B. Discretization

Going back to the definition of $\mathbf{B}_o^{\{\ \}}$ in equation (3.28), it is seen that what is actually inside the average operator is the average of what was defined in equations (3.52) and (3.53) as the interior component of the stresses, $\boldsymbol{\sigma}_{\text{int}}^h$,

$$\left\{\mathbf{C} : \nabla^s \boldsymbol{u}^h\right\} \boldsymbol{n}_K = \left\{\boldsymbol{\sigma}_{\text{int}}^h\right\} \boldsymbol{n}_K = \frac{1}{2}\left(\boldsymbol{\sigma}_{\partial K_1 \text{ int}}^h + \boldsymbol{\sigma}_{\partial K_2 \text{ int}}^h\right) \boldsymbol{n}_K = \mathbf{B}_o^{\{\ \}} \mathbf{a}_o. \tag{B.1}$$

So in the case of two dimensional plane strain, and omitting for simplicity the subindices and superindexes, $\boldsymbol{\sigma}_{\partial K_1 \text{ int}}^h \boldsymbol{n}_K$, is equal to

$$\boldsymbol{\sigma}_{\partial K_1 \text{ int}}^h \boldsymbol{n}_K = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} n_1 + \sigma_{12} n_2 \\ \sigma_{21} n_1 + \sigma_{22} n_2 \\ 0 \end{bmatrix}. \tag{B.2}$$

For plane strain it is known that,

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}, \tag{B.3}$$

being,

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad\qquad \nu = \frac{E}{2(1+\nu)}, \tag{B.4}$$

and with,

$$\varepsilon_{11} = u_{,x} = \begin{bmatrix} N_{1,x} & 0 & N_{2,x} & 0 & N_{3,x} & 0 & N_{4,x} & 0 \end{bmatrix} \mathbf{a}_e, \tag{B.5}$$

$$\varepsilon_{22} = v_{,y} = \begin{bmatrix} 0 & N_{1,y} & 0 & N_{2,y} & 0 & N_{3,y} & 0 & N_{4,y} \end{bmatrix} \mathbf{a}_e, \tag{B.6}$$

$$\begin{aligned} \gamma_{12} &= 2\varepsilon_{12} = u_{,y} + v_{,x} \\ &= \begin{bmatrix} N_{1,y} & N_{1,x} & N_{2,y} & N_{2,x} & N_{3,y} & N_{3,x} & N_{4,y} & N_{4,x} \end{bmatrix} \mathbf{a}_e. \end{aligned} \tag{B.7}$$

From equation (B.3), and using (B.5)-(B.7),

$$\begin{aligned} &\sigma_{11} = c_{11}\varepsilon_{11} + c_{12}\varepsilon_{22} \\ &= \begin{bmatrix} c_{11}N_{1,x} & c_{12}N_{1,y} & c_{11}N_{2,x} & c_{12}N_{2,y} & c_{11}N_{3,x} & c_{12}N_{3,y} & c_{11}N_{4,x} & c_{12}N_{4,y} \end{bmatrix} \mathbf{a}_e, \end{aligned} \tag{B.8}$$

$$\begin{aligned} &\sigma_{22} = c_{21}\varepsilon_{11} + c_{22}\varepsilon_{22} \\ &= \begin{bmatrix} c_{21}N_{1,x} & c_{22}N_{1,y} & c_{21}N_{2,x} & c_{22}N_{2,y} & c_{21}N_{3,x} & c_{22}N_{3,y} & c_{21}N_{4,x} & c_{22}N_{4,y} \end{bmatrix} \mathbf{a}_e, \end{aligned} \tag{B.9}$$

$$
\begin{aligned}
\sigma_{12} &= c_{33}\gamma_{12} \\
&= c_{33} \begin{bmatrix} N_{1,y} & N_{1,x} & N_{2,y} & N_{2,x} & N_{3,y} & N_{3,x} & N_{4,y} & N_{4,x} \end{bmatrix} \mathbf{a}_e. \quad \text{(B.10)}
\end{aligned}
$$

and with equations (B.2), and (B.8)-(B.10) it is obtained,

$$
\begin{bmatrix} \sigma_{11}n_1 + \sigma_{12}n_2 \\ \sigma_{21}n_1 + \sigma_{22}n_2 \end{bmatrix} =
$$
$$
\begin{bmatrix} C_{11}N_{1,x} & C_{12}N_{1,y} & C_{11}N_{2,x} & C_{12}N_{2,y} & C_{11}N_{3,x} & C_{12}N_{3,y} & C_{11}N_{4,x} & C_{12}N_{4,y} \\ C_{21}N_{1,x} & C_{22}N_{1,y} & C_{21}N_{2,x} & C_{22}N_{2,y} & C_{21}N_{3,x} & C_{22}N_{3,y} & C_{21}N_{4,x} & C_{22}N_{4,y} \end{bmatrix} \mathbf{a}_e +
$$
$$
\begin{bmatrix} D_2N_{1,y} & D_2N_{1,x} & D_2N_{2,y} & D_2N_{2,x} & D_2N_{3,y} & D_2N_{3,x} & D_2N_{4,y} & D_2N_{4,x} \\ D_1N_{1,y} & D_1N_{1,x} & D_1N_{2,y} & D_1N_{2,x} & D_1N_{3,y} & D_1N_{3,x} & D_1N_{4,y} & D_1N_{4,x} \end{bmatrix} \mathbf{a}_e, \quad \text{(B.11)}
$$

where,

$$
C_{11} = c_{11}n_1 \qquad C_{12} = c_{12}n_1 \qquad D_1 = c_{33}n_1 \quad \text{(B.12)}
$$
$$
C_{21} = c_{21}n_2 \qquad C_{22} = c_{22}n_2 \qquad D_2 = c_{33}n_2 \quad \text{(B.13)}
$$

and this in compact notation,

$$
\boldsymbol{\sigma}^h_{\partial K_1}\, \boldsymbol{n}_K = \begin{bmatrix} \sigma_{11}n_1 + \sigma_{12}n_2 \\ \sigma_{21}n_1 + \sigma_{22}n_2 \end{bmatrix} = \mathbf{B}_e^{\{\,\}} \mathbf{a}_e. \quad \text{(B.14)}
$$

So finally going back to (B.1),

$$
\begin{aligned}
\left\{ \mathbf{C} : \nabla^s \boldsymbol{u}^h \right\} \boldsymbol{n}_K = \frac{1}{2}\left(\boldsymbol{\sigma}^h_{\partial K_1} + \boldsymbol{\sigma}^h_{\partial K_2} \right) \boldsymbol{n}_K = \mathbf{B}_o^{\{\,\}} \mathbf{a}_o &= \frac{1}{2}\left(\mathbf{B}_{e1}^{\{\,\}} \mathbf{a}_{e1} + \mathbf{B}_{e2}^{\{\,\}} \mathbf{a}_{e2} \right) \\
&= \frac{1}{2} \begin{bmatrix} \mathbf{B}_{e1}^{\{\,\}} & \mathbf{B}_{e2}^{\{\,\}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_{e2} \\ \mathbf{a}_{e2} \end{bmatrix}. \quad \text{(B.15)}
\end{aligned}
$$