Chapter 2
Linear dynamic analysis of a structural system

2.1 Dynamic equilibrium

The dynamic equilibrium analysis of a structure is the most general case that can be studied as it takes into account all the forces acting on it. When the external loads and displacements are applied very slowly, a simplified approach of the problem is done, leading to the static problem in which some of the forces are neglected. This division exclusively depends on the value of the inertia forces, equal to mass times the acceleration according to the Newton’s second law. In the general case four terms are considered in the equilibrium equation:

\[ f_{(t)}I + f_{(t)}D + f_{(t)}S = r_{(t)} \]  

where,

- \( f_{(t)}I \) is the vector of inertia forces
- \( f_{(t)}D \) is the vector of viscous damping forces
- \( f_{(t)}S \) is the vector of internal forces
- \( r_{(t)} \) is the vector of external loads

This equilibrium is valid for both linear and nonlinear systems, if equilibrium is formulated with respect to the deformed geometry of the structure. It is also valid when there is material nonlinearity, and, in all cases, it must be fulfilled at any time \( t \). In order to obtain the finite element formulation of the equilibrium, the behaviour of the structure is assumed to be linear:

- Infinitesimally small displacements
- The nature of the boundary conditions never changes during the analysis
- The material is linearly elastic, with a constitutive law of the form: \( \sigma = D\varepsilon + \sigma_0 \)
According to Bathe (1982), these three assumptions lead to the expression of the vector of internal forces as $Ku$. The general finite element formulation of the inertia forces is $Mu$ and of the damping forces is $Cu$, either for linear or nonlinear analysis.

Now we are able to rewrite the equilibrium expression as the second-order differential equation (2.2).

$$M \ddot{u} + C \dot{u} + K u = r$$  
(2.2)

in which $M$ is the mass matrix, $C$ is the damping matrix that approximates the energy dissipation in the structure, and $K$ is the static stiffness matrix. The time-dependant vectors $u$, $\dot{u}$ and $\ddot{u}$ are the absolute nodal displacements, velocities and accelerations.

For the static approach these last two terms, velocity and acceleration, are considered to be very small with regard to the displacement, therefore the inertia and damping forces are neglected and the equilibrium is reduced to the simplest expression $Ku = r$, which is not time-dependant. If the analysis is quasistatic, $r$ depends on time but loads are applied so slowly that the same approximations can be done, leading to the expression $Ku = r$. The choice for a static or dynamic analysis is usually decided by engineering criteria, with the aim of reducing the analysis effort required. However the assumptions done must be justified, otherwise the results would be meaningless.

The solution to the dynamic equilibrium can be obtained with standard procedures for the solutions of differential equations with constant coefficients, also called Runge-Kutta methods, but they may become very expensive if the order of the matrices is large (unless advantage is taken from any special characteristic of $M$, $C$ or $K$). In the following sections we will concentrate in the effective methods, which can be divided in two methods of solution: direct integration and mode superposition.

The aim of both methods is the resolution of a second-order differential equation, which means that two initial conditions, at time 0, are required. It is important to remark that only for two of the variables ($u_0$, $\dot{u}_0$ and $\ddot{u}_0$) the initial value is imposed, since the third one can be calculated from the other two. For example if the initial acceleration of the structure ($\ddot{u}_0$) is unknown, it can be calculated using the initial displacements, velocities and the requirement of equilibrium fulfilment at time 0, leading to a linear system that has to be solved.

In order to have a less complex notation of the different procedures, the absolute time-dependant displacements will be written as $u$ instead of $u(t)$. The same simplification is made on the velocity and acceleration vectors.
2.2 Direct integration

In direct integration the equations in (2.2) are integrated using a numerical step-by-step procedure. The term direct means that no transformation of the equations is performed before the time integration is carried out. The direct integration is based on two ideas:

- Instead of satisfying equation (2.2) at any time \( t \), it will be satisfied only at discrete intervals \( \Delta t \).
- A variation of displacements, velocities and accelerations within each time interval is assumed. It is the form of this assumption that determines the accuracy, stability and cost of the procedure.

Next, some of the effective methods will be presented. Since all of them calculate the solution of the next time step from the solutions at the previous times considered, algorithms will be derived assuming that the solutions are known at times 0, \( \Delta t \), 2\( \Delta t \), …, \( t \) and the solution at \( t + \Delta t \) is required. Therefore, the first step of the procedure will be calculated from the imposed initial conditions on the structure.

The most important choice in any direct integration method is the value of the time step, due to the cost of the analysis mainly depends on it. On one hand, the time step must be small enough to obtain accuracy in the solution, but, on the other hand, the time step must not be smaller than necessary, because this would mean that the solution is more costly than required. The two fundamental concepts of the integration scheme to be considered are stability and accuracy, for more information on them we refer to Bathe (1982).

2.2.1 The central difference method

In this method the following expressions for acceleration and velocity are assumed:

\[
\begin{align*}
\dot{\ddot{u}} &= \frac{1}{\Delta t^2} \left\{ (t-\Delta t)\dot{u} - 2(\Delta t)\ddot{u} + (t+\Delta t)\dot{u} \right\} \\
\dot{\dddot{u}} &= \frac{1}{2\Delta t} \left\{ -(t-\Delta t)\dot{u} + (t+\Delta t)\dot{u} \right\}
\end{align*}
\]

The error in both expansions is of order \((\Delta t)^2\). Substituting the relations into the equilibrium equation evaluated at time \( t \) the linear system to solve in each step is obtained, from which \( \dot{\ddot{u}} \) can be calculated.

\[
\left( \frac{1}{\Delta t^2} M + \frac{1}{2\Delta t} C \right)^{\Delta t} \dot{u} = \ddot{r} - \left( K - \frac{2}{\Delta t^2} M \right) \dot{u} - \left( \frac{1}{\Delta t^2} M - \frac{1}{2\Delta t} C \right)^{\Delta t} \dot{u}
\]

It has to be remarked that the fact of evaluating the equilibrium equation at \( t \) leads to an explicit integration method, in which no factorization of the stiffness matrix is required. On the other hand, the following methods that will be considered are implicit ones, evaluating equilibrium at time \( t+\Delta t \).
The effectiveness of the procedure depends on the use of a diagonal mass matrix and the neglect of damping forces. This first disadvantage is not very serious, since using a fine-enough mesh, good accuracy of the solution can be obtained.

A more important consideration is that the time step \( \Delta t \) must be smaller than a critical value \( \Delta t_{cr} \), which depends on the mass and stiffness properties of the structure. Therefore this method is conditionally stable. If the used time step is larger than \( \Delta t_{cr} \), the integration is unstable, meaning that any error from round-off or numerical integration grow and make the result worthless.

The following discussed procedures are unconditionally stable. Thus, any time step \( \Delta t \) can be chosen for carrying out the analysis, and, in many cases, \( \Delta t \) can be orders of magnitude larger than what \( \Delta t_{cr} \) would allow. However, as mentioned above, these methods are implicit, so a triangularization of the effective stiffness matrix is required.

2.2.2 The Houbolt method

As in the central difference method, the Houbolt integration scheme uses standard finite difference expressions for the acceleration and velocity. These approaches are backward-difference formulas, with errors of order \((\Delta t)^2\):

\[
\begin{align*}
\frac{t+\Delta t}{\Delta t} \ddot{u} &= \frac{1}{\Delta t^2} \left\{ 2^{t+\Delta t} u - 5^{t+\Delta t} u + 4^{t-\Delta t} u - 2^{-2\Delta t} u \right\} \\
\frac{t+\Delta t}{\Delta t} \dddot{u} &= \frac{1}{6\Delta t} \left\{ 11^{t+\Delta t} u - 18^{t+\Delta t} u + 9^{t-\Delta t} u - 2^{-2\Delta t} u \right\}
\end{align*}
\]

(2.6) \hspace{1cm} (2.7)

The difference from the previous method is that, to solve the displacement field at \( t+\Delta t \), the equilibrium equation (2.2) is evaluated at time \( t+\Delta t \). When substituting the expressions for \( \frac{t+\Delta t}{\Delta t} \ddot{u} \) and \( \frac{t+\Delta t}{\Delta t} \dddot{u} \) the linear system in equation (2.8) has to be solved.

\[
\begin{align*}
\left( \frac{2}{\Delta t^2} M + \frac{11}{6\Delta t} C + K \right)^{t+\Delta t} u &= \frac{t+\Delta t}{\Delta t} r + \left( \frac{5}{\Delta t^2} M + \frac{3}{\Delta t} C \right)^{t+\Delta t} u - \\
&- \left( \frac{4}{\Delta t^2} M + \frac{3}{2\Delta t} C \right)^{t-\Delta t} u + \left( \frac{1}{\Delta t^2} M + \frac{1}{3\Delta t} C \right)^{-2\Delta t} u
\end{align*}
\]

(2.8)

It is shown in equation (2.8) that \( \frac{t-\Delta t}{\Delta t} u \) needs to be known in order to calculate the value of the independent term in the linear system. Although the knowledge of \( \frac{0}{\Delta t} u \), \( \frac{0}{\Delta t} \ddot{u} \) and \( \frac{0}{\Delta t} \dddot{u} \) is useful to start the Houbolt method, usually special starting procedures are employed. An example is to calculate \( \frac{\Delta t}{\Delta t} u \) and \( \frac{2\Delta t}{\Delta t} u \) by means of a conditionally stable scheme, such as the central difference method.

It should be reminded that the evaluation of equation (2.2) at \( t+\Delta t \) yields to an implicit method, in which no restriction for \( \Delta t \) exists but the effective stiffness matrix, defined as the matrix factor of \( \frac{t+\Delta t}{\Delta t} u \), \( \left( \frac{2}{\Delta t^2} M + \frac{11}{6\Delta t} C + K \right) \) in this case, has to be factorized.
It can be seen in equation (2.8) that when neglecting mass and damping effects \((M=0\) and \(C=0\)) the Houbolt method reduces directly to a static analysis for time-depandant loads. In this particular case the central difference method can not be used.

2.2.3 The Wilson \(\theta\) method

In the Wilson \(\theta\) method a linear variation of the acceleration from time \(t\) to time \(t+\theta\Delta t\) is assumed, where \(\theta \geq 1\). If the unconditional stability of the method is required, \(\theta\) must be larger or equal to 1.37; thus, \(\theta=1.40\) is usually employed. Denoting \(\tau\) as the time increase, the previous mentioned linear variation is expressed as:

\[
^{t+\theta}\ddot{u} = ^t\ddot{u} + \frac{\tau}{\theta \Delta t} \left( ^{t+\theta}\ddot{u} - ^t\ddot{u} \right)
\]  

(2.9)

Integrating this equation, formulas for velocity and displacement variation are derived:

\[
^{t+\theta}\ddot{u} = ^t\ddot{u} + ^t\ddot{u}\tau + \frac{\tau^2}{2 \theta \Delta t} \left( ^{t+\theta}\ddot{u} - ^t\ddot{u} \right)
\]  

(2.10)

\[
^{t+\theta}u = ^t u + ^t u\tau + \frac{1}{2} ^t u\tau^2 + \frac{\tau^3}{6 \theta \Delta t} \left( ^{t+\theta}\ddot{u} - ^t\ddot{u} \right)
\]  

(2.11)

Evaluating these expressions at \(\tau = \theta \Delta t\) it is possible to solve \(^{t+\theta}\ddot{u}\) and \(^{t+\theta}\ddot{u}\) in terms of \(^{t+\theta}u\), with the result shown in equations (2.12) and (2.13).

\[
^{t+\theta}\ddot{u} = \frac{6}{\theta^2 \Delta t^2} \left( ^{t+\theta}u - ^t u \right) - \frac{6}{\theta \Delta t} ^t\ddot{u} - 2 ^t\ddot{u}
\]  

(2.12)

\[
^{t+\theta}\ddot{u} = \frac{3}{\theta \Delta t} \left( ^{t+\theta}u - ^t u \right) - 2 ^t\ddot{u} - \frac{\theta \Delta t}{2} ^t\ddot{u}
\]  

(2.13)

To obtain the solution for the displacements, velocities and accelerations at time \(t+\Delta t\), the equilibrium equations (2.2) are considered at \(t+\theta \Delta t\). Consequently, the load vector has to be linearly interpolated.

\[
M^{t+\theta}\dddot{u} + C^{t+\theta}\ddot{u} + K^{t+\theta}u = ^{t+\theta}\dddot{f}
\]  

(2.14)

\[
^{t+\theta}\dddot{f} = ^t r + \theta \left( ^{t+\Delta t}r - ^t r \right)
\]  

(2.15)

Substituting equations (2.12) and (2.13) into equation (2.14) a linear system is obtained and \(^{t+\theta}u\) can be solved. Next, this value is used in equation (2.12) to obtain \(^{t+\theta}\ddot{u}\) which is employed in equations (2.9), (2.10) and (2.11) with \(\tau = \Delta t\).

As mentioned in section 2.2.1, the Wilson \(\theta\) method is an implicit procedure, because the stiffness matrix is a coefficient of the unknown displacement vector \(^{t+\theta}u\). It might also be noted that no starting procedures are needed, since the variables at \(t+\Delta t\) are only expressed in terms of the same quantities at time \(t\).
2.2.4 The Newmark method

In this method the following assumptions for the displacement and velocity vectors are used:

\[
\begin{align*}
\tau^{\Delta t} \ddot{u} &= \dot{u} + \left[ (1 - \delta) \dot{\ddot{u}} + \delta \tau^{\Delta t} \ddot{u} \right] \Delta t \\
\tau^{\Delta t} \dot{u} &= \dot{u} + \left[ \left( \frac{1}{2} - \alpha \right) \dot{\ddot{u}} + \alpha \tau^{\Delta t} \ddot{u} \right] \Delta t^2
\end{align*}
\]

(2.16) \hspace{1cm} (2.17)

where \( \alpha \) and \( \delta \) are parameters that can be determined to obtain integration accuracy and stability. The scheme will be unconditionally stable when \( \delta \geq \frac{1}{2} \) and \( \alpha \geq 0.25(\delta + 0.5)^2 \). Newmark (1959) originally proposed as an unconditionally stable scheme the constant-average-acceleration method (also called trapezoidal rule), in which \( \alpha = \frac{1}{4} \) and \( \delta = \frac{1}{2} \), that has the most desirable accuracy characteristics.

To solve the displacement, velocities and accelerations at time \( t + \Delta t \), the equilibrium equations (2.2) are evaluated at the same time \( t + \Delta t \).

\[
M \tau^{\Delta t} \ddot{u} + C \tau^{\Delta t} \dot{u} + K \tau^{\Delta t} u = \tau^{\Delta t} r
\]

(2.18)

Working with equations (2.16) and (2.17) it is possible to obtain \( \tau^{\Delta t} \ddot{u} \) and \( \tau^{\Delta t} \dot{u} \) in terms of \( \tau^{\Delta t} u \) only. These relations are substituted into equation (2.18) and \( \tau^{\Delta t} u \) is obtained solving a linear system.

The Newmark method has similar characteristics as the Wilson \( \theta \) method, as it is an implicit integration scheme and no starting procedures are needed.

2.3 Mode superposition

A very rough estimation on the number of operations needed to solve one time step in a direct integration method, using a diagonal mass matrix and neglecting damping, is \( 2nm^k \), where \( n \) is the order of \( K \) and \( m^k \) is its half-bandwidth. Moreover, in the implicit methods an initial cost for the factorization of the effective stiffness matrix is added. If a consistent mass matrix is used and damping is taken into account, the total number of required operations is about \( \alpha nm^k s \), where \( \alpha \geq 2 \) and \( s \) is the number of time steps.

Due to this consideration it is expected effectiveness of a direct time integration only for short durations (few time steps). If the integration has to be done over a longer interval it may be more effective to first transform the equilibrium equations (2.2) into a form with less costly step-by-step solution. In particular, a reduction of the half-bandwidth \( m^k \) would be successful. We may note that this bandwidth depends on the nodal numbering, but modifying this feature there is a minimum value for \( m^k \) that can not be reduced.
2.3.1 Change of basis to modal generalized displacements

A more effective form of the equilibrium equations is obtained by using the following transformation on the nodal displacements vector:

\[ u_{(t)} = Px_{(t)} \] (2.19)

where \( P \) is a square matrix, which is still unknown and will be determined, and \( x_{(t)} \) is the time-dependant vector containing the generalized displacements. It is remarkable that the matrix \( P \) must be non-singular, in order to have a unique relation between \( u_{(t)} \) and \( x_{(t)} \). Substituting equation (2.19) into equation (2.2) and pre-multiplying by \( P^T \) the new equilibrium equation (2.20) is derived.

\[
P^T M \ddot{x}_{(t)} + P^T C \dot{x}_{(t)} + P^T K x_{(t)} = P^T r_{(t)}
\]

\[
M \ddot{x}_{(t)} + C \dot{x}_{(t)} + K x_{(t)} = r_{(t)}
\] (2.20)

where the new stiffness, mass and damping matrices have smaller bandwidth than the original ones.

Many different matrices \( P \) can be used, but an effective one is set by using the displacement solutions of the free vibration equilibrium equations with damping neglected.

\[ M \ddot{u}_{(t)} + Ku_{(t)} = 0 \] (2.21)

Considering the solution of the form \( u_{(t)} = \phi \sin(\omega(t-t_0)) \), where \( \phi \) is a vector of order \( n \) and \( \omega \) represents the frequency of vibration (rad/s) of the vector \( \phi \), the generalized eigenproblem is derived:

\[ K \phi = \omega^2 M \phi \] (2.22)

The eigenproblem leads to \( n \) eigensolutions \((\omega_1^2, \phi_1), (\omega_2^2, \phi_2), \ldots, (\omega_n^2, \phi_n)\) where the eigenvectors are \( M \)-orthonormalized:

\[
\phi_i^T M \phi_j = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}
\]

(2.23)

and \( 0 \leq \omega_1^2 \leq \omega_2^2 \leq \ldots \leq \omega_n^2 \)

The vector \( \phi_i \) is called the \( i \)-th-mode shape vector and \( \omega_i \) is the corresponding frequency of vibration (rad/s). With this group of eigenvalues two useful matrices are defined. \( \Phi \) is a matrix whose columns are the eigenvectors \( \phi_i \) and \( \Omega^2 \) is a diagonal matrix storing the values \( \omega_i^2 \).

\[
\Phi = [\phi_1, \phi_2, \ldots, \phi_n] \quad \Omega^2 = \begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \vdots \\ \omega_n^2 \end{bmatrix}
\] (2.24)
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Since the eigenvectors are $\mathbf{M}$-orthonormal, $\Phi^\top \mathbf{M} \Phi = \mathbf{I}$, and writing the $n$ solutions as $\mathbf{K} \Phi = \mathbf{M} \Phi \Omega^2$, the following expression is obtained:

$$\Phi^\top \mathbf{K} \Phi = \Omega^2 \quad (2.25)$$

It is apparent that the matrix $\Phi$ is suitable as the transformation matrix $\mathbf{P}$, $\mathbf{u}_{(t)} = \Phi \mathbf{x}_{(t)}$. This yields to the equilibrium equation for modal generalized displacements:

$$\ddot{\mathbf{x}}_{(t)} + \Phi^\top \mathbf{C} \Phi \dot{\mathbf{x}}_{(t)} + \Omega^2 \mathbf{x}_{(t)} = \Phi^\top \mathbf{r}_{(t)} \quad (2.26)$$

The initial conditions on $\mathbf{x}_{(t)}$ are obtained from $\mathbf{u}_{(t)}$ using the expressions:

$$\mathbf{x}_{(0)} = \Phi^\top \mathbf{M} \mathbf{u}_{(0)} \quad \dot{\mathbf{x}}_{(0)} = \Phi^\top \mathbf{M} \dot{\mathbf{u}}_{(0)} \quad (2.27)$$

### 2.3.2 Analysis with damping neglected

The easiest problem that can be solved is neglecting the damping effects on the structure. Then, the analysis reduces to the solution of the differential equation (2.28).

$$\ddot{\mathbf{x}}_{(t)} + \Omega^2 \mathbf{x}_{(t)} = \Phi^\top \mathbf{r}_{(t)} \quad (2.28)$$

where it must be noted that all equations are decoupled, so they can be solved individually. Each $i$th-equation correspond to the equilibrium equation of a single degree of freedom system with unit mass and stiffness $\omega_i^2$:

$$\ddot{x}_{i(t)} + \omega_i^2 x_{i(t)} = \ddot{r}_{i(t)} = \Phi^\top \mathbf{r}_{(t)} \quad (2.29)$$

Equation (2.29) can be solved using either the previous discussed direct integration procedures or the numerical evaluation of the Duhamel integral:

$$x_{i(t)} = \frac{1}{\omega_i} \int_0^t \ddot{r}_i(\tau) \sin \omega_i (t - \tau) d\tau + \alpha_i \sin \omega_i t + \beta_i \cos \omega_i t \quad (2.30)$$

where $\alpha_i$ and $\beta_i$ have to be determined from the initial conditions on displacement and velocity.

For the complete response of the structure the differential equation (2.29) has to be solved for $i = 1, 2, \ldots, n$. Then, the time-history of the total nodal displacements is obtained by the superposition of each mode as shown in equation (2.31):

$$\mathbf{u}_{(t)} = \sum_{i=1}^n \Phi_i \mathbf{x}_{i(t)} \quad (2.31)$$
2.3.3 Analysis including damping

When damping effects on the structures are considered, it is also desired to calculate having decoupled equilibrium equations. But, it is only possible if the assumption of proportional damping can be made, in which case

$$\phi^T C \phi = 2 \omega_i \xi_i \delta_{ij}$$  \hspace{1cm} (2.32)

where $\xi_i$ is the modal damping parameter and $\delta_{ij}$ is the Kronecker delta. Therefore the eigenvectors $\phi_i$ are also C-orthogonal and the equilibrium equation reduces to $n$ decoupled differential equations of the form:

$$\ddot{x}_{i(t)} + 2 \omega_i \xi_i \dot{x}_{i(t)} + \omega_i^2 x_{i(t)} = \vec{r}_{i(t)}$$  \hspace{1cm} (2.33)

The solution of the differential equation can be calculated using the same procedures as in the neglected damping case. These are either direct time integration, implicit or explicit, or evaluation of the Duhamel integral:

$$x_{i(t)} = \frac{1}{\omega_i} \int_0^{\tau} \vec{f}(\tau) e^{-\xi_i \omega_i (t-\tau)} d\tau + e^{-\xi_i \omega_i t} \left\{ \alpha_i \sin \omega_i t + \beta_i \cos \omega_i t \right\}$$  \hspace{1cm} (2.34)

where $\omega_i$ is defined as $\omega_i = \omega_i \sqrt{1-\xi_i^2}$ and $\alpha_i$ and $\beta_i$ are obtained from the initial conditions.

The assumption of proportional damping leads to two remarkable considerations:

- The total damping in the structure is the “sum” of individual damping in each mode.
- There is no need to calculate the matrix $C$ for the complete structure, since only the modal damping ratio ($\xi_i$) is used in the decoupled equations.

Let's make the assumption that in some particular case the direct step-by-step procedure is more effective, and $p$ modal damping ratios are known ($\xi_1$, ..., $\xi_p$). In consequence the matrix $C$ has to be explicitly evaluated to carry out the analysis. If $p = 2$, Rayleigh damping can be assumed, which is of the form

$$C = \alpha M + \beta K$$  \hspace{1cm} (2.35)

where $\alpha$ and $\beta$ are constants to be determined from two given damping ratios and two associated frequencies of vibration, working with expressions (2.32) and (2.35).

$$\phi^T (\alpha M + \beta K) \phi = \alpha + \beta \omega_i^2 = 2 \omega_i \xi_i$$  \hspace{1cm} (2.36)
which gives two equations to solve the two constants $\alpha$ and $\beta$. If $p > 2$ a possible option to determine the C matrix is to maintain the assumption of Rayleigh damping and use two pairs of average values ($\xi_i, \omega_i$) and ($\xi_j, \omega_j$).

Another possibility is to use a more complicated damping matrix, that satisfies the relation (2.37).

$$C = M \sum_{k=0}^{p-1} a_k \left[ M^{-1} K \right]^k$$  \hspace{1cm} (2.37)

where the coefficients $a_k$, $k = 1, \ldots, p$, are calculated from the $p$ simultaneous equations in (2.38)

$$\xi_i = \frac{1}{2} \left( a_k + a_k \omega_i + a_2 \omega_i^2 + \ldots + a_{p-1} \omega_i^{2p-3} \right)$$  \hspace{1cm} (2.38)

It has to be noted that for $p = 2$ it reduces to the Rayleigh damping. An important consideration is that for $p \geq 3$ the matrix C is, in general, a full matrix, which means that the cost of the analysis is significantly increased. Therefore, in order to work with a banded C-matrix, in most analyses Rayleigh damping is assumed. A disadvantage of the Rayleigh damping is that the higher modes are considerably more damped than the lower modes. The reason is that, once the coefficients have been determined using the lowest modes, any other required damping ratio is obtained from equation (2.39), which provides higher values of the damping ratio for higher eigenfrequencies.

$$\xi_i = \frac{\alpha + \beta \omega_i^2}{2 \omega_i}$$  \hspace{1cm} (2.39)

In practice, approximately the same values for $\alpha$ and $\beta$ are used in similar structures.

However, the damping characteristics of some structures are not well approached by using proportional damping. An example is the analysis of foundation-structure interaction problems, where more damping may be observed in the foundation than in the surface. For a case like this one, it is reasonable to assign different Rayleigh coefficients ($\alpha$ and $\beta$) to different parts of the structure. Another example of nonproportional damping is the existence of concentrated dampers, e.g. at the supports of the structure.

The consideration of nonproportional damping does not affect the solution of the equilibrium equations by a direct integration scheme. No modifications are required in the procedures previously discussed. On the other hand, it originates a big change in the mode superposition analysis, since $\Phi^T C \Phi$ is a full matrix, equation (2.40 a). It means that the equilibrium equations in (2.26) are no longer decoupled. Assuming that the response of the structure is mainly controlled by the first $p$ modes ($\phi_i, \ldots, \phi_p$), only the first $p$ equations need to be considered. Therefore, the assumption that $x_i$, $i=1, \ldots, p$ and $x_j$, $i=p+1, \ldots, n$ are decoupled is done, equation (2.40 b), and the first $p$ equations, shown in expression (2.40 c), can be solved by means of a direct integration procedure.
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\[
\Phi^T C \Phi = \tilde{\Phi} = \begin{pmatrix}
\tilde{c}_{1,1} & \ldots & \tilde{c}_{1,p} & \tilde{c}_{1,p+1} & \ldots & \tilde{c}_{1,n} \\
\vdots & \ddots & \vdots & \vdots & & \vdots \\
\tilde{c}_{p,1} & \ldots & \tilde{c}_{p,p} & \tilde{c}_{p,p+1} & \ldots & \tilde{c}_{p,n} \\
\tilde{c}_{p+1,1} & \ldots & \tilde{c}_{p+1,p} & \tilde{c}_{p+1,p+1} & \ldots & \tilde{c}_{p+1,n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\tilde{c}_{n,1} & \ldots & \tilde{c}_{n,p} & \tilde{c}_{n,p+1} & \ldots & \tilde{c}_{n,n}
\end{pmatrix}
\] (2.40 a)

\[
\tilde{\Phi} = \begin{pmatrix}
\tilde{c}_{1,1} & \ldots & \tilde{c}_{1,p} & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & & \vdots \\
\tilde{c}_{p,1} & \ldots & \tilde{c}_{p,p} & 0 & \ldots & 0 \\
0 & \ldots & 0 & \tilde{c}_{p+1,p+1} & \ldots & \tilde{c}_{p+1,n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \tilde{c}_{n,p+1} & \ldots & \tilde{c}_{n,n}
\end{pmatrix}
\] (2.40 b)

\[
1 \quad 0 \\
\vdots \\
0 \quad 1
\]
\[
\begin{pmatrix}
\tilde{c}_{1,1} & \ldots & \tilde{c}_{1,p} \\
\vdots & \ddots & \vdots \\
\tilde{c}_{p,1} & \ldots & \tilde{c}_{p,p}
\end{pmatrix}
\]
\[
\omega^2 \quad 0 \\
\vdots \\
0 \quad \omega^2
\]
\[
\tilde{r}_i
\]
\[
\begin{pmatrix}
x \\
\dot{x} \\
x
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
\vdots & \ddots \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\tilde{c}_{1,1} & \ldots & \tilde{c}_{1,p} \\
\vdots & \ddots & \vdots \\
\tilde{c}_{p,1} & \ldots & \tilde{c}_{p,p}
\end{pmatrix} \begin{pmatrix}
\omega^2 & 0 \\
\vdots & \ddots \\
0 & \omega^2
\end{pmatrix} \begin{pmatrix}
x \\
\dot{x} \\
x
\end{pmatrix} = \begin{pmatrix}
\tilde{r}_i \\
\tilde{r}_{\dot{i}} \\
\tilde{r}_p
\end{pmatrix}
\] (2.40 c)

2.4 Comparison of solution procedures

The unique difference between a mode superposition and a direct integration analysis is that, before the time integration is done, a change of basis is carried out, from the finite element coordinate basis to the basis of eigenvectors. Since the same space is spanned by the \( n \) eigenvectors as by the \( n \) nodal point finite element displacements, the same solution must be obtained in both cases. Therefore, the choice of method will be only decided by effectiveness criteria. It is for this reason that another aspect of the mode superposition has to be discussed, because it can significantly decrease the cost of the analysis.

Consider an applied load of the form \( r_{i(t)} = M \phi_i f_{i(t)} \), where \( f_{i(t)} \) is an arbitrary function of \( t \), and the initial conditions \( u_{i(0)} = 0, \dot{u}_{i(0)} = 0 \). In such a case, and working with the decoupled equations (2.33), only the component \( x_{i(t)} \) is nonzero. Also the frequency of the loading determines if the \( i \)th equation will have an important contribution to the response. Therefore, \( x_{i(t)} \) is relatively large if the excitation frequency of \( r_i \), defined in (2.29), lies near \( \omega_i \).

The conclusion is that frequently only a small number of the decoupled equations need to be considered in order to obtain a good approximation to the response of the system. The most usual case is to use the equation (2.33) for \( i=1, 2, \ldots p \), where \( p \ll n \), which also means that only the \( p \) lowest eigenvalues and corresponding eigenvectors have to be calculated. This reduction in the number of equations can make the mode superposition much more effective than direct integration, but the mentioned
effectiveness depends on the number of modes included in the analysis, as well as their relation with the applied load (orthogonality and frequency).

When obtaining the behaviour of the structure not only the displacements are obtained, the elastic forces over the structure members are also of interest. These forces are calculated according to expression (2.41).

\[ f_{(t)} = K u_{(t)} = \omega^2 M u_{(t)} = M \sum_{i=1}^{n} \omega_i^2 \phi_i x_{i(t)} \]  

(2.41)

Because each modal contribution is multiplied by the square of the modal frequency, it is evident that the higher modes are of greater significance in defining the forces than in the displacements. As a consequence, it will be necessary to include more modal components to define the forces to any desired degree of accuracy than to define the displacements.

In any numerical analysis it is useful to know the origin of the errors, so the correct decisions can be taken to reduce them. In a direct integration they appear because a too large time step is used, whereas in the mode superposition the reason is that not enough modes have been used, assuming that the \( p \) decoupled equations have been accurately solved.

We may also remark that, when dealing with a nonlinear analysis, the only possibility is to perform a direct time integration. But none of the here discussed step-by-step methods are valid, since they were derived for a linearly elastic material of which the stiffness properties, i.e. \( K \)-matrix, do not vary during the whole analysis. Detailed information in nonlinear dynamic analysis can be found in Bathe (1982) and Clough (1975).