C. Plasticity

This Appendix and Appendix D are intended to establish the theory needed for the development of the return mapping algorithms. They are therefore addressed to those unexperienced readers in the field of plasticity, who wish to know more about the topic. Most of the information appearing in these two appendices has been obtained from [1]. One is referred to that book for further detail.

Plastic deformations can be simulated with the finite element method. Due to the displacement driven nature of the finite element method, special procedures need to be implemented to ensure that the final stress state lies in the yield surface. This is usually done using what is called the return mapping algorithm. But before defining this return mapping algorithm the basis for plasticity have to be set.

To set the basis of elasto-plasticity the following conditions have to be defined,

- A stress-strain relationship before and after the yielding.
- A yield criterion expressing the combination of stresses at which plastic deformations occur.

Through the following sections a generalized three dimensional case will be studied. From there an analysis of the one dimension plasticity and two dimensions plane strain plasticity case will be derived.

C.1. Stress-strain relationship

The stress space, $\mathbb{E}_\sigma$, is defined as,

$$\mathbb{E}_\sigma = \{(\boldsymbol{\sigma}, \boldsymbol{q}) \in \mathbb{S} \times \mathbb{R}^n \mid f(\boldsymbol{\sigma}, \boldsymbol{q}) \leq 0\}, \tag{C.1}$$

where $\mathbb{S}$ is the space of symmetric second order tensors, n the spatial dimension, $\boldsymbol{\sigma}$ the stresses and the variable $\boldsymbol{q}$ expresses the hardening which will be defined later. $f(\boldsymbol{\sigma}, \boldsymbol{q})$ is the yield function, and a definition of it will be given in section C.2.

The interior of $\mathbb{E}_\sigma$ is called the elastic range or domain. As long as $f(\boldsymbol{\sigma}, \boldsymbol{q}) < 0$, the material behaves elastically. Plastic deformations occur whenever the stresses reach the yield surface defined as,

$$\partial\mathbb{E}_\sigma = \{(\boldsymbol{\sigma}, \boldsymbol{q}) \in \mathbb{S} \times \mathbb{R}^n \mid f(\boldsymbol{\sigma}, \boldsymbol{q}) = 0\}. \tag{C.2}$$

A simple relationship can be found between the elastic strain and the stresses,

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}^e, \tag{C.3}$$

where $\mathbf{C}$ is the elastic symmetric forth order tensor and $\boldsymbol{\varepsilon}^e$ is the elastic total strain. In the elastic range this expression can be extended to a relation between total strains and stresses,

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}^e = \mathbf{C} : \boldsymbol{\varepsilon}, \tag{C.4}$$

being $\boldsymbol{\varepsilon}$ is the total strain. However, as the stresses meet the yield surface equation C.4 does not hold. To obtain a new relation valid in the plastic range the stress tensor $\boldsymbol{\varepsilon}$ will be divided in an elastic and a plastic component as,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p. \tag{C.5}$$

Inserting this in equation (C.3),

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}^e = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p), \tag{C.6}$$

and expressing equation (C.6) in a variational form it yields,

$$\dot{\boldsymbol{\sigma}} = \mathbf{C} : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p), \tag{C.7}$$

where again the notation defined in equation (4.13) has been used.

To express the notion of irreversibility of displacement the flow rule is defined as,

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\, \boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{q}), \tag{C.8}$$

where $\dot{\lambda}$ is the rate of the plastic multiplier which determines the magnitude of the plastic flow and $\boldsymbol{r} : \mathbb{S} \times \mathbb{R}^n \to \mathbb{S}$ expresses the direction of the plastic deformations. By definition it is required that $\dot{\lambda} \geq 0$.

As has been stated before, if $f(\boldsymbol{\sigma}, \boldsymbol{q}) < 0$, no plastic deformation occur, that is, $\boldsymbol{\varepsilon}^p$ remains constant. Using the flow rule definition in equation (C.8) this can be written as,

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) < 0 \to \dot{\lambda} = 0. \tag{C.9}$$

However, if $f(\boldsymbol{\sigma}, \boldsymbol{q}) = 0$ plastic deformations might now be different than zero. This can be written as,

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) = 0 \to \dot{\lambda} \geq 0. \tag{C.10}$$

From equations (C.9) and (C.10), the Kuhn-Tucker loading-unloading condition can be derived as,

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) \leq 0, \; \dot{\lambda} \geq 0, \; f(\boldsymbol{\sigma}, \boldsymbol{q})\, \dot{\lambda} = 0. \tag{C.11}$$

To determine whether $\dot{\lambda}$ is equal or different than zero in equation (C.10) whenever $f(\boldsymbol{\sigma}, \boldsymbol{q}) = 0$, an extra condition is required. Imagine that the stresses touch the yield surface and then move immediately back to the elastic range, plastic deformations will then not occur. The stresses need to remain in the yield surface during a certain amount of fictitious time. This can be stated as,

$$\dot{f}(\boldsymbol{\sigma}, \boldsymbol{q}) = 0 \to \dot{\lambda} > 0. \tag{C.12}$$

Now imagine that the stresses are located in the elastic range. If the yield surface is approached within the elastic range the value of $f(\sigma, \boldsymbol{q})$ decreases, so $\dot{f}(\boldsymbol{\sigma}, \boldsymbol{q}) < 0$. But since the stresses move in the elastic range, $\dot{\lambda} = 0$,

$$\dot{f}(\boldsymbol{\sigma}, \boldsymbol{q}) < 0 \rightarrow \dot{\lambda} = 0. \tag{C.13}$$

Equations (C.12) and (C.13) can be combined leading to the expression known as Prager's consistency condition,

$$\dot{\lambda}\, \dot{f}(\boldsymbol{\sigma}, \boldsymbol{q}) = 0. \tag{C.14}$$

The variable $\boldsymbol{q}$ has not been defined yet. It includes components related to the isotropic and kinematic hardening, and can be computed using the hardening law,

$$\dot{\boldsymbol{q}} = -\dot{\lambda}\, \boldsymbol{h}(\boldsymbol{\sigma}, \boldsymbol{q}), \tag{C.15}$$

where $\boldsymbol{h} : \mathbb{S} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ depends on the type of hardening.

With the hardening law in equation (C.15), the flow rule in equation (C.8) and Prager's consistency condition in equation (C.14), it is possible to compute the rate of the plastic multiplier during plastic deformations, that is if $\dot{f}(\boldsymbol{\sigma}, \boldsymbol{q}) = 0$,

$$\begin{aligned} \dot{f}(\boldsymbol{\sigma}, \boldsymbol{q}) &= \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \dot{\boldsymbol{q}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : [\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p] - \dot{\lambda}\, \frac{\partial f}{\partial \boldsymbol{q}} \cdot \boldsymbol{h} \\ &= \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \left[\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \boldsymbol{r} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \boldsymbol{h} \right] = 0, \end{aligned} \tag{C.16}$$

and assuming that the flow rule, hardening law and yield function are so that it holds,

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \boldsymbol{r} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \boldsymbol{h} > 0, \tag{C.17}$$

then from equation (C.16) it yields,

$$\dot{\lambda} = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \dot{\boldsymbol{\varepsilon}}}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \boldsymbol{r} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \boldsymbol{h}}. \tag{C.18}$$

With this result the continuum tangent modulus can be computed. Going back to equation (C.7), and using the flow rule in equation (C.8),

$$\dot{\boldsymbol{\sigma}} = \mathbf{C} : [\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p] = \mathbf{C} : \left[\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\, \boldsymbol{r}\right] = \mathbf{C}^{\text{ep}} : \dot{\boldsymbol{\varepsilon}}, \tag{C.19}$$

and substituting equation (C.18) into the above result an expression of the continuum elasto-plastic tangent modulus, $\mathbf{C}^{\text{ep}}$, can be obtained,

$$\mathbf{C}^{\text{ep}} = \mathbf{C} - \frac{\mathbf{C} : \boldsymbol{r} \otimes \mathbf{C} : \frac{\partial f}{\partial \boldsymbol{\sigma}}}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \boldsymbol{r} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \boldsymbol{h}}. \tag{C.20}$$

Little has been said about $\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{q})$ and $\boldsymbol{h}(\boldsymbol{\sigma}, \boldsymbol{q})$. An important type of plasticity that covers most of the cases is what is called the non-associative plasticity. In non-associative plasticity a function $g : \mathbb{S} \times \mathbb{R}^m \rightarrow \mathbb{R}$ called the plastic potential exists such that,

$$\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{q}) = \frac{\partial g(\boldsymbol{\sigma}, \boldsymbol{q})}{\partial \boldsymbol{\sigma}}, \tag{C.21}$$

$$\boldsymbol{h}(\boldsymbol{\sigma}, \boldsymbol{q}) = \mathbf{H} \frac{\partial g(\boldsymbol{\sigma}, \boldsymbol{q})}{\partial \boldsymbol{q}}, \tag{C.22}$$

where $\mathbf{H}$ is a second order tensor called the hardening modulus. So the flow rule, in equation (C.8), and the hardening law, in equation (C.15), become for non-associative plasticity,

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\, \boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{q}) = \dot{\lambda}\, \frac{\partial g(\boldsymbol{\sigma}, \boldsymbol{q})}{\partial \boldsymbol{\sigma}}, \tag{C.23}$$

$$\dot{\boldsymbol{q}} = -\dot{\lambda}\, \boldsymbol{h}(\boldsymbol{\sigma}, \boldsymbol{q}) = -\dot{\lambda}\, \mathbf{H} \frac{\partial g(\boldsymbol{\sigma}, \boldsymbol{q})}{\partial \boldsymbol{q}}. \tag{C.24}$$

Note that non-associative plasticity results in non-symmetric continuum elasto-plastic modulus, $\mathbf{C}^{\mathrm{ep}}$, see equation (C.20). This can have severe consequences for the numerical stability of the iterative process when using a Newton-Raphson procedure since pivoting strategies are omitted, [14]. Symmetry is only guaranteed if,

$$\boldsymbol{r}(\sigma, q) = \frac{\partial f(\sigma, q)}{\partial \sigma}, \tag{C.25}$$

that is if $f(\boldsymbol{\sigma}, \boldsymbol{q}) = g(\boldsymbol{\sigma}, \boldsymbol{q})$. In that case one talks about associative flow rule and associative plasticity. From now on and for simplicity only associative plasticity will be used.

C.2. Yield criterion

Perfect plasticity is the simplest plasticity model. In this it is assumed that once the value of σ_y, the yield stress, is reached, the stresses can no longer increase and plastic strains continue indefinitely at constant stress. Materials rarely behave as perfect plastic, however, in some cases, where the plastic response of a material might be unclear, perfect plasticity can be used for low risk assumptions.

Unlike perfect plasticity there is often an increase or reduction in the yield stress once this is reached. Now the yield surface does no longer remain constant, there is an expansion (strain hardening) or contraction (strain softening) of the elastic range when plastic deformations occur. If this is the case one talks about isotropic plasticity.

In the case of cyclic loading and unloading conditions, for example in seismic or low cycle fatigue loading, the isotropic linear hardening plasticity fails to give a correct prediction of the material behavior [15]. As seen in figure C.1, there is a shift in the yield function. The initial elastic range (A-A'), turns into the elastic range (B-B') after unloading and reloading. This phenomena is known as Bauschinger's effect or kinematic hardening.

A generalized expression of the yield surface considering isotropic and kinematic hardening and from which later derive the yield surface for one dimension and two dimensions

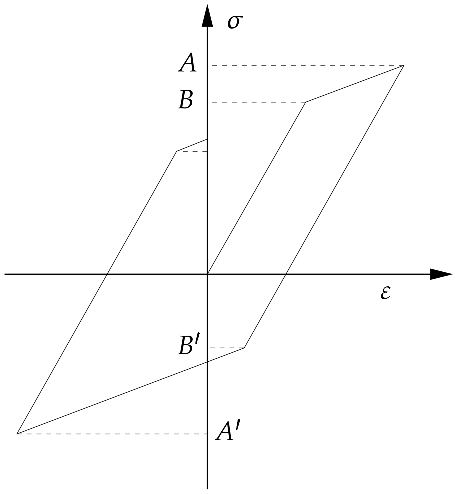

Figure C.1. Shift of the elastic range.

plane-strain can be written as,

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) := \|\boldsymbol{\sigma} + \boldsymbol{q}_{\text{kin}}\| + q_{\text{iso}} - \sigma_y, \tag{C.26}$$

being σ_y the yield stress and q_{iso} and $\boldsymbol{q}_{\text{kin}}$ the isotropic and kinematic hardening respectively.

For associative plasticity equation (C.24) can be rewritten as,

$$\boldsymbol{q} = -\mathbf{H}\,\boldsymbol{\alpha} \qquad \dot{\boldsymbol{\alpha}} = \dot{\lambda}\,\frac{\partial f}{\partial \boldsymbol{q}}, \tag{C.27}$$

so that,

$$\boldsymbol{q} = \begin{bmatrix} q_{\text{iso}} \\ \boldsymbol{q}_{\text{kin}} \end{bmatrix} = -\begin{bmatrix} H_{\text{iso}} & 0 \\ 0 & \mathbf{H}_{\text{kin}} \end{bmatrix} \begin{bmatrix} \alpha_{\text{iso}} \\ \boldsymbol{\alpha}_{\text{kin}} \end{bmatrix} \qquad \begin{bmatrix} \dot{\alpha}_{\text{iso}} \\ \dot{\boldsymbol{\alpha}}_{\text{kin}} \end{bmatrix} = \dot{\lambda} \begin{bmatrix} \partial_{q_{\text{iso}}} f \\ \partial_{\boldsymbol{q}_{\text{kin}}} f \end{bmatrix}, \tag{C.28}$$

where H_{iso} is the isotropic hardening modulus, and $\mathbf{H}_{\text{kin}}$ the kinematic hardening modulus. Both hardening modulus are constant, being H_{iso} a scalar and $\mathbf{H}_{\text{kin}}$ a diagonal matrix with equal non-negative constant diagonal terms. The rate of the isotropic and kinematic hardening can be written together as,

$$\dot{\boldsymbol{q}} = -\dot{\lambda}\,\mathbf{H}\,\partial_{\boldsymbol{q}} f, \tag{C.29}$$

where the following notation has been used,

$$\frac{\partial f}{\partial \boldsymbol{q}} = \partial_{\boldsymbol{q}} f. \tag{C.30}$$

See how the hardening rate depends on $\dot{\lambda}$, equation (C.29), and therefore on the strain history. This type of hardening is therefore called strain hardening. Another possibility could have been to use the work hardening hypothesis where hardening is defined as a function of the total plastic work.

C.2.1. One dimension yield criterion

The yield criterion for the one dimensional case can be defined as,

$$f(\sigma, \boldsymbol{q}) := |\sigma + q_{\text{kin}}| + q_{\text{iso}} - \sigma_y \leq 0, \tag{C.31}$$

where q_{kin} is now a scalar.

The flow rule in equation (C.23) is then,

$$\dot{\varepsilon}^p = \dot{\lambda} \frac{\partial f(\sigma, \boldsymbol{q})}{\partial \boldsymbol{\sigma}} = \dot{\lambda} \operatorname{sign}(\sigma + q_{\text{kin}}), \tag{C.32}$$

and the hardening defined in equation (C.28) turns now into,

$$\begin{bmatrix} q_{\text{iso}} \\ q_{\text{kin}} \end{bmatrix} = - \begin{bmatrix} H_{\text{iso}} & 0 \\ 0 & H_{\text{kin}} \end{bmatrix} \begin{bmatrix} \alpha_{\text{iso}} \\ \alpha_{\text{kin}} \end{bmatrix} \quad \begin{bmatrix} \dot{\alpha}_{\text{iso}} \\ \dot{\alpha}_{\text{kin}} \end{bmatrix} = \dot{\lambda} \begin{bmatrix} 1 \\ \operatorname{sign}(\sigma + q_{\text{kin}}) \end{bmatrix}. \tag{C.33}$$

In one dimension plasticity the notation used to express equation (C.31) is normally,

$$f(\sigma, \alpha_{\text{iso}}, q_{\text{kin}}) := |\sigma + q_{\text{kin}}| - \left(\sigma_y + H_{\text{iso}}\, \alpha_{\text{iso}}\right) \leq 0. \tag{C.34}$$

It is clear that the elastic modulus $\mathbf{C}$ reduces to E, so expression (C.7) is written for the one dimensional case as,

$$\dot{\sigma} = E\left(\dot{\varepsilon} - \dot{\varepsilon}^p\right). \tag{C.35}$$

A summary of the most important expressions for the one dimension rate-independent mixed kinematic and isotropic hardening plasticity is given in Box C.1.

C.2.2. The von Mises yield criterion

Von Mises criterion states that yielding occurs whenever the second invariant of the deviatoric stress tensor, J_2, reaches a critical value given by the following expression,

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) := \sqrt{3J_2} + q_{\text{iso}} - \sigma_y \leq 0, \tag{C.36}$$

where the value of σ_y is obtained through uniaxial experimental yield tension tests on the specific material. The second invariant of the deviatoric stress tensor is defined to be equal to,

$$J_2 = \frac{1}{2}\operatorname{tr}(\mathbf{s}_{\text{kin}}^2), \tag{C.37}$$

where $\mathbf{s}_{\text{kin}}$ stands for the deviatoric part of the stress tensor plus the kinematic hardening,

$$\mathbf{s}_{\text{kin}} = \boldsymbol{\sigma} + \boldsymbol{q}_{\text{kin}} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma} + \boldsymbol{q}_{\text{kin}}). \tag{C.38}$$

1. Elastic stress-strain relationship,
   $\sigma = E\,(\varepsilon - \varepsilon^p)$.

2. Yield condition,
   $f(\sigma, \alpha_{\text{iso}}, q_{\text{kin}}) := |\sigma + q_{\text{kin}}| - [\sigma_y + H_{\text{iso}}\,\alpha_{\text{iso}}] \leq 0$.

3. Flow rule and isotropic and kinematic hardening law,
   $\dot{\varepsilon}^p = \dot{\lambda}\ \text{sign}\,(\sigma + q_{\text{kin}})$,
   $\dot{\alpha}_{\text{iso}} = \dot{\lambda}$,
   $\dot{q}_{\text{kin}} = \dot{\lambda}\ H_{\text{kin}}\ \text{sign}\,(\sigma + q_{\text{kin}})$.

4. Kuhn-Tucker loading unloading conditions,
   $f\,(\sigma, \alpha_{\text{iso}}, q_{\text{kin}}) \leq 0,\ \dot{\lambda} \geq 0,\ f\,(\sigma, \alpha_{\text{iso}}, q_{\text{kin}})\,\dot{\lambda} = 0$.

5. Prager's consistency condition,
   $\dot{\lambda}\,\dot{f}\,(\sigma, \alpha_{\text{iso}}, q_{\text{kin}}) = 0$.

Box C.1: Basic equations for one dimension rate-independent mixed isotropic and kinematic hardening.

Note how the yield criteria should be independent of the orientation of the coordinate system, and should therefore only depend on stress invariants which is the case for equation (C.36). The von Mises criteria is generally used for metals and saturated soils and can be represented in the stress space as a cylinder which projection in the π space is a circle.

A summary of the most important expressions obtained using the von Mises criterion is given in Box C.2.

1. Elastic stress-strain relationship,

$$\boldsymbol{\sigma} = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p).$$

2. Yield condition,

$$s_{\text{kin}} = \boldsymbol{\sigma} + \boldsymbol{q}_{\text{kin}} - \frac{1}{3}\text{tr}(\boldsymbol{\sigma} + \boldsymbol{q}_{\text{kin}}),$$

$$J_2 = \frac{1}{2}\text{tr}(\mathbf{s}^2_{\text{kin}}),$$

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) := \sqrt{3J_2} + q_{\text{iso}} - \sigma_y \leq 0.$$

3. Flow rule and isotropic and kinematic hardening law,

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\,\frac{\partial f(\sigma,q)}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\sqrt{\frac{3}{4}}J_2^{-1/2}\begin{bmatrix} s_{\text{kin}\,xx} & s_{\text{kin}\,yy} & s_{\text{kin}\,zz} & 2s_{\text{kin}\,xy} \end{bmatrix}^T,$$

$$\dot{\boldsymbol{q}} = -\dot{\lambda}\,\mathbf{H}\frac{\partial f(\sigma,q)}{\partial \boldsymbol{q}} = -\dot{\lambda}\,\mathbf{H}\begin{bmatrix} 1 & s_{\text{kin}\,xx} & s_{\text{kin}\,yy} & s_{\text{kin}\,zz} & 2s_{\text{kin}\,xy} \end{bmatrix}^T.$$

4. Kuhn-Tucker loading unloading conditions,

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) \leq 0,\ \dot{\lambda} \geq 0,\ f(\boldsymbol{\sigma}, \boldsymbol{q})\,\dot{\lambda} = 0.$$

5. Prager's consistency condition,

$$\dot{\lambda}\,\dot{f}(\boldsymbol{\sigma}, \boldsymbol{q}) = 0.$$

Box C.2: Basic equations for von Mises rate-independent mixed isotropic and kinematic hardening.