A. Average and jump operators

It can be proved that for any vector $g : \Gamma \cup \Gamma \rightarrow \mathbb{R}^3$ and second order tensor $t : \Gamma \cup \Gamma \rightarrow T$ the following expression holds,

$$\sum_{k \in \Omega} \int_{\partial K} g \cdot t \, n_{\partial K} ds = \int_{\Gamma} [g] \cdot \{t\} \, n_k ds + \int_{\Gamma} [g] \cdot [t] \, n_k ds. \tag{A.1}$$

Remember that $n_{\text{edg}}$ is the total number of shared or interior edges and $n_{\text{ext}}$ the total number of non shared or exterior edges. So the integral on the left hand side of equation A.1 can be divided as,

$$\sum_{k \in \Omega} \int_{\partial K} g \cdot t \, n_{\partial K} ds = \sum_{i=1}^{n_{\text{ext}}} \int_{t_i} g \cdot t n_{\partial K_i} ds + \sum_{i=1}^{n_{\text{edg}}} \left( \int_{t_i} g_1 \cdot t_1 n_{\partial K_i} ds + \int_{t_i} g_2 \cdot t_2 n_{\partial K_i} ds \right), \tag{A.2}$$

where the subindex one and two refer to each of the common sides in every interior boundary, see figure 3.1. It should actually be written as $g_{\partial K_i}, g_{\partial K_i}, t_{\partial K_i}$ and $t_{\partial K_i}$, but to keep the notation simple $g_1, g_2, t_1$ and $t_2$ will be used respectively. It is seen from the definitions of the average and jump operators over $\Gamma$, equations (2.17) and (2.19), that equation (A.2) can be rewritten as,

$$\sum_{k \in \Omega} \int_{\partial K} g \cdot t n_{\partial K} ds = \int_{\Gamma} [g] \cdot \{t\} \, n_{\partial K} ds + \sum_{i=1}^{n_{\text{edg}}} \left( \int_{t_i} g_1 \cdot t_1 n_{\partial K_i} ds + \int_{t_i} g_2 \cdot t_2 n_{\partial K_i} ds \right). \tag{A.3}$$

Knowing that $n_{\partial K_i}|_{t_i} = -n_{\partial K_i}|_{t_i}$, and defining $n_{K_i}|_{t_i} = n_{\partial K_i}|_{t_i}$ over $\Gamma$, the second and third term of the right hand side of the above equation can then be expanded as,

$$\sum_{i=1}^{n_{\text{edg}}} \left( \int_{t_i} g_1 \cdot t_1 n_{\partial K_i} ds + \int_{t_i} g_2 \cdot t_2 n_{\partial K_i} ds \right) = \sum_{i=1}^{n_{\text{edg}}} \left( \int_{t_i} g_1 \cdot t_1 n_{K_i} ds - \int_{t_i} g_2 \cdot t_2 n_{K_i} ds \right)$$

$$= \sum_{i=1}^{n_{\text{edg}}} \int_{t_i} \frac{1}{2} \left( g_1 \cdot t_1 n_K - g_2 \cdot t_2 n_K - g_1 \cdot t_2 n_K + g_2 \cdot t_1 n_K \right) ds$$

$$+ \sum_{i=1}^{n_{\text{edg}}} \int_{t_i} \frac{1}{2} \left( g_1 \cdot t_1 n_K - g_2 \cdot t_2 n_K + g_1 \cdot t_2 n_K - g_2 \cdot t_1 n_K \right) ds$$

$$= \sum_{i=1}^{n_{\text{edg}}} \int_{t_i} \frac{1}{2} (g_1 + g_2) (t_1 - t_2) n_K ds + \sum_{i=1}^{n_{\text{edg}}} \int_{t_i} \frac{1}{2} (g_1 - g_2) (t_1 + t_2) n_K ds$$

$$= \int_{\Gamma} [g] \cdot \{t\} \, n_K ds + \int_{\Gamma} [g] \cdot [t] \, n_K ds. \tag{A.4}$$
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So with this result and going back to equation (A.3), the following result is finally obtained,

$$\sum_{k \in \Omega} \int_{\partial K} g \cdot t n_{\partial K} ds = \int_{\Omega} \{g\} \cdot \{t\} n_{K} ds + \int_{\Omega} \{t\} \cdot \{t\} n_{K} ds.$$  (A.5)