A. Average and jump operators

It can be proved that for any vector $\boldsymbol{g} : \tilde{\Gamma} \cup \Gamma \to \mathbb{R}^3$ and second order tensor $\boldsymbol{t} : \tilde{\Gamma} \cup \Gamma \to \mathbb{T}$ the following expression holds,

$$\sum_{K\in\Omega} \int_{\partial K} \boldsymbol{g} \cdot \boldsymbol{t}\, \boldsymbol{n}_{\partial K} ds = \int_{\tilde{\Gamma}\cup\Gamma} [\![\boldsymbol{g}]\!] \cdot \{\boldsymbol{t}\}\, \boldsymbol{n}_K ds + \int_{\tilde{\Gamma}} \{\boldsymbol{g}\} \cdot [\![\boldsymbol{t}]\!]\, \boldsymbol{n}_K ds. \tag{A.1}$$

Remember that n_{edg} is the total number of shared or interior edges and n_{ext} the total number of non shared or exterior edges. So the integral on the left hand side of equation A.1 can be divided as,

$$\sum_{K\in\Omega} \int_{\partial K} \boldsymbol{g} \cdot \boldsymbol{t}\, \boldsymbol{n}_{\partial K} ds = \sum_{i=1}^{n_{\text{ext}}} \int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{t} \boldsymbol{n}_{\partial K} ds + \sum_{i=1}^{n_{\text{edg}}} \left(\int_{\tilde{\Gamma}_i} \boldsymbol{g}_1 \cdot \boldsymbol{t}_1 \boldsymbol{n}_{\partial K_1} ds + \int_{\tilde{\Gamma}_i} \boldsymbol{g}_2 \cdot \boldsymbol{t}_2 \boldsymbol{n}_{\partial K_2} ds \right), \tag{A.2}$$

where the subindex one and two refer to each of the common sides in every interior boundary, see figure 3.1. It should actually be written as $\boldsymbol{g}_{\partial K_1}$, $\boldsymbol{g}_{\partial K_2}$, $\boldsymbol{t}_{\partial K_1}$ and $\boldsymbol{t}_{\partial K_2}$, but to keep the notation simple $\boldsymbol{g}_1$, $\boldsymbol{g}_2$, $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ will be used respectively. It is seen from the definitions of the average and jump operators over Γ, equations (2.17) and (2.19), that equation (A.2) can be rewritten as,

$$\sum_{K\in\Omega} \int_{\partial K} \boldsymbol{g} \cdot \boldsymbol{t} \boldsymbol{n}_{\partial K} ds = \int_{\Gamma} [\![\boldsymbol{g}]\!] \cdot \{\boldsymbol{t}\}\, \boldsymbol{n}_{\partial K} ds + \sum_{i=1}^{n_{\text{edg}}} \left(\int_{\Gamma^i} \boldsymbol{g}_1 \cdot \boldsymbol{t}_1 \boldsymbol{n}_{\partial K_1} ds + \int_{\Gamma^i} \boldsymbol{g}_2 \cdot \boldsymbol{t}_2 \boldsymbol{n}_{\partial K_2} ds \right). \tag{A.3}$$

Knowing that $\boldsymbol{n}_{\partial K_1}|_{\Gamma^i} = -\boldsymbol{n}_{\partial K_2}|_{\Gamma^i}$, and defining $\boldsymbol{n}_K|_{\Gamma^i} = \boldsymbol{n}_{\partial K_1}|_{\Gamma^i}$ over $\tilde{\Gamma}$, the second and third term of the right hand side of the above equation can then be expanded as,

$$\begin{aligned}
&\sum_{i=1}^{n_{\text{edg}}} \left(\int_{\tilde{\Gamma}_i} \boldsymbol{g}_1 \cdot \boldsymbol{t}_1 \boldsymbol{n}_{\partial K_1} ds + \int_{\tilde{\Gamma}_i} \boldsymbol{g}_2 \cdot \boldsymbol{t}_2 \boldsymbol{n}_{\partial K_2} ds \right) = \sum_{i=1}^{n_{\text{edg}}} \left(\int_{\tilde{\Gamma}_i} \boldsymbol{g}_1 \cdot \boldsymbol{t}_1 \boldsymbol{n}_K ds - \int_{\tilde{\Gamma}_i} \boldsymbol{g}_2 \cdot \boldsymbol{t}_2 \boldsymbol{n}_K ds \right) \\
&= \sum_{i=1}^{n_{\text{edg}}} \int_{\tilde{\Gamma}_i} \frac{1}{2} \left(\boldsymbol{g}_1 \cdot \boldsymbol{t}_1 \boldsymbol{n}_K - \boldsymbol{g}_2 \cdot \boldsymbol{t}_2 \boldsymbol{n}_K - \boldsymbol{g}_1 \cdot \boldsymbol{t}_2 \boldsymbol{n}_K + \boldsymbol{g}_2 \cdot \boldsymbol{t}_1 \boldsymbol{n}_K \right) \\
&+ \sum_{i=1}^{n_{\text{edg}}} \int_{\tilde{\Gamma}_i} \frac{1}{2} \left(\boldsymbol{g}_1 \cdot \boldsymbol{t}_1 \boldsymbol{n}_K - \boldsymbol{g}_2 \cdot \boldsymbol{t}_2 \boldsymbol{n}_K + \boldsymbol{g}_1 \cdot \boldsymbol{t}_2 \boldsymbol{n}_K - \boldsymbol{g}_2 \cdot \boldsymbol{t}_1 \boldsymbol{n}_K \right) ds \\
&= \sum_{i=1}^{n_{\text{edg}}} \int_{\tilde{\Gamma}_i} \frac{1}{2} \left(\boldsymbol{g}_1 + \boldsymbol{g}_2 \right) \left(\boldsymbol{t}_1 - \boldsymbol{t}_2 \right) \boldsymbol{n}_K ds + \sum_{i=1}^{n_{\text{edg}}} \int_{\tilde{\Gamma}_i} \frac{1}{2} \left(\boldsymbol{g}_1 - \boldsymbol{g}_2 \right) \left(\boldsymbol{t}_1 + \boldsymbol{t}_2 \right) \boldsymbol{n}_K ds \\
&= \int_{\tilde{\Gamma}} [\![\boldsymbol{g}]\!] \cdot \{\boldsymbol{t}\}\, \boldsymbol{n}_K ds + \int_{\tilde{\Gamma}} \{\boldsymbol{g}\} \cdot [\![\boldsymbol{t}]\!]\, \boldsymbol{n}_K ds.
\end{aligned} \tag{A.4}$$

So with this result and going back to equation (A.3), the following result is finally obtained,

$$\sum_{K \in \Omega} \int_{\partial K} \boldsymbol{g} \cdot \boldsymbol{t} \boldsymbol{n}_{\partial K} ds = \int_{\tilde{\Gamma} \cup \Gamma} [\![\boldsymbol{g}]\!] \cdot \{\boldsymbol{t}\} \, \boldsymbol{n}_K ds + \int_{\tilde{\Gamma}} \{\boldsymbol{g}\} \cdot [\![\boldsymbol{t}]\!] \, \boldsymbol{n}_K ds. \tag{A.5}$$