2. Discontinuous Galerkin methods for elasticity

In this chapter the general weak form for discontinuous Galerkin methods in the elastic case will be derived. Although the final primal form is only valid for elasticity, it will as well be stated which of the obtained equations are valid for the plastic case, in Chapter 4.

2.1. Domain discretization

Before starting formulating the weak form for the discontinuous Galerkin (dG) method the notation used for the domain as well as for the partitions of this is introduced. The domain of interest is denoted as $\Omega \subset \mathbb{R}^n$, figure 2.1, where n is the spatial dimension. We let $\tilde{\Omega}$ be a partition of the interval Ω into n_{el} elements, K_i, such that,

$$\bigcup_{i=1}^{n_{\text{el}}} K_i = \tilde{\Omega}, \tag{2.1}$$

$$K_i \cap K_j = \emptyset. \tag{2.2}$$

Note that K_i is an open set. The interior boundaries or edges between elements are referred to as ∂K^{int} with n_{edg} being the total number of interior edges, $\tilde{\Gamma}$ can be defined as,

$$\tilde{\Gamma} = \bigcup_{i=1}^{n_{\text{edg}}} \partial K_i^{\text{int}}. \tag{2.3}$$

The external boundary of Ω, denoted as Γ, is composed of the union of the exterior boundaries of each of the subparts, ∂K^{ext},

$$\Gamma = \overline{\bigcup_{i=1}^{n_{\text{ext}}} \partial K_i^{\text{ext}}}. \tag{2.4}$$

On Γ, displacements or tractions are prescribed. Dirichlet boundary conditions are prescribed on Γ^D, and Neumann boundary conditions on Γ^N. The partition is defined such that,

$$\overline{\Gamma^D \cup \Gamma^N} = \Gamma, \tag{2.5}$$

$$\Gamma^D \cap \Gamma^N = \emptyset. \tag{2.6}$$

The prescribed displacement is denoted as g^D and the prescribed tractions as g^N, so that $g^D : \Gamma^D \to \mathbb{R}^n$, and $g^N : \Gamma^N \to \mathbb{R}^n$. To guarantee the uniqueness of the solution $\Gamma^D \neq \emptyset$ will be assumed [9].

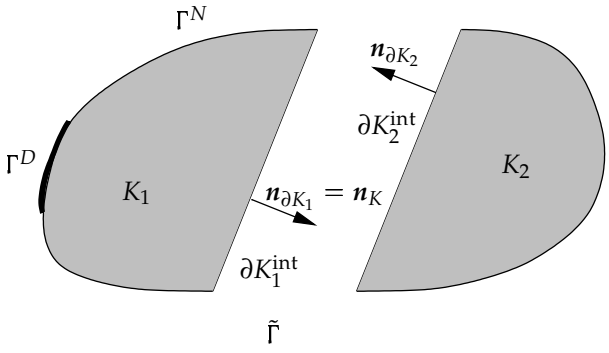

Figure 2.1. Definition of domain Ω.

2.2. Governing form

With the notation introduced above the elasticity problem is rewritten into a first order system as it is usually done in discontinuous Galerkin methods for elliptic problems [10]. So the following boundary value problem is considered: find $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ so that,

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = 0 \qquad \text{in } \Omega, \tag{2.7}$$

$$\boldsymbol{\sigma} = \mathbf{C} : \nabla^s \boldsymbol{u} \qquad \text{in } \Omega, \tag{2.8}$$

$$\boldsymbol{u} = \boldsymbol{g}^D \qquad \text{on } \Gamma^D, \tag{2.9}$$

$$\boldsymbol{\sigma n} = \boldsymbol{g}^N \qquad \text{on } \Gamma^N, \tag{2.10}$$

where $\mathbf{C}$ is a forth-order tensor describing the linear elastic relationship between stress and strain and $\boldsymbol{n}$ is the unit outward normal vector to Γ^N. The displacement field is denoted as $\boldsymbol{u} : \Omega \to \mathbb{R}^n$, the external force applied as $\boldsymbol{f} : \Omega \to \mathbb{R}$ and the stress tensor as $\boldsymbol{\sigma} : \Omega \to \mathbb{S}$, with $\mathbb{S}$ being the space of second-order symmetric tensors.

Note that for plasticity the constitutive equation in (2.8) does no longer hold, instead a relation between stress and strain rate will be used.

2.3. Weak form of the problem

Due to the fact that the element space will consist of discontinuous elements, the trial functions will not have to lie in the Sobolev space $H^1(\Omega)$. It is the restriction to each element K what must lie in the Sobolev space $H^1(K_i)$, [6]. Following [8], the trial functions $\boldsymbol{u}^h$ must now belong to the space $\mathcal{U}^h$ characterized as,

$$\mathcal{U}^h := \left\{ \boldsymbol{u}^h \mid \boldsymbol{u}^h \in \left[L^2(\Omega) \right]^n, \boldsymbol{u}^h \in \left[H^1(\tilde{\Omega}) \right]^n, \boldsymbol{u}^h = \boldsymbol{g}^D \text{ on } \Gamma^D \right\}. \tag{2.11}$$

So $\mathcal{U}^h$ is a collection of functions from $H^1(\tilde{\Omega})$ satisfying Dirichlet boundary conditions. From now on the superscript h will be used to refer to an approximate solution. The space

Σ^h is defined as,

$$\Sigma^h := \left\{ \boldsymbol{\sigma}^h \mid \boldsymbol{\sigma}^h \in \left[L^2\left(\Omega\right) \right]^{n\times n},\ \boldsymbol{\sigma}^h \in \left[H^1\left(\tilde{\Omega}\right) \right]^{n\times n} \right\}. \tag{2.12}$$

The same definition is adopted for the space $\mathcal{W}^h$ of weight functions, as for the continuous case, the value of the weight functions where Dirichlet boundary conditions are found is equal to zero,

$$\mathcal{W}^h := \left\{ \boldsymbol{\omega}^h \mid \boldsymbol{\omega}^h \in \left[L^2\left(\Omega\right) \right]^{n},\ \left[\boldsymbol{\omega}^h \in H^1\left(\tilde{\Omega}\right) \right]^{n},\ \boldsymbol{\omega}^h = \mathbf{0} \text{ on } \Gamma^D \right\}. \tag{2.13}$$

Going back to the two first order governing equations in (2.7) and (2.8), and in order to obtain the weak form, the same procedure followed in the continuous case is considered. Both equations are multiplied by a weight function and integrated over the whole domain. There is however a new complication that does not appear for the continuous case. Now, the domain has been discretized into different subdomains and when applying the integration by parts rule, integrals over the interior edges appear.

Integrating by parts equations (2.7) and (2.8) over one of these subparts K it yields,

$$-\int_K \nabla\boldsymbol{\omega}^h : \boldsymbol{\sigma}^h d\Omega + \int_{\partial K} \boldsymbol{\omega}^h \cdot \boldsymbol{\sigma}^h \boldsymbol{n}_{\partial K} d\Gamma + \int_K \boldsymbol{\omega}^h \cdot \boldsymbol{f} d\Omega = \mathbf{0}, \tag{2.14}$$

and,

$$\int_K \boldsymbol{\tau}^h : \boldsymbol{\sigma}^h d\Omega = -\int_K \left(\nabla^s \cdot \left(\boldsymbol{\tau}^h : \mathbf{C} \right) \right) \cdot \boldsymbol{u}^h d\Omega + \int_{\partial K} \left(\boldsymbol{\tau}^h : \mathbf{C} \right) \boldsymbol{n}_{\partial K} \cdot \boldsymbol{u}^h d\Gamma, \tag{2.15}$$

where $\boldsymbol{\omega}^h \in \mathcal{W}^h$ and $\boldsymbol{\tau}^h \in \Sigma^h$. $\boldsymbol{n}_{\partial K}$ denotes the unit outward normal vector to ∂K^{int}, see figure 2.1.

This weak formulation must now be extended to the domain Ω. Before doing so, some definitions must be introduced in order to cope with the discontinuities between elements. The average operator $\{\bullet\}$ is defined on an interface as, [10],

$$\left\{ \boldsymbol{u}^h \right\} := \frac{1}{2}\left(\boldsymbol{u}_1^h + \boldsymbol{u}_2^h \right) \qquad \text{on } \tilde{\Gamma}, \tag{2.16}$$

$$\left\{ \boldsymbol{u}^h \right\} := \boldsymbol{u}^h \qquad \text{on } \Gamma, \tag{2.17}$$

where the subscript 1 and 2 refer to each of the sides of an interior edge, see figure 2.1. The jump operator $[\![\bullet]\!]$ is defined on an interface as [10],

$$\left[\!\left[\boldsymbol{u}^h \right]\!\right] := \boldsymbol{u}_1^h - \boldsymbol{u}_2^h \qquad \text{on } \tilde{\Gamma}, \tag{2.18}$$

$$\left[\!\left[\boldsymbol{u}^h \right]\!\right] := \boldsymbol{u}^h \qquad \text{on } \Gamma. \tag{2.19}$$

The weak form is now extended to the total domain Ω. To do so, equations (2.14) and (2.15) must be summed over every single element K_i, to obtain,

$$-\int_{\tilde{\Omega}} \nabla\boldsymbol{\omega}^h : \boldsymbol{\sigma}^h d\Omega + \sum_{K\in\Omega} \int_{\partial K} \boldsymbol{\omega}^h \cdot \boldsymbol{\sigma}^h \boldsymbol{n}_{\partial K} d\Gamma + \int_{\tilde{\Omega}} \boldsymbol{\omega}^h \cdot \boldsymbol{f} d\Omega = 0, \tag{2.20}$$

and,

$$\int_{\tilde{\Omega}} \boldsymbol{\tau}^h : \boldsymbol{\sigma}^h d\Omega = -\int_{\tilde{\Omega}} \left(\nabla^s \cdot \left(\boldsymbol{\tau}^h : \mathbf{C}\right)\right) \cdot \boldsymbol{u}^h d\Omega + \sum_{K\in\Omega} \int_{\partial K} \left(\boldsymbol{\tau}^h : \mathbf{C}\right) \boldsymbol{n}_{\partial K} \cdot \boldsymbol{u}^h d\Gamma. \quad (2.21)$$

It can be proved straightforwardly, see Appendix A, that the following expressions hold, see [8],

$$\sum_{K\in\Omega} \int_{\partial K} \boldsymbol{\omega}^h \cdot \boldsymbol{\sigma}^h \boldsymbol{n}_{\partial K} d\Gamma = \int_{\tilde{\Gamma}\cup\Gamma} \left[\!\left[\boldsymbol{\omega}^h\right]\!\right] \cdot \left\{\boldsymbol{\sigma}^h\right\} \boldsymbol{n}_K d\Gamma + \int_{\tilde{\Gamma}} \left\{\boldsymbol{\omega}^h\right\} \cdot \left[\!\left[\boldsymbol{\sigma}^h\right]\!\right] \boldsymbol{n}_K d\Gamma, \quad (2.22)$$

$$\sum_{K\in\Omega} \int_{\partial K} \left(\boldsymbol{\tau}^h : \mathbf{C}\right) \boldsymbol{n}_{\partial K} \cdot \boldsymbol{u}^h d\Gamma = \int_{\tilde{\Gamma}\cup\Gamma} \left\{\boldsymbol{\tau}^h : \mathbf{C}\right\} \boldsymbol{n}_K \cdot \left[\!\left[\boldsymbol{u}^h\right]\!\right] d\Gamma + \int_{\tilde{\Gamma}} \left[\!\left[\boldsymbol{\tau}^h : \mathbf{C}\right]\!\right] \boldsymbol{n}_K \cdot \left\{\boldsymbol{u}^h\right\} d\Gamma, \quad (2.23)$$

where $\boldsymbol{n}_K$, is defined as the unit outward vector such that,

$$\boldsymbol{n}_K = \boldsymbol{n}_{\partial K_1} = -\boldsymbol{n}_{\partial K_2}. \quad (2.24)$$

Using equations (2.22) and (2.23) in equations (2.20),

$$-\int_{\tilde{\Omega}} \nabla \boldsymbol{\omega}^h : \boldsymbol{\sigma}^h d\Omega + \int_{\tilde{\Gamma}\cup\Gamma} \left[\!\left[\boldsymbol{\omega}^h\right]\!\right] \cdot \left\{\boldsymbol{\sigma}^h\right\} \boldsymbol{n}_K d\Gamma + \int_{\tilde{\Gamma}} \left\{\boldsymbol{\omega}^h\right\} \cdot \left[\!\left[\boldsymbol{\sigma}^h\right]\!\right] \boldsymbol{n}_K d\Gamma + \int_{\tilde{\Omega}} \boldsymbol{\omega}^h \cdot \boldsymbol{f} d\Omega = 0, \quad (2.25)$$

$$\int_{\tilde{\Omega}} \boldsymbol{\tau}^h : \boldsymbol{\sigma}^h d\Omega = -\int_{\tilde{\Omega}} \left(\nabla^s \cdot \left(\boldsymbol{\tau}^h : \mathbf{C}\right)\right) \cdot \boldsymbol{u}^h d\Omega + \int_{\tilde{\Gamma}\cup\Gamma} \left\{\boldsymbol{\tau}^h : \mathbf{C}\right\} \boldsymbol{n}_K \cdot \left[\!\left[\boldsymbol{u}^h\right]\!\right] d\Gamma + \int_{\tilde{\Gamma}} \left[\!\left[\boldsymbol{\tau}^h : \mathbf{C}\right]\!\right] \boldsymbol{n}_K \cdot \left\{\boldsymbol{u}^h\right\} d\Gamma. \quad (2.26)$$

The terms $\boldsymbol{\sigma}^h$ and $\boldsymbol{u}^h$ in the integrals over the boundaries $\tilde{\Gamma}$ and Γ of equations (2.25) and (2.26) are now substituted by $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{u}}$, which are known as numerical fluxes. The choice of these numerical fluxes is one of the most important steps of the dG methods, as it can affect the consistency, stability, symmetry and accuracy of the method. Following the replacements the two equations of the weak form in (2.25) and (2.26) are given by,

$$-\int_{\tilde{\Omega}} \nabla \boldsymbol{\omega}^h : \boldsymbol{\sigma}^h d\Omega + \int_{\tilde{\Gamma}\cup\Gamma} \left[\!\left[\boldsymbol{\omega}^h\right]\!\right] \cdot \{\hat{\boldsymbol{\sigma}}\} \boldsymbol{n}_K d\Gamma + \int_{\tilde{\Gamma}} \left\{\boldsymbol{\omega}^h\right\} \cdot [\![\hat{\boldsymbol{\sigma}}]\!] \boldsymbol{n}_K d\Gamma + \int_{\tilde{\Omega}} \boldsymbol{\omega}^h \cdot \boldsymbol{f} d\Omega = 0, \quad (2.27)$$

and,

$$\int_{\tilde{\Omega}} \boldsymbol{\tau}^h : \boldsymbol{\sigma}^h d\Omega = -\int_{\tilde{\Omega}} \left(\nabla^s \cdot \left(\boldsymbol{\tau}^h : \mathbf{C}\right)\right) \cdot \boldsymbol{u}^h d\Omega + \int_{\tilde{\Gamma}\cup\Gamma} \left\{\boldsymbol{\tau}^h : \mathbf{C}\right\} \boldsymbol{n}_K \cdot [\![\hat{\boldsymbol{u}}]\!] d\Gamma + \int_{\tilde{\Gamma}} \left[\!\left[\boldsymbol{\tau}^h : \mathbf{C}\right]\!\right] \boldsymbol{n}_K \cdot \{\hat{\boldsymbol{u}}\} d\Gamma. \quad (2.28)$$

Note that equation (2.27) follows from the local equilibrium equation in (2.7). The constitutive condition, in equation (2.8), has not been considered in the derivation of this equation, therefore expression (2.27) will hold as well in the plastic case. Equation (2.28) has been derived from (2.8) and will not hold for the plastic case. Instead, as will be seen in Chapter 4, a similar expression relating strain and stress rates will be obtained.

Defining $\boldsymbol{\tau}^h$ to be equal to $\nabla^s \boldsymbol{\omega}^h$ and introducing this into expression (2.28),

$$\int_{\tilde{\Omega}} \nabla^s \boldsymbol{\omega}^h : \boldsymbol{\sigma}^h d\Omega = -\int_{\tilde{\Omega}} \left(\nabla^s \cdot \left(\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right)\right) \cdot \boldsymbol{u}^h d\Omega + \int_{\tilde{\Gamma}\cup\Gamma} \left\{\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right\} \boldsymbol{n}_K \cdot [\![\hat{\boldsymbol{u}}]\!]\, d\Gamma + \int_{\tilde{\Gamma}} \left[\!\left[\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right]\!\right] \boldsymbol{n}_K \cdot \{\hat{\boldsymbol{u}}\}\, d\Gamma. \quad (2.29)$$

The same substitution is done with equation (2.23), obtaining,

$$\sum_{K\in\Omega} \int_{\partial K} \left(\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right) \boldsymbol{n}_{\partial K} \cdot \boldsymbol{u}^h d\Gamma = \int_{\tilde{\Gamma}\cup\Gamma} \left\{\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right\} \boldsymbol{n}_K \cdot \left[\!\left[\boldsymbol{u}^h\right]\!\right] d\Gamma + \int_{\tilde{\Gamma}} \left[\!\left[\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right]\!\right] \boldsymbol{n}_K \cdot \left\{\boldsymbol{u}^h\right\} d\Gamma. \quad (2.30)$$

Integrating by parts the second term of equation (2.29) and using the result in (2.30) it yields,

$$\begin{aligned}\int_{\tilde{\Omega}} \nabla^s \boldsymbol{\omega}^h : \boldsymbol{\sigma}^h d\Omega = {} & \int_{\tilde{\Omega}} \nabla^s \boldsymbol{\omega}^h : \mathbf{C} : \nabla^s \boldsymbol{u}^h d\Omega - \int_{\tilde{\Gamma}\cup\Gamma} \left\{\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right\} \boldsymbol{n}_K \cdot \left[\!\left[\boldsymbol{u}^h\right]\!\right] d\Gamma \\ & - \int_{\tilde{\Gamma}} \left[\!\left[\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right]\!\right] \boldsymbol{n}_K \cdot \left\{\boldsymbol{u}^h\right\} d\Gamma + \int_{\tilde{\Gamma}\cup\Gamma} \left\{\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right\} \boldsymbol{n}_K \cdot [\![\hat{\boldsymbol{u}}]\!]\, d\Gamma \\ & + \int_{\tilde{\Gamma}} \left[\!\left[\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right]\!\right] \boldsymbol{n}_K \cdot \{\hat{\boldsymbol{u}}\}\, d\Gamma. \quad (2.31)\end{aligned}$$

This equation will be important in Chapter 3, since it will be used to obtain the expression of the stresses over $\tilde{\Omega}$.

By adding expressions (2.27) and (2.31) and making use of the linear nature of the jump and average operators,

$$[\![\hat{\boldsymbol{u}}]\!] - \left[\!\left[\boldsymbol{u}^h\right]\!\right] = \left[\!\left[\hat{\boldsymbol{u}} - \boldsymbol{u}^h\right]\!\right], \qquad \{\hat{\boldsymbol{u}}\} - \left\{\boldsymbol{u}^h\right\} = \left\{\hat{\boldsymbol{u}} - \boldsymbol{u}^h\right\}, \quad (2.32)$$

and knowing that for a symmetric tensor, $\boldsymbol{\sigma}^h$, it holds [11],

$$\nabla^s \boldsymbol{\omega}^h : \boldsymbol{\sigma}^h = \nabla \boldsymbol{\omega}^h : \boldsymbol{\sigma}^h, \quad (2.33)$$

the weak form for the discontinuous formulation is obtained, reading find $\boldsymbol{u}^h \in \mathcal{U}^h$ such that,

$$\begin{aligned}\int_{\tilde{\Omega}} \nabla^s \boldsymbol{\omega}^h : \mathbf{C} : \nabla^s \boldsymbol{u}^h d\Omega + {} & \int_{\tilde{\Gamma}\cup\Gamma} \left(\left\{\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right\} \boldsymbol{n}_K \cdot \left[\!\left[\hat{\boldsymbol{u}} - \boldsymbol{u}^h\right]\!\right] - \left[\!\left[\boldsymbol{\omega}^h\right]\!\right] \cdot \{\hat{\boldsymbol{\sigma}}\}\, \boldsymbol{n}_K\right) d\Gamma \\ & + \int_{\tilde{\Gamma}} \left(\left[\!\left[\nabla^s \boldsymbol{\omega}^h : \mathbf{C}\right]\!\right] \boldsymbol{n}_K \cdot \left\{\hat{\boldsymbol{u}} - \boldsymbol{u}^h\right\} - \left\{\boldsymbol{\omega}^h\right\} \cdot [\![\hat{\boldsymbol{\sigma}}]\!]\, \boldsymbol{n}_K\right) d\Gamma = \int_{\tilde{\Omega}} \boldsymbol{\omega}^h \cdot \boldsymbol{f} d\Omega, \quad (2.34)\end{aligned}$$

for all $\boldsymbol{\omega}^h \in \mathcal{W}^h$. In compact notation, the above equation will be expressed as,

$$B_h(\boldsymbol{\omega}^h, \boldsymbol{u}^h) = L_h(\boldsymbol{w}^h). \tag{2.35}$$

The obtained expression (2.34) is only valid for elasticity since equation (2.31) has been used in its derivation.

2.4. Consistency of the discontinuous Galerkin methods

The method is consistent if the approximate solution $\boldsymbol{u}^h$ can be substituted by the exact solution $\boldsymbol{u}$ in the weak form [12]. Doing so, the general weak form for dG methods in equation (2.34) yields,

$$\begin{aligned}\int_{\tilde{\Omega}} \nabla^s \boldsymbol{\omega}^h : \mathbf{C} : \nabla^s \boldsymbol{u}\, d\Omega + \int_{\tilde{\Gamma}\cup\Gamma} \left(\left\{ \nabla^s \boldsymbol{\omega}^h : \mathbf{C} \right\} \boldsymbol{n}_K \cdot [\![\hat{\boldsymbol{u}} - \boldsymbol{u}]\!] - \left[\!\left[\boldsymbol{\omega}^h \right]\!\right] \cdot \{\hat{\boldsymbol{\sigma}}\}\, \boldsymbol{n}_K \right) d\Gamma \\ + \int_{\tilde{\Gamma}} \left(\left[\!\left[\nabla^s \boldsymbol{\omega}^h : \mathbf{C} \right]\!\right] \boldsymbol{n}_K \cdot \{\hat{\boldsymbol{u}} - \boldsymbol{u}\} - \left\{ \boldsymbol{\omega}^h \right\} \cdot [\![\hat{\boldsymbol{\sigma}}]\!]\, \boldsymbol{n}_K \right) d\Gamma = \int_{\tilde{\Omega}} \boldsymbol{\omega}^h \cdot \boldsymbol{f}\, d\Omega. \end{aligned} \tag{2.36}$$

For consistency of the numerical fluxes, it is required that given the exact solution, $\boldsymbol{u}$, $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{\sigma}}$ must be equal to $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ respectively, [8]. The numerical flux will therefore be consistent whenever,

$$\hat{\boldsymbol{u}}(\boldsymbol{u}) = \boldsymbol{u} \qquad \text{on } \tilde{\Gamma} \cup \Gamma, \tag{2.37}$$

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{u}) = \boldsymbol{\sigma} = \mathbf{C} : \nabla^s \boldsymbol{u} \qquad \text{on } \tilde{\Gamma} \cup \Gamma. \tag{2.38}$$

Note that for consistency the following is as well required,

$$\hat{\boldsymbol{u}}(\boldsymbol{u}) = \boldsymbol{g}^D \qquad \text{on } \Gamma^D, \tag{2.39}$$

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{u})\boldsymbol{n}_K = \boldsymbol{g}^N \qquad \text{on } \Gamma^N. \tag{2.40}$$

Since by construction the basis and $\hat{\boldsymbol{u}}$ satisfy Dirichlet boundary conditions, for a consistent numerical flux it holds that $\hat{\boldsymbol{u}} - \boldsymbol{u} = \mathbf{0}$ on Γ^D. The same happens for consistent numerical fluxes on $\tilde{\Gamma}$ and Γ^N, see equation (2.37), where $\hat{\boldsymbol{u}} - \boldsymbol{u} = \mathbf{0}$, so it follows,

$$\int_{\tilde{\Gamma}\cup\Gamma} \left\{ \nabla^s \boldsymbol{\omega}^h : \mathbf{C} \right\} \boldsymbol{n}_K \cdot [\![\hat{\boldsymbol{u}} - \boldsymbol{u}]\!]\, d\Gamma = \mathbf{0}. \tag{2.41}$$

Since on Γ^D $\boldsymbol{\omega}^h = 0$,

$$\int_{\Gamma^D} \left[\!\left[\boldsymbol{\omega}^h \right]\!\right] \cdot \{\hat{\boldsymbol{\sigma}}\}\, \boldsymbol{n}_K d\Gamma = 0 \qquad \text{on } \Gamma^D, \tag{2.42}$$

and considering equation (2.40) on Γ^N and the definitions of the average and jump operators on Γ in equations (2.17) and (2.19),

$$\int_{\Gamma^N} \left[\!\left[\boldsymbol{\omega}^h \right]\!\right] \cdot \{\hat{\boldsymbol{\sigma}}\}\, \boldsymbol{n}_K d\Gamma = \int_{\Gamma^N} \boldsymbol{\omega}^h \cdot \boldsymbol{g}^N d\Gamma \qquad \text{on } \Gamma^N. \tag{2.43}$$

So finally, with consistent numerical fluxes equation (2.36), can be reformulated as,

$$\int_{\tilde{\Omega}} \nabla^s \boldsymbol{\omega}^h : \mathbf{C} : \nabla^s \boldsymbol{u}\, d\Omega - \int_{\tilde{\Gamma}} \left[\!\left[\boldsymbol{\omega}^h \right]\!\right] \cdot \{\mathbf{C} : \nabla^s \boldsymbol{u}\}\, \boldsymbol{n}_K d\Gamma - \int_{\tilde{\Gamma}} \left\{ \boldsymbol{\omega}^h \right\} \cdot [\![\mathbf{C} : \nabla^s \boldsymbol{u}]\!]\, \boldsymbol{n}_K d\Gamma = \int_{\tilde{\Omega}} \boldsymbol{\omega}^h \cdot \boldsymbol{f} d\Omega + \int_{\Gamma^N} \boldsymbol{\omega}^h \cdot \boldsymbol{g}^N d\Gamma. \quad (2.44)$$

Now integrating by parts the first term of equation (2.44),

$$\begin{aligned} &- \int_{\tilde{\Omega}} \boldsymbol{\omega}^h \cdot (\nabla^s \cdot (\mathbf{C} : \nabla^s \boldsymbol{u}))\, d\Omega + \int_{\tilde{\Gamma} \cup \Gamma} \left[\!\left[\boldsymbol{\omega}^h \right]\!\right] \cdot \{\mathbf{C} : \nabla^s \boldsymbol{u}\}\, \boldsymbol{n}_K d\Gamma + \int_{\tilde{\Gamma}} \left\{ \boldsymbol{\omega}^h \right\} \cdot [\![\mathbf{C} : \nabla^s \boldsymbol{u}]\!]\, \boldsymbol{n}_K d\Gamma \\ &- \int_{\tilde{\Gamma}} \left[\!\left[\boldsymbol{\omega}^h \right]\!\right] \cdot \{\mathbf{C} : \nabla^s \boldsymbol{u}\}\, \boldsymbol{n}_K d\Gamma - \int_{\tilde{\Gamma}} \left\{ \boldsymbol{\omega}^h \right\} \cdot [\![\mathbf{C} : \nabla^s \boldsymbol{u}]\!]\, \boldsymbol{n}_K d\Gamma = \int_{\tilde{\Omega}} \boldsymbol{\omega}^h \cdot \boldsymbol{f} d\Omega + \int_{\Gamma^N} \boldsymbol{\omega}^h \cdot \boldsymbol{g}^N d\Gamma, \end{aligned} \quad (2.45)$$

and rearranging, it yields,

$$- \int_{\tilde{\Omega}} \boldsymbol{\omega}^h \cdot (\nabla^s \cdot (\mathbf{C} : \nabla^s \boldsymbol{u}))\, d\Omega + \int_{\Gamma^N} \boldsymbol{\omega}^h \cdot \boldsymbol{\sigma} \boldsymbol{n}_K = \int_{\tilde{\Omega}} \boldsymbol{\omega}^h \cdot \boldsymbol{f} d\Omega + \int_{\Gamma^N} \boldsymbol{\omega}^h \cdot \boldsymbol{g}^N d\Gamma, \quad (2.46)$$

which proves consistency of the formulation.

As stated in [8], it can be concluded that discontinuous Galerkin method will be consistent if the used numerical fluxes are consistent. In other words, the consistency of the numerical fluxes guarantees the consistency of the method.

2.5. Stability of the discontinuous Galerkin methods

Stability is related to the uniqueness of the solution. Consistency as well as stability are required for convergence. The stability condition, also known as coercivity condition, can be defined as, see [8],

$$B_h \left(\boldsymbol{\omega}^h, \boldsymbol{\omega}^h \right) \geq \alpha \left\| \boldsymbol{\omega}^h \right\|^2, \quad (2.47)$$

for any value of $\boldsymbol{\omega}^h \in \mathcal{W}^h$, being α a positive constant, and $\left\| \boldsymbol{\omega}^h \right\|$ an associated a norm. It is not the intention of this section to derive the stability condition, instead, the coercivity condition will tried to be related to the concept of strain energy.

The strain energy is defined as the work done by the internal forces as a result of its deformation under external loads. It can be determined for the elastic case as,

$$U = \frac{1}{2} B_h \left(\boldsymbol{\omega}^h, \boldsymbol{\omega}^h \right). \quad (2.48)$$

From the principle of energy conservation this energy is equal to the work done by the external loads, assuming no other energy transfer occurred. The strain energy is therefore required to be positive, that is, the stiffness matrix associated to the problem must be positive definite.