4  Approach to the simple normalization of DSD.

4.1  Introduction

Many studies have proposed analytical expressions to describe the measured DSD. The most common methodology used in the past (coming from Marshall and Palmer 1948 works), consisted on grouping the measured realizations into different classes (usually of rainfall intensity) to compute a mean spectrum for each class. Afterwards, for every mean spectrum a distribution was fitted, and their different parameters were related to the rain intensity through power laws.

Sekhon and Srivastava 1970; Sekhon and Srivastava 1971 applied for the first time a normalization to the DSD study. They suggested a new approach normalizing the diameters and the DSD in order to deal directly with the whole set of the spectra on a unique plot.

Later on, Sempere-Torres et al. 1994 proposed a general formulation of the DSD in terms of a scaling law that was able to reproduce all previous studies dealing with DSDs modelizations. In the following sections, the normalization of DSDs is re-examined. First section is devoted to the presentation of the theory and its implications. Then a methodology to fit the scaling normalization of the DSD from experimental spectra is presented. Section 4.4 analyzes the meaning of the different parameters of the normalization. A review of the previous expressions is presented in terms of the normalization. Finally, conclusions are given in the last section.

4.2  Scaling law for the DSD

Sempere-Torres et al. 1994 proposed a general expression for DSDs in which any DSD is written as a function of the drop diameter (D) and a reference variable (Ψ) in the following way:
\[
\frac{N(D, \Psi)}{\Psi^{\alpha_\Psi}} = g(D \Psi^{-\beta_\Psi})
\]  
(4.1)

where \( \Psi \) can be any integral variable (R is generally used), \( \alpha_\Psi \), and \( \beta_\Psi \) are constants with no functional dependence on \( \Psi \), and \( g \) is called the “general distribution function”, which is independent of the value of \( \Psi \). An important improvement in this methodology is that it is no longer necessary to choose a priori shape of the DSD. See the Appendix B for additional explanation of this expression.

The expression (4.1) is in fact a scaling law that remains constant for a wide range of values of the scaling variable, \( \Psi \) (Figure 2.6). The peculiarity here is that the scaling variable \( \Psi \) is an integral function of \( N(D, \Psi) \), an arbitrary moment of the DSD expressed as follow:

\[
\Psi = M_i = \int N(D)D^i dD
\]  
(4.2)

where “\( i \)” is the order of the reference moment.

### 4.2.1 Implications of the formulation

A number of implications follow from the scaling law formulation

#### 4.2.1.1 Relations between integral moments of DSD

The first one is that the reference variable \( \Psi \) and any other integral moments of \( N(D, \Psi) \) are powerly related. Considering these integral moments as

\[
M_n = \int N(D)D^n dD
\]  
(4.3)

And substituting (4.1) into (4.3), and writing \( x = D \cdot M_i^{-\beta} \), one obtains

\[
M_n = \left[ \left( M_i^{-\beta} g(x) \right) x \right]^{n-1} \left( M_i^{-\beta} dx \right)
\]

\[
= M_i^{n(x+1)-\beta} \int g(x)x^n dx = a_n M_i^{n(x)}
\]  
(4.4)

where

\[
\gamma(n) = \alpha + (n + 1)\beta
\]  
(4.5)

\[
a_n = \int g(x)x^n dx
\]  
(4.6)

#### 4.2.1.2 Self-consistency relationships

A second implication comes from taking in the relations between moments of the DSD the integral rainfall variable \( (M_n, \text{ the moment of order } n) \) as the reference variable \( (M_i, \text{ moment of order } i) \). That is, \( n=i \) in equation (4.4). This leads to constraints on the exponents \( \alpha \) and \( \beta \) and the general function \( g \). These constraints are called self-consistency relationships.

\[
\alpha + (i + 1)\beta = 1
\]  
(4.7)

\[
a_i = \int g(x)x^i dx = 1
\]  
(4.8)
Thus $\alpha$ and $\beta$ are not independent and only one exponent ($\alpha$ or $\beta$) is free. In the same way the general function $g(x)$ is reduced its number of parameters by one. The general expression then, only depends on a single coefficient $\beta$ and a general function $g(x)$ independent of the reference variable, which should be chosen to fit the characteristics of a particular event or a particular climate.

Following we present different examples of self-consistency relationships using different reference variables:

1. $\Psi = W = A_w M_3$. Liquid water content: where $A_w$ is a constant.
   a. The relation between exponents is $\alpha_w + 4\beta_w = 1$
   b. The general function $g(x)$ must satisfy $\int g(x) x^3 dx = 1/A_w$

2. $\Psi = R$. Rainfall intensity (weighted moment of the DSD). Considering the weighting function (the fall velocity) a power law with the diameter ($v(D) = kD^{0.67}$) then

$$R = ctt \int D^3 N(D)v(D)dD = A_R \int D^{3.67} N(D)dD = A_R M_{3.67}$$

(4.9)

where $ctt$ and $A_R$ are constants.
   a. The relation between the exponents is $\alpha_R + 4.67\beta_R = 1$ and
   b. The general function $g(x)$ must satisfy $\int g(x) x^{3.67} dx = 1/A_R$

### 4.3 Methodology to fit a DSD model from experimental spectrum

From the general DSD formulation proposed by Sempere-Torres et al. 1994 derived a methodology to fit $\alpha$, $\beta$ and the general function $g(x)$ was also introduced.

On of the first steps consists on choosing the reference variable, $\Psi$. Any integral rainfall variable can play this role but we must note that it should be selected considering the application for which it is considered. Because of the DSD analysis is usually used to establish $Z$-$R$ relationships rain intensity is commonly used.

#### 4.3.1 Identification of the exponents $\alpha$ and $\beta$

Once the reference variable is selected we start fitting the exponents $\alpha$ and $\beta$, which are not independent, but related through (4.7).

Due to the fact that experimental DSD is discretely measured, relation (4.10) is used to compute the different moments and reference variable for each sampling time recorded during a storm (Figure 4.1).

$$M_n = \sum_p N(D_p)D_p^n \Delta D_p = \sum_p \frac{C_p D_p^n}{v(D_p) \cdot Area_p \cdot \Delta t}$$

(4.10)

where $D_p$ is the mid size of the $p$th bin, $C_p$ is the number of drops collected in the $p$th size in a $\Delta t$ time interval, $Area_p$ is the cross section area [m$^2$], $v(D_p)$ in is the fall speed at a diameter $D_p$ and $\Delta D_p$ is the width of the $p$th channel.
Power functions are fitted over the computed moments and the reference variable as shows Figure 4.2.

Figure 4.1.-a) Experimental one minute spectra, collected in the storm of August 14, 2004, in Wallps Island by an impact type disdrometer. b) One-minute rainfall rate for the same event. c) One-minute moments $M_1$ to $M_6$ calculated from the one minute DSD spectrum.
Figure 4.2.- Scatterplots representing the rain rate (our reference moment; $M_i$ with $i=3.67$) versus the moments $M_1$ to $M_6$ obtained one minute DSD collected in the storm of August 14, 2004, in Wallps Island by an impact type disdrometer.
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As already expressed above (equation (4.4)) from the single normalization model \( M_n = a_n M_i^{\alpha} \). The exponents \( \gamma(n) \) of these power functions follow a linear relation with the order of the moment (see equation (4.5) and Figure 4.3). It makes possible to identify the values of \( \alpha \) and \( \beta \). Considering the self-consistency this regression is constraint to pass over the \([i+1,1]\) (equation (4.7) and Figure 4.3). Therefore, only one of these parameters is required to be fitted, the question is which one should we choose. The first one, \( \alpha \), is the origin ordinate, while \( \beta \) is the slope of the linear relation, usually more stable. This is the reason why \( \beta \) is generally used.

\[
\gamma(n+1) = i = 3.67 \\
\beta = \tan(\delta)
\]

Figure 4.3.- Experimental fit of \( \beta \) according to for the storm of August 14, 2004, in Wallops Island recorded by an impact type disdrometer.

### 4.3.2 Identification of the general function \( g(x) \)

Once \( \beta \) is estimated, \( g(x) \) can be modelled plotting the scaled diameter versus the scaled experimental concentration of drops (Figure 4.4).

- Scaled drop diameter: \( D_p^\gamma(p) = \Psi_t^\beta \)
- Scaled concentration of drops: \( \Psi_t^\alpha \)

Thus, we have \( m \cdot k \) data (being \( m \) the number of the diameter bins of the disdrometer and \( k \) the number of time intervals of the rain event)

According to the shape of the normalized DSD, an analytical model can be selected among different functions and their parameters should be fitted verifying the self-consistency (4.8). Several distribution functions (exponential, gamma, lognormal, Weibull...) have been used in the past as models to describe the DSD.
For example, the exponential and gamma distributions are particular cases of the generalized gamma distribution, which has more flexibility. The generalized gamma that verify the self-consistency could be written as:

\[
g_{GG} = \frac{c \lambda^{\mu+1}}{\Gamma(\mu + \frac{n}{c})} x_1^{\mu-1} \exp\left[-(\lambda x_1)^c\right]
\]  

(4.11)

Exponential DSD model is obtained from (4.11) with \( c = 1 \) and \( \mu = 1 \), and the gamma DSD model is obtained with \( c = 1 \). With this model the multiplicative coefficients of power laws could be obtained as follow.

\[
a_n = \lambda^{\mu-n} \frac{\Gamma(\mu + \frac{n}{c})}{\Gamma(\mu + \frac{1}{c})}
\]  

(4.12)

Figure 4.4.-Scatter plot of the general function for the storm of August 14, 2004, in Wallops Island. The line is the fitted \( g(x) \) for this case (an exponential model has been chosen).

4.4 Interpretation of \( \beta \)

As said, the parameter \( \beta \), is dependent on the chosen reference variable \( \Psi \). However, this parameter is not a universal constant for each selected moment. Some authors (Sempere-Torres et al. 1999, Lee et al. 2004) have shown a dependence of \( \beta \) with the rainfall type. The reason of this dependence is that each rainfall type is related to different processes of drops formation and growth, which determine the final drop size distribution. (Figure 4.5 shows the main processes associated to stratiform and convective rain).
Thus, the physical raindrop growth processes are associated with the scaling parameter $\beta$. Following we give a representation of the modes of DSD growth:

$\beta = 0$ It implies a uniform growth the concentration with no pivoting, which leads to linear relationships between moments of the DSD. This situation is an effect of the equilibrium between coalescence (the merging of drops into a larger drop after collision) and breakup (when a drop achieves a certain size, it becomes unstable and breaks up into smaller drops). This situation is given when deep convective storms.

$0 < \beta < 0.214$ The greater the parameter $\beta$, the higher the pivoting.

In convective phases (lower values of $\beta$) the breakup becomes less important than in the equilibrium, this produces an increase of pivoting of the DSD.

In stratiform phases (higher values of $\beta$) the pivoting is due to the low aggregation (ice particles collect other ice particles) and deposition (growth of an ice particle by diffusion of ambient vapor toward the particles). When increase the riming, the pivoting tends to be combined with a growth in size.

$\beta = 0.214$ The DSD pivots around $D=0$. It is a situation where the growth in size is produced by deposition and the pivoting by aggregation (Marshall-Palmer’s DSD).

$\beta > 0.214$ Dominant aggregation leads to a pivoting distribution, with the pivoting around
a diameter different from zero.

As we have showed $\beta$ describes the growth of the particle sizes, and is dependent on the dominant processes of the rainfall (break-up, coalescence…).

### 4.5 Rewrite previous suggested analytical expressions

The scaling law for the DSD summarized all the previously suggested analytical expressions of DSDs. Following are analyzed some important studies of the normalization of the DSD. The different parameters of the normalization are identified for the different models, which in most of the cases do not verify the self-consistency.

#### 4.5.1 Exponential distribution function

- Marshall and Palmer 1948 proposed the first DSD.
  \[
  N(D,R) = N_0 \cdot \exp[-\Lambda(R)D] \tag{4.13}
  \]
where \( N_0 = 0.08 \text{cm}^{-4} \) and \( \Lambda(R) = 41 \cdot R^{-0.21} \)

Rewriting the previous expression in terms of the scaling normalization parameters we get the following

\[ \Psi = R, \quad \alpha = 0, \quad \beta = 0.21, \]

\[ g(x) = 0.08 \exp(-41x) \]

- Sekhon and Srivastava 1971 proposed the same shape of the DSD with \( N_0 = 0.07R^{0.37} \) and \( \Lambda(R) = 38 \cdot R^{-0.141} \)

The expression could be rewritten with the scaling normalization with the following parameters.

\[ \Psi = R, \quad \alpha = 0.37, \quad \beta = 0.14, \]

\[ g(x) = 0.07 \exp(-38x) \]

### 4.5.2 Gamma distribution function

Willis and Tattelman 1989 proposed

\[ N(D) = N_0 D^{2.16} \exp(-5.59 \frac{D}{D_0}) \]

\[ N_0 = 512.9 \cdot 10^6 W \left( \frac{1}{D_0} \right)^{2.16} \]

\[ D_0 = 0.157W^{0.168} \]

Can be rewritten with

\[ \Psi = D_0, \quad \alpha_{D_0} = 1.95, \quad \beta_{D_0} = 1, \]

\[ g(x) = 31.0 \exp(-5.6x) \]

### 4.5.3 Lognormal distribution function

Feingold and Levin 1986 proposed

\[ N(D) = \frac{N_f}{(2\pi)^{0.5} \ln(\sigma)} \frac{1}{D} \exp \left[ -\ln^2 \left( \frac{D/D_s}{2\ln^2(\sigma)} \right) \right] \]

\[ N_f = 172R^{0.22} \]

\[ D_s = 0.72R^{0.23} \]

With \( \sigma = 1.43 \) the different parameters could be rewritten as

\[ \Psi = R, \quad \alpha = -0.01, \quad \beta = 0.23, \]
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\[ g(x) = 191.85 \exp[-3.91 \cdot \ln^2(1.39x)] \]

4.6 Conclusions

In this chapter the general formulation for DSD formulated by Sempere-Torres et al. 1994 has been presented. This model is a generalization of all previous of DSD models. It is based on proposing a scaling law formulation, which parameters are independent of the values of the reference variable \( \Psi \) that can be any integral variable of the DSD.

The empirical power relations between the integral moments of the DSDs are a consequence of the proposed scaling law.

The exponent \( \beta \) is a descriptor of the growth of the drop size and, thus, it is useful to interpret the physical mechanisms of drop formation.

A methodology to fit the distribution function is presented here. This methodology does not involve setting a priori shape for DSD and the analytical model of the general function is chosen considering the scatter of the normalized DSD.