tilde{u}}{\partial n} = \frac{\partial u}{\partial n} \quad \text{on } \partial \Omega \tag{4}
\]

where \( \frac{\partial}{\partial n} \) is the derivative in the direction normal to the boundary \( \partial \Omega \).

However, with Eq. (3), there is no regularisation on \( \partial \Omega \). Therefore, if the damage starts on the boundary, physically unrealistic results may be obtained. Equation (4) leads to a more realistic behaviour. Nevertheless, in symmetric examples, the results obtained with this equation are not symmetric. This loss of symmetry is not understood yet and must be analysed.

Due to this reason, alternative boundary conditions for this problem are the mixed conditions

\[
\begin{aligned}
\tilde{u} \cdot n &= u \cdot n \quad \text{on } \partial \Omega \\
\nabla (\tilde{u} t) \cdot n &= 0 \quad \text{on } \partial \Omega
\end{aligned} \tag{5}
\]

where \( n \) is the direction normal to the boundary \( \partial \Omega \) and \( t \) is the direction tangent to the boundary \( \partial \Omega \).

With these boundary conditions, the same normal displacements are imposed and a relative slip is allowed.

### 2.2 Numerical examples: validation of the model

The goal of this section is to illustrate that the gradient version of the model based on non-local displacements is an effective technique for regularising two-dimensional boundary
2.2 Numerical examples: validation of the model

value problems, capturing the size effects and avoiding spurious localisation that gives rise to pathological mesh sensitivity in numerical computations. Due to this reason I have implemented in Matlab this model with 2D elements using a given code with 1D elements. Both Mazars and modified von Mises models have been implemented.

In this section, four different examples will be analysed. First of all, in Section 2.2.1, a 2D uniaxial tension test is studied. Although 2D elements are used, the behaviour of this example is one-dimensional because the Poisson coefficient $\nu = 0$ is imposed and the mixed boundary conditions (5) are used. Nevertheless, this example is analysed to be able to compare the qualitative response with a uniaxial tension test with 1D elements, see [1]. In Section 2.2.2, an example with a two-dimensional behaviour is discussed. In particular, a direct tension test on a double-notched specimen is carried out. The structure is partially weakened to represent the notches. In Section 2.2.3, another direct tension test on a double-notched specimen is analysed. In this case, the notches are included in the geometry. Finally, in Section 2.2.4, a three-point bending test is carried out to study the phenomenon called size effect.

2.2.1 2D uniaxial tension test

In this section, a uniaxial tension test with 2D elements is studied, see Figure 2.
CONTINUOUS MODELS OF FAILURE BASED ON NON-LOCAL DISPLACEMENTS

The two-dimensional particularisations of the gradient version, Table 1, and the linear softening law, Eq. (A.7), are used. Both Eq. (A.8) and Eq. (A.9) are discussed, see Appendix A.

The central tenth of the bar is weakened (10% reduction in Young’s modulus) to cause localisation. The dimensionless geometric and material parameters used in the example are summarised in Table 2. The Poisson coefficient is set to $\nu = 0$ to be able to neglect the lateral effect and reproduce the example with 1D elements.

Table 2: Uniaxial tension test: geometric and material parameters.

<table>
<thead>
<tr>
<th>Meaning</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of the bar</td>
<td>$L$</td>
<td>100</td>
</tr>
<tr>
<td>Cross-section of the bar</td>
<td>$A$</td>
<td>1</td>
</tr>
<tr>
<td>Length of weaker part</td>
<td>$L_W$</td>
<td>10</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E$</td>
<td>20 000</td>
</tr>
<tr>
<td>Idem of weaker part</td>
<td>$E_W$</td>
<td>18 000</td>
</tr>
<tr>
<td>Damage threshold</td>
<td>$\varepsilon_i$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Final strain</td>
<td>$\varepsilon_f$</td>
<td>$1.25 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

The comparison of the mixed boundary conditions (5) and the Dirichlet boundary conditions (3) is carried out.

Simplified Mazars model

As a first test, a fixed characteristic length $l_c = \sqrt{5}$ is chosen. The analysis is carried out with four different meshes of 40, 80, 160 and 320 elements in the $x$ axis and one element in the $y$ axis.

The force-displacement curves and the damage profiles are depicted in Figures 3 and 4.

As expected, the responses for the four meshes are very similar. Moreover, exactly the same behaviour as in the example with 1D elements of [1] is obtained.

Note that if Dirichlet boundary conditions (3) are used, there is no regularisation on $\partial \Omega$. As observed in Figure 5, damage profiles do not depend on a characteristic length but on numerical parameters such as $L_W$. Note that since only one element in the $y$ axis is used, all nodes are on the boundary.
2.2 Numerical examples: validation of the model

Figure 3: Force-displacement curves for various meshes.

Figure 4: Damage profiles for various meshes.

Regularisation via non-local displacements with the mixed boundary conditions (5) solves the pathological mesh dependence also in 2D examples with damage starting on the boundary.

As a second test, a fixed mesh with 80 elements in the $x$ axis and one element in the $y$ axis and four different characteristic lengths ($l_c^2 = 1$, $l_c^2 = 2$, $l_c^2 = 5$, $l_c^2 = 10$) are used.

The force-displacement curves and the damage profiles are shown in Figures 6 and 7.

The same response as the one with 1D elements is obtained. Both the ductility in the force-displacement response and the width of the final damage profile increase with the internal length scale.
Modified von Mises model

The same previous tests are carried out, with the modified von Mises model with a linear softening law. As a first test, a fixed characteristic length $l_c = \sqrt{5}$ is chosen. The analysis is carried out again with four different meshes of 40, 80, 160 and 320 elements in the $x$ axis and one element in the $y$ axis. Then, as a second test, a fixed mesh of 80 elements in the $x$ axis and one element in the $y$ axis and four different characteristic lengths ($l_c^2 = 1$, $l_c^2 = 2$, $l_c^2 = 5$, $l_c^2 = 10$) are used.

The damage profiles for these two tests are depicted in Figures 8 and 9.

The force-displacement curves for these two tests are shown in Figure 10.

As observed, the same response as the one with simplified Mazars model is obtained. These results are due to the one-dimensional behaviour of the test.
2.2 Numerical examples: validation of the model

Figure 6: Force-displacement curves for various characteristic lengths.

Figure 7: Damage profiles for various characteristic lengths.

Figure 8: Damage profiles for various meshes.
2.2.2 Direct tension test on double-notched specimen I

Once the uniaxial test is studied, an example with a 2D behaviour is analysed. The goal is to study the response of a structure which is partially weakened (10% reduction in Young’s modulus) to represent the notches. Horizontal displacements are prescribed at the right and left edges, see Figure 11.

As in the previous numerical example, the two-dimensional particularisations of the gradient version, Table 1, and the linear softening law, Eq. (A.7), are used. Both Eq. (A.8) and Eq. (A.9) are discussed. The dimensionless geometric and material parameters used in the example are summarised in Table 3.
2.2 Numerical examples: validation of the model

Table 3: Direct tension test on double-notched specimen: geometric and material parameters.

<table>
<thead>
<tr>
<th>Meaning</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of the bar</td>
<td>$L$</td>
<td>300</td>
</tr>
<tr>
<td>Width of the bar</td>
<td>$A$</td>
<td>150</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E$</td>
<td>20 000</td>
</tr>
<tr>
<td>Idem of weaker part</td>
<td>$E_W$</td>
<td>18 000</td>
</tr>
<tr>
<td>Damage threshold</td>
<td>$\varepsilon_i$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Final strain</td>
<td>$\varepsilon_f$</td>
<td>$1.25 \times 10^{-2}$</td>
</tr>
<tr>
<td>Poisson coefficient</td>
<td>$\nu$</td>
<td>0</td>
</tr>
</tbody>
</table>

**Simplified Mazars model**

As a first test, a fixed characteristic length $l_c = \sqrt{5}$ is chosen. The analysis is carried out with six different meshes of $40 \times 8$, $40 \times 16$, $80 \times 8$, $80 \times 16$, $160 \times 8$ and $320 \times 8$ elements.

The damage profiles for these meshes are shown in Figure 12.
The force-displacement curves are depicted in Figure 13.

![Force-displacement curves for various meshes.](image1)

**Figure 13:** Force-displacement curves for various meshes.

As expected, the responses for these meshes are very similar if the mixed boundary conditions (5) are used. The width of the final damage profile does not depend on the finite element size.

As a second test, a fixed mesh of $80 \times 16$ elements and four different characteristic lengths ($l_c^2 = 1$, $l_c^2 = 2$, $l_c^2 = 5$, $l_c^2 = 10$) are used. The damage profiles and the force-displacement curves are shown in Figures 14 and 15 respectively.

![Damage profiles for various characteristic lengths.](image2)

**Figure 14:** Damage profiles for various characteristic lengths.
As a third test, a fixed mesh of $80 \times 16$ elements and a fixed characteristic length $l_c = \sqrt{5}$ are chosen. The goal is to observe that the response of this test does not depend on the percentage of the reduction in Young’s modulus. Due to this reason two different tests will be carried out. The dimensionless numerical parameters are summarised in Table 4.

<table>
<thead>
<tr>
<th>10% reduction</th>
<th>1% reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus</td>
<td>$E$</td>
</tr>
<tr>
<td>Idem of weaker part</td>
<td>$E_W$</td>
</tr>
</tbody>
</table>

The force-displacement curves and the damage profiles are shown in Figures 16 and 17 respectively.
Figure 16: Force-displacement curves for two different Young’s modulus of weakened regions.

Figure 17: Damage profiles for (a) 10% reduction of Young’s modulus; (b) 1% reduction of Young’s modulus.
2.2 Numerical examples: validation of the model

The responses for these two tests are very similar. The width of the final damage profile
does not depend on the reduction in Young’s modulus, needed to cause localisation.

Finally, a fourth test will be analysed. A fixed mesh of $80 \times 16$ elements and a fixed
characteristic length $l_c = \sqrt{5}$ are chosen again. Two different examples will be studied,
in which the size weakened zone needed to cause localisation will differ, see Figure 18.

![Figure 18: Damage profiles for (a) 5% of weakened elements; (b) 2.5% of weakened
elements.](image)

The force-displacement curves and the damage profiles are depicted in Figures 19 and 20.

![Figure 19: Force-displacement curves for two different sizes of imperfections.](image)

As observed in Figures 19 and 20, the response for this test does not depend on the size
of the imperfection.

**Modified von Mises model**

The same previous tests are carried out, with the modified von Mises model with a linear
softening law.
2.2.3 Direct tension test on double-notched specimen II

The goal of this section is to make a comparative analysis between two tests (direct tension tests on double-notched specimens) to study if the mixed boundary conditions regularise the boundary value problem also with a specimen in which the damage inception occurs
at the boundary of a complicated geometry. The difference between these two tests is the way notches are represented. On the one hand, in the first test, they are represented as a weakened zone. On the other hand, in the second test, the notches are part of the specimen. Horizontal displacements are again prescribed at the right and left edges, see Figure 21.

![Figure 21: Problem statement of the two tests.](image)

As in the previous numerical examples, the two-dimensional particularisations of the gradient version, Table 1, and the linear softening law, Eq. (A.7), are used. A simplified Mazars criterion, Eq. (A.8), is employed.

The dimensionless geometric and material parameters used in this analysis are summarised in Table 5.

**Table 5: Direct tension test on double-notched specimen: geometric and material parameters.**

<table>
<thead>
<tr>
<th>Meaning</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of the bar</td>
<td>$L$</td>
<td>300</td>
</tr>
<tr>
<td>Width of the bar</td>
<td>$A$</td>
<td>150</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E$</td>
<td>20 000</td>
</tr>
<tr>
<td>Idem of weaker part (Test 1)</td>
<td>$E_W$</td>
<td>200</td>
</tr>
<tr>
<td>Damage threshold</td>
<td>$\varepsilon_i$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Final strain</td>
<td>$\varepsilon_f$</td>
<td>$1.25 \times 10^{-2}$</td>
</tr>
<tr>
<td>Poisson coefficient</td>
<td>$\nu$</td>
<td>0</td>
</tr>
</tbody>
</table>
A fixed characteristic length \( l_c = \sqrt{5} \) is chosen. The analysis is carried out with a mesh of \( 40 \times 16 \) elements. The damage profiles and the force-displacement curves are depicted in Figures 22 and 23 respectively.

![Damage profiles for (a) Test 1; (b) Test 2.](image)

As shown in Figures 22 and 23, the qualitative responses are very similar. The mixed boundary conditions allow to regularise the problem, even when the damage inception occurs at the boundary of the specimen. Nevertheless, the peak load for Test 1 is quite higher than the one for Test 2. The reason for this behaviour is that although in Test 1 there is a weakened zone, this is a part of the structure and it is not removed from that (such as in Test 2). That is, the stiffness in Test 1 is higher than in Test 2.
2.2.4 Three-point bending test: size effect

This section deals with the phenomenon called size effect. First of all, this phenomenon and the importance of its control is presented. Then, the Bažant law is introduced. Finally, a three-point bending test is carried out to illustrate that the regularisation technique that has been studied, using non-local displacements, is also valid to describe size effects.

Introduction

When the dimensions of a structure are scaled proportionally, the mechanical properties of the various examples differ. This phenomenon is known as size effect. Since the size of specimens that can be tested experimentally is limited, being able to model size effects is very important.

Size effects occur when characteristic lengths at the structural level and at the material level interact. In standard media, such as elasticity, plasticity or damage, there is not any material length scale, and size effects cannot be described properly. On the other hand, in enhanced media, since an intrinsic length scale is incorporated as an additional material parameter, size effects can be captured.

Bažant law

In the past decades, many formulas have been proposed in order to capture size effects,
most notably the Multi-Fractal Scaling Law (MFSL) by Carpinteri and co-workers and
the Size Effect Law (SEL) by Bažant, which will be used in this work for the numerical
analyses. A simplified version expresses the nominal strength $\sigma$ as

$$\sigma = \frac{B f'_t}{\sqrt{1 + \frac{D}{D_0}}}$$

(6)

where $f'_t$ is the tensile strength of the material, $D_0$ is a characteristic size and $B$ is related
to the geometry. Parameters $B f'_t$ and $D_0$ are fitted from experimental results. For doing
so, it is convenient to rewrite Eq. (6) into

$$\frac{1}{\sigma^2} = a D + c \quad \text{with} \quad a = \frac{1}{(B f'_t)^2 D_0}; \quad c = \frac{1}{(B f'_t)^2}$$

(7)

The fit, see Reference [3], consists in:

1. Determining experimentally the peak load $F_i$ for every size $D_i$.
2. Computing the nominal strength $\sigma_i$ as
   $$\sigma = \frac{F}{D}$$
3. Obtaining the parameters $a$ and $c$ in expression (7) via a linear regression with
   points $(D_i, \sigma_i)$.
4. Computing $D_0 = \frac{c}{a}$ and $B f'_t = \frac{1}{\sqrt{c}}$.

**Description of numerical simulations**

For the size effect analysis, three-point bending specimens are analysed numerically by
means of the finite element method. In particular, a notched beam where the notch
dimensions scale proportionally with the other dimensions of the specimen will be used,
see Figure 24. The size of the specimen ranges of $D = 1\text{mm}$ up to $D = 64\text{mm}$.

For the damage evolution, an exponential law is used

$$\omega = 1 - \frac{Y_0}{Y} e^{-\beta(Y-Y_0)} \quad \text{if} \ Y > Y_0$$
2.2 Numerical examples: validation of the model

The state variable is defined according to the modified von Mises criterion (A.9). The dimensionless geometric and material parameters used in this test, obtained by [4], are summarised in Table 6.

![Figure 24: Problem statement.](image)

Table 6: Size effect: geometric and material parameters.

<table>
<thead>
<tr>
<th>Meaning</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus</td>
<td>$E$</td>
<td>30 000</td>
</tr>
<tr>
<td>Poisson coefficient</td>
<td>$\nu$</td>
<td>0.15</td>
</tr>
<tr>
<td>Characteristic length</td>
<td>$l_c$</td>
<td>0.1</td>
</tr>
<tr>
<td>Damage threshold</td>
<td>$Y_0$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Slope of the stress-strain relation</td>
<td>$\beta$</td>
<td>500</td>
</tr>
<tr>
<td>Compressive-to-tensile strength ratio</td>
<td>$k$</td>
<td>10</td>
</tr>
</tbody>
</table>

For the numerical analyses, finite element meshes consisting of four-node quadrilaterals with linear shape functions are used. To be able to determine the peak load $F_i$ for every size $D_i$, enough-refined meshes must be used, specially near the zone where the damage inception occurs. In view of the symmetry of the problem only one half of the specimen has been discretised.

Numerical results are shown in Figure 25 (load-displacement curves) and Table 7 (load and displacement at peak).
2 CONTINUOUS MODELS OF FAILURE BASED ON NON-LOCAL DISPLACEMENTS

Non-local damage models allow for the description of size effects thanks to the characteristic length $l_c$. In Figure 26 the nominal strength is plotted as a function of the structural dimension $D$ in the usual logarithmic scale. The size effect is clearly visible. For comparison purposes, the theoretical Bažant law is also plotted.

As seen in Figure 26, the size effects shown by proportionally notched beams are in

Figure 25: Load-displacement curves.

Table 7: Size effect: load and displacement at peak.

<table>
<thead>
<tr>
<th>Size $D$ (mm)</th>
<th>Peak load (MPa)</th>
<th>Displacement at peak (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1789</td>
<td>7.5e-04</td>
</tr>
<tr>
<td>2</td>
<td>0.3378</td>
<td>0.0013</td>
</tr>
<tr>
<td>4</td>
<td>0.6398</td>
<td>0.0021</td>
</tr>
<tr>
<td>8</td>
<td>1.1114</td>
<td>0.0034</td>
</tr>
<tr>
<td>16</td>
<td>1.9587</td>
<td>0.0054</td>
</tr>
<tr>
<td>32</td>
<td>3.5120</td>
<td>0.0092</td>
</tr>
<tr>
<td>64</td>
<td>7.6971</td>
<td>0.0205</td>
</tr>
</tbody>
</table>
2.2 Numerical examples: validation of the model

Figure 26: Size effect for proportionally notched beams: numerical experiments versus Size Effect law.

reasonable agreement with the Size Effect Law. The regularisation technique based on non-local displacements is also valid to describe size effects.
3 Continuous-discontinuous failure in standard media

A realistic analysis tool should be able to simulate the inception and growth of damage up to the initiation of a macroscopic crack and its subsequent propagation.

In a numerical context, two approaches for structural simulation of fracture processes can be employed: a continuum and a discrete approach.

As already seen, with a non-local continuum approach, the damage inception and its propagation can be simulated properly and physically realistic results are obtained. Nevertheless, for increasing levels of damage, a continuum local model is not able to simulate the physical discontinuities. For increasing loads, this may lead to an unrealistic spread of damage, [5]. That is, the performance of some non-local models deteriorates in the final stage of failure.

A discontinuous approach solves these problems. Nevertheless, it is unable to describe the first phase of damage and the physical phenomenon of crack inception. For this reason, these two techniques must coexist. In other words, failure can be described as progressive material degradation which develops into a crack. To represent this crack, a discontinuity in the displacement fields must be introduced.

There are different methods for modelling discontinuities, which can be classified into explicit (in which the approximation function is discontinuous) and implicit models (where modifications of the derivative fields and the coefficients of the partial differential equation
lead to a representation of the discontinuity), see [6].

In this work, displacement discontinuities are introduced into the model through a discontinuous interpolation of the problem fields [7]. Within this approach, the standard approximation basis is enriched locally with special functions. This enrichment results in extra degrees of freedom for the nodes in the domain subjected to the enrichment, without modification of the mesh topology.

This section deals with the continuous-discontinuous approach to failure. In Section 3.1, the characterisation of the problem fields for a body crossed by a discontinuity is introduced. The governing equations, the variational formulation and its linearised discrete format are derived in Section 3.2, 3.3 and 3.4 respectively. Some issues related to the finite element technology are discussed in Section 3.6 and some numerical examples are presented in Section 3.7 to validate the discontinuous strategy. Three uniaxial tests are considered: a test with an elastic model and a linear behaviour of the crack (Section 3.7.1), an elastic model and a softening behaviour of the crack (Section 3.7.2) and finally, a test with a local damage model and a softening behaviour of the crack (Section 3.7.3). To finish with, the need for a regularisation is discussed in Section 3.7.4.

3.1 Problem fields

In the body $\Omega$, see Figure (28), the displacement field $\mathbf{u}$ can be decomposed as

$$
\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_1(\mathbf{x}, t) + \mathcal{H}_{\Gamma_d}(\mathbf{x})\mathbf{u}_2(\mathbf{x}, t)
$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are continuous functions on $\Omega$ and $\mathcal{H}_{\Gamma_d}$ is the Heaviside function centered at the discontinuity surface $\Gamma_d$:

$$
\mathcal{H}_{\Gamma_d}(\mathbf{x}) = \begin{cases} 
1 & \text{if } \mathbf{x} \in \Omega^+ \\
-1 & \text{if } \mathbf{x} \in \Omega^-
\end{cases}
$$
3.2 Governing equations

The equilibrium equations and boundary conditions for the body $\tilde{\Omega}$ without body forces can be summarised as

\[
\begin{align*}
\nabla \cdot \sigma &= 0 \quad (10a) \\
\sigma_n &= t \quad \text{on } \Gamma_t \quad (10b) \\
\sigma_m &= t \quad \text{on } \Gamma_d \quad (10c) \\
u &= u^* \quad \text{on } \Gamma_u \quad (10d) \\
u_2 &= 0 \quad \text{on } \Gamma_u \quad (10e)
\end{align*}
\]

where $\sigma$ is the Cauchy stress tensor, $n$ is the outward unit normal to the body, $m$ is the
inward unit normal to $\Omega^+$ on $\Gamma_d$, $\mathbf{u}^*$ is a prescribed displacement, $\mathbf{t}$ is the load on the boundary, $t$ is the load on the discontinuity surface and $\Gamma_t$ and $\Gamma_u$ are the surfaces with Neumann and Dirichlet boundary conditions respectively.

In order to solve this problem, a constitutive equation is needed to characterise the material. Equations (11a) and (11b) are the constitutive equations for an elastic and a damage model.

$$\sigma(x,t) = C : \varepsilon(x,t) \tag{11a}$$
$$\sigma(x,t) = [1 - D(x,t)] C : \varepsilon(x,t) \tag{11b}$$

### 3.3 Variational formulation

In this section, the governing equations (10) will be cast in a weak form. The space of trial local displacements is defined by the function

$$\mathbf{u}(x,t) = u_1(x,t) + \mathcal{H}_{\Gamma_d}(x)u_2(x,t), \quad u_1, u_2 \in \mathcal{U}_u, \tag{12}$$

where

$$\mathcal{U}_u = \{u_{1,j}, u_{2,j} \mid u_{1,j}, u_{2,j} \in H^1(\Omega) \text{ and } u_{1,j}|_{\Gamma_u} = u_{2,j}|_{\Gamma_u} = 0\} \tag{13}$$

with $H^1(\Omega)$ a Sobolev space; $u_{1,j}$ and $u_{2,j}$ indicates the $j^{th}$ component of $\mathbf{u}_1$ and $\mathbf{u}_2$ respectively.

The equilibrium equation (10a) is multiplied by the weight function

$$\mathbf{\omega}(x,t) = \omega_1(x,t) + \mathcal{H}_{\Gamma_d}(x)\omega_2(x,t), \quad \omega_1, \omega_2 \in \mathcal{W}_{u,0} \tag{14}$$

with

$$\mathcal{W}_{u,0} = \{\omega_{1,j}, \omega_{2,j} \mid \omega_{1,j}, \omega_{2,j} \in H^1(\Omega) \text{ and } \omega_{1,j}|_{\Gamma_u} = \omega_{2,j}|_{\Gamma_u} = 0\}, \tag{15}$$

where $\omega_{1,j}$ and $\omega_{2,j}$ are the $j^{th}$ component of $\mathbf{\omega}_1$ and $\mathbf{\omega}_2$ respectively. Then, Eq. (10a) is integrated over the domain $\Omega$ to obtain the weak equilibrium statement:

$$\int_{\Omega} (\mathbf{\omega}_1 + \mathcal{H}_{\Gamma_d}\mathbf{\omega}_2) \cdot (\nabla \cdot \sigma) \, d\Omega = 0 \tag{16}$$
The term related to the continuous part of the displacement field can be expanded using integration by parts, Gauss’ theorem and the boundary condition (10b) to yield

\[
\int_{\Omega} \omega_1 \cdot (\nabla \cdot \sigma) \, d\Omega = \int_{\Gamma_t} \omega_1 \cdot \bar{t} \, d\Gamma - \int_{\Omega} \nabla^s \omega_0 : \sigma \, d\Omega
\]  

(17)

Similarly, the term related to the discontinuous part of the displacement field is expanded using integration by parts, Gauss’ theorem and the boundary conditions (10b) and (10c):

\[
\int_{\Omega} \mathcal{H}_{\Gamma_d} \omega_2 \cdot (\nabla \cdot \sigma) \, d\Omega = \int_{\Gamma_t^+} \omega_2 \cdot \bar{t} \, d\Gamma - \int_{\Gamma_t^-} \omega_2 \cdot t \, d\Gamma - \int_{\Omega^+} \nabla^s \omega_2 : \sigma \, d\Omega
\]

\[
- \int_{\Gamma_t^+} \omega_2 \cdot \bar{t} \, d\Gamma - \int_{\Gamma_t^-} \omega_2 \cdot t \, d\Gamma + \int_{\Omega^-} \nabla^s \omega_2 : \sigma \, d\Omega
\]  

(18)

where \( \Gamma_t^{+/−} \) and \( \Gamma_d^{+/−} \) are the part of the surfaces \( \Gamma_t \) and \( \Gamma_d \) such that \( \mathcal{H}_{\Gamma_d} = ±1 \) respectively.

Therefore, using these expressions in (16), one obtains:

\[
\int_{\Gamma_t} \omega_1 \cdot \bar{t} \, d\Gamma - \int_{\Omega} \nabla^s \omega_0 : \sigma \, d\Omega + \int_{\Gamma_t^+} \omega_2 \cdot \bar{t} \, d\Gamma - \int_{\Gamma_t^-} \omega_2 \cdot t \, d\Gamma - \int_{\Omega^+} \nabla^s \omega_2 : \sigma \, d\Omega
\]

\[
- \int_{\Gamma_t^+} \omega_2 \cdot \bar{t} \, d\Gamma - \int_{\Gamma_t^-} \omega_2 \cdot t \, d\Gamma + \int_{\Omega^-} \nabla^s \omega_2 : \sigma \, d\Omega = 0
\]  

(19)

From the decomposition of the problem fields it follows that any admissible variation \( \omega \) can be regarded as admissible variations \( \omega_1 \) and \( \omega_2 \), thus leading to two variational statements. Taking first variations \( \omega_1 (\omega_2 = 0) \), and then variations \( \omega_2 (\omega_1 = 0) \), leads to:

\[
\int_{\Omega} \nabla^s \omega_1 : \sigma \, d\Omega = \int_{\Gamma_t} \omega_1 \cdot \bar{t} \, d\Gamma \quad \forall \omega_1 \in H^1(\Omega)
\]  

(20a)

\[
\int_{\Omega^+} \nabla^s \omega_2 : \sigma \, d\Omega - \int_{\Omega^-} \nabla^s \omega_2 : \sigma \, d\Omega + \int_{\Gamma_t^+} \omega_2 \cdot \bar{t} \, d\Gamma + \int_{\Gamma_t^-} \omega_2 \cdot t \, d\Gamma
\]

\[
= \int_{\Gamma_t^+} \omega_2 \cdot \bar{t} \, d\Gamma - \int_{\Gamma_t^-} \omega_2 \cdot \bar{t} \, d\Gamma \quad \forall \omega_2 \in H^1(\Omega)
\]  

(20b)
3.4 Discretisation and linearisation

3.4.1 Problem field description

Using a Galerkin approach, Eq. (8) can be discretised in each element affected by the enhancement using

\[
\begin{align*}
    u_1 &= Na \\
    u_2 &= Nb \\
    \nabla^a u_1 &= Ba \\
    \nabla^a u_2 &= Bb \\
    \omega_1 &= Na' \\
    \omega_2 &= Nb' \\
    \nabla^a \omega_1 &= Ba' \\
    \nabla^a \omega_2 &= Bb'
\end{align*}
\]

where \( N \) is the matrix of standard finite element shape functions, \( B \) is the matrix of shape function derivatives, \( a \) are the continuous nodal displacements, \( b \) are the discontinuous nodal displacements, \( a' \) are the admissible variations of \( a \) and \( b' \) are the admissible variations of \( b \).

For elements with standard degrees of freedom \( a \) only, the problem fields can be discretised in a standard way.

Using this discretisation, Eq. (8) reads, for nodes whose support is crossed by \( \Gamma_d \),

\[
    u_h = Na + \mathcal{H}_d Nb
\]  \hspace{1cm} (21)

The support of a node is considered to be the set of elements that share the node.

Note that the array \( N \) multiplying \( a \) and \( b \) are not the same since only part of the degrees of freedom in the array \( b \) are activated. The displacement jump across the discontinuity \( \Gamma_d \) is given by

\[
    [u_h] = Nb|_{\Gamma_d}
\]  \hspace{1cm} (22)

For nodes whose support is not crossed by a discontinuity, the Heaviside function is a constant function over its support. Since there is no enhancement, the standard finite element interpolation is retrieved.

The interpolation in Eq. (8) can be understood as an enrichment of the standard polynomial finite element spaces by a special function, the Heaviside function, see Figure 30.
3.4 Discretisation and linearisation

3.4.2 Discretised and linearised weak governing equations

In FE analysis, the discrete format of the problem fields leads to the two discrete weak governing equations

\[
\int_{\Omega} B^T \sigma \, d\Omega = \int_{\Gamma} N^T \bar{t} \, d\Gamma
\]

\[
\int_{\Omega^+} B^T \sigma \, d\Omega - \int_{\Omega^-} B^T \sigma \, d\Omega + \int_{\Gamma^d_-} N^T \bar{t} \, d\Gamma + \int_{\Gamma^d_+} N^T \bar{t} \, d\Gamma = \int_{\Gamma^+_t} N^T \bar{t} \, d\Gamma - \int_{\Gamma^-_t} N^T \bar{t} \, d\Gamma
\]

from which the equivalent nodal force vector related to admissible variations \( \mathbf{a} \) and \( \mathbf{b} \) results in

\[
f_{\text{int},a} = \int_{\Omega} B^T \sigma \, d\Omega
\]

\[
f_{\text{ext},a} = \int_{\Gamma_t} N^T \bar{t} \, d\Gamma
\]

\[
f_{\text{int},b} = \int_{\Omega^+} B^T \sigma \, d\Omega - \int_{\Omega^-} B^T \sigma \, d\Omega + \int_{\Gamma^d_-} N^T \bar{t} \, d\Gamma + \int_{\Gamma^d_+} N^T \bar{t} \, d\Gamma
\]

\[
f_{\text{ext},b} = \int_{\Gamma^+_t} N^T \bar{t} \, d\Gamma - \int_{\Gamma^-_t} N^T \bar{t} \, d\Gamma
\]

The linearised form of the discretised weak governing equation is obtained by substituting the stress rate in the bulk

\[
\dot{\sigma} = C \ddot{\varepsilon} = C(B \dot{\mathbf{a}} + H_{\Gamma_d} B \dot{\mathbf{b}})
\]
with \( \mathbf{C} \) the tangent matrix for the bulk material and the traction rate at the discontinuity \( \Gamma_d \)

\[
\mathbf{t} = \mathbf{T} |\mathbf{u}| = \mathbf{T} (\mathbf{N}\mathbf{b}) |\Gamma_d|
\]  

(26)

After standard manipulations, the linearised weak form of the governing equations at iteration \( i \) within a time step \( k \) reads

\[
\begin{bmatrix}
K_{a,a}^{k,i-1} & K_{a,b}^{k,i-1} \\
K_{b,a}^{k,i-1} & K_{b,b}^{k,i-1}
\end{bmatrix}
\begin{bmatrix}
\delta a^{k,i} \\
\delta b^{k,i}
\end{bmatrix} =
\begin{bmatrix}
f^{k,i-1}_{\text{ext},a} \\
f^{k,i-1}_{\text{ext},b} \\
f^{k,i-1}_{\text{int},a} \\
f^{k,i-1}_{\text{int},b}
\end{bmatrix} -
\begin{bmatrix}
f^{k,i-1}_{\text{int},a} \\
f^{k,i-1}_{\text{int},b}
\end{bmatrix}
\]  

(27)

where

\[
K_{a,a} := \int_{\Omega} \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega 
\]  

(28a)

\[
K_{a,b} := \int_{\Omega} \mathcal{H}_{\Gamma_d} \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega 
\]  

(28b)

\[
K_{b,a} := K_{a,b}^T = \int_{\Omega} \mathcal{H}_{\Gamma_d} \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega 
\]  

(28c)

\[
K_{b,b} := \int_{\Omega} \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega + \int_{\Gamma_d^+} \mathbf{N}^T \mathbf{T} \mathbf{N} \, d\Gamma + \int_{\Gamma_d^-} \mathbf{N}^T \mathbf{T} \mathbf{N} \, d\Gamma
\]  

(28d)

Note that the arrays \( \mathbf{N} \) and \( \mathbf{B} \) multiplying \( \mathbf{a} \) and \( \mathbf{b} \) are not the same since only part of the degrees of freedom in the arrays \( \mathbf{b} \) are activated.

Some remarks about the tangent matrix (28):

- Matrix \( K_{a,a} \) is the secant tangent matrix.

- Matrix \( K_{a,b} \) may be understood as an enriched secant tangent matrix, since the expression is the same, except for the Heaviside function.

- In Eq. (28d), the property \( \mathcal{H}_{\Gamma_d} \mathcal{H}_{\Gamma_d} = 1 \) is used. If a standard definition of the Heaviside function is used, see Appendix B, \( \mathcal{H}_{\Gamma_d} \mathcal{H}_{\Gamma_d} = \mathcal{H}_{\Gamma_d} \) and the first term of the matrix \( K_{b,b} \) is the matrix \( K_{a,b} \). Note that the integrals over \( \Gamma_d^+/- \) show the contribution of the crack.
3.5 One-dimensional particularisation: uniaxial tension test

In this section, the one-dimensional particularisations of the governing equations, the variational formulation and the discretisation and linearisation for a uniaxial tension test are presented, see Figure 31.

![Figure 31: Problem statement.](image)

### Governing equations

The equilibrium equations and boundary conditions can be summarised as

\[
\begin{align*}
\frac{d\sigma}{dx} &= 0 \quad x \in (0, L) \\
\sigma(x = L/2) &= t \\
u(0) &= 0 \\
u(L) &= u^* 
\end{align*}
\]

(29)

### Variational formulation

The space of trial displacements is defined by the function \( u = u_1 + \mathcal{H}(x)u_2 \), with \( u_1, u_2 \in \mathcal{U}_u \), where

\[
\mathcal{U}_u = \{ u_1 \text{ and } u_2 : u_1, u_2 \in H^1(\Omega) \text{ and } u_1(0) = 0, u_1(L) = u^*, u_2(0) = u_2(L) = 0 \} 
\]

(30)

and

\[
\mathcal{H}(x) = \begin{cases} 
-1 & \text{if } x \in (0, \frac{L}{2}) \\
1 & \text{if } x \in (\frac{L}{2}, L) 
\end{cases}
\]

(31)

Note that the domain \( \Omega^+ \) is given by \( \frac{L}{2} < x < L \), while \( \Omega^- \) is given by \( 0 < x < \frac{L}{2} \).
The space of admissible displacement variations is defined by \( \omega = \omega_1 + \mathcal{H}(x)\omega_2 \), with \( \omega_1, \omega_2 \in W_{u,0} \), where

\[
W_{u,0} = \{ \omega_1, \omega_2 \in H^1(\Omega) \text{ and } \omega_1(0) = \omega_1(L) = \omega_2(0) = \omega_2(L) = 0 \} \tag{32}
\]

and \( \mathcal{H} \) the Heaviside function defined by Eq. (31).

The two variational statements are derived from Eq. (20) and read

\[
\int_0^L \frac{d\omega_1}{dx} \cdot \sigma dx = 0 \quad \forall \omega_1 \in W_{u,0} \tag{33}
\]

\[
\int_0^L \mathcal{H} \frac{d\omega_2}{dx} \cdot \sigma dx = -\omega_2(L/2)\sigma^+(L/2) - \omega_2(L/2)\sigma^-(L/2) \quad \forall \omega_2 \in W_{u,0} \tag{34}
\]

### Discretisation and linearisation

Using

\[
\begin{align*}
    u_1 &= Na \\
    u_2 &= Nb \\
    d\omega_1 &= Ba \\
    d\omega_2 &= Bb
\end{align*}
\]

the discrete format of the problem fields leads to the two discrete weak governing equations

\[
\int_0^L B^T a' \cdot \sigma dx = 0 \quad \forall a' \quad \Rightarrow \quad \int_0^L B^T \sigma dx = 0 \tag{35}
\]

\[
\mathcal{H} B^T b' \cdot \sigma dx = -N^T(L/2)b' (\sigma^+(L/2) + \sigma^-(L/2)) \quad \forall b' \quad \Rightarrow \quad \int_0^L \mathcal{H} B^T \sigma dx = -N^T(L/2) (\sigma^+(L/2) + \sigma^-(L/2)) \tag{36}
\]

from which the equivalent nodal force vector related to admissible variations of \( a \) and \( b \) results in

\[
f_{int,a} = \int_0^L B^T \sigma dx \tag{37}
\]
\[ f_{\text{int,b}} = \int_0^L \mathcal{H} B^T \sigma \, dx + N^T (L/2) \left( \sigma^+ (L/2) + \sigma^- (L/2) \right) \] (38)

Therefore, the expression of \( f_{\text{int,b}} \) depends on the evolution law for the crack.

### 3.6 Finite element technology

#### 3.6.1 Introducing a discontinuity

In a continuous-discontinuous strategy, a continuous technique is used to simulate the first stages of a failure description up to the detection of a critical situation. When a critical situation is detected in a finite element, a discontinuity is introduced and a discontinuous strategy is employed.

A critical situation must be understood depending on the underlying continuous model. In a damage continuum model, for example, we will say that a critical situation is achieved when the damage parameter exceeds a critical damage value set a priori. On the other hand, in an elastic continuum model, a critical situation will be reached when the stress exceeds a value set a priori.

As soon as a discontinuity is introduced, the crack growth direction must be determined. There are different criteria for orienting a discontinuity, see [5] and [7]. Since in this work only one-dimensional examples have been analysed, this issue has not been addressed. This will be relevant when dealing with 2D examples.

#### 3.6.2 Numerical integration in discontinuous elements

In this strategy, computing integrals of discontinuous functions is required. The traditional quadrature rules, for example Gauss quadratures, are designed to integrate polynomials and functions that are well approximated by polynomials. These quadratures are not valid to integrate discontinuous functions properly.

To solve this problem, the domain intersected by a discontinuity is split in subdomains, where the functions are continuous and standard quadratures may be employed, see [8].
3.7 Numerical examples: uniaxial tension test

In this section, three one-dimensional examples are analysed. These examples deal with
the solution of a one-dimensional bar in tension with a discontinuity in the cross section.
The numerical tests are displacement-controlled, see Figure 31.

On the one hand, the first two examples deal with an elastic continuum model. A linear
and a softening behaviour of the crack are considered in the first and in the second test
respectively. On the other hand, in the third example a damage continuum model is used.

3.7.1 Elastic continuum and elastic crack

In this first example, both the bar and the crack are considered to follow an elastic law,
see Figure 32.

As shown in Figure 32,

\[ \begin{align*}
\dot{t} &= T|\dot{u}|, & T > 0 \\
t(0) &= 0
\end{align*} \]

\[ (39) \]

Figure 32: Evolution law for (a) the bar; (b) the crack.
3.7 Numerical examples: uniaxial tension test

Taking into account that $\sigma(x = L/2) = t$, one obtains

$$\sigma^+(L/2) = \sigma^-(L/2) = t
= T \cdot |u|
= T \cdot (u_1 + u_2 - (u_1 - u_2))
= 2T \cdot u_2
= 2TN(L/2)b$$

and therefore,

$$f_{int,b} = \int_0^L HB^T \sigma dx + 4T \cdot N^T(L/2)N(L/2)b =
= \int_0^L HB^T \sigma dx + T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} b$$

(41)

The bar is discretised as depicted in Figure 33. The element types needed for the finite element solution are shown in Figure 34.

Figure 33: Discretisation for the tension test.

Figure 34: Different types of elements.
1. Stiffness matrix computation

The stiffness matrix for standard elements and for those which have an enriched node are calculated.

(a) Standard elements

\[
K = \int_0^h B^T E B \, dx = \int_0^h \left[ \begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right] \frac{1}{h} E \left[ \begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right] \frac{1}{h} \, dx
\]

\[
= \int_0^h \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \frac{1}{h^2} E \, dx = \frac{E}{h} \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right]
\]

(b) Elements with a discontinuity in the middle section

\[
K = \begin{bmatrix}K_{aa} & K_{ab} \\ K_{ba} & K_{bb}\end{bmatrix}
\]

where

\[
K_{aa} = \frac{E}{h} \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right]
\]

\[
K_{ab} = K_{ba} = \int_0^h \mathcal{H} B^T E B \, dx =
\]

\[
= -\int_0^{h/2} B^T E B \, dx + \int_{h/2}^h B^T E B \, dx =
\]

\[
= -\frac{1}{h^2} E \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \frac{h}{2} + \frac{1}{h^2} E \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \frac{h}{2} =
\]

\[
= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
K_{bb} = K_{aa} + T \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]
\]

I.e.

\[
K = \begin{bmatrix}
\frac{E}{h} & -\frac{E}{h} & 0 & 0 \\
-\frac{E}{h} & \frac{E}{h} & 0 & 0 \\
0 & 0 & \frac{E}{h} + T & -\frac{E}{h} + T \\
0 & 0 & -\frac{E}{h} + T & \frac{E}{h} + T
\end{bmatrix}
\]

As shown, there is no interaction between degrees of freedom \( a \) and \( b \). This behaviour is due to the definition of the Heaviside function, see Appendix B.
3.7 Numerical examples: uniaxial tension test

(c) Elements with an extra degree of freedom

If the discontinuity does not cross the element but one of the element nodes has got extra degrees of freedom \( b \), see Figure 35, the element stiffness matrix can be derived following the procedure described above.

![Figure 35: Elements with an extra degree of freedom.](image)

The stiffness matrix for these elements is

\[
K = \frac{E}{h} \begin{bmatrix}
1 & -1 & 1 \\
-1 & 1 & -1 \\
1 & -1 & 1
\end{bmatrix}
\] (48)

where the degrees of freedom have been ordered in the sequence \([a_i \ a_j \ b_j]\) (Fig. 35(a)) and \([a_i \ a_j \ b_i]\) (Fig. 35(b)).

2. Assembly and solution

The global system of equations reads

\[
\begin{bmatrix}
\frac{E}{h} & -\frac{E}{h} & 0 & \frac{E}{h} & 0 & 0 \\
-\frac{E}{h} & \frac{2E}{h} & -\frac{E}{h} & -\frac{E}{h} & 0 & 0 \\
0 & -\frac{E}{h} & \frac{2E}{h} & 0 & \frac{E}{h} & -\frac{E}{h} \\
\frac{E}{h} & -\frac{E}{h} & 0 & \frac{2E}{h} + T & -\frac{E}{h} + T & 0 \\
0 & 0 & \frac{E}{h} & -\frac{E}{h} + T & \frac{2E}{h} + T & -\frac{E}{h} \\
0 & 0 & -\frac{E}{h} & 0 & -\frac{E}{h} & \frac{E}{h}
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
b_2 \\
b_3 \\
a_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\] (49)

Setting \( a_1 = 0, a_4 = u^* \), the solution yields

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
b_2 \\
b_3 \\
a_4
\end{bmatrix}
= \begin{bmatrix}
0 & u^* - \alpha & \alpha & 2u^* - 3\alpha & 2u^* - 3\alpha & u^*
\end{bmatrix}
\] (50)

where

\[
\alpha = u^* \frac{E}{h} + 4T
\] (51)

The displacement field for the one-dimensional tension test is then given by the sum of the continuous and the discontinuous displacement fields, see Figure 36.
To illustrate the behaviour of this example, a particular test is carried out with the dimensionless geometric and material parameters summarised in Table 8. The results are summarised in Figure 37. Note that:

Table 8: Uniaxial tension test with an elastic model and a linear behaviour of the crack: geometric and material parameters.

<table>
<thead>
<tr>
<th>Meaning</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of the bar</td>
<td>$L$</td>
<td>100</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E$</td>
<td>20 000</td>
</tr>
<tr>
<td>Parameter of the crack</td>
<td>$T$</td>
<td>100</td>
</tr>
<tr>
<td>Imposed displacement</td>
<td>$u^*$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

- The force-displacement curve, Fig. 37(a), exhibits the expected slope.

$$\Delta u = L \Delta \varepsilon + ||u|| = L \frac{\Delta \sigma}{E} + \frac{\Delta \sigma}{T} = \left( \frac{L}{E} + \frac{1}{T} \right) \Delta \sigma$$

$$\Delta \sigma = \frac{1}{L} \frac{1}{E + \frac{1}{T}} \Delta u \quad \Rightarrow \quad \Delta F = \frac{1}{L} \frac{1}{E + \frac{1}{T}} \Delta u$$

- The force-jump curve also exhibits the expected slope $T > 0$, Fig. 37(b). Moreover, $u^* = ||u|| + L \varepsilon$.

- The stress profile $\sigma$, Fig. 37(c), is constant, as dictated by equilibrium.

- The strain profile $\varepsilon$, Fig. 37(d), is constant, as there is not any weakened zone.

- The displacement profile, Fig. 37(e), exhibits the expected jump $||u|| = b_2 + b_3$. 
3.7 Numerical examples: uniaxial tension test

Figure 37: Uniaxial tension test with an elastic model and a linear behaviour of the crack:
(a) force-displacement; (b) force-jump; (c) stress $\sigma$; (d) strain $\varepsilon$; (e) displacements.
3.7.2 Elastic continuum and softening crack

This test deals with the elastic solution of a one-dimensional bar in tension. Nevertheless, some softening is introduced via the crack, see Figure 38.

\[
\begin{align*}
\dot{t} &= T |\dot{u}|, \quad T < 0 \\
t(0) &= t_{\text{crit}}
\end{align*}
\]

(53)

In this case, the critical situation means that the stress exceeds a threshold set a priori. As soon as the critical situation is detected in one finite element, a discontinuity is introduced and the discontinuous technique is used. The discretisation (Fig. 33) and the stiffness matrix (Eq. (42), (47) and (48)) used in the previous example are here retrieved.

In this case, taking into account that \( \sigma(x = L/2) = t \), one obtains

\[
\begin{align*}
\sigma^+(L/2) &= \sigma^-(L/2) = t \\
&= t_{\text{crit}} + T \cdot |u| \\
&= t_{\text{crit}} + T \cdot (u_1 + u_2 - (u_1 - u_2)) \\
&= t_{\text{crit}} + 2T \cdot u_2 \\
&= t_{\text{crit}} + 2TN(L/2)b
\end{align*}
\]

(54)
and therefore,
\[ f_{\text{int,b}} = \int_0^L \mathcal{H}^T \mathbf{B}^T \sigma \, dx + \mathbf{N}^T (L/2) [2t_{\text{crit}} + 4T \cdot \mathbf{N}(L/2)b] \] (55)

To illustrate the behaviour of this second example, a particular test is carried out. The dimensionless geometric and material parameters are summarised in Table 9.

Table 9: Uniaxial tension test with an elastic model and a softening behaviour of the crack: geometric and material parameters.

<table>
<thead>
<tr>
<th>Meaning</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of the bar</td>
<td>(L)</td>
<td>100</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>(E)</td>
<td>20000</td>
</tr>
<tr>
<td>Parameter of the crack</td>
<td>(T)</td>
<td>-50</td>
</tr>
<tr>
<td>Critical stress</td>
<td>(t_{\text{crit}})</td>
<td>1.6</td>
</tr>
</tbody>
</table>

The results are summarised in Figure 39. Note that:

- The force-displacement curve, Fig. 39(a), exhibits the expected slope. On the one hand, in the first load increments, the stress is lower than the critical stress set a priori and there is no crack. Therefore, the bar exhibits an elastic behaviour and the force-displacement curve has a slope of \(\frac{E}{L}\). On the other hand, when the stress exceeds \(\sigma_{\text{crit}}\), a crack is introduced. In this case, the slope is given by Eq. (52).

- The force-jump curve also exhibits the expected slope \(T < 0\), Fig. 39(b).

- The stress profiles \(\sigma\), Fig. 39(c), are constant, as dictated by equilibrium. In the first load increments, the stress profiles increase until they reach the critical stress. From that moment on, the stress profiles decrease due to the crack opening.

- The strain profiles \(\varepsilon\), Figure 39(d), are constant, as there is not any weakened zone. They behave as dictated by the elastic constitutive equation.

- The displacement profiles, Fig. 39(e), exhibit the expected jump.
3 CONTINUOUS-DISCONTINUOUS FAILURE IN STANDARD MEDIA

Figure 39: Uniaxial tension test with an elastic model and a softening behaviour of the crack: (a) force-displacement; (b) force-jump; (c) stress $\sigma$; (d) strain $\varepsilon$; (e) displacements.
3.7 Numerical examples: uniaxial tension test

3.7.3 Local damage continuum and softening crack

In this example, a continuum damage model is used, see Eq. (A.7). Moreover, some softening is added via the crack, see Figure 40.

As shown in Figure 40,

\[
\begin{align*}
\dot{t} &= T|\dot{u}|, \\
t(0) &= t_{\text{crit}}
\end{align*}
\]  

(56)

As in the previous example, a continuous technique is employed until a critical situation is detected. Here, the critical situation means that the damage parameter exceeds a threshold set a priori called $D_{\text{crit}}$. As soon as the critical situation is detected in one finite element, a discontinuity is introduced and the discontinuous technique is used. From that moment on, two different models may be considered. On the one hand, damage value is fixed to $D_{\text{crit}}$, see Fig. 41(a), and on the other hand, this parameter is allowed to increase, see Fig. 41(b). These two options are analysed.

The one-dimensional particularisations of the gradient version, Table 1, with $Y(\varepsilon) = \varepsilon$, are employed in this example. Since the behaviour of the crack is the same as in example 3.7.2,

\[
f_{\text{int,b}} = \int_0^L \mathcal{H} \mathbf{B}^T \sigma \, dx + \mathbf{N}^T(L/2) \left[ 2t_{\text{crit}} + 4T \cdot \mathbf{N}(L/2) \mathbf{b} \right]
\]  

(57)
To illustrate the behaviour of this third example, two particular tests are analysed, in which the central tenth of the bar is weakened to cause localisation. In the first test, the damage parameter is set to $D_{\text{crit}}$ and in the second one, the damage value may increase. The dimensionless geometric and material parameters for both tests are summarised in Table 10.

Table 10: Uniaxial tension test with a local damage model and a softening behaviour of the crack: geometric and material parameters.

<table>
<thead>
<tr>
<th>Meaning</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of the bar</td>
<td>$L$</td>
<td>100</td>
</tr>
<tr>
<td>Length of weaker part</td>
<td>$L_W$</td>
<td>$L/7$</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E$</td>
<td>20 000</td>
</tr>
<tr>
<td>Idem of weaker part</td>
<td>$E_W$</td>
<td>18 000</td>
</tr>
<tr>
<td>Damage threshold</td>
<td>$\varepsilon_i$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Final strain</td>
<td>$\varepsilon_f$</td>
<td>$1.25 \times 10^{-2}$</td>
</tr>
<tr>
<td>Parameter of the crack</td>
<td>$T$</td>
<td>$-20$</td>
</tr>
<tr>
<td>Critical damage</td>
<td>$D_{\text{crit}}$</td>
<td>0.9</td>
</tr>
</tbody>
</table>

The results for the first test (Fig. 41(a)) are summarised in Figure 42. Note that:

- The force-displacement curve, Fig. 42(a), exhibits the expected slope. Since in the first load increments the strain is lower than the damage threshold $\varepsilon_i$, the bar exhibits an elastic behaviour and the force-displacement curve has the expected slope (58):
\[ \Delta u = (L - L_W) \frac{\Delta \sigma}{E} + L_W \frac{\Delta \sigma}{E_W} = \left( \frac{L - L_W}{E} + \frac{L_W}{E_W} \right) \Delta \sigma \]

\[ \Delta \sigma = \frac{1}{\frac{L - L_W}{E} + \frac{L_W}{E_W}} \Delta u \quad \Rightarrow \quad \Delta F = \frac{1}{\frac{L - L_W}{E} + \frac{L_W}{E_W}} \Delta u \]

(58)

Once the strain exceeds the damage threshold, the damage continuum model is used and the force-displacement curve has the slope (59):

\[ \Delta u = (L - L_W) \frac{\Delta \sigma}{E} + L_W \frac{\Delta \sigma}{E_{soft}} = \left( \frac{L - L_W}{E} + \frac{L_W}{E_{soft}} \right) \Delta \sigma \]

\[ \Delta \sigma = \frac{1}{\frac{L - L_W}{E} + \frac{L_W}{E_{soft}}} \Delta u \quad \Rightarrow \quad \Delta F = \frac{1}{\frac{L - L_W}{E} + \frac{L_W}{E_{soft}}} \Delta u \]

(59)

where \( E_{soft} = \frac{\sigma_{max}}{\varepsilon_i - \varepsilon_f} \).

Finally, when the damage exceeds the critical damage \( D_{crit} \), a crack is introduced. In this case, since \( D \) is set to \( D_{crit} \), the slope is given by Eq. (60).

\[ \Delta u = (L - L_W) \frac{\Delta \sigma}{E} + L_W \frac{\Delta \sigma}{E_{crit}} + \frac{\Delta \sigma}{T} = \left( \frac{L - L_W}{E} + \frac{L_W}{E_{crit}} + \frac{1}{T} \right) \Delta \sigma \]

\[ \Delta \sigma = \frac{1}{\frac{L - L_W}{E} + \frac{L_W}{E_{crit}} + \frac{1}{T}} \Delta u \quad \Rightarrow \quad \Delta F = \frac{1}{\frac{L - L_W}{E} + \frac{L_W}{E_{crit}} + \frac{1}{T}} \Delta u \]

(60)

where \( E_{crit} = (1 - D_{crit}) E_W \).

- The force-jump curve also exhibits the expected slope \( T < 0 \), Fig. 42(b). Moreover,

\[ u_f = \|u\| \]

(61)

where \( u_f \) is the final displacement.

- The stress profiles \( \sigma \), Fig. 42(c), are constant, as dictated by equilibrium. In the first load increments, the stress profiles increase until the strain reaches the damage threshold. From that moment on, the stress profiles decrease.

- The strain profiles \( \varepsilon \), Figure 42(d), are not constant, since there is a weakened part. Once \( D \) is set to \( D_{crit} \), in the weakened elements, the strain profiles change their qualitative response starting to decrease to satisfy the constitutive equation.

- The damage profiles \( D \), Figure 42(e), behave as dictated by a local damage model. The damage parameter is not higher than \( D_{crit} \).

- The displacement profiles, Fig. 42(f), exhibit the expected jump.
Figure 42: Uniaxial tension test with a local damage model and a softening behaviour of the crack (damage parameter fixed): (a) force-displacement; (b) force-jump; (c) stress $\sigma$; (d) strain $\varepsilon$; (e) damage $D$; (f) displacements.
3.7 Numerical examples: uniaxial tension test

The results for the second test, Fig. 41(b), are shown in Fig. 43.

Note that:

- The force-displacement curve, Fig. 43(a), exhibits the expected slope. While the crack is not introduced into the model, the bar exhibits the same response as in the previous example. Eq. (58) and (59) are here retrieved. However, the qualitative response changes in the last load increments. In the current example, the bar is allowed to download for the softening branch and the force-displacement curve exhibits the slope given by Eq. (62).

\[
\begin{align*}
\Delta u &= (L - L_W) \frac{\Delta \sigma}{E} + L_W \frac{\Delta \sigma}{E_{\text{soft}}} + \frac{\Delta \sigma}{T} = \left( \frac{L - L_W}{E} + \frac{L_W}{E_{\text{soft}}} + \frac{1}{T} \right) \Delta \sigma \\
\Delta \sigma &= \frac{1}{L - L_W/E_{\text{soft}} + L/W} \Delta u \\
\Rightarrow \quad \Delta F &= \frac{1}{L - L_W/E_{\text{soft}} + L/W} \frac{1}{T} \Delta u
\end{align*}
\]

(62)

where \( E_{\text{soft}} = \frac{\sigma_{\text{max}}}{\varepsilon_i - \varepsilon_f} \).

- The force-jump curve also exhibits the expected slope \( T < 0 \), Fig. 43(b). In this case,

\[
\begin{align*}
u_f &= [|u|] + L_W \varepsilon_f
\end{align*}
\]

(63)

- The stress profiles \( \sigma \), Fig. 43(c), are constant, as dictated by equilibrium. They behave in a similar way that in the previous example.

- The strain profiles \( \varepsilon \), Figure 43(d), are now not constant, since there is a weakened part to cause localisation. Note that, as the damage parameter reaches \( D_{\text{crit}} \), the strain profiles do not change their qualitative response, since now the damage parameter increases.

- The damage profiles \( D \), Figure 43(e), behave as dictated by a local damage model.

- The displacement profiles, Fig. 43(f), exhibit the expected jump.
Figure 43: Uniaxial tension test with a local damage model and a softening behaviour of the crack (damage parameter not fixed): (a) force-displacement; (b) force-jump; (c) stress $\sigma$; (d) strain $\varepsilon$; (e) damage $D$; (f) displacements.
3.7.4 The need for a regularisation

As mentioned in Section 2, regularisation techniques must be considered to preserve well-posedness of governing equations, to avoid the pathological mesh dependence and to obtain physically realistic results. In this section, two different examples are presented to illustrate the pathological mesh-dependence exhibited by local models, also in a discontinuous framework.

As a first test, the example in Section 3.7.2 is analysed. To check whether the model regularises softening, two different meshes of 21 and 63 elements are used. The results are summarised in Fig. 44.

As shown in Figure 44, the two curves are overlapped. That is, the element size does not determine the response, because of the zero-dimensional character of the discontinuity.

As a second test, the example in Section 3.7.3 is retrieved. The goal is to observe that the response of this test does depend on numerical parameters, such as the length of the weakened zone needed to cause localisation. Due to this reason two different tests will be carried out, see Table 11.

First of all, the damage parameter is set to $D_{\text{crit}}$ and then, the damage value may increase. The results are shown in Figure 45.

As clearly seen in Figure 45, numerical parameters determine the qualitative response of the test. Due to this reason, a regularisation technique must be employed to avoid a dependence of the solution on the discretisation. In Section 4, a continuous-discontinuous
gradient-enhanced model is presented, which allows a realistic characterisation of the entire failure process.

\begin{table}[ht]
\centering
\caption{Size of weakened region.}
\begin{tabular}{lcc}
\hline
Meaning & Symbol & Value \\
\hline
Length of weaker part & $L_W$ & $L/7$ \\
$L_W = L/7$ Test & & \\
Length of weaker part & $L_W$ & $L/21$ \\
$L_W = L/21$ Test & & \\
\hline
\end{tabular}
\end{table}

Figure 45: Uniaxial tension test with a local damage model and a softening behaviour of the crack: Damage parameter set to $D_{\text{crit}}$: (a) force-displacement; (b) damage $D$; Damage parameter is allowed to increase: (c) force-displacement; (d) damage $D$. 
4 Continuous-discontinuous failure in regularised media

The goal of this section is to present a new continuous-discontinuous strategy, which allows a realistic characterisation of the entire failure process. This strategy is based on a non-local model based on non-local displacements, see Section 2. Within this strategy, the standard approximation basis is enriched locally with special functions.

In Section 4.1, the characterisation of the problem fields is introduced. The governing equations, the variational formulation and its linearised discrete format are derived in Section 4.2, 4.3 and 4.4 respectively. Finally, in Section 4.6, a uniaxial tension test is analysed to illustrate the regularisation capabilities of this new strategy.

4.1 Problem fields

In the gradient-enhanced damage continuum model based on non-local displacements, the problem is characterised by the local displacement field \( u \) and the non-local displacements \( \tilde{u} \), see Table 1.

In the body \( \bar{\Omega} \), the displacement field \( u \) can be decomposed as Eq. (8). Similarly, the non-local displacements \( \tilde{u} \) can be decomposed as

\[
\tilde{u}(x, t) = \tilde{u}_1(x, t) + \mathcal{H}_{\Gamma_d} \tilde{u}_2(x, t)
\]  

where \( \tilde{u}_1 \) and \( \tilde{u}_2 \) are continuous functions on \( \bar{\Omega} \) and \( \mathcal{H}_{\Gamma_d} \) is defined in Eq. (9).

4.2 Governing equations

The equilibrium equations and boundary conditions for the elastic body \( \bar{\Omega} \) are summarised in Eq. (10). The constitutive equation for a damage continuum model is given by Eq. (11b).

Moreover, in the non-local damage model based on non-local displacements \( \tilde{u} \) to local displacements \( u \), see Eq. (1), which is added to the equilibrium equation.
4.3 Variational formulation

Similarly as done in Section 3.3, the space of trial non-local displacements \( \tilde{\mathbf{u}} \) is defined by the function

\[
\tilde{\mathbf{u}}(\mathbf{x}, t) = \tilde{\mathbf{u}}_1(\mathbf{x}, t) + \mathcal{H}_{\Gamma_d} \tilde{\mathbf{u}}_2(\mathbf{x}, t), \quad \tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2 \in \mathcal{U}_u
\]  

(65)

where \( \mathcal{U}_u \) is defined in Eq. (13).

Eq. (1) can be cast in a variational form by multiplication with a vector test function \( \mathbf{\omega}(\mathbf{x}, t) \), see Eq. (15).

Then, Eq. (1) is integrated over the domain \( \Omega \) to obtain the weak equilibrium statement.

\[
\int_{\Omega} (\mathbf{\omega}_1 + \mathcal{H}_{\Gamma_d} \mathbf{\omega}_2) \cdot (\tilde{\mathbf{u}}_1 + \mathcal{H}_{\Gamma_d} \tilde{\mathbf{u}}_2) \, d\Omega = \int_{\Omega} \mathbf{\omega}_1 \cdot \nabla^2 (\tilde{\mathbf{u}}_1 + \mathcal{H}_{\Gamma_d} \tilde{\mathbf{u}}_2) \, d\Omega
\]

\[
\forall \mathbf{\omega}_1, \mathbf{\omega}_2 \in \mathcal{W}_{u,0}
\]  

(66)

From the decomposition of the problem fields it follows that any admissible variation \( \mathbf{\omega} \) can be regarded as admissible variations \( \mathbf{\omega}_1 \) and \( \mathbf{\omega}_2 \), thus leading to two variational statements. Taking first variations \( \mathbf{\omega}_1 (\mathbf{\omega}_2 = 0) \), and then \( \mathbf{\omega}_2 (\mathbf{\omega}_1 = 0) \), leads to:

\[
\int_{\Omega} \mathbf{\omega}_1 \cdot (\tilde{\mathbf{u}}_1 + \mathcal{H}_{\Gamma_d} \tilde{\mathbf{u}}_2) \, d\Omega - l_c^2 \int_{\Omega} \mathbf{\omega}_1 \cdot \nabla^2 (\tilde{\mathbf{u}}_1 + \mathcal{H}_{\Gamma_d} \tilde{\mathbf{u}}_2) \, d\Omega = \int_{\Omega} \mathbf{\omega}_1 \cdot (\mathbf{u}_1 + \mathcal{H}_{\Gamma_d} \mathbf{u}_2) \, d\Omega
\]

\[
\forall \mathbf{\omega}_1 \in \mathcal{W}_{u,0}
\]  

(67a)

\[
\int_{\Omega} \mathcal{H}_{\Gamma_d} \mathbf{\omega}_2 \cdot (\tilde{\mathbf{u}}_1 + \mathcal{H}_{\Gamma_d} \tilde{\mathbf{u}}_2) \, d\Omega - l_c^2 \int_{\Omega} \mathcal{H}_{\Gamma_d} \mathbf{\omega}_2 \cdot \nabla^2 (\tilde{\mathbf{u}}_1 + \mathcal{H}_{\Gamma_d} \tilde{\mathbf{u}}_2) \, d\Omega = \int_{\Omega} \mathcal{H}_{\Gamma_d} \mathbf{\omega}_2 \cdot (\mathbf{u}_1 + \mathcal{H}_{\Gamma_d} \mathbf{u}_2) \, d\Omega
\]

\[
\forall \mathbf{\omega}_2 \in \mathcal{W}_{u,0}
\]  

(67b)

In order to manipulate Eq. (67), the derivative of the Heaviside function must be introduced:

\[
\nabla \mathcal{H}_{\Gamma_d} = 2\delta_{\Gamma_d} \mathbf{m}
\]  

(68)

where \( \delta_{\Gamma_d} \) is the Dirac-delta function centered at the discontinuity surface \( \Gamma_d \) and \( \mathbf{m} \) is the inward unit normal to \( \Omega^+ \) on \( \Gamma_d \). Note that the factor two is a consequence of the definition of the Heaviside function, see Appendix B.

Therefore, using Eq. (68), Green’s theorem and after the application of the boundary
conditions, the variational statements in Eq. (67) can be written as

\[
\int_{\Omega} \omega_1 \cdot \tilde{u}_1 \, d\Omega + \int_{\Omega} \mathcal{H}_{\Gamma_d} \omega_1 \cdot \tilde{u}_2 \, d\Omega + l_c^2 \int_{\Omega} \nabla \omega_1 : \nabla \tilde{u}_1 \, d\Omega + l_c^2 \int_{\Omega} \mathcal{H}_{\Gamma_d} \nabla \omega_1 : \nabla \tilde{u}_2 \, d\Omega \\
- l_c^2 \left[ \int_{\Gamma_d^-} \mathcal{H}_{\Gamma_d} \omega_1 \nabla \tilde{u}_2 \, d\Gamma + 2 \int_{\Omega} \omega_1 \delta_{\Gamma_d}^m \nabla \tilde{u}_2 \, d\Omega + 2 \int_{\Omega} \omega_1 \nabla \cdot (\delta_{\Gamma_d}^m \tilde{u}_2) \, d\Omega \right]
\]

\[
= \int_{\Omega} \omega_1 \cdot u_1 \, d\Omega + \int_{\Omega} \mathcal{H}_{\Gamma_d} \omega_1 \cdot u_2 \, d\Omega \quad \forall \omega_0 \in W_{u,0} \quad (69a)
\]

\[
\int_{\Omega} \mathcal{H}_{\Gamma_d} \omega_2 \cdot \tilde{u}_1 \, d\Omega + \int_{\Omega} \omega_2 \cdot \tilde{u}_2 \, d\Omega + l_c^2 \int_{\Omega} \mathcal{H}_{\Gamma_d} \nabla \omega_2 : \nabla \tilde{u}_1 \, d\Omega + l_c^2 \int_{\Omega} \nabla \omega_2 : \nabla \tilde{u}_1 \, d\Omega \\
+ l_c^2 \int_{\Gamma_d^+} \omega_2 \nabla \tilde{u}_1 \, m \, d\Gamma - l_c^2 \int_{\Gamma_d^-} \omega_2 \nabla \tilde{u}_1 \, m \, d\Gamma - 4l_c^2 \int_{\Omega} \mathcal{H}_{\Gamma_d} \omega_2 \delta_{\Gamma_d}^m \nabla \tilde{u}_2 \, d\Omega \\
- 4l_c^2 \int_{\Omega} \mathcal{H}_{\Gamma_d} \omega_2 \nabla \cdot (\delta_{\Gamma_d}^m \tilde{u}_2) \, d\Omega = \int_{\Omega} \mathcal{H}_{\Gamma_d} \omega_2 \nabla \cdot u_1 \, d\Omega + \int_{\Omega} \omega_2 \cdot u_2 \, d\Omega \quad \forall \omega_2 \in W_{u,0} \quad (69b)
\]

As shown in Eq. (69), the integrals evaluated on the discontinuities are multiplied by the characteristic length. That is, some diffusion is introduced into the crack. The reason of this behaviour is that Eq. (1) has been introduced into all the domain, including into the crack. I.e., the PDE which is added to the equilibrium equation consists of

\[
\tilde{u} (x, t) - l_c^2 \nabla^2 \tilde{u} (x, t) = u (x, t) \quad \text{in } \Omega \quad (70)
\]

A different option, which has not been addressed, is to consider that the regularisation PDE is not introduced into the crack, that is:

\[
\tilde{u} (x, t) - l_c^2 \nabla^2 \tilde{u} (x, t) = u (x, t) \quad \text{in } \Omega^+ \cup \Omega^- \quad (71)
\]

Moreover, it is important to note that during the manipulations in Eq. (69) the property \( \mathcal{H}_{\Gamma_d} \mathcal{H}_{\Gamma_d} = +1 \) is used.

### 4.4 Discretisation and linearisation

#### 4.4.1 Problem field description

Using a Galerkin approach, the local and non-local displacements can be discretised in each element with extra degrees of freedom using

\[
\begin{align*}
\mathbf{u}_1 &= N\mathbf{a} & \mathbf{u}_2 &= N\mathbf{b} \\
\tilde{\mathbf{u}}_1 &= N\tilde{\mathbf{a}} & \tilde{\mathbf{u}}_2 &= N\tilde{\mathbf{b}} \\
\omega_1 &= N\omega_1 & \omega_2 &= N\omega_2 \\
\nabla \mathbf{u}_1 &= B\mathbf{a} & \nabla \mathbf{u}_2 &= B\mathbf{b} \\
\nabla \tilde{\mathbf{u}}_1 &= B\tilde{\mathbf{a}} & \nabla \tilde{\mathbf{u}}_2 &= B\tilde{\mathbf{b}} \\
\nabla \omega_1 &= B\omega_1 & \nabla \omega_2 &= B\omega_2
\end{align*}
\]
Similarly as in Section 3.4.1, $\tilde{a}$ and $\tilde{b}$ are the continuous and the discontinuous non-local nodal displacements respectively.

With this discretisation, Eq. (64) reads, for nodes whose support is crossed by a discontinuity $\Gamma_d$,

$$\tilde{u}_h = N\tilde{a} + H_{\Gamma_d}N\tilde{b} \quad (72)$$

For nodes whose support is not crossed by a discontinuity, since there is no enhancement, the standard finite element interpolations are retrieved,

$$\tilde{u}_h = N\tilde{a} \quad (73)$$

### 4.4.2 Discretised and linearised weak governing equations

The discrete format of the problem fields leads to the two discrete weak governing equations

\begin{align*}
(M + l_c^2 D)\tilde{a} &+ (M_{\mathcal{H}_d} + l_c^2 (D_{\mathcal{H}_d} + K^1))\tilde{b} = Ma + M_{\mathcal{H}_d} b \quad (74a) \\
(M_{\mathcal{H}_d} + l_c^2 (D_{\mathcal{H}_d} + K^2))\tilde{a} &+ (M + l_c^2 (D + K^3))\tilde{b} = M_{\mathcal{H}_d} a + Mb \quad (74b)
\end{align*}

where

\begin{align*}
M &= \int_\Omega N^T N \, d\Omega \\
M_{\mathcal{H}_d} &= \int_\Omega H_{\Gamma_d} N^T N \, d\Omega \\
D &= \int_\Omega \nabla N^T \nabla N \, d\Omega \\
D_{\mathcal{H}_d} &= \int_\Omega H_{\Gamma_d} \nabla N^T \nabla N \, d\Omega
\end{align*}

and matrices $K^1$, $K^2$ and $K^3$ are referred to the integrals on the boundary $\Gamma_d$. The expression of these matrices for a one-dimensional particular problem will be addressed in detail.

In summary, the finite element discretisation results in

\begin{align*}
\mathbf{r}_{\text{regu},a} &:= f_{\text{int},a} - f_{\text{ext},a} = 0 \quad (76a) \\
\mathbf{r}_{\text{regu},b} &:= f_{\text{int},b} - f_{\text{ext},b} = 0 \quad (76b) \\
\mathbf{r}_{\text{regu},\tilde{a}} &:= (M + l_c^2 D)\tilde{a} + (M_{\mathcal{H}_d} + l_c^2 (D_{\mathcal{H}_d} + K^1))\tilde{b} - Ma - M_{\mathcal{H}_d} b = 0 \quad (76c) \\
\mathbf{r}_{\text{regu},\tilde{b}} &:= (M_{\mathcal{H}_d} + l_c^2 (D_{\mathcal{H}_d} + K^2))\tilde{a} + (M + l_c^2 (D + K^3))\tilde{b} - M_{\mathcal{H}_d} a - Mb = 0 \quad (76d)
\end{align*}
where \( f_{\text{int,a}}, f_{\text{ext,a}}, f_{\text{int,b}}, f_{\text{ext,b}} \) are defined in Eq. (24).

The consistent tangent matrix is

\[
\begin{bmatrix}
  K_{a,a} & K_{a,b} & K_{a,a} & K_{a,b} \\
  K_{b,a} & K_{b,b} & K_{b,a} & K_{b,b} \\
  K_{a,a} & K_{a,b} & K_{a,a} & K_{a,b} \\
  K_{b,a} & K_{b,b} & K_{b,a} & K_{b,b}
\end{bmatrix}
\]

(77)

with \( K_{a,a}, K_{a,b}, K_{b,a}, K_{b,b} \) defined in Eq. (28) and the matrices involving regularised degrees of freedom defined in Eq. (78).

\[
K_{a,a} := \frac{\partial r_{\text{regu,a}}}{\partial a} = -M (78a)
\]

\[
K_{a,b} := \frac{\partial r_{\text{regu,a}}}{\partial b} = -M H_d (78f)
\]

\[
K_{a,a} := \frac{\partial r_{\text{regu,a}}}{\partial a} = M + l^2 D (78g)
\]

\[
K_{a,b} := \frac{\partial r_{\text{regu,a}}}{\partial b} = M H_d + l^2 c (D_{H_d} + K^1) (78h)
\]

\[
K_{b,a} := \frac{\partial r_{\text{regu,b}}}{\partial a} = -M H_d (78i)
\]

\[
K_{b,b} := \frac{\partial r_{\text{regu,b}}}{\partial b} = -M (78j)
\]

\[
K_{b,a} := \frac{\partial r_{\text{regu,b}}}{\partial a} = M H_d + l^2 c (D_{H_d} + K^2) (78k)
\]

\[
K_{b,b} := \frac{\partial r_{\text{regu,b}}}{\partial b} = M + l^2 (D + K^3) (78l)
\]

So the linearised weak form at iteration \( i \) within a time step \( k \) reads

\[
\begin{bmatrix}
  K_{a,a}^{k,i-1} & K_{a,b}^{k,i-1} & K_{a,a}^{k,i-1} & K_{a,b}^{k,i-1} \\
  K_{b,a}^{k,i-1} & K_{b,b}^{k,i-1} & K_{b,a}^{k,i-1} & K_{b,b}^{k,i-1} \\
  \delta a^{k,i} & \delta b^{k,i} & \delta a^{k,i} & \delta b^{k,i}
\end{bmatrix}
\begin{bmatrix}
  -r_{\text{equil,a}}^{k,i} \\
  -r_{\text{equil,b}}^{k,i}
\end{bmatrix}
\]

(79)
Note that the arrays $N$ and $B$ multiplying $a$, $b$, $\tilde{a}$ and $\tilde{b}$ are not the same since only part of the degrees of freedom in the arrays $b$ and $\tilde{b}$ are activated.

Some remarks about the tangent matrix (77):

- Matrices $K_{a,\bar{a}}$ and $K_{b,\bar{b}}$ are the local tangent matrices already obtained in [1]. Matrices $K_{a,b}$ and $K_{b,\bar{a}}$ can be understood as enriched local tangent matrices, since the expression is the same, except for the Heaviside function.
- The mass matrix $M$ and the diffusivity matrix $D$ are constant.
- Once the critical situation is reached and the crack is introduced, the matrices $M_{\Gamma_d}$, $D_{\Gamma_d}$, $K^1$, $K^2$ and $K^3$ are also constant. Note that $M_{\Gamma_d}$ and $D_{\Gamma_d}$ can be understood as enriched mass and diffusivity matrices respectively.
- Thanks to the linear relation between the degrees of freedom $a$, $b$, $\tilde{a}$ and $\tilde{b}$, $r_{\text{regu},\bar{a}}$ and $r_{\text{regu},\bar{b}}$ are zero.
- Note again that, during the manipulations in Eq. (78), the property $H_{\Gamma_d}H_{\Gamma_d} = +1$ is used.

In summary:

1. **Elements with standard degrees of freedom:**
   
   For elements with standard degrees of freedom $a$ and $\tilde{a}$, the tangent matrix can be discretised in a standard way:
   
   $$K_{\text{tan}} = \begin{bmatrix} K_{k,i-1}^{a,a} & K_{k,i-1}^{a,\bar{a}} \\ K_{k,i-1}^{\bar{a},a} & K_{k,i-1}^{\bar{a},\bar{a}} \end{bmatrix}$$

2. **Elements with extra degrees of freedom:**
   
   On the other hand, for elements with extra degrees of freedom $b$ and $\tilde{b}$,
   
   $$K_{\text{tan}} = \begin{bmatrix} K_{k,i-1}^{b,a} & K_{k,i-1}^{b,b} & K_{k,i-1}^{b,\bar{a}} & K_{k,i-1}^{b,\bar{b}} \\ K_{k,i-1}^{\bar{a},b} & K_{k,i-1}^{\bar{a},\bar{b}} & K_{k,i-1}^{\bar{a},a} & K_{k,i-1}^{\bar{a},\bar{a}} \\ K_{k,i-1}^{\bar{b},a} & K_{k,i-1}^{\bar{b},b} & K_{k,i-1}^{\bar{b},\bar{a}} & K_{k,i-1}^{\bar{b},\bar{b}} \end{bmatrix}$$

   If the discontinuity does not cross the element but one of the element nodes has got extra degrees of freedom $b$ and $\tilde{b}$, $K_{\text{tan}}$ can be derived following the procedure described above.
4.5 One-dimensional particularisation: uniaxial tension test

The example analysed in Section 3.5 is retrieved. The one-dimensional particularisations of the governing equations for the regularised problem, the variational formulation and the discretisation and linearisation are presented.

Governing equations

The equilibrium equations and boundary conditions, summarised by Eq. (29), are coupled with

\[ \ddot{u} - \ell_c^2 \nabla^2 \bar{u} = u \quad x \in (0, L) \]
\[ u = \bar{u} \quad \text{on } x = \{0, L\} \]

That is,

\[ u(0) = \bar{u}(0) = 0 \quad \text{(83a)} \]
\[ u(L) = \bar{u}(L) = u^* \quad \text{(83b)} \]

Variational formulation

As done in Section 3.5, the space of trial local displacements is defined by the function

\[ u = u_1 + \mathcal{H}(x)u_2, \] with \( u_1, u_2 \in U_u \), where \( U_u \) and \( \mathcal{H}(x) \) are defined in Eq. (30) and (31) respectively.

Similarly, the space of trial non-local displacements is defined by the function

\[ \bar{u} = \bar{u}_1 + \mathcal{H}(x)\bar{u}_2, \] with \( \bar{u}_1, \bar{u}_2 \in U_u \).

The space of admissible displacement variations is defined by

\[ \omega = \omega_1 + \mathcal{H}(x)\omega_2, \] with \( \omega_1, \omega_2 \in W_{u,0} \), where \( W_{u,0} \) is defined in Eq. (32).
The two variational statements are derived from Eq. (69) and read

\[ \int_0^L \omega_1 \tilde{u}_1 \, dx + \int_0^L \mathcal{H} \omega_1 \tilde{u}_2 \, dx + l_c^2 \int_0^L \frac{d\tilde{u}_1}{dx} \, d\omega_1 + l_c^2 \int_0^L \mathcal{H} \frac{d\tilde{u}_2}{dx} \, d\omega_1 \\
+ 2l_c^2 \left( \frac{\tilde{u}_2(L/2)}{2} \right) \frac{d\omega_2}{dx} \left( \frac{L}{2} \right) + 2l_c^2 \frac{d\tilde{u}_2}{dx} \omega_1(L/2) \]
\[ = \int_0^L \omega_1 u_1 \, dx + \int_0^L \mathcal{H} \omega_1 u_2 \, dx \quad \forall \omega_1 \in \mathcal{W}_{u,0} \quad (84a) \]

\[ \int_0^L \mathcal{H} \omega_2 \tilde{u}_1 \, dx + \int_0^L \omega_2 \tilde{u}_2 \, dx + l_c^2 \int_0^L \mathcal{H} \frac{d\tilde{u}_1}{dx} \, d\omega_2 + l_c^2 \int_0^L \frac{d\tilde{u}_2}{dx} \, d\omega_2 \\
+ 2l_c^2 \left( \frac{\tilde{u}_2(L/2)}{2} \right) \frac{d\omega_1}{dx} \left( \frac{L}{2} \right) = \int_0^L \mathcal{H} \omega_2 u_1 \, dx \\
+ \int_0^L \omega_2 u_2 \, dx \quad \forall \omega_2 \in \mathcal{W}_{u,0} \quad (84b) \]

**Discretisation and linearisation**

Using

\[ u_1 = Na \quad u_2 = Nb \quad \frac{du_1}{dx} = Ba \quad \frac{du_2}{dx} = Bb \]

\[ \tilde{u}_1 = N\tilde{a} \quad \tilde{u}_2 = N\tilde{b} \quad \frac{d\tilde{u}_1}{dx} = B\tilde{a} \quad \frac{d\tilde{u}_2}{dx} = B\tilde{b} \]

\[ \omega_1 = Na' \quad \omega_2 = Nb' \quad \frac{d\omega_1}{dx} = Ba' \quad \frac{d\omega_2}{dx} = Bb' \]

the discrete format of the problem fields leads to the two discrete weak governing equations

\[ (M + l_c^2 D)\tilde{a} + (M_H + l_c^2 (D_H + 2(N^T(L/2)B(L/2) + B^T(L/2)N(L/2))) \tilde{b} \]
\[ = Ma + M_H b \quad (85) \]

\[ (M_H + l_c^2 (D_H + 2(N^T(L/2)B(L/2))) \tilde{a} + (M + l_c^2 D)\tilde{b} \\
= M_H a + Mb \quad (86) \]

where the matrices are defined in Eq. (75).
4.6 Numerical example: uniaxial tension test

4.6.1 Non-local damage model and softening crack

The particular test analysed in Section 3.7.3 is retrieved, see Table 10. As explained, when a critical situation is reached, damage parameter may be set to $D_{\text{crit}}$ (Fig. 41(a)) or may increase (Fig. 41(b)). Both options are here analysed.

The results for these two models, with a mesh of 105 elements and a characteristic length $l_c = \sqrt{5}$ are summarised in Figures 46 and 47 respectively.

Note that:

- The force-jump curve exhibits the expected slope $T < 0$.
- The stress profiles $\sigma$ are constant, as dictated by equilibrium.
- The strain profiles $\varepsilon$ are now not piecewise constant, since some diffusion has been incorporated into the model, also in a discontinuous setting. In Figure 46(d), the strain profiles change their qualitative response in those elements which have reached $D_{\text{crit}}$ in order to satisfy the constitutive equation.
- The damage profiles $D$ are also not piecewise constant. The width of the damage profiles does not depend on $L_W$. In Figure 46(e), damage cannot be higher than $D_{\text{crit}}$ while in Figure 47(e) it does.

To be able to study the regularisation capabilities of these two models, the following tests are considered.

As a first test, a fixed characteristic length $l_c = \sqrt{5}$ is chosen. The analysis is carried out with five different meshes. The force-displacement curves, the force-jump curves and the damage profiles are shown in Figure 48. As desired, the responses for this test do not depend on finite element sizes.

As a second test, a fixed mesh of 105 elements is considered and four different internal lengths are used, $l_c = \sqrt{1}, \sqrt{2}, \sqrt{5}, \sqrt{10}$. The results are depicted in Figure 49. The ductility in the force-displacement response and the width of the final damage profile increase with the internal length scale.
Figure 46: Uniaxial tension test with a non-local damage model and a softening behaviour of the crack (elastic unloading in damaged zone): (a) force-displacement; (b) force-jump; (c) stress $\sigma$; (d) strain $\varepsilon$; (e) damage $D$; (f) displacements.
4.6 Numerical example: uniaxial tension test

Figure 47: Uniaxial tension test with a non-local damage model and a softening behaviour of the crack (additional softening in damaged zone): (a) force-displacement; (b) force-jump; (c) stress $\sigma$; (d) strain $\varepsilon$; (e) damage $D$; (f) displacements.
Figure 48: Fixed characteristic length with various meshes. Damage parameter set to $D = D_{crit}$: (a) force-displacement; (c) force-jump; (e) damage profiles; Damage parameter may increase: (b) force-displacement; (d) force-jump; (f) damage profiles.
4.6 Numerical example: uniaxial tension test

Figure 49: Fixed mesh with various characteristic lengths. Damage parameter set to $D = D_{\text{crit}}$: (a) force-displacement; (c) force-jump; (e) damage profiles; Damage parameter may increase: (b) force-displacement; (d) force-jump; (f) damage profiles.
Finally, as a third test, a fixed mesh of 105 elements and a fixed characteristic length $l_c = \sqrt{5}$ are chosen. Two different examples will be studied, see Table 11. Results are shown in Figure 50.

Figure 50: Uniaxial tension test with a non-local damage model and a softening behaviour of the crack: Damage parameter set to $D_{\text{crit}}$: (a) force-displacement; (b) damage $D$; Damage parameter is allowed to increase: (c) force-displacement; (d) damage $D$.

It is important to note the differences between Figure 45 (with a local damage model) and Figure 50 (with a non-local damage model). On the one hand, with a local model, both the slopes of force-displacement curves and the width of damage profiles depend on the weakened length $L_W$. On the other hand, in Figure 50, the obtained results are physically realistic: they do not depend, pathologically, on a numerical parameter such as $L_W$. 
To sum up: a new continuous-discontinuous strategy based on non-local displacements has been developed. This technique is valid to simulate both the inception of damage and its propagation. For increasing levels of damage, this strategy is able to simulate the physic discontinuities. Note that this strategy regularises the boundary problem, also in a discontinuous setting. The numerical responses do not depend on numerical parameters, such as the finite element size or the weakened zone $L_W$ but do depend on an internal material parameter, $l_c$. 
CONTINUOUS-DISCONTINUOUS FAILURE IN REGULARISED MEDIA
5 Concluding remarks and future work

5.1 Concluding remarks

• In this work, a new continuous-discontinuous strategy to model failure phenomena is presented. This technique consists of a continuous damage model to simulate the first stages of failure. Also there is a discontinuous model to reproduce the stages with increasing levels of damage.

• In order to simulate the first stages of damage, a gradient version of a non-local damage model based on non-local displacements is used. The implementation of this model with 2D elements is carried out. As shown in this work, the definition of appropriate boundary conditions for the regularisation equation is not a simple task. If only Dirichlet boundary conditions are imposed, there is no regularisation on $\partial \Omega$. Therefore, if damage starts on the boundary, physically unrealistic results may be obtained. Neumann boundary conditions are not expected to regularise the problem neither. Due to this reason, mixed conditions must be imposed. With these conditions, non-local displacements models are able to regularise softening damage models, not only with 1D elements but also with 2D.

• Since an internal length scale is incorporated into the model via the characteristic length $l_c$, the interaction between characteristic lengths at the structural level and at material level may be controlled. That is, non-local displacements models are able to describe size effects.

• In order to simulate the entire failure process, discontinuities in the displacement field must be incorporated into the model. In this work, a technique that consists of an enrichment of the standard polynomial finite element spaces by the Heaviside function is considered. This enrichment involves extra degrees of freedom for the nodes in the domain subjected to the enrichment. A one-dimensional example has been implemented to validate the strategy. All the numerical results are in agreement with the analytical ones. Similarly, as in a continuous model, if there is no regularisation, this model exhibits a pathological mesh dependence and leads to physically unrealistic results. Due to this behaviour, a regularisation technique must be incorporated into the model. A technique based on non-local displacements is used.
• The continuous-discontinuous technique based on non-local displacements is formulated. Non-local displacements can be used to regularise discontinuous damage models.

5.2 Future work

• The use of two displacement fields

As explained in Section 2, the second-order PDE (1) relates non-local displacements \( \tilde{u} \) to local displacements \( u \). Taking advantage of this fact, the idea is to use the non-local displacements only in the damaged zone, either in a continuous or a discontinuous setting.

• Boundary conditions at the discontinuity surface

As shown in Section 4, in enhanced media, the enrichment needed to simulate discontinuities involves extra degrees of freedom for the nodes in the domain subjected to the enrichment. In the damage model based on non-local displacements, the second-order PDE (1) is added to the equilibrium equation. To complete the coupled system of equations, appropriate boundary conditions must be defined. Until now, these boundary conditions are only imposed on \( \partial \Omega \). The idea is to impose these conditions also into the discontinuity surface in order to establish relationships between the nodal unknown values to reduce the extra degrees of freedom. This will lead to a very attractive strategy from a computational viewpoint.

• Incorporating \( n \) non-intersecting discontinuities

Moreover, a more general formulation is going to be studied. If the body \( \Omega \) is crossed by \( n \) non-intersecting discontinuities, the displacement field \( u \) can be decomposed as

\[
 u(x, t) = u_0(x, t) + \sum_{i=1}^{n} H_i(x) u_i(x, t) 
\]

(87)

where \( u_0 \) and \( u_i, \forall i = 1 \div n \), are continuous functions on \( \Omega \) and \( H_i \) are the Heaviside functions centered at the discontinuity surface \( \Gamma_i \):

\[
 H_i(x) = \begin{cases} 
 1 & \text{if } x \in \Omega^+_i \\
 -1 & \text{if } x \in \Omega^-_i 
\end{cases} 
\]

(88)
Similarly, the non-local displacements $\tilde{u}$ can also be decomposed as

$$
\tilde{u}(x, t) = \tilde{u}_0(x, t) + \sum_{i=1}^{n} \mathcal{H}_i \tilde{u}_i(x, t)
$$

(89)

where $\tilde{u}_0$ and $\tilde{u}_i$, $\forall i = 1 \div n$, are continuous functions on $\bar{\Omega}$.

With this formulation, more complicated problems will be analysed.

- **2D and 3D applications**

  Furthermore, two and three-dimensional applications are going to be analysed. This examples will require to study techniques to determine the crack growth direction.

- **Plasticity models**

  By now, only the damage model has been used. The idea is to develop a valid technique with any underlying continuous model. Due to this reason, plasticity models are going to be studied.
A Overview of damage models

This appendix deals with the basic features of damage models. On the one hand, the standard local damage model is presented in Section A.1. First, the generic equations are reviewed in Section A.1.1. The definition of the evolution law for the damage parameter and the state variable are discussed in Sections A.1.2 and A.1.3 respectively. Two particular models are summarised in Section A.1.4 (Mazars model) and A.1.5 (modified von Mises model). On the other hand, the basic features of gradient non-local damage models are described in Section A.2.

For simplicity, only isotropic elastic-damage models are considered here. However, the concept of the continuous-discontinuous strategy based on non-local displacements can be extended to more complex models.

A.1 Local damage models

A.1.1 Generic equations

A generic local damage model consists of the following equations:

- Constitutive equation

\[ \sigma(x, t) = (1 - D(x, t)) C : \varepsilon(x, t) \]  
(A.1)

where \( \sigma \) is the Cauchy stress tensor, \( \varepsilon \) the small strain tensor, \( C \) the tensor of elastic moduli and \( D \) the damage parameter. \( D \) ranges between 0 (undamaged material) and 1 (completely damaged material).

- Strains

\[ \varepsilon(x, t) = \nabla^s u(x, t) \]  
(A.2)

- Damage evolution

It is assumed that \( D \) depends on a state variable \( Y \).

\[ D(x, t) = D(Y(x, t)) \]  
(A.3)

- Local state variable

It is also assumed that \( Y \) depends on the strains.

\[ Y(x, t) = Y(\varepsilon(x, t)) \]  
(A.4)
To define a particular model, the damage evolution, Eq. (A.3), and the local state variable, Eq. (A.4), must be defined.

### A.1.2 Damage evolution

In damage models, damage starts above a threshold $Y_0$ (that is, $D = 0$ for $Y \leq Y_0$). Moreover, damage cannot decrease (that is, $\dot{D} \geq 0$). Therefore, a particular expression for $Y > Y_0$ must be given.

The most common expressions are the following:

- **Exponential law**
  
  $$D = 1 - \frac{Y_0 (1 - A)}{Y} - A e^{-B(Y - Y_0)}$$  
  \hspace{1cm} (A.5)

- **Polynomial law**
  
  $$D = 1 - \frac{1}{1 + B (Y - Y_0) + A (Y - Y_0)^2}$$  
  \hspace{1cm} (A.6)

  where the parameters $A$ and $B$ control the residual strength and the slope of the softening branch at the peak respectively.

- **Linear softening branch**
  
  $$D = \frac{Y_f}{Y_f - Y_0} \left(1 - \frac{Y_0}{Y}\right)$$  
  \hspace{1cm} (A.7)

  where $Y_f$ is the maximum admissible value for the state variable.

### A.1.3 Definition of the state variable

$Y$ should account for those features of the strain field which are responsible for damage inception and propagation. Moreover, $Y$ should be more sensitive to positive strains than to negative strains.

In the Mazars model, for example,

$$Y = \sqrt{\sum_i \left[\max(0, \varepsilon_i)^2\right]}$$  
  \hspace{1cm} (A.8)

where $\varepsilon_i$ are the principal strains.
On the other hand, in the modified von Mises model,

\[
Y = \frac{k - 1}{2k (1 - 2\nu)} I_1 + \frac{1}{2k} \sqrt{\left( \frac{k - 1}{1 - 2\nu} I_1 \right)^2 + \frac{12k}{(1 + \nu)^2} J_2}
\]  \hspace{1cm} (A.9)

where \( k \) is the ratio of compressive strength to tensile strength, \( I_1 \) is the first invariant of the strain tensor and \( J_2 \) is the second invariant of the deviatoric strain tensor.

### A.1.4 Mazars model

The definition of \( Y \) of Eq. (A.8) and the exponential law for the evolution of damage are combined. In fact, the damage parameter \( D \) is expressed as a linear combination of the tensile damage \( D_t \) and the compressive damage \( D_c \),

\[
D = \alpha_t D_t + \alpha_c D_c
\]  \hspace{1cm} (A.10)

where for \( D_t \) and \( D_c \) an exponential law is assumed.

There is also a simplified Mazars model, where the damage does not split into tensile and compressive components.

The equations for these two versions of the Mazars model are summarised in Tables A.1 and A.2.

### Table A.1: Full Mazars model.

<table>
<thead>
<tr>
<th>Local state variable</th>
<th>( Y = \sqrt{\sum_i \left[ \max (0, \varepsilon_i)^2 \right]} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damage evolution</td>
<td>( D = \alpha_t D_t + \alpha_c D_c )</td>
</tr>
<tr>
<td></td>
<td>( D_t = 1 - \frac{Y_0(1-A_t)}{Y} - A_t e^{-B_t(Y-Y_0)} )</td>
</tr>
<tr>
<td></td>
<td>( D_c = 1 - \frac{Y_0(1-A_c)}{Y} - A_c e^{-B_c(Y-Y_0)} )</td>
</tr>
</tbody>
</table>

### A.1.5 Modified von Mises model

This model is characterised only by the definition of the state variable, Eq. (A.9). Both the exponential and the polynomial laws are used, see Tables A.3 and A.4.
Table A.2: Simplified Mazars model.

<table>
<thead>
<tr>
<th>Local state variable</th>
<th>$Y = \sqrt{\sum_i \left[ \max(0, \varepsilon_i) \right]^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damage evolution</td>
<td>$D = 1 - \frac{Y_0(1-A)}{Y} - Ae^{-B(Y-Y_0)}$</td>
</tr>
</tbody>
</table>

Table A.3: Modified von Mises model with exponential law.

<table>
<thead>
<tr>
<th>Local state variable</th>
<th>$Y = \frac{k-1}{2k(1-2\nu)} I_1 + \frac{1}{2k} \sqrt{\left(\frac{k-1}{1-2\nu} I_1 \right)^2 + \frac{12k}{(1+\nu)^2} J_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damage evolution</td>
<td>$D = 1 - \frac{Y_0(1-A)}{Y} - Ae^{-B(Y-Y_0)}$</td>
</tr>
</tbody>
</table>

### A.2 Gradient non-local damage models

The non-locality needed to achieve physically realistic results can be incorporated into the model in two different ways (integral, see Reference [1], and gradient type models). Since only gradient non-local damage models have been employed in this work, only these are reviewed.

The gradient regularisation can be explicit or implicit. The implicit one provides a better approximation to the integral-type non-local model than the explicit regularisation. In the implicit gradient models, the non-local state variable $\tilde{Y}$ is the solution of the partial differential equation

$$\tilde{Y}(\mathbf{x}, t) - l_c^2 \nabla^2 \tilde{Y}(\mathbf{x}, t) = Y(\mathbf{x}, t) \quad (A.11)$$

where $l_c$ is the characteristic length of the non-local damage model, see Table A.5.

The regularisation PDE is in this case a diffusion-reaction equation. To solve it, appropriate boundary conditions must be imposed. The definition of these boundary conditions is not a simple task.

Note that although this model is non-local, it is mathematically local, because non-local
A.2 Gradient non-local damage models

Table A.4: Modified von Mises model with polynomial law.

<table>
<thead>
<tr>
<th>Local state variable</th>
<th>$Y = \frac{k-1}{2k(1-2\nu)} I_1 + \frac{1}{2k} \sqrt{\left( \frac{k-1}{1-2\nu} I_1 \right)^2 + \frac{12k}{(1+\nu)^2} J_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damage evolution</td>
<td>$D = 1 - \frac{1}{1+B(Y-Y_0)+A(Y-Y_0)^2}$</td>
</tr>
</tbody>
</table>

Table A.5: General expression of a gradient non-local damage model.

<table>
<thead>
<tr>
<th>Constitutive equation</th>
<th>$\sigma(x, t) = (1 - D(x, t)) C : \varepsilon(x, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strains</td>
<td>$\varepsilon(x, t) = \nabla^s u(x, t)$</td>
</tr>
<tr>
<td>Local state variable</td>
<td>$Y(x, t) = Y(\varepsilon(x, t))$</td>
</tr>
<tr>
<td>Non-local state variable</td>
<td>$\tilde{Y}(x, t) - t_e^2 \nabla^2 \tilde{Y}(x, t) = Y(x, t) + \text{b. cond.}$</td>
</tr>
<tr>
<td>Damage evolution</td>
<td>$D(x, t) = D(\tilde{Y})$</td>
</tr>
</tbody>
</table>

interaction is accounted for locally via higher-order spatial derivatives.
B  Heaviside function

As seen in this work, discontinuous displacement fields across $\Gamma_d$ can be modelled by enriching the interpolation field with the Heaviside function. The standard definition of the Heaviside function centered at the discontinuity surface $\Gamma_d$ reads as

$$H_{\Gamma_d}(x) = \begin{cases} 1 & \text{if } x \in \bar{\Omega}^+ \\ 0 & \text{if } x \in \bar{\Omega}^- \end{cases} \quad (B.1)$$

In fact, in [7], this definition is used.

Nevertheless, other definitions of this function may be considered. In this work, the alternative definition

$$\hat{H}_{\Gamma_d}(x) = \begin{cases} 1 & \text{if } x \in \bar{\Omega}^+ \\ -1 & \text{if } x \in \bar{\Omega}^- \end{cases} \quad (B.2)$$

is used. In the literature, this definition is also considered, see [5].

Independently of the definition of this function, the same numerical results must be obtained, since it is a numerical tool needed to introduce discontinuities into the model. Nevertheless, as seen in Sections 3 and 4, some differences in the formulation are observed depending on the chosen definition. When dealing with derivatives, for example, it is important to note that if Eq. (B.1) is used, Eq. (B.3a) is obtained, while the derivative of Eq. (B.2) is Eq. (B.3b).

$$\nabla H_{\Gamma_d} = \delta_{\Gamma_d}(x) \mathbf{m} \quad (B.3a)$$
$$\nabla \hat{H}_{\Gamma_d} = 2\delta_{\Gamma_d}(x) \mathbf{m} \quad (B.3b)$$

where $\mathbf{m}$ is the inward unit normal to $\Omega^+$ on $\Gamma_d$ and $\delta_{\Gamma_d}(x)$ is the Dirac-delta function centered at the discontinuity surface $\Gamma_d$.

In spite of the fact that Eq. (B.2) is a non-standard definition, there is an advantage of choosing it. To analyse it, the example in Section 3.7.1 is retrieved and the stiffness matrix of an element with a discontinuity in the middle section is computed with both definitions. The tangent matrix computed with Eq. (B.1) is shown in Eq. (B.4a), while the matrix computed with Eq. (B.2) can be seen in (B.4b).
As shown in Eq. (B.4b), the matrices $K_{ab}$ and $K_{ba}$ are equal to zero. This fact can be exploited to reduce the memory required for data storage. Due to this reason, in this work, Eq. (B.2) is used.
References


