Chapter 3
The seismic loading case

3.1 Modification of the equilibrium equations

It has been mentioned in section 2.1 that the linear dynamic analysis of the structure consists in solving the second order differential equation:

\[ M\ddot{u} + Cu + Ku = r(t) \]  

(3.1)

In the particular case of seismic loading, the vector of external forces \( r(t) \) is equal to zero and the displacements and stresses on the structure are due to the free-field ground displacement \( u_{fg} \). It is a three components motion, which is known or assumed at some point below the structure. As a consequence the absolute displacements, velocities and accelerations can be removed from equation (3.1) by means of the simple relations in (3.2), leading to the final equilibrium equation in (3.3)

\[ u(t)_a = u(t) + I_i u_{fg}\;\text{sg} + I_j u_{fg}\;\text{yg} + I_z u_{fg}\;\text{yg} \]

\[ \dot{u}(t)_a = \dot{u}_a + I_i \dot{u}_a\;\text{sg} + I_j \dot{u}_a\;\text{yg} + I_z \dot{u}_a\;\text{yg} \]

\[ \ddot{u}(t)_a = \ddot{u}_a + I_i a\;\text{sg} + I_j a\;\text{yg} + I_z a\;\text{yg} \]  

(3.2)

Where \( I_i \) is a vector with ones in the “i” directional degrees-of-freedom and zero in all other positions.

\[ M\ddot{u} + Cu + Ku = -MI_x a\;\text{sg} - MI_y a\;\text{yg} - MI_z a\;\text{yg} \]  

(3.3)

It should be remembered that this simplification is possible since the rigid body velocities and displacements associated with the base motions do not cause any additional damping or structural forces.

It is important to realize that the displacements are relative ones, and that the loading on the structure is a foundation motion and not externally applied loads. To solve the differential equation we refer to the general procedures previously discussed in sections 2.2 and 2.3.
3.2 Application of the seismic action

From equation (3.3) it is seen that the most natural way of considering the ground motions in the analysis is using three accelerograms, $\ddot{u}(t)x$, $\ddot{u}(t)y$, and $\ddot{u}(t)z$. An accelerogram is as simple as the time-evolution of the ground accelerations in a specific direction during the earthquake. This is the type of data which can be recorded from real earthquakes, as an example the accelerogram for the Ardal earthquake (Iran, 6/4/1977) is shown in figure 3.1, which was obtained from the Internet-Site for European Strong-Motion Data (2001).

![Figure 3.1 Accelerogram registered during the Ardal earthquake (6/4/1977), Internet-Site for European Strong-Motion Data (2001)](image)

Using this information and any of the previous numerical procedures, the time-history behaviour of the structure is calculated. Since it usually requires a big computational effort and in most cases only the maximum structural response is of interest, a second way of considering the seismic excitation was derived. It is called response spectrum, and it will be fully described in the next section. Its main characteristics are:

- It is a function of the frequency ($\omega$) and damping ratio of the structure ($\xi$)
- It is calculated from the ground acceleration $a_g(t)$
- The analysis works with the mode superposition method
- Only the maximum response of the structure is obtained, therefore the computational cost and time are significantly reduced.
Moreover, the response spectrum is the information given by the building codes in order to design an earthquake-resisting structure. The specific code used in further calculations is Eurocode 8.

### 3.3 Response spectrum

The response spectrum was derived due to the need of more precise characterizations of ground shaking than the existing ones. The response spectrum is defined as the maximum response of the one degree of freedom (DOF) system, as a function of $\omega$ or $T$, for a certain ground acceleration ($a_{g(t)}$) and damping ratio ($\xi$). This maximum response can be obtained either in relative displacements, velocities or accelerations. The evaluation of the Duhamel’s integral, equation (2.34), leads to the spectral relative displacement, denoted as $S_{d(\omega, \xi)}$.

$$S_{d(\omega, \xi)} = \left[ \frac{1}{m\omega} \int_0^t ma_{g(t)} e^{-\xi\omega(t-\tau)} \sin \omega(t-\tau) d\tau \right]_{\max}$$  \hspace{1cm} (3.4)

Considering that usually the damping ratio of the structures is small ($2\% < \xi < 20\%$), the difference between the damped and undamped frequency is neglected. It also has to be noted that the negative sign has no importance in determining the maximum of the integral. With these two consideration equation (3.4) can be reduced to

$$S_{d(\omega, \xi)} = \left[ \frac{1}{\omega} \int_0^t a_{g(t)} e^{-\xi\omega(t-\tau)} \sin \omega(t-\tau) d\tau \right]_{\max}$$  \hspace{1cm} (3.5)

The spectral relative velocity, $S_{r(\omega, \xi)}$, is obtained by means of the first time derivative of the relative displacement expression in equation (2.34).

$$S_{r(\omega, \xi)} = \left[ \int_0^t a_{g(t)} e^{-\xi\omega(t-\tau)} \cos \omega(t-\tau) d\tau - \xi \int_0^t a_{g(t)} e^{-\xi\omega(t-\tau)} \sin \omega(t-\tau) d\tau \right]_{\max}$$  \hspace{1cm} (3.6)

The expression for the total acceleration is derived using equations (2.34), its first time derivative and the damped vibration equation with zero external force, written in the form

$$\ddot{x}_{(t)} = -2\omega\xi\dot{x}_{(t)} - \omega^2 x_{(t)}.$$

$$S_{a(\omega, \xi)} = \left[ \omega(2\xi^2 - 1) \int_0^t a_{g(t)} e^{-\xi\omega(t-\tau)} \sin \omega(t-\tau) d\tau - 2\omega\xi \int_0^t a_{g(t)} e^{-\xi\omega(t-\tau)} \cos \omega(t-\tau) d\tau \right]_{\max}$$  \hspace{1cm} (3.7)

The discussion above clearly shows that the evaluation of the response spectrums is complex and some simplification would be useful in order to obtain simpler expressions. As a consequence, a new function called pseudo-velocity response spectrum is defined by
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\[
S_{p\nu(\omega, \xi)} = \left| \int_0^t a_g(\tau) e^{-\xi \omega (t-\tau)} \sin \omega (t-\tau) d\tau \right|_{\text{max}}
\]  

(3.8)

This new expression (3.8) has some significant properties. For example it is seen from equation (3.5) that

\[
S_{d(\omega, \xi)} = \frac{1}{\omega} S_{p\nu(\omega, \xi)}
\]  

(3.9)

and from equations (3.6) and (3.8) that for \( \xi = 0 \)

\[
S_{v(\omega, 0)} = \left| \int_0^t a_g(\tau) \cos \omega (t-\tau) d\tau \right|_{\text{max}}
\]  

(3.10)

\[
S_{p\nu(\omega, 0)} = \left| \int_0^t a_g(\tau) \sin \omega (t-\tau) d\tau \right|_{\text{max}}
\]  

(3.11)

which are identical except for the trigonometric functions. It has been demonstrated that \( S_{v(\omega, 0)} \) and \( S_{p\nu(\omega, 0)} \) differ very little numerically, except in the case of very small frequency values. For damped systems, the difference between \( S_v \) and \( S_{p\nu} \) is considerably large and can differ by as much as 20 percent for \( \xi = 0.20 \). Note also from equation (3.7) for \( \xi = 0 \) that

\[
S_{d(\omega, 0)} = \left| \omega \int_0^t a_g(\tau) \sin \omega (t-\tau) d\tau \right|_{\text{max}}
\]  

(3.12)

therefore, from equation (3.11)

\[
S_{d(\omega, 0)} = \omega S_{p\nu(\omega, 0)}
\]  

(3.13)

It can be shown that equation (3.13) is nearly satisfied for damping values over the range \( 0 < \xi < 0.20 \), thus, we are able to use the approximate relation

\[
S_{d(\omega, \xi)} \approx \omega S_{p\nu(\omega, \xi)}
\]  

(3.14)

with little error being introduced. The quantity on the right hand side of equation (3.14) is called the pseudo-acceleration response spectrum and is denoted as \( S_{pa(\omega, \xi)} \). This quantity is particularly significant since it is a measure of the maximum elastic force developed in the oscillator

\[
f_{r,\text{max}} = k S_{d(\omega, \xi)} = \omega^2 m S_{d(\omega, \xi)} = m S_{pa(\omega, \xi)}
\]  

(3.15)

As a result of the discussion above, only the pseudo-velocity response spectrum needs to be generated, the other desired response spectra can be easily obtained using equations (3.9) and (3.14).

Since we are only interested in the calculation of the pseudo-spectrums for seismic analysis, from now on we will refer to them as the response spectrums, with the
The seismic loading case notation $S_{d(\omega, \xi)}$, $S_{v(\omega, \xi)}$ and $S_{a(\omega, \xi)}$. It is important to always keep in mind that these values are an approximation to the real functions.

The complete spectrum for a certain ground acceleration consists in a family of curves, each of them corresponding to a different value of the damping ratio. If plotted in linear form, each of the three response spectrums has an associated family of curves. But, we notice that the three spectrums can be plotted in the same graphic using a single group of curves. This is possible with the use of a trilogarithmic scale (base 10), as shown in figure 3.2, and the relations

\[
\log S_{d(\omega, \xi)} = \log S_{v(\omega, \xi)} - \log \omega \tag{3.16}
\]
\[
\log S_{a(\omega, \xi)} = \log S_{v(\omega, \xi)} + \log \omega \tag{3.17}
\]

![Figure 3.2 Example of a set of response spectrum curves in a trilogarithmic graphic, Clough (1975)](image)

If the plot is made with $\log S_{v(\omega, \xi)}$ as the ordinate and $\log T$ as the abscissa, then equation (3.16) is a straight line with slope of $-45^\circ$ for a constant value of $\log S_{d(\omega, \xi)}$ and equation (3.17) is a straight line with slope of $+45^\circ$ for a constant value of $\log S_{a(\omega, \xi)}$.

When interpreting such plots, it is important to note the following limiting values:

\[
\lim_{\omega \to 0} S_{d(\omega, \xi)} = \left|\frac{d}{g(t)}\right|_{\text{max}} \tag{3.18}
\]
\[
\lim_{\omega \to \infty} S_{a(\omega, \xi)} = \left|\frac{d}{g(t)}\right|_{\text{max}} \tag{3.19}
\]

These conditions mean that the response spectrum curves approach asymptotically the maximum ground displacement with decreasing values of frequency and the maximum ground acceleration with increasing values of frequency. This behaviour can be guaranteed for typical damping ratios, $\xi < 0.20$.

It is important to note that the response spectrum curves represent the properties of the earthquake at a specific site, and are not a function of the properties of the
structural system. After an estimation of the damping properties of the structure is made, a specific curve, or the proper interpolation, is selected from the set of curves calculated to carry out the dynamic analysis.

If sets of response spectrum curves are generated for ground motions recorded at different locations, large variations will be observed in both the response spectral values and the shape of the curves. These variations depend on many factors, such as energy release mechanism, epicentral distance, focal depth, geology and its variations along the energy transmission path, Richter magnitude and soil conditions. The effects of most of them are either not well understood, and as a consequence not quantifiable, or neglected. An exception is the soil condition, which is taken into account when defining the intensity and shape of the design response spectrum. A longer discussion on this factors is done by Clough (1975).

### 3.4 Analysis with response spectrum

The use of the response spectrum is another method for solving the decoupled equilibrium equations (2.33). But there is a big difference between it and the employment of a direct time integration or Duhamel’s integral, no time-history response is obtained. Only the maximum values for displacements, velocities and accelerations are calculated.

According to Barbat-Canet (1994) the expression for the maximum generalized displacement of the \( i \)th-mode is:

\[
\left| x_{(i)} \right|_{\text{max}} = \frac{\phi_i^T \mathbf{M} j \left( S_a \right)_i}{\phi_i^T \mathbf{M} \phi_i \omega_i^2}
\] (3.20)

where,

- \( j \) is the vector representing the displacement resulting from a unit support displacement
- \( \mathbf{M} \) is the mass-matrix
- \( \phi_i \) and \( \omega_i \) are the eigenvector and eigenfrequency associated to the \( i \)th mode of vibration.
- \( (S_a)_i \) is the value of the response spectrum for \( \omega_i \) and \( \xi_i \)

Therefore, using the basis transformation in equation (2.19), the maximum displacements for each mode are of the form

\[
\left| u_{(i)} \right|_{\text{max}} = \phi_i^T \mathbf{M} j \left( S_a \right)_i = A_i \frac{(S_a)_i}{\omega_i^2}
\] (3.21)

where \( A_i \) is the vector storing the modal participation coefficients. As a result, the maximum elastic internal force is obtained using equation (2.22).
Once the maximum responses for all the considered modes have been evaluated, \( i = 1, 2, \ldots, p \), we are interested in the maximum global response of the structure. A good approximation to this required value is obtained by a proper modal combination. It is not a unique procedure, since different assumptions can be made in the behaviour of the modes. In the next section, these options are discussed.

### 3.5 Modal combination of the results

The most conservative method of obtaining the maximum displacement or stress in a structure is the sum of the absolute modal response values. This procedure assumes that for all modes the maximum nodal values occur at the same time, which is certainly not true.

\[
\begin{align*}
\mathbf{u}_{\text{max}} &= \sum_{i=1}^{p} |u_{i\text{max}}| \\
\end{align*}
\]  

(3.23)

A second approach that is commonly used is the Square Root of the Sum of the Squares (SRSS) on the maximum modal values. The assumption of this method is that all of the maximum modal values are statically independent.

\[
\begin{align*}
\mathbf{u}_{\text{max}} &= \sqrt{\sum_{i=1}^{p} (u_{i\text{max}})^2} \\
\end{align*}
\]  

(3.24)

For three dimensional structures with a large number of almost identical frequencies, this assumption is not justified.

The last method here presented is the Complete Quadratic Combination (CQC), based on random vibration theories, that was first published in the 1980 and has been widely accepted by most engineers. It has also been incorporated in computer programs, as it has advantages compared with the SRSS method.

The peak value of a force can be estimated, from the maximum modal values, by the following equation:

\[
\begin{align*}
\mathbf{f}_{\text{max}} &= \sqrt{\sum_{n=1}^{p} \sum_{m=1}^{p} f_{n} \rho_{nm} f_{m}} \\
\end{align*}
\]  

(3.25)

where \( f_{n} \) is the modal force associated with mode \( n \). The summation is done over all modes considered in the analysis. The same equation can be applied to find node displacements, base shears and moments.

The expression of the cross-modal coefficient, \( \rho_{nm} \), depends on the assumption made on the kind of excitation signal. This input can be of two different types:
• White Noise (WN) excitation: The signal has a constant power spectral density.

\[ G_{F(\omega)} = G_0 \]  \hspace{1cm} (3.26)

• Filtered White Noise (FWN): It is the response of an oscillator to a white noise input. It is often used to represent the input into a structure supported by a single-degree primary system which itself is subjected to a white-noise excitation. A common example is the base input into a structures situated on a soil layer, which is excited by earthquake motions. In this case the power spectral density is of the form:

\[ G_{F(\omega)} = \frac{\omega_g^2 + 4\xi_g^2\omega_g^2\omega^2}{\left(\omega_g^2 - \omega^2\right)^2 + 4\xi_g^2\omega_g^2\omega^2} G_0 \]  \hspace{1cm} (3.27)

in which \( \omega_g \) and \( \xi_g \) are constants. For modelling ground acceleration during earthquakes they have been suggested to have the values \( \omega_g = 5\pi \) and \( \xi_g = 0.6 \).

For the white-noise excitation the simplest expression of \( \rho_{nm} \) is obtained, which is shown in equation (3.28).

\[ \rho_{nm} = \frac{8\sqrt{\xi_i\xi_j\omega_i\omega_j\left(\xi_i\omega_i + \xi_j\omega_j\right)}\omega_i\omega_j}{\left(\omega_i^2 - \omega_j^2\right)^2 + 4\xi_i\xi_j\omega_i\omega_j\left(\omega_i^2 + \omega_j^2\right) + 4\left(\xi_i^2 + \xi_j^2\right)\omega_i^2\omega_j^2} \]  \hspace{1cm} (3.28)

As a consequence, the cross-modal coefficient array has the following characteristics:

• It is symmetric (\( \rho_{nm} = \rho_{mn} \)).
• All terms are positive.
• All diagonal terms equal 1.
• Large off-diagonal terms indicate that those two modes are coupled.

The coefficient expression for the filtered white noise is much longer, and we refer to Der Kiureghian (1980) for the complete equation.

One possible simplification, commonly accepted and therefore also used in this thesis analyses, is to use the expression of \( \rho_{nm} \) given in equation (3.28) as an approximation to the exact value. For \( \xi_g = 0.6 \), which best represents earthquake type excitations, comparisons between the approximate value and the exact one are shown in figure 3.3. Note that the approximate expression is reasonably accurate, particularly for small damping.

It has to be noted that, if the assumption of statistically independent modes is done, the CQC leads to the SRSS method, since the coefficients \( \rho_{nm} \) are equal to the Kronecker’s Delta (\( \delta_{nm} \)). As a consequence, less terms are considered in the double summation which may lead the reader to the erroneous idea that the result using the CQC will always be higher than with the SRSS combination. But this is not true in most
cases, because the CQC takes into account the relative sign of the variable under evaluation.

To summarize, we may say that this ability to recognize the sign and the consideration of coupling between vibration modes, are the keys to eliminate the errors in the SRSS method.

Figure 3.3 Comparison of $\rho_{nm}$ for two different types of excitations. Der Kiureghian (1980)