D. Computational plasticity

Once the basis for plasticity have been established, it is necessary to determine the required steps to define the integration or return mapping algorithm. As has been explained in chapter 4, first a trial displacement increment is used, step one of sections 4.1.3 and 4.2.3, from which an updated strain is obtained, $\boldsymbol{\varepsilon}^{(k+1)}$. The question is then, how to calculate from there the stresses corresponding to the updated strains.

First, it is assumed that the given trial strain increment is purely elastic. The new trial stress is completely characterized in terms of the initial value of the plastic deformations, $\boldsymbol{\varepsilon}^{p\ (k)}$, and the updated trial strain, $\boldsymbol{\varepsilon}^{(k+1)}$,

$$\boldsymbol{\sigma}_{\text{trial}}^{(k+1)} = \mathbf{C} : \left(\boldsymbol{\varepsilon}^{(k+1)} - \boldsymbol{\varepsilon}^{p\ (k)}\right). \tag{D.1}$$

With the calculated trial stress $\boldsymbol{\sigma}_{\text{trial}}^{(k+1)}$ the value of,

$$f_{\text{trial}}^{(k+1)} = \left\|\boldsymbol{\sigma}_{\text{trial}}^{(k+1)} + q_{\text{kin}}^{(k)}\right\| + \boldsymbol{q}_{\text{iso}}^{(k)} - \sigma_y, \tag{D.2}$$

can be obtained.

Note that elastic behavior has been assumed so if these trial stresses are found to lie within the established limits of the yield surface, $f_{\text{trial}}^{(k+1)} \leq 0$, the starting hypothesis of elastic behavior is correct and there is no need for correction,

$$\boldsymbol{\sigma}^{(k+1)} = \boldsymbol{\sigma}_{\text{trial}}^{(k+1)}, \tag{D.3}$$

$$\boldsymbol{\varepsilon}^{p\ (k+1)} = \boldsymbol{\varepsilon}^{p\ (k)}, \tag{D.4}$$

$$\boldsymbol{q}^{(k+1)} = \boldsymbol{q}^{(k)}. \tag{D.5}$$

However, if the stresses happen to lie outside the yield surface, $f_{\text{trial}}^{(k+1)} > 0$, unless a procedure is addopted to return the stresses to the yield surface, error will accumulate. The stresses need to be mapped back to the yield function. The procedure to be followed for the one dimension and two dimensions plane-strain case will be explained in section D.1 and D.2 respectively.

D.1. One dimension case

The main objective is to find the values of $\varepsilon^{p\ (k+1)}$, $\alpha_{\text{iso}}^{(k+1)}$, $q_{\text{kin}}^{(k+1)}$ which satisfy the condition $f^{(k+1)} = 0$. To start with, the final stress $\sigma^{(k+1)}$ will be expressed in terms of $\sigma_{\text{trial}}^{(k+1)}$ and $\Delta\dot{\lambda}$.

$$\begin{aligned}\sigma^{(k+1)} = E\left(\varepsilon^{(k+1)} - \varepsilon^{p\ (k+1)}\right) &= E\left(\varepsilon^{(k+1)} - \varepsilon_n^p\right) - E\left(\varepsilon^{p\ (k+1)} - \varepsilon^{p\ (k)}\right)\\ &= \sigma_{\text{trial}}^{(k+1)} - E\,\Delta\dot{\lambda}\ \text{sign}\left(\sigma^{(k+1)} + q_{\text{kin}}^{(k+1)}\right).\end{aligned} \tag{D.6}$$

In order to express $\varepsilon^{p\ (k+1)}$ as a function of $\varepsilon^{p\ (k)}$, an implicit Euler integration scheme will be used, as done in [1],

$$\varepsilon^{p\ (k+1)} = \varepsilon^{p\ (k)} + \dot{\varepsilon}^{p\ (k+1)} = \varepsilon^{p\ (k)} + \Delta\dot{\lambda}\ \text{sign}\left(\sigma^{(k+1)} + q_{\text{kin}}^{(k+1)}\right). \tag{D.7}$$

The same can be done with $\alpha_{\text{iso}}^{(k+1)}$ and $q_{\text{kin}}^{(k+1)}$ to obtain,

$$\begin{aligned} f^{(k+1)} &= |\xi^{(k+1)}| - \left[\sigma_y + H_{\text{iso}}\,\alpha_{\text{iso}}^{(k+1)}\right] = 0, && \text{(D.8)} \\ \sigma^{(k+1)} &= \sigma_{\text{trial}}^{(k+1)} - E\,\Delta\dot{\lambda}\ \text{sign}\left(\xi^{(k+1)}\right), && \text{(D.9)} \\ \varepsilon^{p\ (k+1)} &= \varepsilon^{p\ (k)} + \Delta\dot{\lambda}\ \text{sign}\left(\xi^{(k+1)}\right) && \text{(D.10)} \\ \alpha_{\text{iso}}^{(k+1)} &= \alpha_{\text{iso}}^{(k)} + \Delta\dot{\lambda}, && \text{(D.11)} \\ q_{\text{kin}}^{(k+1)} &= q_{\text{kin}}^{(k)} - H_{\text{kin}}\,\Delta\dot{\lambda}\ \text{sign}\left(\xi^{(k+1)}\right), && \text{(D.12)} \end{aligned}$$

where $\xi^{(k+1)} = \sigma^{(k+1)} + q_{\text{kin}}^{(k+1)}$. As done in [1] equation (D.12) can be subtracted from equation (D.9) to obtain,

$$\xi^{(k+1)} = \left(\sigma_{\text{trial}}^{(k+1)} + q_{\text{kin}}^{(k)}\right) - \Delta\dot{\lambda}\,(E + H_{\text{kin}})\ \text{sign}\left(\xi^{(k+1)}\right), \tag{D.13}$$

and using the introduced notation, $\xi_{\text{trial}}^{(k+1)} = \sigma_{\text{trial}}^{(k+1)} + q_{\text{kin}}^{(k)}$, it yields,

$$\xi^{(k+1)} = \xi_{\text{trial}}^{(k+1)} - \Delta\dot{\lambda}\,(E + H_{\text{kin}})\ \text{sign}\left(\xi^{(k+1)}\right), \tag{D.14}$$

which rearranging,

$$\left[\left|\xi^{(k+1)}\right| + \Delta\dot{\lambda}\,(E + H_{\text{kin}})\right]\ \text{sign}\left(\xi^{(k+1)}\right) = \left|\xi_{\text{trial}}^{(k+1)}\right|\ \text{sign}\left(\xi_{\text{trial}}^{(k+1)}\right). \tag{D.15}$$

Since $E > 0$, $\Delta\dot{\lambda} \geq 0$ and $H_{\text{kin}} \geq 0$, from equation (D.15)

$$\begin{aligned} \text{sign}\left(\xi^{(k+1)}\right) &= \text{sign}\left(\xi_{\text{trial}}^{(k+1)}\right), && \text{(D.16)} \\ \left[\left|\xi^{(k+1)}\right| + \Delta\dot{\lambda}\,(E + H_{\text{kin}})\right] &= \left|\xi_{\text{trial}}^{(k+1)}\right|. && \text{(D.17)} \end{aligned}$$

Now using these last two expression into equation (D.8),

$$\begin{aligned} f^{(k+1)} &= \left|\xi_{\text{trial}}^{(k+1)}\right| - \Delta\dot{\lambda}\,(E + H_{\text{kin}}) - \left[\sigma_y + H_{\text{iso}}\,\alpha_{\text{iso}}^{(k+1)}\right] \\ &= \underbrace{\left|\xi_{\text{trial}}^{(k+1)}\right| - \left(\sigma_y + H_{\text{iso}}\,\alpha_{\text{iso}}^{(k)}\right)}_{f_{\text{trial}}^{(k+1)}} - \Delta\dot{\lambda}\,(E + H_{\text{kin}}) - H_{\text{iso}}\underbrace{\left(\alpha_{\text{iso}}^{(k+1)} - \alpha_{\text{iso}}^{(k)}\right)}_{\Delta\dot{\lambda}} = 0, \end{aligned} \tag{D.18}$$

and finally rearranging,

$$\Delta\dot{\lambda} = \frac{f_{\text{trial}}^{(k+1)}}{E + H_{\text{iso}} + H_{\text{kin}}}. \tag{D.19}$$

Using equations (D.16), (D.17) and (D.19), equations (D.8)-(D.12) can be solved, and the values of $\varepsilon^{p\ (k+1)}$, $\alpha_{\text{iso}}^{(k+1)}$ and $q_{\text{kin}}^{(k+1)}$ calculated.

It is clear that the tangent elastic modulus will be equal to E. However to calculate the plastic tangent modulus the general form of the plastic tangent in equation (C.20) is considered. It has been said that the tensor $\mathbf{C}$ reduces to E, so for one dimensional plasticity equation (C.20) turns into,

$$\mathbf{C}^{\text{ep}} = E - \frac{E\, \partial_\sigma f\; E\, \partial_\sigma f}{\partial_\sigma f\; E\, \partial_\sigma f + \partial_{\boldsymbol{q}} f\; \mathbf{H}\, \partial_{\boldsymbol{q}} f}, \tag{D.20}$$

where from equations (C.32) and (C.33),

$$\partial_\sigma f = \operatorname{sign}(\sigma + q_{\text{kin}}), \tag{D.21}$$

$$\partial_{\boldsymbol{q}} f = \begin{bmatrix} 1 & \operatorname{sign}(\sigma + q_{\text{kin}}) \end{bmatrix}^T, \tag{D.22}$$

so finally,

$$\mathbf{C}^{\text{ep}} = E - \frac{E^2}{E + H_{\text{iso}} + H_{\text{kin}}} = \frac{E(H_{\text{kin}} + H_{\text{iso}})}{E + H_{\text{iso}} + H_{\text{kin}}}. \tag{D.23}$$

A summary of the most important expressions for the one dimension return mapping algorithm is given in Box D.1.

D.2. Two dimension plane strain case using von Mises yield criterion

The procedure to follow is basically the same as in one dimension. The problem is solved again by applying an implicit backward difference scheme. A Forward Euler scheme at the Gauss point level would lead to an incorrect drift from the yield surface, leading to a violation of the the yield criterion [15],

$$f^{(k+1)} = \sqrt{3J_2^{(k+1)}} + q_{\text{iso}}^{(k)} - \sigma_y = 0, \tag{D.24}$$

$$\boldsymbol{\sigma}^{(k+1)} = \boldsymbol{\sigma}_{\text{trial}}^{(k+1)} - \Delta\dot{\lambda}\, \mathbf{C} : \partial_{\boldsymbol{\sigma}} f^{(k+1)}, \tag{D.25}$$

$$\boldsymbol{\varepsilon}^{p\ (k+1)} = \boldsymbol{\varepsilon}^{p\ (k)} + \Delta\dot{\lambda}\, \partial_{\boldsymbol{\sigma}} f^{(k+1)}, \tag{D.26}$$

$$\boldsymbol{q}^{(k+1)} = \boldsymbol{q}^{(k)} - \Delta\dot{\lambda}\, \mathbf{H}\, \partial_{\boldsymbol{q}} f^{(k+1)}. \tag{D.27}$$

where as seen in Box C.2, [16],

$$\partial_{\boldsymbol{\sigma}} f = \sqrt{\frac{3}{4}} J_2^{-1/2} \begin{bmatrix} s_{\text{kin}\ xx} & s_{\text{kin}\ yy} & s_{\text{kin}\ zz} & 2s_{\text{kin}\ xy} \end{bmatrix}^T, \tag{D.28}$$

$$\partial_{\boldsymbol{q}} f = \begin{bmatrix} 1 & s_{\text{kin}\ xx} & s_{\text{kin}\ yy} & s_{\text{kin}\ zz} & 2s_{\text{kin}\ xy} \end{bmatrix}^T. \tag{D.29}$$

The Newton-Raphson procedure used to solve equations (D.24)-(D.27) is called the closest point projection. The index k indicates the iteration number in the global incremental procedure and will be omitted for simplicity. The index t will express the iteration number in the Newton-Raphson procedure, within every iteration k.

So for every iteration k the following four steps will be followed repetively to solve equations (D.24)-(D.27).

1. Initial data for each quadrature point,
   $\varepsilon^{(k+1)},\ \varepsilon^{p\ (k)},\ \alpha_{\text{iso}}^{(k)},\ q_{\text{kin}}^{(k)}.$

2. Compute trial stresses,
   $\sigma_{\text{trial}}^{(k+1)} = E\left(\varepsilon^{(k+1)} - \varepsilon^{p\ (k)}\right).$

3. Test plastic loading,
   $\xi_{\text{trial}}^{(k+1)} = \sigma_{\text{trial}}^{(k+1)} + q_{\text{kin}}^{(k)},$
   $f_{\text{trial}}^{(k+1)} = |\xi_{\text{trial}}^{(k+1)}| - \left[\sigma_y + H_{\text{iso}}\, \alpha_{\text{iso}}^{(k)}\right].$

4. If $f_{\text{trial}}^{(k+1)} \leq 0$, elastic range,
   4.1 Update stresses,
   $\sigma^{(k+1)} = \sigma_{\text{trial}}^{(k+1)}.$

   4.2 Tangent modulus,
   $D_{n+1} = E.$

5. If $f_{\text{trial}}^{(k+1)} > 0$, plastic range.
   5.1 Calculate plastic multiplier increment,
   $$\Delta\dot{\lambda} = \frac{f_{\text{trial}}^{(k+1)}}{E + H_{\text{iso}} + H_{\text{kin}}}.$$

   5.2 Update stresses,
   $\sigma^{(k+1)} = \sigma_{\text{trial}}^{(k+1)} - \Delta\dot{\lambda}\, E \,\text{sign}\left(\xi_{\text{trial}}^{(k+1)}\right).$

   5.3 Update plastic strain,
   $\varepsilon^{p\ (k+1)} = \varepsilon_n^p + \Delta\dot{\lambda}\,\text{sign}\left(\xi_{\text{trial}}^{(k+1)}\right).$

   5.4 Update hardening parameter,
   $\alpha_{\text{iso}}^{(k+1)} = \alpha_{\text{iso}}^{(k)} + \Delta\dot{\lambda}.$

   5.5 Update back stress,
   $q_{\text{kin}}^{(k+1)} = q_{\text{kin}}^{(k)} - \Delta\dot{\lambda}\, H_{\text{kin}}\,\text{sign}\left(\xi_{\text{trial}}^{(k+1)}\right).$

   5.6 Tangent modulus,
   $$D_{n+1} = \frac{E\,[H_{\text{iso}} + H_{\text{kin}}]}{E + H_{\text{iso}} + H_{\text{kin}}}.$$

Box D.1: Mixed kinematic and isotropic rate-independent return mapping algorithm for each quadrature point.

1. Define the residual stresses,

$$\mathbf{R}_{\sigma}^{(t)} = \boldsymbol{\sigma}^{(t)} - \boldsymbol{\sigma}_{\text{trial}}^{(t)} + \dot{\lambda}^{(t)}\,\mathbf{C} : \partial_{\sigma} f^{(t)}, \tag{D.30}$$

$$\mathbf{R}_{f}^{(t)} = f(\boldsymbol{\sigma}^{(t)}, \boldsymbol{q}^{(t)}). \tag{D.31}$$

These are residuals and must be driven to zero using the Newton Raphson procedure.

2. In order to solve the above equations, linearize these about the current state,

$$\mathbf{R}_{\sigma}^{(t)} + \left[\mathbf{Q}^{(t)}\right]^{-1} : \Delta\boldsymbol{\sigma}^{(t)} + \Delta\dot{\lambda}^{(t)}\,\mathbf{C} : \partial_{\sigma} f^{(t)} = 0, \tag{D.32}$$

$$\mathbf{R}_{f}^{(t)} + \partial_{\sigma} f^{(t)} : \Delta\boldsymbol{\sigma}^{(t)} - \dot{\lambda}^{(t)}\,\partial_{q} f^{(t)} : \mathbf{H} : \partial_{q} f^{(t)} = 0, \tag{D.33}$$

where $\mathbf{Q}^{(t)} = \left[\mathbf{I} + \dot{\lambda}^{(t)}\mathbf{C} : \partial^2_{\sigma\sigma} f^{(t)}\right]^{-1}$. For the von Mises yield criterion $\partial^2_{\sigma\sigma} f =$ is equal to, [16],

$$\partial^2_{\sigma\sigma} f = \sqrt{\frac{3}{4}} J_2^{-1/2} \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} + \frac{\sqrt{3}}{4} J_2^{-3/2} \boldsymbol{a}^T \boldsymbol{a}, \tag{D.34}$$

being,

$$\boldsymbol{a} = \begin{bmatrix} s_{\text{kin } xx} & s_{\text{kin } yy} & s_{\text{kin } zz} & 2s_{\text{kin } xy} \end{bmatrix}^T. \tag{D.35}$$

3. Solve the linearized problem to obtain $\Delta\dot{\lambda}^{(t)}$ and $\Delta\boldsymbol{\sigma}^{(t)}$,

$$\Delta\dot{\lambda}^{(t)} = \frac{\mathbf{R}_{f}^{(t)} - \mathbf{R}_{\sigma}^{(t)} : \mathbf{Q}^{(t)} : \partial_{\sigma} f^{(t)}}{\partial_{\sigma} f^{(t)} : \boldsymbol{\Xi}^{(t)} : \partial_{\sigma} f^{(t)} + \partial_{q} f^{(t)} : \mathbf{H} : \partial_{q} f^{(t)}}, \tag{D.36}$$

$$\Delta\boldsymbol{\sigma}^{(t)} = \left[-\Delta\dot{\lambda}^{(t)}\,\mathbf{C} : \partial_{\sigma} f^{(t)} - \mathbf{R}_{\sigma}^{(t)}\right] : \mathbf{Q}^{(t)}, \tag{D.37}$$

where $\boldsymbol{\Xi} = \mathbf{Q} : \mathbf{C}$.

4. Update the plastic multiplier, the stress and the plastic strain,

$$\dot{\lambda}^{(t+1)} = \dot{\lambda}^{(t)} + \Delta\dot{\lambda}^{(t)}, \tag{D.38}$$

$$\boldsymbol{\sigma}^{(t+1)} = \boldsymbol{\sigma}^{(t)} + \Delta\boldsymbol{\sigma}^{(t)}, \tag{D.39}$$

$$\boldsymbol{\varepsilon}^{p\,(t+1)} = \boldsymbol{\varepsilon}^{p\,(t)} + \Delta\dot{\lambda}^{(t)}\,\partial_{\sigma} f^{(t)}. \tag{D.40}$$

The name closest point projection comes from the fact that for a convex and associative yield surface the final computed stresses, $\boldsymbol{\sigma}^{(k+1)}$, correspond to the point that minimizes the distance from the trial stresses $\boldsymbol{\sigma}_{\text{trial}}^{(k+1)}$.

D.2.1. Consistent tangent modulus

An advantage of the closest point projection algorithm is that an elasto-plastic tangent modulus that is fully consistent with the implicit Euler scheme can be derived. Using a standard $\mathbf{C}^{\text{ep}}$ matrix would lead to a lower convergence rate. If the consistent tangent modulus is used the convergence of the Newton-Raphson is quadratic, and the number of iterations is reduced. To derive the consistent tangent modulus equation (C.7) will be considered in a discretized form as,

$$\mathrm{d}\boldsymbol{\sigma} = \mathbf{C} : (\mathrm{d}\boldsymbol{\varepsilon} - \mathrm{d}\boldsymbol{\varepsilon}^{p}) \,. \tag{D.41}$$

Going back to the flow rule in (C.23), dicretizing and using the chain rule to obtain an increment,

$$\mathrm{d}\boldsymbol{\varepsilon}^{p} = \mathrm{d}\dot{\lambda}\, \partial_{\sigma} f + \dot{\lambda}\, \partial^{2}_{\sigma\sigma} f : \mathrm{d}\boldsymbol{\sigma}. \tag{D.42}$$

Combining equation (D.41) and (D.42),

$$\mathrm{d}\boldsymbol{\sigma} = \boldsymbol{\Xi} : \left(\mathrm{d}\boldsymbol{\varepsilon} - \mathrm{d}\dot{\lambda}\, \partial_{\sigma} f\right) . \tag{D.43}$$

The consistency condition in (C.16) yields using equation (C.24),

$$\mathrm{d}f = \partial_{\sigma} f : \mathrm{d}\boldsymbol{\sigma} + \partial_{q} f : \mathrm{d}\dot{\boldsymbol{q}} = \partial_{\sigma} f : \mathrm{d}\boldsymbol{\sigma} - \mathrm{d}\dot{\lambda}\, \partial_{q} f : \mathbf{H} : \partial_{q} f = 0. \tag{D.44}$$

Introducing equation (D.43) into (D.44) yields,

$$\mathrm{d}\dot{\lambda} = \frac{\partial_{\sigma} f : \boldsymbol{\Xi} : \mathrm{d}\boldsymbol{\varepsilon}}{\partial_{\sigma} f : \boldsymbol{\Xi} : \partial_{\sigma} f + \partial_{q} f : \mathbf{H} : \partial_{q} f}, \tag{D.45}$$

(D.46)

and introducing this in (D.43),

$$\mathrm{d}\boldsymbol{\sigma} = \left[\boldsymbol{\Xi} - \frac{\boldsymbol{\Xi} : \partial_{\sigma} f \otimes \partial_{\sigma} f : \boldsymbol{\Xi}}{\partial_{\sigma} f : \boldsymbol{\Xi} : \partial_{\sigma} f + \partial_{q} f : \mathbf{H} : \partial_{q} f}\right] : \mathrm{d}\boldsymbol{\varepsilon}. \tag{D.47}$$

The return mapping algorithm for rate-independent plasticity with isotropic and kinematic hardening using the von Mises criterion is summarized in Box D.2

1. Initial data for each quadrature point,
   $\boldsymbol{\varepsilon}^{(k+1)}, \ \boldsymbol{\varepsilon}^{p\ (k)}, \ \boldsymbol{q}^{(k)}$.
2. Compute trial stresses,
   $\boldsymbol{\sigma}_{\text{trial}}^{(k+1)} = \mathbf{C} : \left(\boldsymbol{\varepsilon}^{(k+1)} - \boldsymbol{\varepsilon}^{p\ (k)}\right)$.
3. Test plastic loading,
   $\boldsymbol{s}_{\text{kin trial}}^{(k+1)} = \boldsymbol{\sigma}_{\text{trial}}^{(k+1)} + \boldsymbol{q}_{\text{kin}}^{(k)} - \frac{1}{3}\text{tr}(\boldsymbol{\sigma}_{\text{trial}}^{(k+1)} + \boldsymbol{q}_{\text{kin}}^{(k)})$,
   $J_{2\ \text{trial}}^{(k+1)} = \frac{1}{2}\text{tr}\left(\left(\boldsymbol{s}_{\text{kin trial}}^{(k+1)}\right)^2\right)$,
   $f_{\text{trial}}^{(k+1)} = \sqrt{3J_{2\ \text{trial}}^{(k+1)}} + q_{\text{iso}}^{(k)} - \sigma_y$.
4. If $f_{\text{trial}}^{(k+1)} \leq 0$, elastic range.
   4.1 Update stresses,
   $\boldsymbol{\sigma}^{(k+1)} = \boldsymbol{\sigma}_{\text{trial}}^{(k+1)}$.
   4.2 Tangent modulus,
   $\mathbf{C}^{\text{ep}\ (k+1)} = \mathbf{C}$.
5. If $f_{\text{trial}}^{(k+1)} > 0$, plastic range.
   5.1 Start values for iteration,
   $\dot{\lambda}^{(0)} = 0, \ \boldsymbol{\sigma}^{(k+1)\ (0)} = \boldsymbol{\sigma}_{\text{trial}}^{(k+1)}$.
   5.2 Compute stress residuals,
   $\mathbf{R}_{\sigma}^{(t)} = \boldsymbol{\sigma}^{(k+1)\ (t)} - \boldsymbol{\sigma}_{\text{trial}}^{(k+1)\ (t)} + \dot{\lambda}^{(t)}\ \mathbf{C} : \partial_{\sigma} f^{(k+1)\ (t)}$,
   $\mathbf{R}_{f}^{(t)} = f^{(k+1)\ (t)}$.
   If $\mathbf{R}_{\sigma}^{(t)} <$ tol and $\mathbf{R}_{\sigma}^{(t)} <$ tol go to 5.7.
   5.3 Compute $\mathbf{Q}^{(t)}$ and $\Xi^{(t)}$,
   $\mathbf{Q}^{(t)} = \left[\mathbf{I} + \dot{\lambda}^{(t)}\ \mathbf{C} : \partial_{\sigma\sigma}^{2} f^{(k+1)\ (t)}\right]^{-1}$,
   $\Xi^{(t)} = \mathbf{Q}^{(t)} : \mathbf{C}$.
   5.4 Compute plastic multiplier,
   $\Delta\dot{\lambda}^{(t)} = \frac{\mathbf{R}_{f}^{(t)} - \mathbf{R}_{\sigma}^{(t)} : \mathbf{Q}^{(t)} : \partial_{\sigma} f^{(k+1)\ (t)}}{\partial_{\sigma} f^{(k+1)\ (t)} : \Xi^{(t)} : \partial_{\sigma} f^{(k+1)\ (t)} + \partial_{q} f^{(k+1)\ (t)} : \mathbf{H} : \partial_{q} f^{(k+1)\ (t)}}$,
   $\dot{\lambda}^{(t+1)} = \dot{\lambda}^{(t)} + \Delta\dot{\lambda}^{(t)}$.
   5.5 Update stresses,
   $\boldsymbol{\sigma}^{(k+1)} = \boldsymbol{\sigma}^{(k)} + \left[-\Delta\dot{\lambda}^{(t)}\ \mathbf{C} : \partial_{\sigma} f^{(k+1)\ (t)} - \mathbf{R}_{\sigma}^{(t)}\right] : \mathbf{Q}^{(t)}$.
   5.6 Update hardening,
   $\boldsymbol{q}^{(k+1)} = \boldsymbol{q}^{(k)} - \Delta\dot{\lambda}^{(t)}\ \mathbf{H}\ \partial_{q} f^{(k+1)\ (t)}$.
   Set $k = k + 1$ and go to 5.2.
   5.7 Update plastic strain,
   $\boldsymbol{\varepsilon}^{p\ (k+1)} = \boldsymbol{\varepsilon}^{p\ (k)} + \dot{\lambda}^{(t)}\ \partial_{\sigma} f^{(k+1)\ (t)}$.
   5.8 Consistent tangent modulus,
   $\mathbf{C}^{\text{ep}\ (k+1)} = \left[\Xi^{(t)} - \frac{\Xi^{(t)} : \partial_{\sigma} f^{(k+1)\ (t)} \otimes \partial_{\sigma} f^{(k+1)\ (t)} : \Xi^{(t)}}{\partial_{\sigma} f^{(k+1)\ (t)} : \Xi^{(t)} : \partial_{\sigma} f^{(k+1)\ (t)} + \partial_{q} f^{(k+1)\ (t)} : \mathbf{H} : \partial_{q} f^{(k+1)\ (t)}}\right]$.

Box D.2: Return mapping algorithm for non-linear plasticity with isotropic hardening.