Chapter 6

Cone calculation

6.1 Introduction

The soil is assumed to be a linear perfectly elastoplastic material. Only in the plastic zone pore pressures will be generated, hence, the first step is to determine the extent (depth) of the plastic zone.

The known parameters are:

- Cone’s geometry: height $H$, opening angle $\alpha$, pile radius (disk radius) $R_0$.
- Force applied $F$
- Soil properties: elastic $E_e$ and plastic $E_p$ moduli. It is also required to know the yield stress or elastic limit of the soil, $\sigma_e$.

![Figure 6.1: Cone geometry](image)

The elastic and plastic lines intersect at the elastic limit of the soil, $\sigma_e$. This limiting stress will define the extent of the plastification. As the cone expands in depth, the stresses will progressively diminish (the same applied force but over larger areas); the area for which the corresponding stress is $\sigma_e$ indicates the limit depth between plastified and elastic domains.
The procedure will be in the coming order:

1. Model of an elastic cone (before plasticity starts to occur, the full cone is elastic). Derivation of the expression for the vertical displacement of the elastic cone.

2. Model of a elastoplastic cone. When the applied stress exceeds the elastic limit, plastification will begin in the top of the cone. As the loading process goes on, the plastified section of the cone increases, therefore the elastic limit is progressively found at larger depths. When the full test load is applied, find out the depth at which the stress level corresponds to the elastic limit. This depth is the limit of the plastic zone.

3. When the full load is applied, find out at which depth corresponds the elastic limit

6.2 Elastic cone

6.2.1 Model definition

The deformation of the cone follows from fig (6.3). Hence, to define the cone model, one needs to consider an infinitesimal slice of the cone (fig.6.4). The model is defined by three equations:

1. Equilibrium of the infinitesimal slice:

   \[ N = N + \frac{\partial N}{\partial z} \Delta z \Rightarrow \frac{\partial N}{\partial z} \Delta z = 0 \]  

   (6.1)

2. Constitutive equation: Elasticity:

   \[ \sigma = E \varepsilon \]  

   (6.2)
3. **Geometric equation:**

\[ \epsilon = \frac{\partial u}{\partial z} \]  \hspace{1cm} (6.3)

Rewriting (6.1) as a function of the cross-section of the cone \( N(z) = A(z)\sigma(z) \), it can be obtained:

\[ \frac{\partial A}{\partial z} \sigma + A \frac{\partial \sigma}{\partial z} = 0 \]  \hspace{1cm} (6.4)

The derivative of \( \sigma \) with respect to \( z \) can be found from eq. (6.2):

\[ \frac{\partial \sigma}{\partial z} = E \frac{\partial \epsilon}{\partial z} \]  \hspace{1cm} (6.5)

with eq. (6.3):

\[ \frac{\partial \sigma}{\partial z} = E \frac{\partial^2 u}{\partial z^2} \]  \hspace{1cm} (6.6)
Finally the governing differential equation of the static cone model in elasticity to solve is:

\[ A \frac{\partial^2 u}{\partial z^2} + \frac{\partial A}{\partial z} \frac{\partial u}{\partial z} = 0 \quad (6.7) \]

One can explicitly express it as a function only of \( z \). The cone has a circular cross-section:

\[ A = \pi r^2 = \pi (R_0 + z \tan \alpha)^2 \quad (6.8) \]

and consequently:

\[ \frac{\partial A}{\partial z} = 2 \pi \tan \alpha (R_0 + z \tan \alpha) \quad (6.9) \]

where \( R_0 \) is the radius of the pile and \( \alpha \) is the opening angle of the cone, that is only function of the Poisson ratio of the soil.

Substituting into eq.(6.7):

\[ \pi (R_0 + z \tan \alpha)^2 \frac{\partial^2 u}{\partial z^2} + 2 \pi \tan \alpha (R_0 + z \tan \alpha) \frac{\partial u}{\partial z} = 0 \quad (6.10) \]

### 6.2.2 Solution of the differential equation

Eq.(6.7) is a second order differential equation. It is derived only as a function of \( z \), so the partial derivatives can be turned into total derivatives. It can be rewritten:

\[ \frac{d}{dz}(A \frac{du}{dz}) = 0 \quad (6.11) \]

This equation has a solution of the shape:

\[ A \frac{du}{dz} = K \quad (6.12) \]

where \( K \) is a constant to be determined and \( A = \pi (R_0 + \tan \alpha)^2 \). Eq.(6.12) is the new equation to be solved. It is now a first order differential equation. With a change of variable it can be obtained a function of linear coefficients. Define the following change of variable:

\[ u = \frac{X(z)}{C_0 + C_1 z + C_2 z^2} \quad (6.13) \]

\[ \frac{du}{dz} = \frac{-C_1 - 2C_2}{(C_0 + C_1 z + C_2 z^2)^2} X(z) + \frac{1}{C_0 + C_1 z + C_2 z^2} \frac{dX(z)}{dz} \quad (6.14) \]

with:

\[ \begin{cases} 
  C_0 = R_0 \\
  C_1 = \tan \alpha \\
  C_2 = 0 
\end{cases} \]

Substituting into eq.(6.12) follows:

\[ -\pi \tan \alpha X(z) + \pi (R_0 + z \tan \alpha) \frac{dX(z)}{dz} = K \quad (6.15) \]

Eq.(6.15) is a linear first order differential equation with non-constant coefficients of the form:

\[ a X(z) + b(z) \frac{dX(z)}{dz} = K \quad (6.16) \]

or, in a more general shape:

\[ \frac{dX}{dz} + p(z) X = g(z) \quad (6.17) \]
with the coefficients:
\[
\begin{align*}
 p(z) &= \frac{a}{b(z)} = \frac{-\pi \tan \alpha}{\pi (R_0 + z \tan \alpha)} = \frac{-\tan \alpha}{R_0 + z \tan \alpha} \\
 g(z) &= \frac{K}{H(z)} = \frac{K}{\pi (R_0 + \tan \alpha)}
\end{align*}
\]

Once the standard shape has been achieved, the next step is to convert it into an integrable equation. For this, eq(6.17) must be multiplied by an integration factor. The next developments will show how this factor can be found.

Eq.(6.17) can be rewritten:
\[
\mu(z)X' + \mu(z)p(z)X = \mu(z)g(z) \tag{6.18}
\]

The left term of eq.(6.18) shall be recognized as the derivative of some function; the most general approach is to consider it to be the derivative of a function of style \( \mu(z)X \). Then the second part of the left term can be associated:
\[
\mu(z)p(z)X = \mu'(z)X \Rightarrow \mu'(z) = p(z)\mu(z) \tag{6.19}
\]
Assuming \( \mu(z) > 0 \):
\[
\frac{\mu'(z)}{\mu(z)} = p(z) \Rightarrow \frac{d}{dz} \ln \mu(z) = p(z) \tag{6.20}
\]
Integrating,
\[
\ln \mu(z) = \int p(z)dz + Y \tag{6.21}
\]
Choosing the integration constant \( Y \) to be 0, the integration factor may be expressed:
\[
\mu(z) = e^{\int p(z)dz} \tag{6.22}
\]
Once the integration factor is known, eq.(6.19) can be rewritten:
\[
[\mu(z)X]' = \mu(z)g(z) \tag{6.23}
\]
Integrating the former expression:
\[
\mu(z)X = \int \mu(z)g(z)dz + C \tag{6.24}
\]
And the solution of our differential equation is the quotient:
\[
X = \frac{\int \mu(z)g(z)dz + C}{\mu(z)} \tag{6.25}
\]
To get the exact form our particular case we only need to substitute the notation previously defined:
\[
g(z) = \frac{K}{\pi (R_0 + \tan \alpha)} \tag{6.26}
\]
The integration factor is:
\[
\mu(z) = e^{\int p(z)dz} = e^{\int \frac{a}{b(z)}dz} = e^{\int \frac{-\tan \alpha}{R_0 + z \tan \alpha}dz}
\]
\[
= e^{\int \frac{1}{\exp[\ln(R_0 + z \tan \alpha)]} - \frac{1}{R_0 + z \tan \alpha}} \tag{6.27}
\]
Then:
\[
\mu(z) = \frac{1}{R_0 + z \tan \alpha} \tag{6.28}
\]
And the product $\mu(z)g(z)$ is expressed:

$$\mu(z)g(z) = \frac{K}{\pi(R_0 + z \tan \alpha)^2} \tag{6.29}$$

Integrating:

$$\int \mu(z)g(z)\,dz = \frac{K}{\pi} \int \frac{dz}{(R_0 + z \tan \alpha)^2} = -\frac{K}{\pi \tan \alpha(R_0 + z \tan \alpha)} + C \tag{6.30}$$

The first solution for the modified differential equation is:

$$X = -\frac{K + C\pi \tan \alpha(R_0 + z \tan \alpha)}{\pi \tan \alpha} \tag{6.31}$$

were $C$ is an integration constant to be determined from the boundary conditions, like $K$.

Finally one has to undo the change of variable to get the expression for the displacement. Substituting expression (6.31) into eq.(6.13) we get the definitive solution:

$$u(z) = C - \frac{K}{\pi \tan \alpha(R_0 + z \tan \alpha)} \tag{6.32}$$

### 6.2.3 Demonstration of the correctness of the solution

If the expression (6.32) is a correct solution it should be possible to put it back in eq.(6.7) and satisfy this equation. Eq.(6.7) was:

$$A \frac{\partial^2 u}{\partial z^2} + \frac{\partial A}{\partial z} \frac{\partial u}{\partial z} = 0 \tag{6.33}$$

First the derivatives of the solution need to be calculated:

$$\frac{du}{dz} = \frac{K}{\pi(R_0 + z \tan \alpha)^2} \tag{6.34}$$

$$\frac{d^2 u}{dz^2} = -\frac{2K \tan \alpha}{\pi(R_0 + z \tan \alpha)^3} \tag{6.35}$$

The area of the circular cone:

$$A = \pi(R_0 + z \tan \alpha)^2 \tag{6.36}$$

And its derivative:

$$\frac{dA}{dz} = 2\pi \tan \alpha(R_0 + z \tan \alpha) \tag{6.37}$$

The first part of the left term is expressed then:

$$A \frac{d^2 u}{dz^2} = -\frac{2K \tan \alpha}{(R_0 + z \tan \alpha)} \tag{6.38}$$

And the second part of the left term:

$$\frac{dA}{dz} \frac{du}{dz} = \frac{2K \tan \alpha}{(R_0 + z \tan \alpha)} \tag{6.39}$$

That is exactly the same as the first part of the left term but with opposite sign, they cancel one another. It has been shown that eq.(6.32) is a solution of eq.(6.7).

**NOTE:** The equation $X' + p(z)X = g(z)$ does indeed have a solution. Since $p(z)$ is
continuous for a general interval \( \alpha < X < \beta \), \( \mu \) is defined in this interval and is a nonzero differentiable function. Both \( \mu \) and \( g \) are continuous, then the function \( \mu g \) is integrable and the integral of the function is differentiable, so the solution for \( X \) in the shape of equation (6.25) does exist and is differentiable throughout the interval \( \alpha < X < \beta \). That the solution verifies the differential equation has been demonstrated. Moreover, the boundary conditions will define constant \( \mathcal{C} \) uniquely, so there is only one solution of the problem. In other words, the solution of the problem is characterized both by its existence and uniqueness.

### 6.2.4 Boundary conditions

The constants \( K, \mathcal{C} \) can be determined with the boundary conditions. The boundary conditions of the cone problem are two:

1. The bottom of the cone corresponds to the bottom of the calibration chamber, thus it is fixed and fully rigid and no displacement is possible there:

\[
z = H \Rightarrow u(H) = 0 \tag{6.40}
\]

2. At the top of the cone a force is applied over a circular area in an elastic material, so the force-displacement relationship may be expressed:

\[
z = 0 \Rightarrow EA \frac{du}{dz} = F \tag{6.41}
\]

From boundary condition [1]:

\[
\mathcal{C} = \frac{K}{\pi \tan \alpha (R_0 + H \tan \alpha)} \tag{6.42}
\]

Applying boundary condition [2] and substituting the result obtained above one can define the two constants as a function only known inputs of the problem:

\[
K = \frac{F}{E} \tag{6.43}
\]

and

\[
\mathcal{C} = \frac{F}{E \pi \tan \alpha (R_0 + H \tan \alpha)} \tag{6.44}
\]

Finally, the definitive solution for the displacement of the cone as a function of depth can be achieved substituting the expressions (6.43) and (6.44) for the constants into eq.(6.32):

\[
u(z) = \frac{F(z - H)}{E \pi (R_0 + z \tan \alpha)(R_0 + H \tan \alpha)} \tag{6.45}
\]

where everything is known except the parameter \( z \). Eq.(6.45) defines the displacement in an axially loaded elastic cone.

**NOTE:** According to Fig.(6.4), the normal force that generates the displacement of eq.(6.32) and (6.49) goes in the upward direction, the negative one, while the displacement follows the positive \( z \) direction. Keeping this idea in mind, one can expect that if the force applied is positive the displacement computed shall be negative or, the other way around, if to be coherent with fig.4, one can use as input \( F = -N \), negative force and the result should be the same value but now a positive displacement.
6.3 Elastoplastic cone

It has been argued previously that if the loading process continues so that the stresses are larger than the elastic or yield stress, the soil will plastify. Plastification will initially take place at the top of the cone, this is the area where the load is imposed. However, the plastification will propagate downwards, increasing the plastified area (or volume in 3d) inside the cone. Logically, the test load will largely exceed the yield criteria and a the cone deformation will follow the pattern shown in fig.(6.1). It is then of crucial importance to determine the extent of the plastification as it is in this area where the excess pore pressures will generate. First, an expression equivalent to eq. (6.45) must be obtained for the case of an elastoplastic cone.

6.3.1 Elastoplastic modeling

Roughly defined, plasticity introduces two transcendental modifications with respect to the previous elastic case:

1. Loss of linearity: tensions are no longer proportional to deformations

2. Introduction of the permanent deformation concept; part of the deformations generated during loading are not recovered during the unloading process.

In this case, the constitutive equation:

\[ \sigma - \sigma_e = E_p(\epsilon - \epsilon_e) \] (6.46)

where \( \sigma_e \) and \( \epsilon_e \) are the elastic limit and the correspondent deformation. These values are properties of a determined soil or material type, this meaning that they are given constants.

The total strength after an elastoplastic loading process is then:

\[ \epsilon(z) = \frac{\sigma(z) - \sigma_e}{E_p} + \epsilon_e \] (6.47)

Considering the cone geometry it can also be expressed:

\[ \epsilon(z) = \epsilon_e(1 - E_e/E_p) + \frac{F}{\pi E_p(R_0 + z \tan \alpha)^2} \] (6.48)

The total displacement caused by the application of the external force in the plastified zone may be derived integrating the deformation over the extent of the plastic zone:

\[ u_p = \int_{0}^{z_p} \epsilon(z)dz = \left[ \epsilon_e(1 - E_e/E_p)z + \frac{-F}{\pi E_p \tan \alpha(R_0 + z \tan \alpha)} \right]^{z_p}_{0} \] (6.49)

The total displacement of an elastoplastic cone comes from the contribution of both the displacement in the plastic area (given by eq.(6.49)) and the displacement in the elastic area. The displacement in the elastic area is the displacement of an elastic cone of height \( \bar{H} = H - z_p \). Note that eq.(6.45) is always applied on the top of the cone, thus, the following variables need to be introduced:

\[ \begin{cases} \bar{H} = H - z_p \\ \bar{R}_0 = R_0 + z_p \tan \alpha \\ \bar{z} = z - z_p \end{cases} \]

\[ u_e = \frac{F(\bar{z} - \bar{H})}{E_e \pi (\bar{R}_0 + \bar{z} \tan \alpha)(\bar{R}_0 + \bar{H} \tan \alpha)} \] (6.50)
Undoing the change of variable the expression for the elastic displacement, for \( z = z_p \):

\[
u_e = \frac{F(z_p - H)}{E_e \pi (R_0 + z_p \tan \alpha)(R_0 + H \tan \alpha)} \tag{6.51}
\]

Finally, if equation (6.45) was the total displacement of an elastic cone, the equivalent expression for an elastoplastic cone follows:

\[
u(z) = u_e(z) + u_p(z) = \frac{F(z_p - H)}{E_e \pi (R_0 + z_p \tan \alpha)(R_0 + H \tan \alpha)} + \frac{F}{\pi E_p \tan \alpha} \left[ \frac{1}{R_0} - \frac{1}{R_0 + z_p \tan \alpha} \right] - \epsilon e z_p \left( \frac{E_e}{E_p} - 1 \right) \tag{6.52}
\]

### 6.3.2 Load-displacement curve

After the calculations, the load-displacement curve for the cone can be plotted. However, the calculations up to now have been derived for a linear elastoplasticity. In reality soil does not behave linearly, for sure not once plastified. The Dutch code presents load-displacement curves without linearity. To be able to compare the obtained one with the standardized, an extrapolation must be made: suppose the load-displacement curve obtained for the cone is also non-linear for plasticity. For the derived equations, i.e. (6.52) to be suitable, the hypothesis that all the plastic history and non-linearity is recorded in the plastic modulus \( E_p \) must be made. Then the same equations derived can be used even taking into consideration that were derived for a linear case, just supposing \( E_p \) records the translation into non-linear.

Then, the coordinates for point 1 are \( z = 0 \) for the top of the elastic cone):

\[
\begin{align*}
F_1 & = \sigma_e \pi R_0^2 \\
\frac{u_1}{F_1} & = \frac{(-H)}{E_e \pi R_0 (R_0 + H \tan \alpha)}
\end{align*}
\]

And the coordinates for point 2 are (in this case the top of the elastic cone is at \( z = z_p \)):

\[
\begin{align*}
F_2 & : \text{any input force larger than } F_1 \\
\frac{u_2}{F_2} & = \frac{F(z_p - H)}{E_e \pi (R_0 + z_p \tan \alpha)(R_0 + H \tan \alpha)} + \frac{F}{\pi E_p \tan \alpha} \left[ \frac{1}{R_0} - \frac{1}{R_0 + z_p \tan \alpha} \right] - \epsilon e z_p \left( \frac{E_e}{E_p} - 1 \right)
\end{align*}
\]

![Figure 6.5: Cone load-displacement curve](image-url)
6.4 Excess pore pressures generated

6.3.3 Extent of the plastic zone

This load-displacement curve that has been obtained for the cone model can now be compared to the NEN 6473 code (see fig.(6.6)). The maximum displacement at the top of an elastic cone is $0.04R_0$ and occurs when the stress level at the top is equal to the elastic limit (yield stress). When the total force is applied, there still exists an elastic cone as it has been derived above, however, its top is placed at a certain depth $z_p$. This is expressed:

$$0.04R_0 = \frac{-F(z_p - H)}{E_e \pi (R_0 + z_p \tan \alpha)(R_0 + H \tan \alpha)}$$

(6.53)

Estimating $E_e = 50000 kN/m^2$ and, for the coming sections also $E_p = 30000 kN/m^2$, this defines the extent of the plastic zone at a depth:

$$z_p = 12 cm$$

(6.54)

According to the cone geometry, this corresponds to a yield stress:

$$\sigma_e = 300 kN$$

(6.55)

6.4 Excess pore pressures generated

The main assumption is that excess pore pressures are only generated within the plastified area; outside it, at the elastic part, no generation takes place. With this approach, pore
pressures do not dissipate instantly after removing the load because they are related to plastic deformation, but instead they will suffer consolidation with time.

As a first approach, let’s evaluate which excess pore pressures would be generated if the loading process was fast enough as not to let drain at all and considering completely incompressible water filling the pores. In this case the generated excess pore pressure equals the given load. The explanation is that in the case of an incompressible pore fluid there can be no immediate volume change. Therefore there can be no vertical strain, if considering only 1D axial deformation (and later 1D axial pore water flow), without any lateral deformation. In this case, there can be no vertical strain at the moment of application of the load, and consequently the effective stress can not increase instantly. This is a situation where all the entire load is carried by the water in the pores.

\[
\epsilon^T = \epsilon^p + \epsilon^e
\]  

(6.56)

The stresses are known and so are the total and elastic deformations:

\[
\begin{cases} 
\epsilon^T = \frac{(\sigma - \sigma_e)}{E_e} \\
\epsilon^e = \frac{(\sigma - \sigma_e)}{E_e}
\end{cases}
\]

And the plastic deformation is:

\[
\epsilon_p = \frac{(\sigma - \sigma_e)(E_e - E_p)}{E_e E_p}
\]  

(6.57)

This value of deformation corresponds to a certain stress level \(\sigma_p\). This stress level is the pore pressures at that depth:

\[
\sigma_p = E_p \epsilon_p = \frac{(\sigma(z) - \sigma_e)(E_e - E_p)}{E_e}
\]  

(6.58)

Finally, the generated excess pore pressures follow the distribution:

\[
\bar{u} = \begin{cases} 
(\sigma - \sigma_e)(1 - \frac{E_p}{E_e}) & \text{if } z \leq z_p \\
0 & \text{if } z > z_p
\end{cases}
\]

This would lead to a value of the excess pore pressures of:

\[
\bar{u} = 5,2MPa
\]  

(6.59)

Which is exaggeratedly large compared to the experimental values (around 0.002MPa for static and 0.03MPa for pseudostatic). The difference between values may be understood recapitulating the main simplification made: unidimensional deformation and flow in the direction of loading. However, in reality it is a three-dimensional case, where lateral deformations can occur and, in fact, do occur; thus, an immediate deformation is possible, although the volume must remain constant if the water is still considered incompressible. In reality then, there can be immediate change in effective stresses. It has been demonstrated that not all the load is sustained by the pore water in the tests. More insight will be provided in the consolidation analysis. First, according to the developed cone model, 1D axial consolidation will be evaluated and the results will be discuss as well as the suitability to take into account radial and even 3D consolidation or coupled loading and consolidation.

Instead, the excess pore pressures may be more correctly estimated when relating them to the volumetric strain:

\[
\bar{u} = k_f \Delta \epsilon_{vol}
\]  

(6.60)


where \( k_f \) is the compressibility of the fluid and \( \Delta \epsilon_{vol} \) is the volumetric strain defined as:

\[
\Delta \epsilon_{vol} = \int_0^z \epsilon_p(z) A(z) dz = \int_0^z \epsilon_p(z) \pi (R_0 + z \tan \alpha)^2 dz
\] (6.61)

The plastic deformation has been found in the previous subsection (see eq. (6.48)). Integrating eq. (6.61) and approximating \( k_f \approx E_p \), the excess pore pressures:

\[
\Delta P_w = 0.003 MPa
\] (6.62)

Which fits perfectly into the range of excess pore pressures generated for the static test.

6.5 Consolidation

6.5.1 Introduction

\(^1\) ‘A decrease of water content of a saturated soil without replacement of the water by air is called a process of consolidation’ (Terzaghi, 1943)

In the first instance it has been considered that the loading process is fully undrained and only after the load has been removed the excess pore pressures start to dissipate and consolidation take place. To sum up, the key assumption in the development of the analytical model is that there are two differentiate steps:

1. **Undrained loading process**: It is that process in which the variation of the load or of the boundary conditions takes place in a time frame very small compared to that necessary for the dissipation of the excess pore pressures. Hence, when saturated low permeability soil is subjected to compressional stress, the pore pressures will increase immediately but they will not dissipate immediately because of the low permeability. It can even be the case that at \( t = 0 \) all stress increase is taken by the pore pressure and none by the soil skeleton, taking into account water incompressibility.

   It was previously assumed that this was the case to model analytically, eventhough this study deals with sand, as it is certainly what happens under fully dynamical loading (one of the main risks of earthquakes for example is that of sand liquefaction). Despite the fact that the pseudostatic methodology is not supposed to introduce stress-waves into the soil, it still is a rapid load test, thus it seems logical to expect the loading process to be undrained, even the relatively high permeability of the soil.

2. **Consolidation**: Once the loading process is finished, the water will start to flow due to the gradient in excess pore pressures and there will be a variation in the volume of the soil. Namely, during consolidation it occurs simultaneous deformation of the porous material and flow of pore fluid.

   There will be an increase in bearing capacity of the pile controlled by the dissipation of excess pore pressures and the increase in effective stresses. Note that as the excess pore pressures diminishes, the hydraulic gradients and the rates of flow also diminish, so that the volume changes and effective stress increments continue at reducing rates.

\(^1\)Theoretical concepts extracted from Lambe [27], Verruijt [28], Das [29] and Atkinson [30].
6.5.2 Certainty or uncertainty of the hypothesis of undrained loading

It has been argued that when first thinking of the problem it seems justified to expect the loading process to be undrained. However, there are two results that should make the reader be doubtful:

- **Experimental testing:** The results of the tests done with E.Archeewa demonstrated the certainty about the excess pore pressure generation. For the rapid test, excess pore pressures up to 10 times larger than found in static results were generated. Surely then the loading process is not drained and there is some effect of the loading rate in the saturated material behavior. However, the same test series showed no difference is found in the bearing capacity values when using pseudostatic test or static one, which is an accordance with Dijkstra’s results for dry sand. If the loading process was fully undrained it should be reflected in lower bearing capacities for the pseudostatic test, as the excess pore pressures reduce the effective stresses acting on the pile. However, experimental results prove the fully undrained loading hypothesis wrong.

- **Numerical results:** The analytical modeling has been carried out simultaneously with a numerical one. It will be presented later in this report that a fully undrained calculation in PLAXIS gives soil failure for too low loads when compared to the experimental values. Once more, results are not precisely pointing towards the case of undrained loading process.

The available results seem to indicate that the loading process is not fully undrained, nor fully drained. It is probably the case of coupled loading and consolidation, partial drainage.

It seems logical to finally check analytically the accuracy of the above statement. Some more theoretical considerations related to drainage and consolidation will follow in the next lines.

It is crucial to make clear that when distinguishing between drained and undrained loading it is the relative rates of loading and seepage that are important and not the absolute rate of loading. In pseudostatic tests loading rates as high as 250 mm/s occur. Besides, the flow rate is determined by Darcy’s law, with the permeability as a key governing parameter. The value of $k$ (permeability) is the seepage velocity of water through soil with unit hydraulic gradient and this value for sands typically falls into the range $10^{-1} - 10^{-5} (m/s)$. Hence, loading rate and seepage rate are probably of the same order; it is coherent to expect the dissipation of excess pore pressures to start while loading process is going on.

Another crucial concept to introduce is that of the **coefficient of consolidation**. It is defined:

$$c_v = \frac{k}{\gamma_w m_v}$$  \hspace{1cm} (6.63)

where $k$ is the coefficient of permeability, $\gamma_w$ is the water specific weight and $m_v$ is the coefficient of volumetric variation ($m_v = \Delta \epsilon_v / \Delta \sigma_v$). The coefficient of consolidation can be determined performing an eodometric test.

Related to the coefficient of consolidation a **time factor** can be introduced as the ratio:

$$T_v = \frac{c_v t}{h^2}$$  \hspace{1cm} (6.64)

where $c_v$ is the coefficient of consolidation, $t$ is time and $h$ is the average draining length. Note that we are considering 1D consolidation. To know whether we need to take into account consolidation or not we need to evaluate this time factor. For $T > 1$ the process
6.5 Consolidation

can be considered fully drained, thus consolidation can be disregarded. In general, the time necessary for fully complete consolidation is proportional to $\frac{h^2 m}{k}$ although according to Verruijt [28] the consolidation can be considered finished when:

$$T_v = \frac{c_v t}{h^2} > 2$$

(6.65)

Then, the time required for the consolidation process to be finished can be calculated:

$$t_{99\%} = \frac{2h^2}{c_v}$$

(6.66)

Note that the consolidation process is governed by the factor $c_v t / h^2$ so its duration can be shortened considerably reducing the drainage length.

Moreover a degree of consolidation can be presented, which indicates how far the consolidation process has reached at a certain time, hence, relates the current excess pore pressure to the original one:

$$U = 1 - \frac{\bar{u}}{u_0}$$

(6.67)

where $\bar{u}$ is the excess pore pressure at that given time and $u_0$ is the maximum initial excess pore pressure. It can be expressed as well as a function of the time factor, for small values of time:

$$U = \frac{2}{\sqrt{\pi}} \sqrt{\frac{c_v t}{h^2}}$$

(6.68)

It can be estimated how short must be the loading time to be considered instantaneous.

$$t_{1\%} = 10^{-4} \frac{h^2}{c_v}$$

(6.69)

A load that is applied faster than this $t_{1\%}$ can be considered an instantaneous load.

No oedometric results are available. Any further investigation on this topic should have good soil data available. If the following approximation is made:

$$m_v \approx \frac{1}{E}$$

(6.70)

and taking average sand permeability values, the coefficient of consolidation is around $0.3 m^2/s$. The pseudostatic tests took some $0.04 - 0.06 s$. Therefore, if the consolidation started at the same time as the load application -which does not- by the end of the test $T_v \approx 0.85$, in 1D axial consolidation in a 12cm thick draining stratum. For $T_v$ values larger than 1 it is the fully drained case, so 0.85 should correspond to partial drainage, for sure not undrained case. Of course the consolidation does not start immediately when the load is applied, there must be a time lapse. This is maybe the most remarkable unknown question: it will be proved that the consolidation process does take place when loading, not after as would correspond to the undrained case, but, when does the consolidation process start. Surely the fact that is starts later in the loading time should make it also finish later, but one must remember all these calculations are for 1D and in reality there would be 3D consolidation which would occur naturally much faster.

6.5.3 Consolidation model

Problem definition

In the previous subsection it has been demonstrated that the loading process is not fully undrained but instead the loading and the consolidation processes take place simultaneously. Also it has been seen that not all the load is carried by the water instantly,
as it corresponds to a more complicated multidimensional deformation and consolidation problem. By now, two of the principal assumptions of the model, namely 1D case (for simplification reasons) and undrained loading (due to the rapid load application it could have been expected), have been demonstrated to be wrong.

However, to model analytically loading, generation of excess pore pressures and consolidation all almost in the same time frame and 2 or 3D is complex. What interests the engineer is to obtain a simple and straightforward manner to model analytically the pseudostatic test or, at least, give insight in the behavior of the soil under those conditions. Then, to account for the generation of excess pore pressures the simplest available approach is that of undrained loading and subsequent consolidation, as it was the original idea. The calculations for this model are be computed despite the previous statements, and in the end the correctness and suitability of the results is to be discussed. It can be interesting to see which results the first idea of the problem may give, keeping in mind which error we might me introducing and why. Further on more attention to the coupled loading-consolidation equation can be paid if necessary.

The next step is then to model the consolidation process. Remember the another of the key assumptions: excess pore pressures are only generated in the plastic zone. Thus, the flow will be towards the elastic area below and also to the laterals of the cone. It would be a 2D consolidation problem. This is a difficult situation to model analytically and even numerical models seem more suitable. Besides, one of the definition statements of Wolf’s cone model is that the soil outside the cone can be neglected. Nevertheless, different authors have studied excess pore pressures generated during pile driving and have explained the flow to be mainly radial (Randolph, Wroth [26]). To sum up: it is a very complicated problem to solve analytically with high accuracy. It is very important to understand this difficulties and why they arise. Once this is understood, some assumptions will require to be made in order to propose a simplified model. Again, it is important to understand the shortcomings and limitations this assumptions may introduce. In the end, this considerations should introduce a degree of perspective and relativity in whatever the results are and their application and correctness.

They main problems to face in the consolidation modelling are:

1. It is not really an undrained loading process but instead a coupled loading-consolidation one. 
   Even separating the processes of loading and consolidation, to make the model more simple, one finds new trouble in the consolidation modelling:

2. Consolidation, and deformation, will be 2-dimensional, this is, axially and radially. 
   Even more, in practice it should be 3D, but axisymmetry can simplify it to 2D. 
   Even considering only 1D consolidation, either axial or radial, one finds new trouble in the consolidation modelling:

3. Excess pore pressure variation with radius has not been obtained in the previous cone model. 
   Even considering only 1D axial consolidation, one finds new trouble in the consolidation modelling:

4. Excess pore pressure distribution is not constant, not even linear within the plastic zone. 
   The initial condition is coupled with the boundary condition.

5. Area of the cone is not constant with depth.

6. Boundary condition in the plastic-elastic boundary: in $t = 0$, $\bar{u} = 0$. However once the consolidation process starts for $t > 0$, $\bar{u} \neq 0$ until the consolidation has finished. 
   To put it in words, it is not really a fully drained boundary.
To define the problem in a simple way, the following assumptions may be made:

1. Undrained loading process + consolidation after the load has been removed.
2. 1D consolidation only in the direction of the application of the load (axially).
3. Soil behaves elastically during consolidation.
4. Plastic-elastic limit is a full drained boundary for any time.

Assumption (1) solves problem (1); assumption (2) solves problems (2) and (3). Assumption (3) allows for the application of Terzaghi’s equation and the classical consolidation theory. Besides it has been proved not to be so inaccurate as during consolidation soil mainly moves backward toward the pile, undergoing an unloading process in shear (Ran- dolph, Wroth [26]). Thanks to assumption (3) problem (5) can be disregarded as the area does not appear in the consolidation equation of Terzaghi, it disappeared when combining volume variation with Darcy’s law. Last assumption (4) solves problem (6). Just problem (4) remains unsolved by now, but it will be dealt with during the calculations.

Finally, the consolidation problem to analyze is:

The equation to solve:

\[
\frac{\partial \bar{u}}{\partial t} = c_v \frac{\partial^2 \bar{u}}{\partial z^2} \tag{6.71}
\]

which is the equation of consolidation of Terzaghi.

Two boundary conditions:

1. At the lower limit of the drainage area (plastic-elastic limit) the excess pore pressure is always 0 (fully drained boundary):

\[
z = z_p \Rightarrow \bar{u}(z_p) = 0 \tag{6.72}
\]

2. At the upper limit of the drainage area there is the pile and this one is not removed after the test and is impermeable (fully undrained boundary):

\[
z = 0 \Rightarrow \frac{\partial \bar{u}}{\partial z} = 0 \tag{6.73}
\]

And one initial condition:

1. At the time of loading an immediate increase in excess pore pressure is generated, which has been discussed before. It follows:

\[
t = 0 \Rightarrow \bar{u} = \bar{u}(z) \tag{6.74}
\]

**Equation solution**

The consolidation equation is a 2nd order partial differential equation that can be solved by means of separation of variables or Laplace transform. Here the first procedure will be considered.

The separation of variables technique makes one defining assumption and that is that the solution \( \bar{u} \) is a product of two functions, one in \( z \) and one in \( t \):

\[
\bar{u} = Z(z)T(t) \tag{6.75}
\]
The partial derivatives follow:

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial t} &= Z(z)T'(t) \\
\frac{\partial^2 \bar{u}}{\partial z^2} &= Z''(z)T'(t)
\end{align*}
\]

and then the equation of consolidation can be rewritten:

\[
Z(z)T'(t) = c_v Z''(z)T'(t)
\]

or also:

\[
\frac{Z''(z)}{Z(z)} = \frac{1}{c_v} \frac{T'(t)}{T'(t)}
\]

where the left side term is independent of \( t \) and the right-hand one is independent of \( z \).

Then, the derivatives and original functions can be related to each other by a constant:

\[
\begin{align*}
Z''(z) &= -B^2 Z(z) \\
T'(t) &= -B^2 c_v T'(t)
\end{align*}
\]

These two equations have respectively solutions of the shape:

\[
\begin{align*}
Z(z) &= A_1 \cos Bz + A_2 \sin Bz \\
T(t) &= A_3 e^{-B^2 c_v t}
\end{align*}
\]

And combining the solutions that of the main equation can be found:

\[
\bar{u} = Z(z)T(t) = (A_4 \cos Bz + A_5 \sin Bz)e^{-B^2 c_v t}
\]

with \( A_4 = A_1 A_3 \) and \( A_5 = A_2 A_3 \). The constants have to be determined with the boundary conditions. From the second boundary condition:

\[
\frac{\partial \bar{u}}{\partial z} = (-A_4 B \sin 0 + A_5 B \cos 0)e^{-B^2 c_v t} = 0 \Rightarrow A_5 = 0
\]

From the first boundary condition:

\[
0 = A_4 \cos B z_p e^{-B^2 c_v t}
\]

which only has two possible solutions for all \( t \), \( A_4 = 0 \) or \( B z_p = n \frac{\pi}{2} \) that would mean:

\[
B = \frac{n \pi}{2 z_p}
\]
Substituting into (6.65):

\[
\ddot{u} = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{2z_p}z\right) \exp\left(-\frac{n^2\pi^2}{2z_p}vt\right) \tag{6.82}
\]

According to Das [29], it can also be rewritten:

\[
\ddot{u} = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{2z_p}z\right) \exp\left(-\frac{n^2\pi^2T_v}{4}\right) \tag{6.83}
\]

where \(T_v\) is an adimensional factor equal to \(c_v t/H^2\) and \(H\) is half the total thickness in a two-way drainage condition. In this case there is only a one way drainage condition, hence, the longest drainage path possible equals the thickness of the draining layer, namely, the plastic zone, thus \(H = z_p\). Applying the initial condition \((t = 0 \Rightarrow u = u_i)\):

\[
u_i = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{2z_p}z\right) \tag{6.84}
\]

Eq. (6.78) is a Fourier cosine series, then A:

\[
A_n = \frac{1}{z_p} \int_0^{z_p} u_i \cos\left(\frac{n\pi z}{2z_p}\right) dz
\tag{6.85}
\]

Combining eq.(6.77) and eq.(6.79):

\[
\ddot{u} = \sum_{n=1}^{\infty} \frac{1}{z_p} \int_0^{z_p} u_i \cos\left(\frac{n\pi z}{2z_p}\right) dz \cos\left(\frac{n\pi}{2z_p}z\right) \exp\left(-\frac{n^2\pi^2T_v}{4}\right) \tag{6.86}
\]

In his book Das [29] developed achieved an equivalent expression to (6.80) but for a two-way drainage \((H = z/2)\); the only difference between his solution and (6.80) is due to the different boundary condition at the top of the layer. This leads to finding \(\sin\) where the current solution has a \(\cos\). He developed the solution to his consolidation problem for several cases, among them:

- **Linear variation of** \(u_i: u_i = u_0 - u_1 \frac{H-z}{H}\):

\[
\ddot{u} = \sum_{n=1}^{\infty} \left[ \frac{1}{H} \int_0^{2H} \left( u_0 - u_1 \frac{H-z}{H} \right) \sin\left(\frac{n\pi z}{2H}\right) dz \right] \sin\left(\frac{n\pi}{2H}z\right) \exp\left(-\frac{n^2\pi^2T_v}{4}\right) \tag{6.87}
\]

- **Sinusoidal variation of** \(u_i: u_i = u_0 \sin\frac{\pi z}{2H}\):

\[
\ddot{u} = \sum_{n=1}^{\infty} \left[ \frac{1}{H} \int_0^{2H} u_0 \sin\left(\frac{\pi z}{2H}\right) \sin\left(\frac{n\pi z}{2H}\right) dz \right] \sin\left(\frac{n\pi}{2H}z\right) \exp\left(-\frac{n^2\pi^2T_v}{4}\right) \tag{6.88}
\]

Parallelly, in the one-way drainage equation for the cone consolidation we can express:

- **Linear variation of** \(u_i: u_i = u_0 - u_1 \frac{z_p-z}{z_p}\):

\[
\ddot{u} = \sum_{n=1}^{\infty} \left[ \frac{1}{z_p} \int_0^{z_p} \left( u_0 - u_1 \frac{z_p-z}{z_p} \right) \cos\left(\frac{n\pi z}{2z_p}\right) dz \right] \cos\left(\frac{n\pi}{2z_p}z\right) \exp\left(-\frac{n^2\pi^2T_v}{4}\right) \tag{6.89}
\]

- **Sinusoidal variation of** \(u_i: u_i = u_0 \sin\frac{\pi z}{z_p}\):

\[
\ddot{u} = \sum_{n=1}^{\infty} \left[ \frac{1}{z_p} \int_0^{z_p} u_0 \sin\left(\frac{\pi z}{z_p}\right) \cos\left(\frac{n\pi z}{2z_p}\right) dz \right] \cos\left(\frac{n\pi}{2z_p}z\right) \exp\left(-\frac{n^2\pi^2T_v}{4}\right) \tag{6.90}
\]
According to the figure, the excess pore pressure distribution in the plastic area can be approximated as the difference between a linear function and a sinusoidal one, thus, subtracting (6.84) to (6.83). This solves the last remaining problem of how to apply the initial excess pore pressure non-linear distribution with depth.

\[
\bar{u} = \sum_{n=1}^{n=\infty} \left[ \int_0^{z_p} \left( u_0 - u_1 \frac{z_p - z}{z_p} - u_0 \sin\left( \frac{\pi z}{2z_p} \right) \right) \cos\left( \frac{n\pi z}{2z_p} \right) dz \right] \cos\left( \frac{n\pi}{2z_p} \right) \exp\left( -\frac{n^2\pi^2 T_v}{4} \right)
\]

Eq.(6.88) may be easier solved considering first the two separate components (linear and sinusoidal distributions). The linear distribution follows:

\[
u_i(z) = u_0 - \frac{u_0}{z_p} z
\]

where \( u_0 \) is the maximum excess pore pressure, located at the top of the cone. Substituting into eq.(6.83) for the two-way drained cone and performing the required calculations:

\[
\bar{u}_{lin}(z, t) = 4 \frac{u_0}{\pi^2} \cos\left( \frac{\pi z}{2z_p} \right) \exp\left( -\frac{\pi^2 T_v}{4} \right)
\]

On the other hand, the sinusoidal distribution for the cone case would be:

\[
u_i(z) = u_{1c} \sin\left( \frac{\pi z}{z_p} \right)
\]

where \( u_{1c} = u_{1b} - u_{1a} \) (see figure (6.8)). Substituting into the general sinusoidal equation (6.84) and integrating:

\[
\bar{u}_{sin}(z, t) = \frac{4u_{1c}}{3\pi} \cos\left( \frac{\pi z}{2z_p} \right) \exp\left( -\frac{\pi^2 T_v}{4} \right)
\]

Finally, the solution to the axial consolidation in the cone is the combination of the two obtained expressions eq.(6.89)-eq.(6.87).

\[
\bar{u}(z, t) = \frac{4}{\pi} \left( \frac{u_0}{\pi} - \frac{u_{1a}}{4} \right) \cos\left( \frac{\pi z}{2z_p} \right) \exp\left( -\frac{\pi^2 T_v}{4} \right)
\]

Degree of consolidation

Pore pressures are directly linked to deformations. To describe the deformation as a function of time, the degree of consolidation proves useful. The degree of consolidation, at any
time and at any depth, was defined as the relation between the excess pore pressure that has been dissipated and the initial excess pore pressure, this is, how far the consolidation has progressed. It may be the case that what is of interest is the average degree of consolidation over an entire layer, that can be expressed according to Das [29]:

\[ U_{av} = \frac{(1/H_t) \int_0^{H_t} u_t dz - (1/H_t) \int_0^{H_t} \bar{u} dz}{(1/H_t) \int_0^{H_t} u_t dz} \]  \hspace{1cm} (6.97)

where \( H_t \) is the total thickness of the layer, \( u_t \) the initial excess pore pressure and \( u \) the actual excess pore pressure. In the cone model case it is interesting to see the evolution in time of the degree of consolidation of the plastic zone, that acts as the layer in question. Once more Das [29] proposed some solutions for general cases:

- **Linear variation of \( u_t \):**
  \[ U_{av} = 1 - \sum_{m=0}^{\infty} \frac{2}{M^2} \exp(-M^2 T_v) \]  \hspace{1cm} (6.98)
  where \( M = (2m + 1)\pi/2 \).

- **Sinusoidal variation of \( u_t \):**
  \[ U_{av} = 1 - \exp\left(-\frac{\pi^2 T_v}{4}\right) \]  \hspace{1cm} (6.99)

Following the approximation of the initial excess pore pressure distribution as the difference between a linear one and a sinusoidal one, the average degree of consolidation in the plastic zone may be estimated:

\[ U_{av}(T_v) = \frac{U_{linear}(T_v)A_1 - U_{sin}(T_v)A_2}{A_1 - A_2} \]  \hspace{1cm} (6.100)

where \( A_1 \) and \( A_2 \) are respectively the areas of the linear and the sinusoidal diagrams.

\[
\{ \\
A_1 = \frac{1}{2} z_p u_0 = 312kN/m^2 \\
A_2 = \sum_{z=0}^{z=z_p} u_{1c}\sin\left(\frac{\pi z}{z_p}\right)dz = \int_{z=0}^{z=z_p} u_{1c}\sin\left(\frac{\pi z}{z_p}\right)dz = 181.8kN/m^2 \\
\}
\]

The values of the degree of consolidation for both linear and sinusoidal distributions are tabulated for different time factors. It is especially interesting to evaluate the case for \( t = t_{test} \approx 0.04s \). The corresponding \( T_v \) is \( T_v = c_v t/z_p^2 = 0.85 \). It can be seen that, mainly due to the large permeability of sand, \( U_{av}(T_v = 0.85) = 92.23\% \), this is, the consolidation process is almost finished by the end of the loading process.

### 6.5.4 Radial consolidation

#### Problem definition

The equation for radial consolidation was derived by Scott in 1963:

\[ c_r \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) = \frac{\partial u}{\partial t} \]  \hspace{1cm} (6.101)

Radial consolidation normally occurs in axisymmetrical problems where there is radial transitory flux but the axial flux is nil. In the case we are studying it seems logical to expect consolidation to happen both radially and axially. Up to now, the evaluation of a simplified case where only axial consolidation takes place has been developed. The next step is, still in the 1D assumption, to consider the fluid to flow radially from the center of the cone outwards, neglecting the axial flux toward the elastic part.

The problem is fully defined with the boundary and initial conditions and the radial consolidation equation. Two boundary conditions are required:
1. There is no flow between the two symmetrical halves of the cone, the axis of symmetry may be modeled as a fully undrained boundary: \( r = 0 \Rightarrow \frac{\partial \bar{u}}{\partial r} = 0 \)

2. At the lateral boundary of the cone the excess pore pressure is always kept to zero. By definition, in the cone model of Wolf, the soil outside the cone can be neglected. Now this soil is supposed to act as a fully drained boundary: \( r = \bar{r} = R_{0} + z \tan \alpha \Rightarrow \bar{u} = 0 \)

And one initial condition. The problem of defining the initial excess pore pressure distribution as a function of the radius arises due to the fact that the excess pore pressures in the cone have only been obtained as a function of depth. The deformation of the cone has been derived only in \( z \), the excess pore pressures are directly related to the plastic deformation. Therefore, it should be possible to approximate the radial distribution by relating the horizontal deformation to the vertical one by means of the Poisson ratio (\( \epsilon_r = \nu \epsilon_z \)).

However, looking more into the geometry one could expect the radial excess pore pressures to be constant, for a certain given depth, throughout the pile section and then start to decrease as the cone expands as shown in the figure. This gives complicated expressions to work with. As a simplification, the radial excess pore pressures will be assumed to be constant with the radius. In this way, the calculations, when related to the Bessel functions are easier. Then, the initial condition:

\[ t = 0 \Rightarrow \bar{u} = \bar{u}, \forall r \]

where \( \bar{u} \) is a constant value.

**Equation solution**

Eq.(6.101) is more difficult to solve than the equation for axial consolidation. Randolph and Wroth [26] proposed an analytical solution for the consolidation around a driven pile, where the gradients were mainly radial. By means of separation of variables and Bessel functions, the solution has the general form:

\[ \bar{u} = Be^{-\alpha^2 t} \left[ J_0(\lambda r) + \mu Y_0(\lambda r) \right] \] (6.102)

where \( -\alpha^2 \) is a separation constant. \( J_0 \) and \( Y_0 \) are Bessel functions of first and second kind. The linear combination of \( J_0(\lambda r) + \mu Y_0(\lambda r) \) is a cylinder function \( C_0(\lambda r) \). Boundary condition [1] implies that:

\[ C_1(0) = J_1(0) + \mu Y_1(0) = 0 \] (6.103)

The Bessel function of second order and second kind (\( Y_1(x) \)) presents a singularity for \( x = 0 \). Hence, the only option remaining is: \( \mu = 0 \).

Form boundary condition [2]:

\[ C_0(\lambda \bar{r}) = J_0(\lambda \bar{r}) = 0 \] (6.104)

The values of \( \lambda \bar{r} \) such that they make the Bessel function of first order and first kind equal to zero are tabulated.

In the end, what we get is:

\[ \bar{u} = Be^{-\alpha^2 t} J_0(\lambda r) \] (6.105)

Also applying the initial condition:

\[ u_0 = \sum_{n=1}^{\infty} B_n J_0(\lambda_n r) \] (6.106)

and then:

\[ \int_0^r u_0 r J_0(\lambda_n r) dr = \frac{B_n}{2} [\bar{r}^2 J_1^2(\lambda_n \bar{r})] \] (6.107)
Integrating the left term of eq.(6.98), according to the rules of integration of the Bessel functions the formal solution for the radial consolidation problem can be obtained. First, the values of the constant $B_n$ can be expressed:

$$B_n = \frac{2u_0}{\lambda_n^2} \left[ \frac{\lambda_n/2J_1(\lambda_n r) + J_0(\lambda_n r) - 1}{r^2J_1^2(\lambda_n r)} \right]$$

(6.108)

So the formal solution for the radial consolidation:

$$\bar{u}(r, t) = \frac{2u_0}{\lambda_n^2} \left[ \frac{\lambda_n/2J_1(\lambda_n r) + J_0(\lambda_n r) - 1}{r^2J_1^2(\lambda_n r)} \right] e^{-\alpha^2 t}J_0(\lambda_n r)$$

(6.109)

What follows is a very complicated evaluation to do by hand and is best computed numerically. No more detail is provided here.

### 6.5.5 Coupled loading and pseudodimensional consolidation

The reasons for the incorrectness of the undrained loading followed by a consolidation process have been explained in detail and supported by analytical results. It has been justified why in reality what takes place is both loading and consolidation simultaneously. Besides, this consolidation is not unidimensional but both radial and axial. Therefore, it can be stated that pseudostatic tests in saturated sand generate instantly excess pore pressures that dissipate during the time lapse in which the load is applied; the loading-consolidation for a pseudostatic test in saturated sand is governed by the equation:

$$c_v \left( \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} \right) = \frac{\partial u}{\partial t} - \frac{\partial \sigma}{\partial t}$$

(6.110)

Eq.(6.110) has to be solved numerically, however this falls beyond the scope of this thesis. Nevertheless, the effects could be roughly appreciated with a pseudobidimensional analysis. Considering the total degree of consolidation it is possible to picture out an idea of the rate of dissipation of excess pore pressures. The results from the unidimensional analysis in $z$ and $r$ may be combined to finally estimate which would the degree of consolidation be if the radial problem was solved:

$$U_{rz} = 1 - (1 - U_z)(1 - U_r)$$

(6.111)