Abstract

In the world of engineering appear a lot of differential problems that need some numerical strategy to be resolved such as the finite difference method, the finite elements method and the spectral methods. During the last years the idea of tackling those problems directly in a discrete medium, which is approximated to the continuum has appeared with great strength. This new conception is known as “mimetic discretizations of the continuum mechanics”, and its philosophy consists of getting the formulations of the differential problems directly in the discrete case like we do in the continuous case, so the target is making a discretization of all the continuum mechanics. Thus, the differential problems become systems of algebraic equations.

This work belongs to this new conception and follows its theory approach. The idea of the study is to establish a useful calculus on a discrete medium, paying special attention to the algorithms’ construction. This way the uniform grid is defined as the model of the discrete medium and basically consists of a finite set of nodes and a connection function between them. This connection function allows us to establish the concepts and techniques from the Algebraic Topology, very useful to define the boundary and the interior nodes.

The kind of uniform grids that are analyzed in this study is very general and brings together many cases. This way, uniform grids in n-dimensional spaces with an arbitrary connection function are studied. The rectangular grids in 2D and 3D, the grid of triangles in 2D or the grid of tetrahedrons in 3D are particular cases of them.

The vectorial discrete calculus is used for the theoretical formulation since it provides a powerful tool to describe the differential problems with enough precision. To do this, we interpret uniform grids as discrete manifolds and tangent space are used as key concept to establish the vectorial calculus. After that, following the way of the continuum medium, functions, vectors and matrices are defined on the grids as well as the respective inner products. So the first order differential operators are provided with composition and duality techniques from the gradient operator, which represents the basic reference operator of the analysis. This is the case of the divergence operator, for instance. The discrete form of the Laplace operator is provided as the composition of the discrete operators divergence and gradient. So once the discrete form of the operators is found, the pde’s on the grids are obtained by replacing the respective expressions.

In this research the elliptic pde’s with constant coefficients whose expression is $L(u) = -div \; K \nabla u + (k, \nabla u) + k_0 u$ are studied, where $K$ is a symmetric matrix, $k$ is a vector and $k_0$ is a parameter, although according to the theoretical formulation is not necessary to restrict it to this kind of equations, since the concepts and techniques are the same for the general second order pde’s. The fact that this kind of pde’s has a discrete version on uniform grids given by the expression $L_h(u) = -div \; A \nabla u + (b, \nabla u) + qu$ is proved in the research, where $A$ is a matrix, $b$ is a vector and $q$ is a parameter. Moreover the fact that the discrete operator $L_h(u)$ formally matches a difference scheme is also deduced. This allows us to relate the existing difference schemes and their properties with particular elections of $A$, $b$ and $q$, and a new technique to generate a lot of them.

The research specially focuses on the consistency of the method, as it informs us about the approximation error that we are making in the discretization. So a generic procedure is provided to calculate the consistency and as a result we get the generic equations of the consistency for any kind of uniform grid. With these equations we get the order of consistency of our operator only by replacing the values.

Finally all the study is characterized for the triangular grids in 2D, getting some useful difference schemes for a computational implementation, and the results and conclusion derived from the study are discussed.