Chapter 2

General concepts

2.1 The Navier-Stokes equations

The Navier-Stokes equations model the fluid mechanics. This set of differential equations describes the motion of a fluid. In the present work the fluid is assumed Newtonian, that is, the stress tensor $\boldsymbol{\sigma}$ is linearly related to the strain-rate tensor. These equations represent the effect of both external and internal forces in a fluid. These equations comprise the equations expressing conservation of mass (2.1) and linear momentum (2.2):

$$\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{2.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \frac{d\boldsymbol{v}}{dt} \tag{2.2}$$

where ρ is the fluid density, $\boldsymbol{v}$ the velocity vector and $\boldsymbol{b}$ the volume force per unit of mass. The Navier-Stokes equations are obtained combining both sets of balance equations. They read:

$$\begin{aligned} \rho(\boldsymbol{v}_t + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}) &= \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \boldsymbol{b} \qquad \text{in } \Omega \times]0, T[\qquad (2.3) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0 \qquad \text{in } \Omega \times]0, T[\qquad (2.4) \end{aligned}$$

These equations cannot be solved unless an extra condition is specified. All materials are expected to satisfy the previous balance equations (2.3) and (2.4). In order to describe the behavior of the different materials, a constitutive law has to be introduced. In this work, the constitutive law used to model the Navier-Stokes problem is the Stokes' law:

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu \boldsymbol{\nabla}^s \boldsymbol{v}, \tag{2.5}$$

where p is the hydrostatic pressure, μ the dynamic viscosity, $\boldsymbol{v}$ the velocity and $\boldsymbol{\nabla}^s\boldsymbol{v}$ the symmetric part of the velocity gradient. Combining the Stokes' law (2.5) with the general Navier-Stokes differential equations (2.3), the most known version of Navier-Stokes equations is obtained:

$$\boldsymbol{v}_t + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - 2\nu\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}^s\boldsymbol{v} + \boldsymbol{\nabla}p = \boldsymbol{b} \tag{2.6}$$

Here, p is the kinematic pressure, that is $p = \mathrm{p}/\rho$ and ν the kinematic viscosity, $\nu = \mu/\rho$.

The incompressibility condition (2.4) expresses the fact that in an incompressible fluid the mass density does not vary along the time. Therefore, applying (2.1) this relation is easily derived. In this case, the Navier-Stokes equations can be transformed in a more convenient form,

$$\boldsymbol{v}_t + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \nu\boldsymbol{\nabla}^2\boldsymbol{v} + \boldsymbol{\nabla}p = \boldsymbol{b} \tag{2.7}$$

Some initial and boundary conditions have to be added in order to solve the system of differential equations. The most common boundary conditions consist on prescribing the value of the velocity on the total, or a part, of the boundary Γ. These are called Dirichlet conditions:

$$\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{v}_D(\boldsymbol{x},t), \qquad \boldsymbol{x}\in\Gamma \tag{2.8}$$

Another type of boundary conditions prescribes the traction on the total, or a part, of the boundary. These are called Neumann conditions:

$$\boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{x},t) = \boldsymbol{t}(\boldsymbol{x},t), \qquad \boldsymbol{x}\in\Gamma \tag{2.9}$$

where $\boldsymbol{n}$ is the normal vector at every point of the boundary.

The initial conditions needed for the time integration are represented by the value of the velocity field at every point of the fluid at a given time, normally $t_0 = 0$. But not every initial condition is possible. The velocity at t_0 has to be irrotational (see Huerta, 2003 for more details):

$$\boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x}), \qquad \boldsymbol{x}\in\Omega \tag{2.10}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v}_0(\boldsymbol{x}) = 0, \qquad \boldsymbol{x}\in\Omega \tag{2.11}$$

Note that the initial conditions have only to be prescribed in velocities because there is no time derivative of the pressure. If the boundary conditions are just of the Dirichlet type, the pressure can be determined only up to a constant that is found imposing the value of the pressure at one point.

2.2 Stationary Stokes problem

The stationary Stokes problem is a particular case of the Navier-Stokes equations where the convective terms and the time dependency is neglected. This case appears when studying highly viscous flows, that is flows with a small Reynolds number. The resulting equations are called the Stokes problem (2.12):

$$-\nu\boldsymbol{\nabla}^2\boldsymbol{v} + \boldsymbol{\nabla}p = \boldsymbol{b} \tag{2.12}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \tag{2.13}$$

Here the problem is formulated in terms of velocity and pressure using the Stokes' law (2.5) and the incompressibility condition (2.4).

2.2.1 Strong form

The complete expression of the strong form of the stationary Stokes problem reads:

$$-\nu\boldsymbol{\nabla}^2\boldsymbol{v} + \boldsymbol{\nabla}p = \boldsymbol{b} \quad \text{in } \Omega \tag{2.14}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \quad \text{in } \Omega \tag{2.15}$$

$$\boldsymbol{v} = \boldsymbol{v}_D \quad \text{on } \Gamma_D \tag{2.16}$$

$$-p\boldsymbol{n} + \nu(\boldsymbol{n}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{t} \quad \text{on } \Gamma_N \tag{2.17}$$

2.2.2 Weak form

The weak form of the problem is obtained multiplying the equation of the stationary Stokes problem by the velocity test function $\boldsymbol{w}$ and applying the divergence theorem to the pressure gradient and to the viscous term. Then, the weak formulation of the stationary Stokes problem can be defined as: find $(\boldsymbol{v}, p) \in \boldsymbol{\mathcal{S}} \times \mathcal{V}$, such that

$$a(\boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{w}, p) + b(\boldsymbol{v}, q) = (\boldsymbol{w}, \boldsymbol{b}) + (\boldsymbol{w}, \boldsymbol{t})_{\Gamma_N} \quad \forall (\boldsymbol{w}, q) \in \boldsymbol{\mathcal{V}} \times \mathcal{Q} \tag{2.18}$$

where,

$$a(\boldsymbol{w}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\nabla}\boldsymbol{w} : \nu\boldsymbol{\nabla}\boldsymbol{v}\, d\Omega \tag{2.19}$$

$$b(\boldsymbol{v}, q) = -\int_\Omega q\boldsymbol{\nabla}\cdot\boldsymbol{v}\, d\Omega \tag{2.20}$$

$$(\boldsymbol{w}, \boldsymbol{b}) = \int_\Omega \boldsymbol{w}\cdot\boldsymbol{b}\, d\Omega \tag{2.21}$$

The spaces introduced in this formulation are the spaces of trial solutions for the velocity $\boldsymbol{v}$, the weighting functions of the velocity $\boldsymbol{w}$ and the pressure p, respectively, $\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{V}}$ and $\mathcal{Q}$. The definitions of these spaces are:

$$\begin{aligned} \boldsymbol{\mathcal{S}} &:= \{\boldsymbol{v} \in \boldsymbol{\mathcal{H}}^1(\Omega) \mid \boldsymbol{v} = \boldsymbol{v}_D \quad on \quad \Gamma_D\} && (2.22) \\ \boldsymbol{\mathcal{V}} &:= \boldsymbol{\mathcal{H}}^1_{\Gamma_D}(\Omega) = \{\boldsymbol{w} \in \boldsymbol{\mathcal{H}}^1(\Omega) \mid \boldsymbol{w} = 0 \quad on \quad \Gamma_D\} && (2.23) \\ \mathcal{Q} &:= \mathcal{L}_2(\Omega) && (2.24) \end{aligned}$$

2.2.3 Galerkin formulation

The Galerkin formulation of the stationary Stokes problem is based on a mixed finite element method. It is needed to use local approximations for the velocity, the pressure and their weighting functions. The velocity approximation $\boldsymbol{v^h}$ can be expressed as

$$\boldsymbol{v}^h = \boldsymbol{u}^h + \boldsymbol{v}^h_D \tag{2.25}$$

where $\boldsymbol{v}^h_D$ satisfies the Dirichlet conditions on Γ_D. Thus the Galerkin formulation can be expressed as: given a field of external forces $\boldsymbol{b}$ and the boundary conditions $\boldsymbol{v}_D$ and $\boldsymbol{t}$, find the velocity $\boldsymbol{u}^h$ and the pressure p^h such that,

$$\begin{aligned} a(\boldsymbol{w}^h, \boldsymbol{u}^h) + b(\boldsymbol{w}^h, p^h) &= (\boldsymbol{w}^h, \boldsymbol{b}^h) + (\boldsymbol{w}^h, \boldsymbol{t}^h)_{\Gamma_N} - a(\boldsymbol{w}^h, \boldsymbol{v}^h_D) && (2.26) \\ b(\boldsymbol{u}^h, q^h) &= -b(\boldsymbol{v}^h_D, q^h) && (2.27) \end{aligned}$$

Here $\boldsymbol{w}^h$ is the weighting function for the velocity and q^h the one for the pressure.

The interpolation of the velocity needs to satisfy an extra condition. It has to be continuous at the interface between elements. This condition is not necessary for the pressure due to the fact that there is no pressure gradient in this formulation.

In order to solve the problem every velocity component v^h_i has to be approximated. This is done using the shape functions and their nodal values. It is important to make a distinction between the number of total nodes in the finite element mesh and the number of nodes with Dirichlet conditions. The difference between both number of nodes is the total number of nodal velocity unknowns. Therefore, some notation has to be introduced. The total nodes in the mesh is denoted by $\eta = 1, 2, ..., n_{np}$. Similarly, $\eta_D \subset \eta$ the subset of nodes where the Dirichlet conditions are imposed. Using this

notation,

$$\boldsymbol{u}^h(\boldsymbol{x}) = \sum_{A\in\eta\setminus\eta_D} N_A(\boldsymbol{x})\boldsymbol{u}_A + \sum_{A\in\eta_D} N_A(\boldsymbol{x})\boldsymbol{u}_D(\boldsymbol{x}_A) \tag{2.28}$$

where N_A is the shape function at node number A and $\boldsymbol{u}_A$ its associated velocity unknown. Now it is possible to rewrite this expression for every component:

$$u_i^h(\boldsymbol{x}) = \sum_{A\in\eta\setminus\eta_{iD}} N_A(\boldsymbol{x})u_{iA} \tag{2.29}$$

$$v_{iD}^h(\boldsymbol{x}) = \sum_{A\in\eta_{iD}} N_A(\boldsymbol{x})v_{iD}(\boldsymbol{x}_A) \tag{2.30}$$

The interpolation for the pressure can be made using the same set of nodes and shape functions but this is not mandatory. Thus in general,

$$p^h(\boldsymbol{x}) = \sum_{\hat{A}\in\hat{\eta}} \hat{N}_{\hat{A}}(\boldsymbol{x})p_{\hat{A}} \tag{2.31}$$

2.2.4 Matrix problem

The implementation of the finite element method consists on separate the global problem in smaller problems defined in every element and then assembly all of these problems in the global one. In order to make it simpler to calculate, every problem in a element is transformed into a master element with normalized coordinates. This process is called isoparametrization. The physical coordinates can be expressed in terms of the reference coordinates as:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \sum_i N_i(\xi,\eta) \begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} \tag{2.32}$$

Moreover,

$$\begin{bmatrix} \frac{dN}{dx} \\ \frac{dN}{dy} \end{bmatrix} = \begin{bmatrix} \frac{d\xi}{dx} & \frac{d\eta}{dx} \\ \frac{d\xi}{dy} & \frac{d\eta}{dy} \end{bmatrix} \begin{bmatrix} \frac{dN}{d\xi} \\ \frac{dN}{d\eta} \end{bmatrix} \tag{2.33}$$

where the gradient of the shape functions is expressed through the Jacobian matrix.
The elemental matrices for the discretization of the problem are (see details

in Huerta(2003),

$$\boldsymbol{K}^e = a(\boldsymbol{w},\boldsymbol{u})_{\Omega^e} = \int_{\Omega^e} \boldsymbol{\nabla} N \cdot \boldsymbol{\nabla} N d\Omega \tag{2.34}$$

$$\boldsymbol{G}^e = b(\boldsymbol{w},p)_{\Omega^e} = -\int_{\Omega^e} \boldsymbol{\nabla} N \cdot \hat{N} d\Omega \tag{2.35}$$

$$\boldsymbol{f}^e = (\boldsymbol{w},\boldsymbol{b})_{\Omega^e} + (\boldsymbol{w},\boldsymbol{t})_{\Gamma^e_N} - a(\boldsymbol{w},\boldsymbol{v}_D)_{\Omega^e} \tag{2.36}$$

$$\boldsymbol{h}^e = -b(\boldsymbol{v}_D,q)_{\Omega^e} \tag{2.37}$$

The global matrices can be represented through the action of an assembly operator $\mathbf{A}^e$ applied on the local matrices:

$$\boldsymbol{K} = \mathbf{A}^e \boldsymbol{K}^e \tag{2.38}$$

$$\boldsymbol{G} = \mathbf{A}^e \boldsymbol{G}^e \tag{2.39}$$

$$\boldsymbol{f} = \mathbf{A}^e \boldsymbol{f}^e \tag{2.40}$$

$$\boldsymbol{h} = \mathbf{A}^e \boldsymbol{h}^e \tag{2.41}$$

Then, the matrix system associated to the stationary Stokes problem can be written as,

$$\begin{pmatrix} \boldsymbol{K} & \boldsymbol{G} \\ \boldsymbol{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{h} \end{pmatrix} \tag{2.42}$$

The resulting matrix system is, in general, singular because the boundary conditions are not imposed yet. There are several options to introduce them in order to make the system solvable. In this work, the implementation using Lagrange multipliers is the chosen solution. The prescribed velocity values on the Dirichlet portion of the boundary are then expressed with these Lagrange multipliers. This method implies additional equations to the matrix system. The final system can be expressed as,

$$\begin{pmatrix} \boldsymbol{K} & \boldsymbol{B}^T & \boldsymbol{G} \\ \boldsymbol{B} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{G}^T & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{c} \\ \boldsymbol{h} \end{pmatrix} \tag{2.43}$$

where $\boldsymbol{B}^T\boldsymbol{\lambda}$ is the vector of reaction forces that enforce the constraints, $\boldsymbol{B}$ a rectangular matrix, $\boldsymbol{c}$ a vector with the prescribed values of the constraints and $\boldsymbol{\lambda}$ a vector containing the Lagrange multipliers.

2.2.5 The LBB condition

The particular form of the system matrix associated to the stationary Stokes problem, see (2.43), with null submatrixes on the diagonal, implies that the

matrix may be singular in certain situations. In order to have $\boldsymbol{u}$ and p uniquely defined the matrix has to be non-singular. That is, the kernel of matrix $\boldsymbol{G}$ has to be zero. Thus, to have ker $\boldsymbol{G} = 0$, the interpolation spaces for velocity and pressure have to satisfy a compatibility condition called the LBB condition (Ladyzhenskaya - Babuška - Brezzi), see Huerta,2003 for further details. This condition reads:

The existence of a stable finite element approximate solution (u^h, p^h) to the steady Stokes problem depends on choosing a pair of spaces $\mathcal{V}^h$ and $\mathcal{Q}^h$, such that the following inf-sup condition holds:

$$inf_{q^h \in \mathcal{Q}^h} \quad sup_{\boldsymbol{w}^h \in \mathcal{V}^h} \frac{(q^h, \boldsymbol{\nabla} \cdot \boldsymbol{w}^h)}{||q||_0 ||\boldsymbol{w}^h||_1} \geq \alpha > 0, \tag{2.44}$$

where α is independent of the mesh size h.

If the LBB condition is satisfied, then ker $\boldsymbol{G} = 0$ and there exists a unique $\boldsymbol{u}^h \in \boldsymbol{\mathcal{V}}^h$ and a $p^h \in \mathcal{Q}^h$.

In figure 2.1 there are some elements in 2D that pass and some that do not pass the LBB condition as an example.

2.2.6 Stabilization of the stationary Stokes problem

In the last decades a lot of effort has been focused to find stabilization methods for the stationary Stokes problem in order to use the velocity-pressure pairs that are instable in the standard Galerkin formulation. One of the most used methods in this area is the Galerkin Least-Squares (GLS).

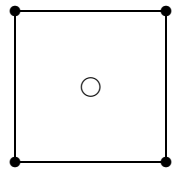
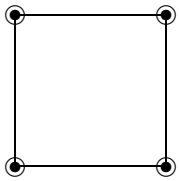
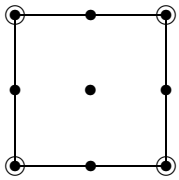
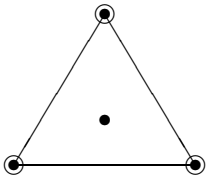
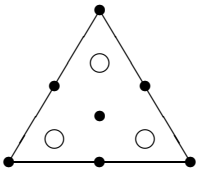
The main idea behind this method resides in enforcing the positive definiteness of the matrix in equation (2.43). This is done by adding a non-null submatrix in the diagonal throughout the weak form of the problem. The method consists on modifying the variational form, see (2.14) with the addition of the minimization of the least-squares form:

$$L_S(\boldsymbol{v}, p) := (-\nu \boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{\nabla} p - \boldsymbol{b}, -\nu \boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{\nabla} p - \boldsymbol{b}) \tag{2.45}$$

$$\frac{dL_S(\boldsymbol{v} + \epsilon \boldsymbol{w}, p + \epsilon q)}{d\epsilon}\bigg|_{\epsilon=0} = 0 \tag{2.46}$$

The application of some algebraic operations in the derivative term (2.46) leads to,

$$(-\nu \boldsymbol{\nabla}^2 \boldsymbol{w} + \boldsymbol{\nabla} q, -\nu \boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{\nabla} p - \boldsymbol{b}) = 0 \qquad \forall (\boldsymbol{w}, q) \in \mathcal{V} \times \mathcal{Q} \tag{2.47}$$

Element type	Element name	Pass LBB condition
	Q1P0	NO
	Q1Q1	NO
	Q2Q1	YES
	Mini	YES
	Crouziex-Raviart	YES

• Velocity nodes ○ Pressure nodes

Figure 2.1: Examples of several 2D elements under LBB condition

Equation (2.47) can be separated into the velocity and pressure components:

$$(-\nu\boldsymbol{\nabla}^2\boldsymbol{w}, -\nu\boldsymbol{\nabla}^2\boldsymbol{v} + \boldsymbol{\nabla}p - \boldsymbol{b}) = 0 \qquad \forall \boldsymbol{w} \in \mathcal{V} \tag{2.48}$$

$$(-\boldsymbol{\nabla}q, -\nu\boldsymbol{\nabla}^2\boldsymbol{v} + \boldsymbol{\nabla}p - \boldsymbol{b}) = 0 \qquad \forall q \in \mathcal{Q} \tag{2.49}$$

Then, the stabilized Galerkin formulation of the problem can be written as:

$$a(\boldsymbol{w}^h, \boldsymbol{v}^h) + b(\boldsymbol{w}^h, p^h) + \sum_{e=1}^{n_{el}} \tau_e(-\nu\boldsymbol{\nabla}^2\boldsymbol{w}^h, -\nu\boldsymbol{\nabla}^2\boldsymbol{v}^h + \boldsymbol{\nabla}p^h - \boldsymbol{b}^h)_{\Omega^e} = (\boldsymbol{w}^h, \boldsymbol{b}^h) + (\boldsymbol{w}^h, \boldsymbol{t}^h)_{\Gamma_N} \tag{2.50}$$

$$b(\boldsymbol{v}^h, q^h) - \sum_{e=1}^{n_{el}} \tau_e(\boldsymbol{\nabla}q^h, -\nu\boldsymbol{\nabla}^2\boldsymbol{v}^h + \boldsymbol{\nabla}p^h - \boldsymbol{b}^h)_{\Omega^e} = 0 \tag{2.51}$$

where τ_e is the stabilization parameter. Note that the non-null submatrix in the diagonal is obtained with the term $(\boldsymbol{\nabla}q^h, \boldsymbol{\nabla}p^h)$ in equation (2.51).

The above formulation can be simplified in the case of using linear elements to approximate the solution. Then, the variational form derives into (see more details in Huerta(2003)):

$$a(\boldsymbol{w}^h, \boldsymbol{v}^h) + b(\boldsymbol{w}^h, p^h) = (\boldsymbol{w}^h, \boldsymbol{b}^h) + (\boldsymbol{w}^h, \boldsymbol{t}^h)_{\Gamma_N} \tag{2.52}$$

$$b(\boldsymbol{v}^h, q^h) - \sum_{e=1}^{n_{el}} \tau_e(\boldsymbol{\nabla}q^h, \boldsymbol{\nabla}p^h)_{\Omega^h} = -\sum_{e=1}^{n_{el}} \tau_e(\boldsymbol{\nabla}q^h, \boldsymbol{b}^h)_{\Omega^e} \tag{2.53}$$

The stabilization parameter τ_e is chosen as:

$$\tau_e = \alpha_0 \frac{h_e^2}{4\nu} \tag{2.54}$$

where h_e is the measure of the element size in the computational mesh and α_0 can be estimated by $\alpha_0 = 1/3$ in the case of linear elements.

The stabilization term added in (2.52) and (2.53) modifies the matrix system (2.43) as,

$$\begin{pmatrix} \boldsymbol{K} & \boldsymbol{B}^T & \boldsymbol{G} \\ \boldsymbol{B} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{G}^T & \boldsymbol{0} & \boldsymbol{G}_{LS} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{b}_{cd} \\ \boldsymbol{h}^* \end{pmatrix} \tag{2.55}$$

where,

$$\boldsymbol{G}^e_{LS} = -\sum_{e=1}^{n_{el}} \tau_e(\boldsymbol{\nabla} q^h, \boldsymbol{\nabla} p^h) = -\int_{\Omega^e} \boldsymbol{\nabla}\hat{N} \cdot \boldsymbol{\nabla}\hat{N} d\Omega \tag{2.56}$$

$$\boldsymbol{h}^{*e} = \boldsymbol{h}^e - \sum_{e=1}^{n_{el}} \tau_e(\boldsymbol{\nabla} q^h, \boldsymbol{b}^h)_{\Omega^e} \tag{2.57}$$

Note that a non-zero matrix, G_{LS}, is added in the diagonal improving the condition number of the global matrix. That makes the global matrix non-singular and therefore, the system has a unique solution.