MASTER'S DEGREE THESIS

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Author: Víctor González Alonso

Advisor: Maria Alberich-Carramiñana

Department: Departament de Matemàtica Aplicada I

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Facultat de Matemàtiques i Estadística

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Introduction.

The study of singularities of algebraic and analytic varieties is an old and still very attractive field of research. It combines a wide range of ideas and techniques arising from Algebra, Analysis, Geometry and Topology.

Amongst all singularities, those of plane curves are the most studied ones, and there is a well-established theory for its analysis and classification, mostly due to Noether, Zariski and Enriques.

In this work, we will follow the geometrical approach of Enriques' infinitely near points, developed and synthesized by Casas in its book 'Singularities of Plane Curves' [3].

Polar germs are one of the main tools to analyze plane curve singularities, because they carry very deep analytical information of the singularity. There have been lots of efforts in the literature (for example, in works of Le, Teissier, Merle and Kuo-Lu) trying to distinguish which of this information is in fact purely topological.

One of the main problems in this direction is recovering the cluster of singular points of a curve from some invariant associated to the polar germs. Most of the approaches of the last twenty years involve the use of a generic polar (works of Teissier [9, 10] and Merle [8]), but this is not the right invariant to look at and the problem remained unsolved. As far as the we are aware, the first positive answer in this direction has been given by Casas in [3], Theorem 8.6.4, where it is shown that the cluster of base points of the generic polars uniquely determines the singular points of the curve. It turns out that it is not enough to consider the singular points of a generic polar, since it is also necessary to take into account the non-singular points shared by all generic polars. It is a sort of local version of the known fact in projective geometry that the proper singular points of plane projective algebraic curves are exactly the proper base points of its polar curves.

The proof of Casas' result works int two steps. The first one is to recover the polar invariants (which are topological invariants of the singularity computable from the polar germs), and the second one is a procedure involving a careful tracking of the Newton polygon of the iterated strict transforms of a generic polar.

The aim of this work is to obtain an algorithm which explicitly recovers the cluster of singular points from the cluster of base points of the polar germs. In particular, we obtain an alternative proof of Theorem 8.6.4 in [3] using just the Enriques' diagram of the generic polar at the origin, avoiding the blowing-ups appearing in Casas' method. To do this, we reinterpret the problem in terms of the theory of planar analytic morphisms, recently developed by Casas in [4].

This work achieves the goals of a Master's Thesis. On the one hand, we have developed skills to study plane curve singularities with a deep knowledge of polar germs and analytic morphisms. On the other hand, most of the results in chapters 2 and 3 are either new or refinements of previous ones in the literature.

This memory is structured in three chapters. The first one is a survey of the theory needed to understand the rest of the work, summarizing the definitions and results needed to understand the next chapters. It is separated in three parts, devoted to infinitely near points, polar germs of singular curves and germs of planar analytic morphisms, respectively.

The second chapter contains all the technical results needed to obtain the algorithm, and although they are motivated from this specific problem, they are interesting on their own. It is also separated in three sections with very different contents. The first one explains how the problem can be stated in terms of morphisms between surfaces, the second one contains a detailed study of the invariants $I_\xi(p)$, and the last one relates the values of the curve with the heights of the iterated trunks of a concrete morphism.

The third and last chapter is devoted to solve the problem, justifying the final algorithm. The results of this chapter are developed in the two first sections, which are summarized in the third one giving rise to the algorithm.

However, to obtain such an explicit algorithm has not been easy at all, and we needed to work in detail with lots of specific examples, which have been included in an appendix because we consider they are also of interest.

Finally, I would like to end this introduction with some acknowledgements to all the people who have contributed somehow to the development of this Master Thesis.

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Chapter 1

Preliminaries.

1.1 Infinitely near points.

In this section we introduce infinitely near points, which are the basic tool we will use in the study of plane curve singularities. We start with its modern definition via blowing-up, explaining also what it means for a curve to have certain multiplicity at such a point. Then we define free and satellite points, and we introduce an associated binary relation called proximity. The next topic in the section is the definition of equisingularity type of a germ of curve, introducing a graphical tool to represent them: Enriques' diagrams. In the final part we deal with weighted clusters of infinitely near points, and, as especial examples, we study the clusters of base points of pencils and linear systems of germs of curves.

Almost all the results in this section have been extracted from [3], Chapters 3, 4 and 7, where the reader may find the proofs of the statements and more detailed explanations.

We assume the reader to be familiar with the notion of intersection multiplicity $[\xi.\zeta]_O$ between two plane curves ξ, ζ at a point O. We will omit the point O if there is no possible confusion. We will also assume the Halphen-Zeuthen formula:

Proposition 1.1.1. *If $s_1, \ldots, s_k$ are the Puiseux series of the branches of ξ, and $s'_1, \ldots, s'_{k'}$ are the Puiseux series of the branches of ζ, then*

$$[\xi.\zeta] = \sum_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant k'}} o_x(s_i(x) - s'_j(x)),$$

where $o_x(f(x))$ denotes the vanishing order of the (probably fractional) power series $f(x)$.

It will be sometimes useful to use the Halphen-Zeuthen formula under the next simplified form.

Corollary 1.1.2. *If $s_1, \ldots, s_k$ are the Puiseux series of the branches of ξ, and $g(x, y)$ is an equation for ζ, then*

$$[\xi.\zeta] = \sum_{1 \leqslant i \leqslant k} o_x(g(x, s_i(x))).$$

1.1.1 Blowing-up. Transforming germs of curves.

Let S be a smooth complex surface, and $O \in S$ a fixed point. The *blowing-up* of S at O is a smooth complex surface $\bar{S}$ together with a (dominant, analytic) morphism $\pi : \bar{S} \longrightarrow S$ such that

- π induces an (analytic) isomorphism $\bar{S} - \pi^{-1}(O) \xrightarrow{\cong} S - \{O\}$, and
- $E_O = \pi^{-1}(O)$ is isomorphic to the projective line $\mathbb{P}^1_{\mathbb{C}}$.

The line E_O is called the *exceptional divisor* of the blowing-up, and is naturally identified with the tangent directions of S at O. We will denote E_O just by E when no confusion may arise with the point O which has been blown up.

The blowing-up $\bar{S}$ may be constructed explicitly, proving thus its existence (see for example [3], Chapter 3). Despite the construction seems to depend on the choice of some local coordinates in a neighbourhood of O, it is actually intrinsic. In fact, if $\pi' : \bar{S}' \longrightarrow S$ is another blowing-up, there exists a unique isomorphism $\varphi : \bar{S} \longrightarrow \bar{S}'$ such that $\pi' \circ \varphi = \pi$. Moreover, φ induces a linear projectivity between the two exceptional divisors.

Let ξ be a curve on S. The pull-back $\bar{\xi} = \pi^*(\xi)$ is called the *total transform* of ξ by π. It may be written as $\tilde{\xi} + e_O(\xi)E$, where $e_O(\xi)$ is the multiplicity of ξ at O and $\tilde{\xi}$ is a curve on $\bar{S}$ not containing the exceptional divisor E, called the *strict transform* of ξ by π.

Moreover, $\tilde{\xi}$ is the closure of $\pi^{-1}(\xi - \{O\})$ and intersects E at a finite number of points. If the intersection $\tilde{\xi} \cap E$ is considered as a divisor on $E \cong \mathbb{P}^1_{\mathbb{C}}$, it can be identified with the tangent cone of ξ at O, and so we have

$$e_O(\xi) = \sum_{p \in E} [\tilde{\xi}.E]_p$$

In particular, $e_p(\tilde{\xi}) \leq e_O(\xi)$ for any $p \in E$.

Note that if ξ is irreducible, then so is $\tilde{\xi}$, and that if ξ is smooth and contains O, then $\tilde{\xi}$ is also smooth, isomorphic to ξ via (the restriction of) π, and intersects E transversally at a single point (which corresponds to the tangent line to ξ at O).

Note that this notions may also be defined for ξ a germ of curve at O, and then $\bar{\xi}$ and $\tilde{\xi}$ are germs of curve in a neighbourhood of the exceptional divisor E. From now on, we will make no difference between curves and germs of curve at O (obviously, if a curve ξ does not contain O, it defines the empty germ at O).

1.1.2 Infinitely near points. Multiplicities and values.

The exceptional divisor E_O of the blowing-up of O is called the *first infinitesimal neighbourhood* of O, and its points are called *points in the first (infinitesimal) neighbourhood* of O. Inductively, for any $i > 1$, we can define the *points in the i-th neighbourhood* of O to be the points in the first neighbourhood of any point in the $(i-1)$-th neighbourhood of O. By convention, O is said to be in the zeroth infinitesimal neighbourhood of O. The points in any of this neighbourhoods of O are called *points infinitely near* to O, and the set of all of them is denoted by $\mathcal{N}_O$.

For any $p, q \in \mathcal{N}_O$, we write $p \leqslant q$ (and read it p *precedes* q) if and only if q is infinitely near p. This gives a partial order relation in $\mathcal{N}_O$. We also write $p < q$ if $p \leqslant q$ but $p \neq q$.

By its own definition, a point p infinitely near to O is a point lying in a surface S_p, which may be obtained by successively blowing-up the points preceding p. We denote by $\pi_p : S_p \longrightarrow S$ the composition of these blowing-ups.

Now let ξ be a (germ of) curve and $p \in \mathcal{N}_O$ any point infinitely near to O. We define the *total* (resp. *strict*) *transform* of ξ at p, denoted $\bar{\xi}_p$ (resp. $\tilde{\xi}_p$), as the iterated total (resp. strict) transform by the sequence of blowing-ups giving rise to the surface S_p where p lies as an ordinary point. The total transform may also be defined as the pull-back $\pi_p^*(\xi)$. Note that the strict transform of the empty germ (given by any equation invertible at O) at any infinitely near point p is also empty.

We define $e_p(\xi)$, the *multiplicity* of ξ at p (or the *multiplicity* of p on ξ), as the multiplicity $e_p(\xi_p)$ of the strict transform at p. Taking the total transform $\bar{\xi}_p$ instead of the strict one, we may define $v_p(\xi) = e_p(\bar{\xi}_p)$, the *value* of ξ at p. Note that both multiplicities and values are a generalization of the multiplicity at O. Note also that since they are defined via multiplicities at p, both of them are additive, that is, for any two curves ξ and ζ it holds

$$e_p(\xi + \zeta) = e_p(\xi) + e_p(\zeta) \qquad \text{and} \qquad v_p(\xi + \zeta) = v_p(\xi) + v_p(\zeta).$$

We say that a point $p \in \mathcal{N}_O$ *lies* on ξ if and only if $e_p(\xi) > 0$. The set of infinitely near points lying on ξ is denoted by $\mathcal{N}_O(\xi)$. A point $p \in \mathcal{N}_O(\xi)$ is said to be *simple* (resp. *multiple*) on ξ if and only if $e_p(\xi) = 1$ (resp. $e_p(\xi) > 1$).

If ξ is irreducible, there is only one point on ξ in the first neighbourhood of any $p \in \mathcal{N}_O(\xi)$. Thus, the set $\mathcal{N}_O(\xi)$ is totally ordered by $\leqslant$, and the sequence of multiplicities $e_p(\xi)$ is non-increasing. In particular, ξ is smooth at O if and only if $e_p(\xi) = 1$ for any $p \in \mathcal{N}_O(\xi)$.

The first property of infinitely near points and multiplicities is that they may be used to compute the intersection multiplicity of two germs of curve:

Theorem 1.1.3. *(Noether's Formula). Let ξ and ζ be two curves defined in a neighbourhood of O. The intersection multiplicity at O, $[\xi.\zeta]_O$, is finite if and only if ξ and ζ share finitely many points infinitely near to O, and in such a case*

$$[\xi.\zeta]_O = \sum_{p \in \mathcal{N}_O(\xi) \cap \mathcal{N}_O(\zeta)} e_p(\xi) e_p(\zeta)$$

1.1.3 Proximity. Free and satellite points.

In addition to the partial order $\leqslant$, there is another relation between points infinitely near to O, called proximity, which we describe now.

Let $p, q \in \mathcal{N}_O$. We say that q is *proximate* to p, denoted $q \to p$, if and only if q belongs (as an ordinary or infinitely near point) to the exceptional divisor E_p obtained by blowing-up the point p. It may be equivalently defined that the points proximate to p are those points in the first neighbourhood of p or in any of its strict transforms.

Note that the points proximate to p are all points in the first neighbourhood of p and all points in the successive neighbourhoods of p which lie on one of the successive strict transforms of E_p. Moreover, since the exceptional divisor E_p is irreducible (in fact, it is smooth), in the first neighbourhood of any point proximate to p there is only one point proximate to p. Finally, note that $q \to p$ implies $q > p$, but not conversely.

Proximity enables us to state an important relation between the multiplicities of a curve at the infinitely near points lying on it:

Theorem 1.1.4. *(Proximity equalities) For any curve ξ and any point $p \in \mathcal{N}_O(\xi)$, it holds*

$$e_p(\xi) = \sum_{q \to p} e_q(\xi).$$

Corollary 1.1.5. *With the notation of the preceding theorem, we have*

$$e_p(\xi) \geqslant \sum_{q \in E_p} e_q(\xi).$$

In the case $\xi = \gamma$ is an irreducible curve, 1.1.4 allows us to give a more precise description of its sequence of multiplicities:

Theorem 1.1.6. *Assume γ is an irreducible (germ of) curve, and let p, q be points on γ, q in the first neighbourhood of p. Write $n = e_p(\gamma)$ and $n' = e_q(\gamma)$ and perform the Euclidean division*

$$n = hn' + r, \qquad 0 \leqslant r < n'.$$

Then the points $q_1, \ldots, q_h$ in the first, ..., h-th neighbourhoods of q and proximate to p belong to γ with multiplicities

$$\begin{aligned} e_{q_i}(\gamma) &= n', \quad i = 1, \ldots, h-1 \\ e_{q_h}(\gamma) &= r. \end{aligned}$$

Moreover, no point proximate to p other than $q, q_1, \ldots, q_h$ belongs to γ.

Let $\pi : S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S$ be a finite composition of blowing-ups (π_i being the blowing-up of $p_{i-1} \in S_{i-1}$. The *exceptional divisor* of π is defined as the set-theoretical union $F = \bigcup_{i=1}^{n} (\pi_i \circ \cdots \circ \pi_n)^{-1}(p_{i-1})$ of the (total or strict) transforms of the exceptional divisors of each π_i. It is known that F is the union of finitely many smooth curves (each isomorphic to $\mathbb{P}^1_{\mathbb{C}}$). Furthermore, any two of this curves either do not intersect or intersect transversally at a single point, and no three of them have a common point.

In particular, if p is any point infinitely near to O, we can consider $\pi_p : S_p \longrightarrow S$ the composition of the blowing-ups of the points preceding p. The preceding considerations imply that the point p belongs to either one or two components of the exceptional divisor $F = \pi_p^{-1}(O)$. In the former case we say p is a *free* point, and in the latter, p is a *satellite* point. We also say that a satellite point p is *satellite of* the last free point q preceding it, or that p is a *q-satellite point*. Sometimes it is useful to consider a free point p to be satellite of itself.

It follows straightforward from the definitions that all points in the first neighbourhood of O are free and that in the first neighbourhood of a free (resp. satellite) point there is exactly one (resp. two) satellite points. Proximity equalities 1.1.4 imply that a satellite point on a germ ξ must be proximate to some multiple point of ξ, and in particular there are no satellite points lying on a smooth germ.

Proximity equalities are relations between the multiplicities of a curve ξ at the points infinitely near to O. There are also some relations between the values $v_p(\xi)$, relating them to the multiplicities:

- If p is a free point, proximate to q, then

$$v_p(\xi) = e_p(\bar{\xi}_p) = e_p(\tilde{\xi}_p) + e_q(\bar{\xi}_q) = e_p(\xi) + v_q(\xi). \tag{1.1}$$

- If q is a satellite point, proximate to q_1 and q_2, then

$$v_p(\xi) = e_p(\bar{\xi}_p) = e_p(\tilde{\xi}_p) + e_{q_1}(\bar{\xi}_{q_1}) + e_{q_2}(\bar{\xi}_{q_2}) = e_p(\xi) + v_{q_1}(\xi) + v_{q_2}(\xi). \tag{1.2}$$

1.1.4 Equisingularity. Enriques Diagrams.

One of the main applications of blowing-ups is the embedded resolution of plane curve singularities. That is, if ξ is a reduced curve contained in a surface S, there exists a composition of a finite number of blowing-ups, $\tilde{\pi} : \tilde{S} \longrightarrow S$ such that the strict transform $\tilde{\xi}$ of ξ is a smooth curve on $\tilde{S}$. By blowing-up the points of intersection of the strict transform $\tilde{\xi}$ with the exceptional divisor of π, it is possible to obtain another composition of blowing-ups $\bar{\pi} : \bar{S} \longrightarrow S$ such that the set-theoretical preimage $\bar{\pi}^{-1}(\xi)$ has at most double nodes as singularities. This is called a *embedded resolution of singularities* of ξ. Such a resolution is called *minimal* if the number of blowing-ups is minimal.

The minimal embedded resolution of singularities is really easy to understand from the viewpoint of infinitely near points. Let ξ be a germ of curve at O. A point $p \in \mathcal{N}_O(\xi)$ is called *singular* if and only if either

- p is multiple on ξ, or
- p is a satellite point, or
- p precedes a satellite point.

Equivalently, p is non-singular if and only if it is free and there is no satellite point $q > p$ on ξ.

Singular points may also be characterized as follows:

Lemma 1.1.7. *Let p be an infinitely near point, and E_p the germ of the exceptional divisor $\pi_p^{-1}(O)$ at $p \in S_p$. The point p is a singular point of a germ ξ if and only if $[\tilde{\xi}_p.E_p] > 1$.*

Let $\mathcal{S}(\xi)$ be the set of singular points of ξ. Let $\gamma_1, \ldots, \gamma_s$ be the branches of ξ, and for each i let p_i be the first point on γ_i which is non-singular on ξ. Denote by $\tilde{\mathcal{S}}(\xi)$ the set $\mathcal{S}(\xi) \cup \{p_1, \ldots, p_s\}$. This sets convey a lot of information about the singularity of ξ at O, motivating the definition of equisingularity.

Two germs of curve ξ and ζ are *equisingular* if both are reduced and non-empty, and there exists a bijection $\varphi : \tilde{\mathcal{S}}(\xi) \longrightarrow \tilde{\mathcal{S}}(\zeta)$ such that both φ and φ^{-1} preserve natural ordering and proximity relations on infinitely near points. The bijection φ is called an *equisingularity (map)* between ξ and ζ. It is clear that being equisingular is an equivalence relation on the set of non-empty reduced germs of curve, and its corresponding classes are called *equisingularity classes* or *equisingularity types*.

Equisingularity might also be defined in terms of $\mathcal{S}(\xi)$ and $\mathcal{S}(\zeta)$, just by adding the condition that φ preserves multiplicities of the maximal points of $\mathcal{S}(\zeta)$, (note that proximity equalities allows us to compute the multiplicities of ξ at every point $p \in \tilde{\mathcal{S}}(\xi)$.)

Thus, the equisingularity class of ξ is determined either by the set $\tilde{\mathcal{S}}(\xi)$, or by the set $\mathcal{S}(\xi)$ together with the multiplicities of ξ at the maximal singular points.

Equisingularity of a (germ of) curve ξ is closely related to the topology of the inclusion $\xi \subseteq S$. Two reduced curves ξ and ζ through O are *topologically equivalent* (at O) if there exist two neighbourhoods $U, V \subseteq S$ of U and an homeomorphism $h : U \xrightarrow{\cong} V$ restricting to an homeomorphism $U \cap \xi \xrightarrow{\cong} V \cap \zeta$. Despite the differences in the definitions of equisingularity and topological equivalency, they are the same relation, i.e., two curves are equisingular if and only if they are topologically equivalent.

The sets $\mathcal{S}(\xi)$ and $\tilde{\mathcal{S}}(\xi)$ are examples of a class of special subsets of $\mathcal{N}_O$: clusters.

A *cluster* of points infinitely near to O is a finite subset $K \subseteq \mathcal{N}_O$ such that if p belongs to K, so does any point $q < p$. O is called the *origin* of the cluster. The *depth* of K is the maximum of the orders of the neighbourhoods of O which contain some point of K.

There are many ways of representing a cluster K. One of the possibilities is a tree (graph without cycles), whose vertices are identified with the points of K, and there is and edge between two vertices if and only if one of them lies in the first neighbourhood of the other one. This tree is in fact a rooted tree, with root at the vertex corresponding to the origin O.

However, this representation only encodes the natural order $<$, forgetting proximity. This problem is solved by the *Enriques diagram* of K (the reason for this name is that they were first introduced by Enriques in [5]). It is also a tree like the one described before, but the edges are drawn in two ways, curved or straight, according to the following rules:

- If q is free and proximate to p, then the edge joining p and q is a smooth curve which if $p \neq O$ has the same tangent at p as the edge ending at p.

- If points p and q (q in the first neighbourhood of p) have been represented, the rest of the points proximate to p in the successive neighbourhoods of q (and the corresponding edges) are represented on a straight half-line starting at q and orthogonal to the edge joining it with p. To avoid self-intersections in the diagram, such half-lines are drawn alternatively to the right and to the left of the preceding one.

In particular, the Enriques diagram of $\tilde{\mathcal{S}}(\xi)$ will be called the *Enriques diagram* of ξ, and is a representation of the singularity of ξ at O.

1.1.5 Weighted clusters and virtual transforms.

A *system of virtual multiplicities* for a cluster K is a map $\nu : K \longrightarrow \mathbb{Z}$, i.e., an assignation of an integer $\nu_p = \nu(p)$ to each point $p \in K$. A *weighted cluster* is a pair $\mathcal{K} = (K, \nu)$, where K is a cluster and ν is a system of virtual multiplicities for it.

The points of a weighted cluster $\mathcal{K} = (K, \nu)$ are by definition the points of K, and we will sometimes write $p \in \mathcal{K}$ meaning that $p \in K$. If n is a positive integer, we denote by $n\mathcal{K}$ the weighted cluster $(K, n\nu)$ consisting of the same points as $\mathcal{K}$, but whose virtual multiplicities are multiplied by n.

A (germ of) curve ξ *goes through* O *with virtual multiplicity* ν_O (or through the weighted cluster (O, ν_O)) if $e_O(\xi) \geqslant \nu_O$. In this case, the *virtual transform* of ξ relative to the virtual

multiplicity ν_O is the curve

$$\check{\xi} = \bar{\xi} - \nu_O E = \tilde{\xi} + (e_O(\xi) - \nu_O)E$$

We may understand the virtual transform as a generalized strict transform, considering O with multiplicity ν_O instead of $e_O(\xi)$.

We may extend this definition to any weighted cluster of positive depth as follows. Let $p_1, \ldots, p_s$ be the points of $\mathcal{K}$ in the first neighbourhood of O, and let K_i be the cluster with origin at p_i that contains p_i and all points infinitely near to it in K. The restriction of ν to K_i is a system of virtual multiplicities ν_i on K_i, and gives rise to a weighted cluster $\mathcal{K}_i = (K_i, \nu_i)$.

Since the depth of the clusters $\mathcal{K}_i$ is strictly less than the depth of $\mathcal{K}$, we can make the following inductive definition. A curve ξ *goes through* $\mathcal{K}$ (or through K with virtual multiplicities ν) if and only if

1. ξ goes through O with virtual multiplicity ν_O, and

2. the virtual transform of ξ relative to the virtual multiplicity ν_O goes through $\mathcal{K}_i$ for $i = 1, \ldots, s$.

In this case, we can define the virtual transform $\check{\xi}_p$ at a point $p \in K$ relative to the virtual multiplicities ν as the iterated virtual transform at the points $q \leqslant p$.

A curve ξ *goes through* $\mathcal{K}$ *with effective multiplicities equal to the virtual ones* if and only if $e_p(\xi) = \nu_p$ for all points $p \in K$.

A curve ξ may go through $\mathcal{K}$ although $e_p(\xi) \neq \nu_p$, or even $e_p(\xi) < \nu_p$, for some $p \in K$. In fact, it is possible that there is no germ going through $\mathcal{K}$ with effective multiplicities equal to the virtual ones, it may happen because the multiplicities of a curve satisfy the proximity equalities, while we have imposed no such conditions on the system of virtual multiplicities. This motivates the next definition.

A cluster $\mathcal{K} = (K, \nu)$ is *consistent* if and only if

$$\nu_p \geqslant \sum_{\substack{q \to p \\ q \in K}} \nu_q$$

for every point $p \in K$. This definition solves the preceding problem, as the following result asserts.

Theorem 1.1.8. *If there is a curve going through $\mathcal{K}$ with effective multiplicities equal to the virtual ones, then $\mathcal{K}$ is consistent. Conversely, if $\mathcal{K}$ is consistent and T is a finite set of points infinitely near to O not in K, then there exists a curve going through $\mathcal{K}$ with effective multiplicities equal to the virtual ones and missing all points in T.*

In particular, it follows that the curves going through a consistent cluster $\mathcal{K}$ with effective multiplicities equal to the virtual ones share no points other than those in K.

We study now some special curves going through a consistent cluster $\mathcal{K}$.

We say that a curve ξ *goes sharply* through $\mathcal{K}$ if and only if it goes through $\mathcal{K}$ with effective multiplicities equal to the virtual ones and has no singular points outside of K.

For each point $p \in K$ we define the *excess* of $\mathcal{K}$ at p as

$$\rho_p = \nu_p - \sum_{\substack{q \to p \\ q \in K}} \nu_q.$$

Note that $\mathcal{K}$ is consistent if and only if $\rho_p \geqslant 0$ for every $p \in K$. In such a case, the germs going sharply through $\mathcal{K}$ have exactly ρ_p points in the first neighbourhood of p not in K, and this points are non-singular. In fact, much more can be proved.

Proposition 1.1.9. *If $\mathcal{K} = (K, \nu)$ is a consistent cluster, the following claims hold.*

1. *All germs going sharply through $\mathcal{K}$ are reduced.*
2. *If ξ goes sharply through $\mathcal{K}$, then, for each $p \in K$, ξ has just ρ_p branches through p missing all points after p in K. Hence ξ has a total of $\sum_{p \in K} \rho_p$ branches.*
3. *Any two curves going sharply through $\mathcal{K}$ are equisingular.*

If $\mathcal{K} = (K, \nu)$ is a consistent weighted cluster and $p \in K$, we will say that p is a *dicritical* point of $\mathcal{K}$ if and only if p has positive excess.

We say that a consistent cluster $\mathcal{K}$ is *irreducible* if the curves going sharply through it are irreducible as well. Equivalently, $\mathcal{K}$ is irreducible if its set of points is totally ordered (we say that such a cluster is *unibranched*) and has excess zero at every point but the maximal one, which has excess one.

We say that two (not necessarily weighted) clusters K, K' are *similar* if and only if there is a bijective map $\varphi : K \longrightarrow K'$ (called *similarity*) so that both φ and φ^{-1} preserve ordering and proximity. In the weighted case, we demand that φ and φ^{-1} also preserve virtual multiplicities. Clearly, excesses are also preserved by similarity, so germs going sharply through similar consistent clusters are equisingular.

Let us introduce some notations which will be useful in the sequel.

If $\mathcal{K} = (K, \nu)$ is a consistent cluster with origin at O, and ξ is a germ of curve at O, we define the *intersection multiplicity* of $\mathcal{K}$ and ξ as

$$[\mathcal{K}.\xi] = [\xi.\mathcal{K}] = \sum_{p \in K} \nu_p e_p(\xi).$$

Clearly, $[\mathcal{K}.\xi]$ equals the intersection multiplicity of ξ with any germ going through $\mathcal{K}$ with virtual multiplicities equal to the virtual ones and sharing no point with ξ outside of K.

If $\mathcal{K}' = (K', \nu')$ is a second weighted cluster with origin at O, we define its intersection

$$[\mathcal{K}.\mathcal{K}'] = \sum_{p \in K \cap K'} \nu_p \nu_{p'}.$$

In particular, we define the *self-intersection* of $\mathcal{K}$ as

$$\mathcal{K}^2 = [\mathcal{K}.\mathcal{K}] = \sum_{p \in K} \nu_p^2.$$

Obviously, $\mathcal{K}^2 = [\xi.\zeta]$ for any couple of germs ξ, ζ going sharply through $\mathcal{K}$ and sharing no points outside it.

1.1.6 Base points of pencils and linear systems.

Let $\mathcal{O} = \mathcal{O}_{S,O}$ be the local ring of S at O, that is, the ring of germs of holomorphic functions at O, and let $\mathcal{M} = \mathcal{M}_{S,O}$ be its maximal ideal, consisting of the functions vanishing at the origin. Once a system of local coordinates (x, y) centered at O is fixed, $\mathcal{O}$ is naturally isomorphic to $\mathbb{C}\{x, y\}$, the ring of convergent power series in two variables, and $\mathcal{M} = (x, y)$ is the ideal generated by the two coordinate functions.

A *linear family* of curves at O is a set $\mathcal{L}$ of curves whose equations are the non-zero elements of a linear subspace $F \subseteq \mathcal{O}$, i.e, a set of the form $\mathcal{L} = \{\xi : f = 0 \,|\, f \in F - \{0\}\}$. If $u \in \mathcal{O}$ is an invertible germ of function (that is, $u(0) \neq 0$), F and uF determine the same linear family, so the linear family $\mathcal{L}$ does not determine the subspace F.

Examples:

- If the subspace F is an ideal I, the linear family is called a *linear system*. In this case I is uniquely determined by $\mathcal{L}$, since it is the minimal ideal containing all the equations of each germ in $\mathcal{L}$. Note that there is a well defined minimal linear system containing a finite number of germs $\xi_i : f_i = 0, i = 1, \ldots, r$, namely, the linear system determined by the ideal $(f_1, \ldots, f_r)$. It is called the linear system *generated* by $\xi_1, \ldots, \xi_r$.

- The linear system defined by the whole ring $\mathcal{O}_{S,O}$ is called the *irrelevant system*.

- A linear system of the form $\mathcal{P} = \{\xi_\lambda : \lambda_1 f_1 + \lambda_2 f_2 = 0 \,|\, (\lambda_1, \lambda_2) \in \mathbb{C}^2 - \{(0, 0)\}\}$, where $f_1 = 0$ and $f_2 = 0$ are different non-empty germs, is called a *pencil*. It has the structure of a projective line with projective coordinates $[\lambda_1 : \lambda_2]$.

- A pencil $\mathcal{L}$ of smooth germs with variable tangent is called a *pencil of lines*. In suitable coordinates (x, y), we may assume it is given by $\mathcal{L} = \{\lambda_1 x - \lambda_2 y = 0\}$.

If g is the greatest common divisor of all elements in F, then we may write $F = gF'$ where F' is a linear subspace and the greatest common divisor of its elements is 1. The germ $\zeta : g = 0$ is contained in every germ $\xi \in \mathcal{L}$, and is called the *fixed part* of $\mathcal{L}$. If $g = 1$ we say that $\mathcal{L}$ has no fixed part. The linear family $\mathcal{L}'$ defined by F' is the *variable part* of $\mathcal{L}$, and has no fixed part.

If $F = I$ is an ideal, so is F', and hence $\mathcal{L}$ and $\mathcal{L}'$ are both linear systems. In this case, the condition of having no fixed part is obviously equivalent to say that there is no principal proper ideal containing I, that is, $I = \mathcal{O}$ is the whole ring (and so $\mathcal{L}$ is the irrelevant system) or I is $\mathcal{M}$-primary (that is, I contains a power $\mathcal{M}^e$ of the maximal ideal).

We list now some basic properties of pencils which will be useful later on.

Lemma 1.1.10.

- *If two germs in a pencil have the same multiplicity but different tangent cones, then all germs in the pencil have the same multiplicity and no two have the same tangent cone.*

- *If two different germs in a pencil either have the same tangent cone or different multiplicities, then all germs in the pencil have the same multiplicity and the same tangent cone, but for a single germ whose multiplicity is higher.*

- *If $\mathcal{P}$ is a pencil without fixed part and γ is an irreducible germ, then there is a germ $\xi_0 \in \mathcal{P}$ such that $[\gamma.\xi] = [\gamma.\xi'] < [\gamma.\xi_0]$ for any $\xi, \xi' \in \mathcal{P}$, $\xi, \xi' \neq \xi_0$, i.e., all the intersection multiplicities $[\xi.\gamma]$ are equal but for a single germ ξ_0, whose intersection with γ is higher than those of the other germs in $\mathcal{P}$.*

- *If $\mathcal{P}$ is any pencil, all intersection multiplicities $[\xi.\zeta]$, $\xi \neq \zeta \in \mathcal{P}$, are equal.*

We now focus on linear systems and their base points. Recall that there is a topology on $\mathcal{O}$, the *Zariski topology*, whose closed subsets are those of the form

$$\{f = \sum_{i,j \geqslant 0} a_{i,j} x^i y^j \mid P(a_{i,j}) = 0, \, \forall P \in \mathcal{A}\}.$$

Here an isomorphism $\mathcal{O} \cong \mathbb{C}\{x, y\}$ has been fixed, and $\mathcal{A}$ is a subset of the ring of polynomials in the (infinitely many) variables $X_{i,j}, i, j \geqslant 0$. Thus, a Zariski-closed set is the set of all convergent series whose coefficients satisfy some algebraic relations. Since with this topology the ideals of $\mathcal{O}$ are irreducible subsets, it is possible to give a precise meaning to the sentence "generic germ in a linear system $\mathcal{L}$": it means "germ in a non-empty Zariski-open subset of the ideal I defining $\mathcal{L}$".

For the rest of this section, $\mathcal{L}$ will be a non-irrelevant linear system without fixed part defined by the ideal I. These linear systems are called *neat* linear systems.

Lemma 1.1.11. *Let ζ be a fixed germ at O, and let $k = \min\{[\zeta.\zeta'] \mid \zeta' \in \mathcal{L}\}$. Then the set U of all elements $g \in I$ for which $\xi : g = 0$ has $[\zeta.\xi] = k$ is a non-empty Zariski-open subset of I.*

Thus, the intersection multiplicity with a fixed germ ζ is an upper-semicontinuous function on any ideal I.

In order to define the cluster of base points of $\mathcal{L}$, we have to define some sort of virtual transform of $\mathcal{L}$ at the origin, and then extend this definition inductively to the whole infinitesimal neighbourhood $\mathcal{N}_O$. Let $e = e_O(\mathcal{L}) = \min\{e_O(\xi) \mid \xi \in \mathcal{L}\}$ be the *multiplicity* of $\mathcal{L}$ at O. Thus, I is contained in $\mathcal{M}^e$, but not in $\mathcal{M}^{e+1}$. Clearly, if $\Xi \subseteq \mathcal{L}$ is a set of germs generating $\mathcal{L}$, then also $e_O(\mathcal{L}) = \min\{e_O(\xi) \mid \xi \in \Xi\}$.

If p is any point in the first neighbourhood of O, let $\mathcal{O}_p = \mathcal{O}_{S_p,p}$ be the local ring at p in the surface S_p, and let $\varphi_p : \mathcal{O} \longrightarrow \mathcal{O}_p$ be the morphism of local rings induced by the blowing-up of O. Let $z \in \mathcal{O}_p$ be an equation at p for the exceptional divisor. The ideal $I_p \subseteq \mathcal{O}_p$ generated by $z^{-e}\varphi_p(I)$ is clearly independent of the choice of z and defines a linear system $\mathcal{L}_p$ at p. This system is by definition the *transform of $\mathcal{L}$ with origin at p*. This definition is extended inductively to the points in successive neighbourhoods of O. For any $p \in \mathcal{N}_O$ the multiplicity $e_p(\mathcal{L})$ of $\mathcal{L}$ at p is by definition the multiplicity of its transform $\mathcal{L}_p$.

Now we have defined a system of virtual multiplicities $e_p(\mathcal{L})$ in the whole $\mathcal{N}_O$. But if $\mathcal{L}$ is a linear system without fixed part, then for all but finitely many points p infinitely near to O, the transform $\mathcal{L}_p$ is irrelevant. Thus there is only a finite set $K \subseteq \mathcal{N}_O$ such that $e_p(\mathcal{L}) > 0$ if and only if $p \in K$. Since any transform of an irrelevant system is again irrelevant, K is a cluster. We denote by $BP(\mathcal{L})$ the weighted cluster whose underlying set of points is K, with virtual multiplicities $e_p(\mathcal{L})$. This cluster is the *cluster of base points of $\mathcal{L}$*.

This definition may be extended to arbitrary linear families $\mathcal{L}$, just applying them to the linear system $\bar{\mathcal{L}}$ generated by $\mathcal{L}$, and in particular to pencils $\mathcal{P}$. We define the cluster of base points of any linear family, $BP(\mathcal{L})$, as the cluster $BP(\bar{\mathcal{L}})$ of base points of the liner system it defines.

The main properties of base points of pencils and linear systems without fixed part are listed in the next theorems.

Theorem 1.1.12. *If $\mathcal{P}$ is a pencil of germs at O without fixed part, then*

1. *All germs in $\mathcal{P}$ go through $BP(\mathcal{P})$.*
2. *All but finitely many germs in $\mathcal{P}$ go sharply through $BP(\mathcal{P})$.*
3. *No point besides the base ones lies on two different germs in $\mathcal{P}$ going sharply through $BP(\mathcal{P})$.*
4. *All but finitely many germs in $\mathcal{P}$ are reduced and have the same equisingularity type.*

Theorem 1.1.13. *Let $\mathcal{L}$ be a neat linear system of germs of curve at O, and T a finite set of points infinitely near to O, no one a base point of $\mathcal{L}$. All germs in $\mathcal{L}$ go through $BP(\mathcal{L})$. Generic germs in $\mathcal{L}$ go sharply through $BP(\mathcal{L})$ and miss all points in T and, in particular, they are reduced and have the same equisingularity type.*

As immediate consequences, we see that all clusters of base points of pencils or neat linear systems are consistent clusters. Also, a linear system $\mathcal{L}$ may be generated by a finite set of germs going sharply through $BP(\mathcal{L})$, any two of them sharing no point but the base ones.

Remark 1.1.14. *If $\mathcal{K} = BP(\mathcal{P})$ is the cluster of base points of a pencil $\mathcal{P}$ without fixed part, then lemma 1.1.10 implies that its self-intersection is equal to $\mathcal{K}^2 = [\mathcal{K}.\zeta] = [\xi.\zeta]$ for* any $\xi, \zeta \in \mathcal{P}, \xi \neq \zeta$.

1.2 Polar germs.

In this section we summarize the basic results about polar germs of curve. We start with its definition and its most basic properties. Then we define the polar invariants, which are topological invariants of the singularity which are related with its polar germs. We close this section with a summary of the properties of the cluster of base points of the jacobian system and its relation with the singular points of the curve.

1.2.1 Definition and first properties.

Let $\xi : f = 0$ be a non-empty germ of curve at O. Let g an analytic (germ of) function such that the curve $\eta : g = 0$ is smooth (and non-empty) at O, that is, such that at least one of the derivatives $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$ does not vanish at O.

The *g-polar (germ)* of ξ relative to the equation f is the curve $P_g(f)$ defined by the jacobian determinant

$$P_g(f) : \frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = 0.$$

A polar of ξ is just a $P_g(f)$ for some g defining a smooth curve at O and some equation f of ξ.

The definition of $P_g(f)$ does not depend on the choice of coordinates, so it is always possible to take $x = g$ (since g defines a smooth germ) and then the equation of the g-polar is just the derivative $\frac{\partial f}{\partial y}$.

If ξ is a multiple of the curve η, the polar germ is undefined because the jacobian identically vanishes. Then, we will always assume that ξ contains some irreducible component different from η. Furthermore, the case η being a component of ξ has little interest. Suppose $\xi = r\eta + \xi'$ for some $r > 0$ (or $f = g^r f'$ in terms of equations), then $P_g(f) = r\eta + P_g(f')$, i.e., the polar of ξ is composed by the same multiple of η and the polar of ξ'. So we will in fact assume that η is not a component of the curve ξ.

In general, the polar $P_g(f)$ depends on the equations f and g, and not only on the germs ξ and η, but the properties of $P_g(f)$ we are interested in are in fact independent of the equations chosen, so we will adopt the notation $P_g(\xi)$ when no confusion may arise.

The next proposition summarizes some basic properties of polar germs.

Proposition 1.2.1. *Assume ξ is not a multiple of η and denote $\zeta = P_g(\xi)$. Then we have:*

1. $[\zeta.\eta] = [\xi.\eta] - 1$.
2. $e_O(\zeta) \geqslant e_O(\xi) - 1$ *and the inequality is strict if and only if all branches of ξ are tangent to η.*
3. *If no branch of ξ is tangent to η, then no branch of the polar ζ is tangent to η either.*

We say that ζ is a *transverse polar* if and only if no branch of ξ is tangent to η. Otherwise we say that ζ is a *non-transverse polar*.

We now look at the polar germs from the point of view of linear systems. Consider the ideal $\mathbf{J}(\xi) = \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \subseteq \mathcal{O}$, which is called the *jacobian ideal* of ξ, and is independent of the choice of coordinates. It defines thus a linear system $\mathcal{J}(\xi)$, the *jacobian system* of ξ. This system is generated by the equations of the polar germs of ξ, and in fact, every $h \in \mathbf{J}(\xi)$ of the form

$$h = u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} + u_3 f$$

with either u_1 or u_2 invertible defines a polar of ξ.

It is clear from its definition and Proposition 1.2.1 that every germ $\zeta \in \mathcal{J}(\xi)$ has multiplicity $e_O(\zeta) \geqslant e_O(\xi) - 1$, and that those for which the equality holds are polar germs. Hence, there is a non-empty Zariski-open subset of $\mathbf{J}(\xi)$, namely $\mathbf{J}(\xi) - \mathcal{M}^{e_O(\xi)}$, whose elements define polar germs. Thus generic germs in $\mathcal{J}(\xi)$ can be assumed to be polar germs, and are called *generic polar germs* of ξ.

1.2.2 Polar invariants.

We now introduce the polar invariants of a non-empty reduced (germ of) curve ξ and derive some important results about polar germs. Although they hold true for any polar germ, the statements are much simpler if we restrict ourselves to the case of transverse polar germs. So we assume $\eta : g = 0$ to be non-tangent to any branch of ξ.

We say that a point $p \in \mathcal{N}_O(\xi)$ is a *rupture point* of ξ if either

1. p is a satellite point and there is at least one free point on ξ in its first neighbourhood, or

2. p is either O or a free point and there are at least two free points on ξ in its first neighbourhood.

We denote by $\mathcal{R}(\xi)$ the set of rupture points of ξ, and by $\mathcal{R}^p(\xi)$ the subset of rupture points which are satellite of the free point p (note that if p is itself a rupture point, we consider $p \in \mathcal{R}^p(\xi)$).

Remark 1.2.2. *Note that the maximal points of $\mathcal{S}(\xi)$ (with respect to the natural order $\leqslant$) are rupture points. Moreover, if $p \in \mathcal{N}(\xi)$ is any free point lying on ξ and q is another p-satellite point on ξ, then there exists some p-satellite rupture point which is infinitely near to q.*

For any point $q \in \mathcal{N}_O$, we denote by γ^q an irreducible germ through q whose point in the first neighbourhood of q is free and does not belong to ξ. We may also add the condition of such point to be non-singular on γ^q (or equivalently, $e_q(\gamma^q) = 1$), but it is not essential and in general we will not assume it. Note that in any case, the multiplicities of γ^q at the points preceding q are determined by the multiplicity $e_q(\gamma^q)$ and the Proximity equalities 1.1.4.

We define the rational number

$$I(q) = I_\xi(q) = \frac{[\xi.\gamma^q]}{e_O(\gamma^q)}$$

which is independent of the choice of the curve γ^q. The *polar invariants* of ξ are the quotients $I(q)$, where q a rupture point of ξ.

This definition of the polar invariants (and in general, of the invariants $I(q)$) is the classical one, but they may be defined without any reference to the curves γ^q. In fact, since the points on γ^q infinitely near to q do not lie on ξ, they are irrelevant to compute $I(q)$ (by the Noether's Formula 1.1.3) as long as the point in the first neighbourhood of q is free. Thus, if $\mathcal{K}(q)$ denotes the irreducible cluster whose last point is q, we can equivalently define

$$I(q) = I_\xi(q) = \frac{[\xi.\mathcal{K}(q)]}{\nu_O(\mathcal{K}(q))}.$$

The main result about (transverse) polar invariants is the next one.

Theorem 1.2.3. *Let $\zeta = P_g(\xi)$ be a transverse polar of a non-empty reduced germ of curve ξ, and let $\gamma_1, \ldots, \gamma_l$ be the branches of ζ. Then*

$$\left\{ \frac{[\xi.\gamma_i]}{e_O(\gamma_i)} \right\}_{i=1,\ldots,l} = \{I(q)\}_{q \in \mathcal{R}(\xi)}$$

Furthermore, if $p \in \mathcal{N}_O(\xi)$ is either O or any free point on ξ, the set of quotients $\frac{[\xi.\gamma]}{e_O(\gamma)}$, for γ a branch of ζ going through p and missing all free points after p on ξ is just $\{I(q)\}_{q \in \mathcal{R}^p(\xi)}$.

Note that the polar invariants can be computed from an Enriques diagram of ξ, so they are equisingularity invariants, while the polar germs of ξ depend on the analytic type of ξ. In fact, there are lots of examples (some of them appear in the last chapter) of equisingular germs whose polar germs (even the generic ones) are non-equisingular. So the polar germs of ξ carry very deep analytic information, and the point of the preceding result is that some of this information is in fact purely topological.

1.2.3 Base points of polar germs.

We close this section about polar germs summarizing some properties of the jacobian system $\mathcal{J}(\xi)$ of a non-empty singular reduced germ ξ and stating the most relevant property for our purposes (Theorem 1.2.7): the base points of the jacobian system $\mathcal{J}(\xi)$ uniquely determine the singular points of the curve ξ.

First of all, since ξ is reduced, $\mathcal{J}(\xi)$ has no fixed part, and we can apply Theorem 1.1.13 to obtain the next result.

Proposition 1.2.4. *If ξ is a singular reduced germ of curve, its jacobian system is neat and there is a non-empty Zariski-open subset of the jacobian ideal of ξ whose elements define polar germs of ξ which are all reduced and have the same equisingularity type.*

In fact, Theorem 1.1.13 also says that generic polars are equisingular and go sharply through $BP(\mathcal{J}(\xi))$, and in particular, they share all their singular points.

Recall that, by definition, $\mathcal{J}(\xi)$ is the linear system associated to $\mathbf{J}(\xi) = \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, but this definition is not easy to handle. The next result really simplifies the use (and the computation) of $BP(\mathcal{J}(\xi))$:

Lemma 1.2.5. *The cluster of base points of the jacobian system $BP(\mathcal{J}(\xi))$ is equal to the cluster of base points $BP(\partial f/\partial x, \partial f/\partial y)$ of the pencil spanned by the two partial derivatives of any equation f of ξ.*

We will also need the following result, characterizing the singular free points of ξ in terms of the jacobian system $\mathcal{J}(\xi)$.

Lemma 1.2.6. *A free point p is a singular point of a reduced singular germ of curve ξ if and only if p lies on ξ and belongs to $BP(\mathcal{J}(\xi))$.*

Note that in particular, every free singular point of ξ is a base point of the jacobian system.

Finally, we state the main result about polar germs, which is the core of this memory.

Theorem 1.2.7. *Let ξ_1 and ξ_2 be germs of curve, both reduced and singular:*

1. *If $BP(\mathcal{J}(\xi_1)) = BP(\mathcal{J}(\xi_2))$, then $\mathcal{S}(\xi_1) = \mathcal{S}(\xi_2)$.*
2. *If $BP(\mathcal{J}(\xi_1))$ and $BP(\mathcal{J}(\xi_2))$ are similar weighted clusters, then ξ_1 and ξ_2 are equisingular.*

Chapters 2 and 3 of this memory will be devoted to give a new proof of this result.

1.3 Germs of morphisms between surfaces.

In this section we summarize some concepts and results about analytic morphisms between surfaces which are fundamental for our purposes. After some basic definitions, we introduce the main tools in our analysis: the trunks of a morphism and its heights. Finally, we give the results that will relate these invariants with the jacobian of the morphism.

All the concepts in this section were first introduced by Casas in [4], where the reader is referred for more detailed explanations and the proofs of the results.

1.3.1 Basic definitions. Fundamental points.

Let S and T be two smooth complex surfaces, and $\varphi : S \longrightarrow T$ an analytic morphism between them. Our main interest is to study the local behaviour of φ, that is, its action near a point $O \in S$ and its image $O' = \varphi(O) \in T$.

We now define how φ acts on germs of curves at O and O'. On the one hand, if $\zeta : f = 0$ is a germ of curve at O', its *inverse image* is just the germ $\varphi^*(\zeta)$ given by the equation $\varphi^*(f) = f \circ \varphi = 0$. On the other hand, we can define the *direct image* of a germ at O as follows. If γ is an irreducible germ (at O) parameterized by σ, the composition $\varphi \circ \sigma$ either is constant or defines an irreducible germ γ' at O'. In the first case, we say that γ is a *contracted germ* and that its *direct image* $\varphi_*(\gamma)$ is just the point O'. In the other case, we take $\varphi_*(\gamma) = d\gamma'$, where d is the degree of the map $\varphi \circ \sigma$ (we must take into account folding phenomena). For any germ, not necessarily irreducible, we extend this definition by linearity.

The next useful result follows directly from these definitions:

Proposition 1.3.1. *(Projection formula.) Keeping notations as above, we have the equality*

$$[\varphi_*(\gamma).\zeta]_{O'} = [\gamma.\varphi^*(\zeta)]_O$$

in the sense that one side is finite if and only if the other side is, and in this case both coincide.

The pull-back operation is in fact a morphism $\varphi^* : \mathcal{O}_{T,O'} \longrightarrow \mathcal{O}_{S,O}$. From now on, we assume this morphism is a monomorphism, or equivalently, no analytic curve in a neighbourhood of O' contains the image of a neighbourhood of O. In this case, we say that φ is *dominant* in a neighbourhood of O, meaning that the image of a neighbourhood of O is dense in a neighbourhood of O'.

Since we are interested in the local properties of φ, we can fix systems of local coordinates (x, y) and (u, v) centered at O and O' respectively, and then identify the rings of germs of analytic functions with rings of convergent power series: $\mathcal{O}_{S,O} \cong \mathbb{C}\{x, y\}$ and $\mathcal{O}_{T,O'} \cong \mathbb{C}\{u, v\}$. With this identifications, given a function $h \in \mathcal{O}_{S,O}$ we denote by $\widehat{h} \in \mathbb{C}[x, y]$ its initial form, and by $o(h) = o_O(h) = \deg \widehat{h}$ its order. Note that o_O is the $\mathcal{M}_{S,O}$-adic valuation in $\mathcal{O}_{S,O}$.

In this setting, the morphism φ is defined by two non-invertible convergent series $f, g \in \mathbb{C}\{x, y\}$, called the *equations* of φ, mapping the point $(x, y) \in S$ to the point $(u, v) = (f(x, y), g(x, y)) \in T$.

Injectivity of φ^* is equivalent to the functional independence of the equations f and g. So if φ is dominant, the jacobian determinant

$$J(\varphi) = \frac{\partial(f,g)}{\partial(x,y)}$$

is not identically zero. It defines thus a curve at O, the *jacobian germ* (or just *jacobian*) of φ, which we denote by $\mathbf{J} = \mathbf{J}(\varphi)$. This will be a key object in our study, since a polar germ may be interpreted as the jacobian of a particular morphism. As in the case of the polar germs, the jacobian germ of φ at O is independent of the choice of coordinates.

Consider now the pencil $\mathcal{P}$ of germs at O generated by the equations f and g, that is, the family of germs $\mathcal{P} = \{\xi_{\lambda,\mu} : \lambda f + \mu g = 0 \,|\, [\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}\}$. Clearly, $\mathcal{P}$ is the pencil of inverse images of the pencil of lines $\mathcal{L} = \{l_{\lambda,\mu} : \lambda u + \mu v = 0 \,|\, [\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}\}$ at O'. By lemma 1.1.10, all germs in $\mathcal{P}$ but at most one have the same multiplicity at O, which is the minimum of $o(f)$ and $o(g)$. This multiplicity is called the *multiplicity* of φ at O, and it is denoted by $e_O(\varphi)$ or simply $e(\varphi)$. The case in which all the germs in $\mathcal{P}$ have the same multiplicity $e(\varphi)$ is the easiest one, and again by lemma 1.1.10, it occurs if and only if the initial forms $\widehat{f}$ and $\widehat{g}$ have the same degree and are linearly independent. In this case, φ is said to be *dicritical*. Otherwise, φ is *non-dicritical*, and all germs in $\mathcal{P}$ but one have the same multiplicity and tangent cone, given by the initial form of minimal degree.

Let $d = gcd(f,g)$ be the greatest common divisor of the local equations f, g in $\mathbb{C}\{x,y\}$ (which is a unique factorization domain, by Corollary 1.8.6 of [3]). The pencil $\mathcal{P}$ has fixed part $\Phi : d = 0$ and variable part $\mathcal{P}' = \{\xi_{\lambda,\mu} : \lambda\frac{f}{d} + \mu\frac{g}{d} = 0 \,|\, [\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}\}$. The germ Φ (which might be empty) is called the *contracted germ* by φ (because it is contracted to the point O') and is contained in the jacobian germ $\mathbf{J}$. If both $\frac{f}{d}$ and $\frac{g}{d}$ are non-invertible (in particular, if $\Phi = \emptyset$), the variable part $\mathcal{P}'$ is a pencil of germs at O without fixed part. Then it makes sense to consider its (weighted) cluster of base points, which by definition is the *cluster of base points of* φ, and is denoted by $BP(\varphi)$. Otherwise, if $\frac{f}{d}$ or $\frac{g}{d}$ is invertible, $\mathcal{P}'$ is a family whose germs but one are empty, and we just take $BP(\varphi) = \emptyset$.

Now we can define $e_p(\varphi)$, the *multiplicity* of φ at any point p infinitely near to O, as the sum of the multiplicity $e_p(\Phi)$ of the contracted germ at p, and the virtual multiplicity of p in $BP(\varphi)$ (or zero if $p \notin BP(\phi)$). This definition obviously extends the former one of $e_O(\varphi)$, and for any p equal or infinitely near O, all but finitely many germs in $\mathcal{P}$ have effective multiplicity $e_p(\varphi)$ at p. A point p will be called a *fundamental point* of φ if and only if $e_p(\varphi) > 0$, that is, if p either is a base point of φ or lies on the contracted germ Φ.

There is another possible generalization of the multiplicity $e(\varphi) = e_O(\varphi)$ to the points infinitely near to O. For any $p \in \mathcal{N}_O$ let $\pi_p : S_p \longrightarrow S$ be the composition of blowing-ups giving rise to p as an ordinary point on the surface S_p, and consider the composite morphism $\varphi_p = \varphi \circ \pi_p : S_p \longrightarrow T$ (more precisely, the germ of this morphism at $p \in S_p$). For $p \neq O$, these morphisms have always non-empty contracted germ Φ_p (since the germ of the exceptional divisor at p is contracted to O by π_p), and its base points $BP(\varphi_p)$ are exactly the base points of φ infinitely near to p with the same virtual multiplicities. The multiplicity of φ_p at p is denoted by $e(\varphi_p) = e_p(\varphi_p)$, and also by n_p if the morphism φ is clear by the context.

Note that the two possible generalizations of multiplicity at any point p infinitely near to O, namely $e_p(\varphi)$ and $e(\varphi_p)$, are somehow analogous to the definitions of multiplicity and value of a curve ξ at p, respectively. In fact, $e_p(\varphi)$ and $e_p(\xi)$ only take into account φ

or ξ, discarding the exceptional divisors at each blowing-up, while $e(\varphi_p)$ and $v_p(\xi)$ both accumulate the contribution of the exceptional divisors.

This analogy implies the following results, which are similar to the relations between the multiplicities and the values of a germ of curve. Their proofs can be found in [4].

Proposition 1.3.2. *If p is free, proximate to a single point p', then*

$$e(\varphi_p) = e_p(\varphi) + e(\varphi_{p'}).$$

If p is satellite, proximate to p' and p'', then

$$e(\varphi_p) = e_p(\varphi) + e(\varphi_{p'}) + e(\varphi_{p''}).$$

Corollary 1.3.3. *For all $p \in \mathcal{N}_O$, $e(\varphi_p) > 0$. If p lies in the first neighbourhood of p', then $e(\varphi_p) \geqslant e(\varphi_{p'})$ and equality holds if and only if p is free and non-fundamental.*

1.3.2 Trunks.

We now define the trunks of a morphism ϕ and its heights, which are fundamental objects in our study. We start with the definition of the main trunk of ϕ, and then extend this definition to any point infinitely near to O.

Let $\mathcal{L} = \{l_\alpha : \alpha \in \mathbb{P}^1_{\mathbb{C}}\}$ be a pencil of lines at O, and let $\gamma_\alpha = \varphi_*(l_\alpha)$ be its direct images. Assume also that (u, v) is a system of coordinates at O' such that φ is given by (f, g) and $n = e_O(\varphi) = o(f) \leqslant o(g)$. In [4], section 4, it is shown that all but a finite number of the germs γ_α may be parameterized as

$$(u, v) = \left(t^n, \sum_{i \geqslant n} a_i t^i\right)$$

where the coefficients $a_i = a_i(\alpha)$ are rational functions on α. Note that γ_α might not be irreducible: if n' is the minimal common denominator of the $\frac{i}{n}$ such that $a_i \neq 0$, γ_α is $\frac{n}{n'}$ times the irreducible germ with Puiseux series $v = \sum_{i \geqslant n} a_i t^i$.

Now, since φ is dominant, at least one coefficient a_i really depends on α (if it is not the case, the germs γ_α would always be the same, and the pull-back of its equation would be identically zero). Let m be the lowest index i for which a_i is non-constant.

Since the coefficients of the Puiseux series of a germ determine the position of its free points (see [3] Chapter 5), all these germs γ_α share all points depending on the coefficients a_i for $i < m$ (both free and satellite) and no further point. Moreover, the multiplicities of γ_α at this points are independent of α.

We define the *(main) trunk* $\mathcal{T} = \mathcal{T}(\varphi)$ of φ to be the weighted cluster of all points shared by all but finitely many of the γ_α, with virtual multiplicities $e_p(\gamma_\alpha)$. It is independent of the choice of the pencil of lines $\mathcal{L}$. Note that $n = e_O(\varphi)$ is the virtual multiplicity of O' in $\mathcal{T}$. By construction, $\mathcal{T} = r\mathcal{T}_{\text{red}}$, where $\mathcal{T}_{\text{red}}$ is an irreducible cluster and r is the *multiplicity* of the trunk. The *top* of $\mathcal{T}$ is its maximal point, and the *height* of $\mathcal{T}$ is the integer m. The (finite) series

$$\mathcal{S}(u, \theta) = \sum_{n \leqslant i < m} a_i u^{\frac{i}{n}} + \theta u^{\frac{m}{n}}$$

is called the *Puiseux series of* $\mathcal{T}$.

One of the most basic properties of the trunk is the next

Theorem 1.3.4. *If ζ is any germ of curve at O', then*

$$e_O(\varphi^*(\zeta)) = [\zeta.\mathcal{T}].$$

The trunk of a morphism is difficult to compute directly from the definition, because it involves the computation of the direct image of a generic line. However, there is an algorithm which computes it in an indirect way, obtaining a pencil $\mathcal{Q}$ of germs at O such that $BP(\mathcal{Q}) = \mathcal{T}_{\text{red}}$.

We need to introduce some definitions and notations before presenting the algorithm. We say that two homogeneous polynomials $F, G \in \mathbb{C}[x, y]$ (not necessarily of the same degree) are *homothetical* if there exist positive integers d, d' and a non-zero complex number $a \in \mathbb{C}^*$ such that $F^d = aG^{d'}$. Equivalently, F and G are homothetical if the divisors they define in $\mathbb{P}^1_{\mathbb{C}}$ are proportional (with rational coefficients). For any finite set of positive integers $n_1, \ldots, n_k$, we denote by $\langle n_1, \ldots, n_k\rangle$ the semigroup $\{\sum_{i=1}^k a_i n_i \,|\, a_i \in \mathbb{Z}, a_i \geqslant 0\}$ spanned by them.

With this conventions, the algorithm works as follows (recall that we are assuming $o(f) \leqslant o(g)$):

1. Start by taking $i = 0, j = 1, P_0 = u, h_{0,1} = v$.

2. Assume we have determined germs of function $P_0, \ldots, P_i, h_{i,1}, \ldots, h_{i,j} \in \mathcal{O}_{T,O'}$ for $i \geqslant 0, j \geqslant 1$ such that the initial forms of the pull-backs $\widehat{\varphi^*(P_0)}, \ldots, \widehat{\varphi^*(P_i)}$, $\widehat{\varphi^*(h_{i,1})}, \ldots, \widehat{\varphi^*(h_{i,j-1})}$ are all homothetical. Write $\check{m}_l = o(\varphi^*(P_l))$ for $l = 0, \ldots, i$ and $\Gamma_i = \langle \check{m}_0, \ldots, \check{m}_i\rangle$. Then:

   - If $o(\varphi^*(h_{i,j})) \in \Gamma_i$, then determine non negative integers $c_0, \ldots, c_i$ so that

     $$o(\varphi^*(h_{i,j})) = \sum_{l=0}^{i} c_l \check{m}_l$$

     and take

     $$Q_{i,j} = \prod_{l=0}^{i} P_l^{c_l}.$$

     – If $\widehat{\varphi^*(h_{i,j})}$ is homothetical to $\widehat{\varphi^*(P_0)}$, then take

     $$h_{i,j+1} = h_{i,j} - aQ_{i,j}$$

     where $a \in \mathbb{C}$ is chosen so that $\widehat{\varphi^*(h_{i,j})} - a\widehat{\varphi^*(Q_{i,j})} = 0$, and increase j by one.

     – If, otherwise, the initial form $\widehat{\varphi^*(h_{i,j})}$ is not homothetical to $\widehat{\varphi^*(P_0)}$, end by taking $\mathcal{Q} = \{\alpha_1 h_{i,j} + \alpha_2 Q_{i,j} = 0\}$.

   - If $o(\varphi^*(h_{i,j})) \notin \Gamma_i$, take $P_{i+1} = h_{i,j}, \check{m}_{i+1} = o(\varphi^*(P_{i+1})), \Gamma_{i+1} = \langle \check{m}_0, \ldots, \check{m}_{i+1}\rangle$ and $n' = \gcd(\check{m}_0, \ldots, \check{m}_i)/\gcd(\check{m}_0, \ldots, \check{m}_{i+1})$. Determine non negative integers $c_0, \ldots, c_i$ so that

     $$n' o(\varphi^*(P_{i+1})) = \sum_{l=0}^{i} c_l \check{m}_l$$

and take

$$Q_{i+1} = \prod_{l=0}^{i} P_l^{c_l}.$$

– If $\widehat{\varphi^*(P_{i+1})}$ is homothetical to $\widehat{\varphi^*(P_0)}$, then take

$$h_{i+1,1} = P_{i+1}^{n'} - aQ_{i+1}$$

where $a \in \mathbb{C}$ is chosen so that $\widehat{\varphi^*(P_{i+1})}^n - a\widehat{\varphi^*(Q_{i+1})} = 0$, increase i by one and set $j = 1$.

– If, otherwise, the initial form $\widehat{\varphi^*(P_{i+1})}$ is not homothetical to $\widehat{\varphi^*(P_0)}$, end by taking $\mathcal{Q} = \{\alpha_1 P_{i+1}^{n'} + \alpha_2 Q_{i+1} = 0\}$.

This algorithm computes a pencil $\mathcal{Q}$ such that $BP(\mathcal{Q}) = \mathcal{T}_{\text{red}}$. In order to recover $\mathcal{T}$, we only need to know the multiplicity r of $\mathcal{T}$, which can be easily computed as the quotient of $e_O(\varphi)$ and the multiplicity at O of a generic member of $\mathcal{Q}$ (which is the minimum of the multiplicities of any two generators).

Remark 1.3.5. *In fact, we can extract a little bit more information from this algorithm. For each (i, j) we can consider the pencil*

$$\mathcal{Q}_{i,j} = \begin{cases} \{\alpha h_{i,j} + \beta Q_{i,j} = 0\} & \text{if } o(\varphi^*(h_{i,j})) \in \Gamma_i, \\ \{\alpha P_{i+1}^{n'} + \beta Q_{i+1} = 0\} & \text{if } o(\varphi^*(h_{i,j})) \notin \Gamma_i, \end{cases}$$

and in section 10 of [4] it is proved that the clusters $\mathcal{K}_{i,j} = BP(\mathcal{Q}_{i,j})$ of base points of these pencils are irreducible and consist of the points in $\mathcal{T}$ up to some $q_{i,j} \in \mathcal{T}$. Moreover, the virtual multiplicities of $\mathcal{K}_{i,j}$ are proportional to those of $\mathcal{T}$. Indeed, if we define the integers $n_i = \gcd(\check{m}_0, \ldots, \check{m}_i)$, the virtual multiplicities of $\mathcal{K}_{i,j}$ are ν_p/n_i (resp. ν_p/n_{i+1}) if $o(\varphi^*(h_{i,j})) \in \Gamma_i$ (resp. if $o(\varphi^*(h_{i,j})) \notin \Gamma_i$), where ν_p denotes the virtual multiplicity of $p \in \mathcal{K}_{i,j}$ in $\mathcal{T}$.

In this setting, the algorithm ends when $q_{i,j}$ is the top of $\mathcal{T}$, and each step consists precisely in choosing the germ of $\mathcal{Q}_{i,j}$ which goes through more free points of $\mathcal{T}$ than $\{h_{i,j} = 0\}$. So at each step, the point $q_{i,j}$ is strictly greater than the previous one, and thus the clusters $\mathcal{K}_{i,j}$ form some sort of increasing sequence of clusters whose last term is $\mathcal{T}_{\text{red}}$. But we can multiply each $\mathcal{K}_{i,j}$ by n_i or n_{i+1} (depending on wether $o(\varphi^*(h_{i,j})) \in \Gamma_i$ or not) to obtain a sequence of clusters $\widetilde{\mathcal{K}}_{i,j}$ which are successive "truncations" of $\mathcal{T}$ up to the point $q_{i,j}$.

If now we define the height of any multiple of an irreducible cluster as it has been done for the trunk of a morphism (i.e. as the exponent of the first term in a Puiseux parameterization which depends on the element of the pencil), the heights of these clusters $\widetilde{\mathcal{K}}_{i,j}$ form a sequence of strictly increasing integers ending at m, the height of $\mathcal{T}$.

Similar to the case of the multiplicities, the definition of the main trunk of φ is extended to any point p infinitely near to O by considering the morphism $\varphi_p = \varphi \circ \pi_p$. More precisely, the trunk $\mathcal{T}(\varphi_p)$ of φ_p is called the *p-trunk* of φ, and is also denoted by $\mathcal{T}_p(\varphi)$ and by $\mathcal{T}_p$ if there is no confusion with the morphism. The multiplicity and the height of $\mathcal{T}_p$ are denoted by r_p and m_p, respectively.

Note that the algorithm presented to compute the main trunk $\mathcal{T}$ may also be applied to compute every trunk $\mathcal{T}_p$, just considering the morphism φ_p instead of φ.

The corresponding generalization of theorem 1.3.4 is the next

Proposition 1.3.6. *For any germ ξ at O' and any p equal or infinitely near to O,*

$$e_p(\varphi^*(\xi)) = [\xi.\mathcal{T}_p(\varphi)] - \sum_{p \to p'} [\xi.\mathcal{T}_{p'}(\varphi)].$$

Remark 1.3.7. *Note that this proposition may be equivalently stated in terms of values as $v_p(\varphi^*(\xi)) = [\xi.\mathcal{T}_p(\varphi)]$.*

1.3.3 Multiplicities of the jacobian.

The last section of this chapter is devoted to explain the relation between the trunks and the jacobian of a morphism, which (as we have already said a hundred of times) is the key fact of our study. More precisely, for every point $p \in \mathcal{N}_O$, there is a simple formula relating the multiplicity $e_p(\mathbf{J}(\varphi))$ of the jacobian germ at p with the multiplicities $e(\varphi_q)$ (for q equal or proximate to p) and with the heights of the trunks at the same points. The explicit result is the next

Theorem 1.3.8. *For any point p infinitely near O, we have*

$$e_p(\mathbf{J}(\varphi)) = \begin{cases} m + n - 2 & \text{if } p = O, \\ m_p + n_p - m_{p'} - n_{p'} - 1 & \text{if } p \text{ is free, proximate to } p', \\ m_p + n_p - m_{p'} - n_{p'} - m_{p''} - n_{p''} & \text{if } p \text{ is satellite, proximate to } p' \text{ and } p''. \end{cases}$$

Note that in this implies that the multiplicities of the jacobian germ $\mathbf{J}(\varphi)$ are uniquely determined by the trunks of φ.

The main application of this theorem in this work is the next

Corollary 1.3.9. *If p is a non-fundamental point of φ, $m_p = m_{p'} + e_p(\mathbf{J}(\varphi)) + 1$ if p is free, and $m_p = m_{p'} + m_{p''} + e_p(\mathbf{J}(\varphi))$ if p is satellite. In any case, $m_p > m_{p'}$.*

The inequality $m_p > m_{p'}$ need not hold if p is a fundamental point. In fact, it is even possible to get the reversed inequality $m_p < m_{p'}$ (see Example 14.5 in [4]).

Chapter 2

The problem.

2.1 Translating the problem to a morphism.

Our aim is to give an explicit algorithm which computes the cluster $\mathcal{S}(\xi)$ of singular points of a singular and reduced germ of curve ξ from the cluster of base points of the jacobian system $BP(\mathcal{J}(\xi))$. In particular, we shall obtain a new proof of Theorem 1.2.7. To achieve this, we reinterpret the problem in terms of the theory of planar analytic morphisms as follows.

Choose a system of local coordinates (x, y) in a neighborhood U of O, and take f to be some equation for the germ ξ. Take also $\eta : g = 0$ to be a smooth germ at O such that

- $\zeta = P_g(f) : \frac{\partial(f,g)}{\partial(x,y)} = 0$ is a generic transverse polar of ξ, and
- the point on η in the first neighbourhood of O is not in $BP(\mathcal{J}(\xi))$.

Note that generic smooth germs through O satisfy the first condition, and the second one exclude finitely many tangent directions at O, so the existence of such a η is guaranteed. Note also that the first condition implies that η is not tangent to ξ.

We now observe that we can think of the polar ζ as the jacobian germ of the morphism $\varphi : U \longrightarrow \mathbb{C}^2$ defined by the equations

$$(x, y) \mapsto (u, v) = \varphi(x, y) = (f(x, y), g(x, y))$$

and this is the key consideration is which follows.

Let us first study the fundamental points of φ. First of all, since we are assuming ζ to be a generic polar, we know that f and g share no factors, and so φ has no contracted germ. Thus the only fundamental points of φ are its base points $BP(\varphi)$. By definition, $BP(\varphi)$ is the cluster of base points of the pencil $\mathcal{P} = \{\xi_\lambda : \lambda_1 f + \lambda_2 g = 0\}$ (where $\lambda = [\lambda_1 : \lambda_2] \in \mathbb{P}^1_{\mathbb{C}}$). Note that $\xi_0 = \xi$ and $\xi_\infty = \eta$. We have $e_O(\xi_\lambda) = 1$ for $\lambda \neq 0$, and so $\nu_O(BP(\varphi)) = 1$. Now, since the cluster of base points of a pencil is consistent, the only possibility for $BP(\varphi)$ is to be irreducible and to have only free points with virtual multiplicity one. Moreover, by remark 1.1.14, the self-intersection of $BP(\varphi)$ is

$$BP(\varphi)^2 = [\xi.\eta] = e_O(\xi)e_O(\eta) = e_O(\xi)$$

(since η is smooth and non-tangent to ξ) and this implies that $BP(\varphi)$ has exactly $e_O(\xi)$ points, which must lie on η again by 1.1.14.

We have thus proved the following

Lemma 2.1.1. *The fundamental points of φ are exactly the first $e_O(\xi)$ points in $\mathcal{N}_O(\eta)$. In particular, there are no fundamental points in $BP(\mathcal{J}(\xi))$ but the origin O.*

Now it is easy to compute the multiplicities n_p of the composite morphisms $\varphi_p = \varphi \circ \pi_p$ for every point $p \in BP(\mathcal{J}(\xi))$. Indeed, if $p = O$ we already know that $n_O = 1$, and otherwise p is non-fundamental, and applying the previous lemma and Proposition 1.3.2 we obtain

$$n_p = \sum_{p \to q} n_q \tag{2.1}$$

(note that $e_p(\varphi) = 0$ because p is non-fundamental). Note that this formula works as long as p is non-fundamental, not necessarily lying on $BP(\mathcal{J}(\xi))$, and in particular for any q satellite of some $p \in BP(\mathcal{J}(\xi))$.

We will consider the trunks of φ. We are only interested in the heights m_p of the trunks $\mathcal{T}_p$ for either p equal to O or non-fundamental, and in particular for the base points of $\mathcal{J}(\xi)$ and their satellites. It is worth to notice that these quantities can be computed using Corollary 1.3.9, and this will be enough for our purposes.

2.2 Behaviour of $I_\xi(p)$.

This section is the most technical part of the whole memory. It is devoted to prove some results on the behaviour of the invariants $I_\xi(p)$ (already defined in section 1.2) when p ranges over the set $\mathcal{N}_O$ of points infinitely near to O.

First of all, let us recall their definition. Let p be any point infinitely near O, and let γ^p be an irreducible curve going through p and whose point in the first neighbourhood of p is free and does not belong to ξ. Then we define

$$I_\xi(p) = \frac{[\xi.\gamma^p]}{e_O(\gamma^p)} = \frac{[\xi.\mathcal{K}(p)]}{\nu_O(\mathcal{K}(p))},$$

where $\mathcal{K}(p)$ is the irreducible cluster whose last point is p (see section 1.2.2). Recall also that this definition is independent of the choice of the branch γ^p.

Under the assumptions made on γ^p, the numerator $[\xi.\gamma^p]$ is $v_p(\xi)e_p(\gamma^p)$, the value $v_p(\xi)$ of ξ at the point p multiplied by the multiplicity of γ^p at p. Indeed, if $\pi_p : S_p \longrightarrow S$ is the composition of blowing-ups giving rise to p, then $\gamma^p = \pi_{p*}(l_p)$ for some irreducible curve l_p at p of multiplicity $3_p(l_p) = e_p(\gamma^p)$ and non-tangent to $\bar{\xi}_p$. Then, by the projection formula, we have

$$[\xi.\gamma^p] = [\xi.\pi_{p*}(l_p)] = [\pi_p^*(\xi).l_p] = [\bar{\xi}_p.l_p] = e_p(\bar{\xi}_p)e_p(l_p) = v_p(\xi)e_p(\gamma^p).$$

As for the denominator $e_O(\gamma^p)$, it depends only on the point p and the multiplicity $e_p(\gamma^p)$, and if we add the extra assumption that γ^p is non-tangent to the germ η introduced in section 2.1 (note that this hypothesis is always satisfied if p is satellite of some base point of $\mathcal{J}(\xi)$), then we have $e_O(\gamma^p) = n_p e_p(\gamma^p)$. Indeed, under these assumptions we can think of $e_O(\gamma^p)$ as the intersection multiplicity $[\gamma^p.\eta]$, and using the projection formula as above, we have

$$e_O(\gamma^p) = v_p(\eta)e_p(\gamma^p).$$

But the values $v_p(\eta)$ also satisfy the recursive formula (2.1) with the same initial value $n_O = 1 = e_O(\eta)$, and hence $e_O(\gamma^p) = n_p e_p(\gamma^p)$ as long as γ^p is not tangent to η.

Therefore, we have proved the following

Lemma 2.2.1. *If $p \in \mathcal{N}_O$ is any point infinitely near to O and such that γ^p is not tangent to η, then*

$$I_\xi(p) = \frac{v_p(\xi) e_p(\gamma^p)}{n_p e_p(\gamma^p)} = \frac{v_p(\xi)}{n_p}.$$

Now we introduce some notations and definitions which will be useful in the sequel.

Definition 2.2.2. *If ξ is any germ of curve at O and $p \in \mathcal{N}_O$ is any point infinitely near to O, we define the* normalized multiplicity *$e'_p(\xi)$ of ξ at the point p as the quotient*

$$e'_p(\xi) = \frac{e_p(\xi)}{e_O(\xi)}.$$

If $p \neq O$ lies in the first neighbourhood of q, we denote by $b_p(\xi)$ (resp. $b'_p(\xi)$) the multiplicity (resp. normalized multiplicity) of ξ at q. That is

$$b_p(\xi) = e_q(\xi) \text{ and } b'_p(\xi) = e'_p(\xi).$$

Remark 2.2.3. *It follows immediately from the definition that $0 \leqslant e'_p(\xi) \leqslant 1$ and that $e'_p(\xi) = 0$ if and only if p does not lie on ξ. It is also obvious that $e_p(\xi) \leqslant b_p(\xi)$ and $e'_p(\xi) \leqslant b'_p(\xi)$, with equality if and only if either q does not belong to ξ or p is the only point on ξ proximate to q.*

In the case $p \in \mathcal{N}_O(\xi)$ is a free point, it is obvious (for example, by the Proximity Equalities) that $e_p(\xi) = b_p(\xi)$ (or equivalently $e'_p(\xi) = b'_p(\xi)$) if and only if the points on ξ in the first neighbourhood of p are all free. The case $e_p(\xi) < b_p(\xi)$ (the opposite inequality is impossible) is not so straightforward, but if the curve ξ is irreducible, the quotient $e_p(\xi)/b_p(\xi) = e'_p(\xi)/b'_p(\xi)$ determine the structure of the set of p-satellite points on ξ as follows.

Set $r_0 = b_p(\xi)$ and $r_1 = e_p(\xi)$, and perform Euclid's algorithm as $r_0 = a_1 r_1 + r_2$, $r_1 = a_2 r_2 + r_3, \ldots, r_{k-1} = a_k r_k$. Then, by iterated use of theorem 1.1.6, we obtain that the set of p-satellite points on ξ consists of the points

$$p_{1,1} = p < \ldots < p_{1,a_1} < p_{2,1} < \ldots < p_{2,a_2} < \ldots < p_{r,1} < \ldots < p_{r,a_r}$$

where each point is in the first neighbourhood of the preceding one and has multiplicity $e_{p_{i,j}}(\xi) = r_i$. We also know that $p_{i,j+1}$ is proximate to $p_{i,j}$ and $p_{i-1,a_{i-1}}$ (where we set $a_0 = 0$ and $p_{0,0} = q$ the point which p is proximate to) and that $p_{i+1,1}$ is proximate to p_{i,a_i} and $p_{i-1,a_{i-1}}$. We will say that these points form the *stair of satellite points of p on C*, and that the points $p_{i,j}$ *lie on an odd* (resp *even*) *step* of this stair if the index i is odd (resp. even).

So the quotients a_i determine the structure of the stair of satellite points of p on ξ. These quotients may also be obtained as the coefficients of the continued fraction of $e_p(\xi)/b_p(\xi)$. More precisely, we have

$$\frac{e_p(\xi)}{b_p(\xi)} = \frac{e'_p(\xi)}{b'_p(\xi)} = [0, a_1, a_2, \ldots, a_n],$$

where we use the notation

$$[a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

for any set of integers $a_0 \geqslant 0$ and $a_1, \ldots, a_n > 0$.

We also need to introduce a new ordering of all the points infinitely near to O, different from the natural one defined above. Recall that for any point $p \in \mathcal{N}_O$ we have defined γ^p to be an irreducible germ going through p and having a free point in its first neighbourhood (we have also required that this free point does not lie on ξ, but this condition makes no sense in this context because we have fixed no germ ξ).

Definition 2.2.4. *Let $q_1 \neq q_2$ be two points infinitely near to O, satellite of the free points p_1 and p_2 respectively. We say that q_1 is* smaller than q_2 *(or q_2 is* bigger than q_1*), and denote it $q_1 \prec q_2$ (or $q_2 \succ q_1$) if $p_1 \leqslant p_2$ (with the usual order) and $e'_{p_1}(\gamma^{q_1}) \leqslant e'_{p_1}(\gamma^{q_2})$. Obviously, we denote by $q_1 \preceq q_2$ the situation in which $q_1 \prec q_2$ or $q_1 = q_2$, and similarly for $q_2 \succeq q_1$.*

Note that this definition is independent of the choice of γ^{q_1} and γ^{q_2}, since the normalized multiplicities of γ^p in the points $q \leqslant p$ are uniquely determined.

Remark 2.2.5. *If p is any point infinitely near to O and q is a free point in its first neighbourhood, then we can take γ^q as a γ^p, and it follows that $p \prec q$. Moreover, if q' is any point infinitely near to q (not necessarily satellite of q), the same argument applies to give $p \prec q'$.*

This definition is closely related to the theory of valuations. Indeed, each point $q \in \mathcal{N}_O$ determines a divisorial valuation ν_q of the local ring $\mathcal{O}_{S,O}$, given by

$$f \mapsto \nu_q(f) = \frac{[\gamma^p.\{f^{-1}(0)\}]}{e_O(\gamma^p)} = \frac{[\mathcal{K}(p).\{f^{-1}(0)\}]}{\nu_O(\mathcal{K}(p))} = I_{\{f^{-1}(0)\}}(p), \tag{2.2}$$

where as always, γ^p is a curve going through p and whose point in the first neighbourhood of p is free and does not belong to $\{f^{-1}(0)\}$. These valuations are normalized in such a way that the minimum value in the maximal ideal $\mathfrak{m}_{S,O}$ is 1. This normalization allows divisorial valuations to take non-integral values, while in [3] they are normalized to have always values in $\mathbb{Z}$. In this setting, we have the following result, which may be found in [1], Proposition 2.4.9.

Proposition 2.2.6. *If ν_p and ν_q are the divisorial valuations associated to the points p and q, then $\nu_p \leqslant \nu_q$[1] if and only if $p \preceq q$.*

As an immediate consequence, we obtain the following

Corollary 2.2.7. *If $p \prec q$ are two points infinitely near to O and ξ is any germ of curve at O, then*

$$I_\xi(p) \leqslant I_\xi(q).$$

[1]Given two valuations ν_1, ν_2 of $\mathcal{O}_{S,O}$, we say that $\nu_1 \leqslant \nu_2$ if $\nu_1(f) \leqslant \nu_2(f)$ for all $f \in \mathcal{O}_{S,O}$.

Proof. If we take into account the definitions of ν_p and ν_q by formula (2.2) and Proposition 2.2.6, we obtain

$$I_\xi(p) = \nu_p(f) \leqslant \nu_q(f) = I_\xi(q)$$

(where f is any equation for ξ), as claimed. □

For further information, see [3] chapter 8, and also [1] chapter 2, [6] and [7] chapter 1.

It is also worth noting that this ordering coincides with the ordering of the exceptional divisors of the dual graph of a composition of blowing-ups. More precisely, let $\pi : \bar{S} \longrightarrow S$ be the composition of blowing up all the points in some cluster containing q_1 and q_2, let Γ be the dual graph of the exceptional locus (that is, a tree having one vertex V_p for each exceptional component E_p of $\pi^{-1}(O)$, and one edge $L_{p,q}$ from V_p to V_q if the exceptional components E_p and E_q intersect). Then $q_1 \prec q_2$ if and only if the vertex V_{q_1} belongs to the minimal path from V_O to V_{q_2}, and in particular it is closer to V_O than V_{q_2} (where Γ is equipped with the usual distance of a graph).

Furthermore, we can relate the approaches of the valuations and of the dual graphs, just noting that in [7] chapter 6, it is proved that the dual graph of any resolution can be embedded as a subtree of the valuative tree.

As a final remark, which is a consequence of all the previous considerations, note also that if p is any free point proximate to p', then every p-satellite point q satisfies $p' \prec q \prec p$.

Before going deeper in the relation between the order $\prec$ and the invariants $I_\xi(p)$, it will be useful to study the induced order in set of satellite points of a free point p. We will need the following useful lemma about continued fractions.

Lemma 2.2.8. *Let m, n be positive integers, and let $a_0, \ldots, a_n, b_0, \ldots, b_m$ be integers such that $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{m-1} \geqslant 1$ and $a_n, b_m \geqslant 2$. Assume that $a_i = b_i$ for all $i < k \leqslant n, m$ and that $a_k > b_k$. Then*

$$\begin{cases} [a_0, \ldots, a_n] > [b_0, \ldots, b_m] \text{ if } k \text{ is even, and} \\ [a_0, \ldots, a_n] < [b_0, \ldots, b_m] \text{ if } k \text{ is odd.} \end{cases}$$

Proof. First of all, note that the assumption $a_n, b_m \geqslant 2$ is imposed only to assure that the expression of $[a_0, \ldots, a_n]$ and $[b_0, \ldots, b_m]$ as continued fractions is unique (it is always true that $[a_0, \ldots, a_n, 1] = [a_0, \ldots, a_n + 1]$).

Let us consider first the case $k = 0$. Under the assumptions on $a_1, \ldots, a_n, b_1, \ldots b_m$, the fractional parts

$$\cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} \quad \text{and} \quad \cfrac{1}{b_1 + \cfrac{1}{\ddots + \cfrac{1}{b_m}}}$$

of $[a_0, \ldots, a_n]$ and $[b_0, \ldots, b_m]$ are strictly positive and less than 1, so

$$[a_0, \ldots, a_n] > a_0 \geqslant b_0 + 1 > [b_0, \ldots, b_m] > b_0$$

as claimed.

Now suppose that $k \geqslant 1$ is even (so in fact $k \geqslant 2$). We have $[a_0, \ldots, a_n] > [b_0, \ldots, b_m]$ if and only if

$$\cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} > \cfrac{1}{b_1 + \cfrac{1}{\ddots + \cfrac{1}{b_m}}}$$

which in turn is equivalent to

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} < b_1 \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_m}}}.$$

So we have proved that $[a_0, \ldots, a_n] > [b_0, \ldots, b_m]$ if and only if $[a_1, \ldots, a_n] < [b_1, \ldots, b_m]$, and applying the same argument k times we obtain that

$$[a_0, \ldots, a_n] > [b_0, \ldots, b_m] \Leftrightarrow [a_k, \ldots, a_n] > [b_k, \ldots, b_m],$$

and this holds by the case $k = 0$.

The case k odd is analogous. □

Theorem 2.2.9. *Let $p \in \mathcal{N}_O$ be any free point different from O, proximate to p'. Then:*

1. *The satellite point q in the first neighbourhood of p satisfies $p' \prec q \prec p$.*
2. *If $q \neq p$ is a p-satellite point, the two satellite points q_1, q_2 in its first neighbourhood may be ordered as $p' \prec q_1 \prec q \prec q_2 \prec p$. We will call q_1 (resp. q_2) the* first (resp. second) satellite *of q. Moreover, every p-satellite point q' infinitely near to q_1 (resp. q_2) satisfies $q' \prec q$ (resp. $q' \succ q$).*

Proof. 1. In this case, the stair of p-satellite points on γ^q consists just on the points p and q, so we have

$$\frac{e'_p(\gamma^q)}{b'_p(\gamma^q)} = [0, 2] = \frac{1}{2} < 1 = \frac{e'_p(\gamma^p)}{b'_p(\gamma^p)}.$$

Since $b'_p(\gamma^p) = b'_p(\gamma^q) = \left(\frac{1}{e_O(\gamma^{p'})} > 0\right)$, we obtain that $e'_p(\gamma^q) < e'_p(\gamma^p)$, and so $q \prec p$. The inequality $p' \prec q$ follows from Remark 2.2.5.

2. Let γ^q be defined as usual, and write $\frac{e_p(\gamma^q)}{b_p(\gamma^q)} = [0, a_1, \ldots, a_n]$ as a continued fraction with $a_n \geqslant 2$. So, the stair of p-satellite points on γ^q is given by n steps of lengths $a_1, \ldots, a_n$.

   Now let q' be one of the two satellite points in the first neighbourhood of q. Depending on the choice of q', the stair of p-satellite points of $\gamma^{p'}$ may have n or $n+1$ steps, and the corresponding values of $\frac{e_p(\gamma^{q'})}{b_p(\gamma^{q'})}$ are $[0, a_1, \ldots, a_n + 1]$ and $[0, a_1, \ldots, a_n - 1, 2]$. Now using Lemma 2.2.8, one of this values is strictly smaller than $[0, a_1, \ldots, a_n]$ and the other one is greater (depending on the parity of n). Therefore, we have proved that we may order the two satellite points q_1, q_2 in the first neighbourhood of q as

$$q_1 \prec q \prec q_2.$$

By Remark 2.2.5, we also have $p' \prec q_1$, and $q_2 \prec p$ follows from

$$\frac{e'_p(\gamma^{q_2})}{b'_p(\gamma^{q_2})} \leqslant 1 = \frac{e'_p(\gamma^p)}{b'_p(\gamma^p)}$$

(since p is free).

Now assume q' is a p-satellite point infinitely near to q_1, and suppose, to fix ideas, that n is even. Thus q_1 is the point such that

$$\frac{e_p(\gamma^{q_1})}{b_p(\gamma^{q_1})} = [0, a_1, \ldots, a_n - 1, 2]$$

and we may write

$$\frac{e_p(\gamma^{q'})}{b_p(\gamma^{q'})} = [0, a_1, \ldots, a_n - 1, b_{n+1}, \ldots, b_m]$$

for suitable integers $b_{n+1}, \ldots, b_{m-1} \geqslant 1, b_m \geqslant 2$. Now Lemma 2.2.8 implies that $q' \prec q$.

The rest of the cases are analogous.

□

Remark 2.2.10. *For the sake of simplicity, the satellite point q in the first neighbourhood of a free point p will also be called the* first *satellite of p.*

It will also be useful to study how a satellite point is ordered with respect to the two points which it is proximate to.

Proposition 2.2.11. *Let p be a satellite point, proximate to q and q', and assume $q \prec q'$. Then*

$$q \prec p \prec q'.$$

Proof. Let p' be the free point of which p is satellite and write $\frac{e_{p'}(\gamma^p)}{b_{p'}(\gamma^p)} = [0, a_1, \ldots, a_n]$ with $a_n \geqslant 2$. Suppose to fix ideas that n is even. Then we have

$$\frac{e_{p'}(\gamma^q)}{b_{p'}(\gamma^q)} = [0, a_1, \ldots, a_n - 1] \text{ and } \frac{e_{p'}(\gamma^{q'})}{b_{p'}(\gamma^{q'})} = [0, a_1, \ldots, a_{n-1}]$$

(to apply Lemma 2.2.8 it might be necessary to normalize the fractions in order to assure that the last integer is greater or equal than 2, but the claim is true in any case). And again by Lemma 2.2.8 we have

$$[0, a_1, \ldots, a_n - 1] < [0, a_1, \ldots, a_n] < [0, a_1, \ldots, a_{n-1}]$$

which gives $q \prec p \prec q'$ as wanted. □

We introduce now the last definition of the section, which will make much easier the statements of the main results in this section.

Definition 2.2.12. *Let γ be any irreducible germ, let q be a p-satellite point for some $p \in \mathcal{N}_O(\gamma)$, and let q' be the last p-satellite point on γ. We say that q is* smaller than γ *(or that γ is* bigger *than q) if $q \prec q'$, and it will be denoted simply as $q \prec \gamma$ or $\gamma \succ q$. On the contrary, q is said to be* bigger than γ *(or γ is* smaller than q*) if $q' \prec q$, and the obvious notation is $q \succ \gamma$ or $\gamma \prec q$.*

Remark 2.2.13. *Every time we write $q \prec \gamma$ (or say "q is smaller than γ"), we will implicitly assume that γ goes through the free point q is satellite of.*

If, as usual, we denote by γ^q some irreducible germ going through q and having a free point in the first neighbourhood of q which does not lie on γ, then $q \preceq \gamma$ if and only if $e'_p(\gamma^q) \leqslant e'_p(\gamma)$.

As well as the order $\prec$ between infinitely near points, this order between points and branches is also closely related to the theory of valuations as follows. Every irreducible curve γ defines a valuation (a *curve valuation*) $\nu_\gamma : \mathcal{O}_{S,O} \longrightarrow \mathbb{Q} \cup \{\infty\}$ given by

$$f \mapsto \nu_\gamma(f) = \frac{[\gamma.\{f^{-1}(0)\}]}{e_O(\gamma)},$$

and for any point q infinitely near to O, the relation $q \prec \gamma$ is equivalent to $\nu_q \leqslant \nu_\gamma$ (this assertion also follows from Proposition 2.2.6). However, it is no longer true that $q \succ \gamma$ if and only if $\nu_q \geqslant \nu_\gamma$. In fact, curve valuations are maximal in the set of normalized valuations, so they can never be smaller than a divisorial valuation (see [6] or [7]).

Example: In figure 2.1 we have represented the Enriques' diagram of a curve with branches $\gamma_1 : y = 0$, $\gamma_2 : y^4 - x^7 = 0$, $\gamma_3 : y^5 - x^8 = 0$ and $\gamma_4 : y^7 - x^9 = 0$.

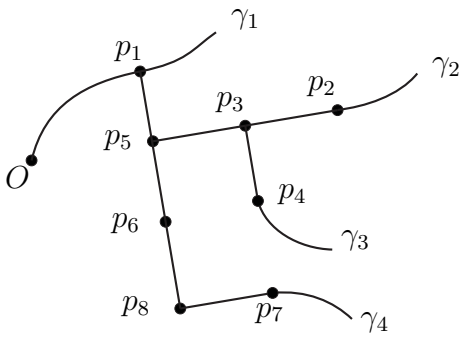


Figure 2.1: Comparison of points and branches.

The points $p_1, \ldots, p_8$ have been numbered in such a way that $p_1 \succ p_2 \succ \ldots \succ p_8$. Therefore

- γ_1 is bigger than any p_i, $i \neq 1$.
- γ_2 is bigger than $p_3, \ldots, p_8$ and smaller than p_1.
- γ_3 is bigger than $p_5, \ldots, p_8$ and smaller than $p_1, \ldots, p_3$.
- γ_4 is bigger than p_8 and smaller than $p_1, \ldots, p_6$.

We now state the most useful result of the section.

Proposition 2.2.14. *Let γ and η be two distinct irreducible germs at O. Let p be the maximal free point lying both in γ and η. Then*

$$\frac{[\gamma.\eta]}{e_O(\gamma)e_O(\eta)} = \sum_{q<p} e'_q(\gamma)^2 + b'_p(\gamma)\min\{e'_p(\gamma), e'_p(\eta)\}$$

Proof. Although the statement is slightly different, this is essentially the proposition 1.3.3 in [1], and the proof is basically the same. □

We present now the main results in this section. They deal with how the order $\prec$ translates into the invariants $I_\xi(p)$. The first one treats the case of free points and is easier than the other ones, which give precise descriptions of the behaviour of $I_\xi(q)$ at the set of p-satellite points for some free point p. All of these results are refinements and generalizations of some results in chapter 7 of [3]. Similar results about the growth of the invariants $I_\xi(p)$ can be found in [11] as long as p is a singular point of ξ. Notice that our results hold for any infinitely near point p hence they generalize also the ones appearing in [11]. As a final remark, note that all of these results cover different cases of 2.2.7, but they give precise characterizations of the cases in which equality holds.

Theorem 2.2.15. *Let p be any point infinitely near to O, and let q be a free point in the first neighbourhood of p. If ξ is any germ of curve at O, then*

$$I_\xi(p) \leqslant I_\xi(q).$$

Equality holds if and only if $q \notin \mathcal{N}_O(\xi)$.

Proof. Let γ^q be any irreducible germ going through q with multiplicity one and such that its point in the first neighbourhood of q is free and does not belong to ξ. Then we have (by the Noether's formula 1.1.3) that

$$[\xi.\gamma^q] = \sum_{q' \leqslant q} e_{q'}(\xi)e_{q'}(\gamma^q) = \sum_{q' \leqslant p} e_{q'}(\xi)e_{q'}(\gamma^q) + e_q(\xi)e_q(\gamma^q). \tag{2.3}$$

Now, let γ^p defined in the same way as γ^q, as usual. Since

$$e_p(\gamma^p) = 1 = e_q(\gamma^q) = e_p(\gamma^q)$$

(where the last equality is a consequence of the proximity equalities 1.1.4), then

$$e_{q'}(\gamma^p) = e_{q'}(\gamma^q) \quad \forall q' \leqslant p$$

(again by 1.1.4), and so the first sum in (2.3) equals $[\xi.\gamma^p]$, proving that

$$[\xi.\gamma^q] = [\xi.\gamma^p] + e_q(\xi).$$

Now, $e_O(\gamma^p) = e_O(\gamma^q)$ implies that

$$I_\xi(q) = \frac{[\xi.\gamma^q]}{e_O(\gamma^q)} = \frac{[\xi.\gamma^p]}{e_O(\gamma^p)} + \frac{e_q(\xi)}{e_O(\gamma^q)} = I_\xi(p) + \frac{e_q(\xi)}{e_O(\gamma^q)} \geqslant I_\xi(p)$$

and the claim follows. □

Remark 2.2.16. *This result is a generalization of Proposition 7.6.5 in [3].*

Theorem 2.2.17. *Let $p \in \mathcal{N}_O$ be any free point different from O, let p' be the point which p is proximate to, and let ξ be any germ of curve at O. Then for any p-satellite point q we have*

$$I_\xi(p') \leqslant I_\xi(q)$$

with equality if and only if ξ does not go through p.

Proof. As above, let γ^q be any irreducible germ going through q with multiplicity one and such that its point in the first neighbourhood of q is free and does not belong to ξ, and let $\gamma^{p'}$ be defined analogously.

Now by Noether's formula 1.1.3 we have

$$[\xi.\gamma^q] = \sum_{q' \leqslant q} e_{q'}(\xi)e_{q'}(\gamma^q) = \sum_{q' \leqslant p'} e_{q'}(\xi)e_{q'}(\gamma^q) + \sum_{q' \geqslant p} e_q(\xi)e_q(\gamma^q). \tag{2.4}$$

Since for any $q' \leqslant p'$ we obviously have $e_{q'}(\gamma^q) = \frac{e_O(\gamma^q)}{e_O(\gamma^{p'})} e_{q'}(\gamma^{p'})$ (this is equivalent to say that $e'_{q'}(\gamma^q) = e'_{q'}(\gamma^{p'})$), the first sum in (2.4) is just

$$\frac{e_O(\gamma^q)}{e_O(\gamma^{p'})}[\xi.\gamma^{p'}],$$

while the second sum is obviously non-negative. So we have

$$I_\xi(p') = \frac{[\xi.\gamma^{p'}]}{e_O(\gamma^{p'})} = \frac{\frac{e_O(\gamma^q)}{e_O(\gamma^{p'})}[\xi.\gamma^{p'}]}{e_O(\gamma^q)} \leqslant \frac{[\xi.\gamma^q]}{e_O(\gamma^q)} = I_\xi(q)$$

with equality if and only if the second sum in (2.4) is zero, which holds if and only if $e_p(\xi) = 0$, that is, ξ does not go through p. □

As far as the author is aware, there is no result similar to Theorem 2.2.17 in the literature.

Theorem 2.2.18. *Let $p \in \mathcal{N}_O$ be any free point different from O, let $q_1 \prec q_2$ be two p-satellite points, and let ξ be any germ at O. Then*

$$I_\xi(q_1) \leqslant I_\xi(q_2),$$

with strict inequality if and only if there exists some branch of ξ bigger than q_1 (in particular, such a branch should go through p).

Proof. As usual, for each $i = 1, 2$ let γ^{q_i} be any irreducible germ going through q_i and such that its point in the first neighbourhood of q_i is free and does not belong to ξ. Then by definition of $q_1 \prec q_2$ we have that $e'_p(\gamma^{q_1}) < e'_p(\gamma^{q_2})$, and we want to show the inequality

$$\frac{[\xi.\gamma^{q_1}]}{e_O(\gamma^{q_1})} \leqslant \frac{[\xi.\gamma^{q_2}]}{e_O(\gamma^{q_2})}$$

characterizing when there is equality.

Let $\xi_1, \ldots, \xi_k$ be the branches of ξ. Then for any $i = 1, 2$ we have

$$I_\xi(q_i) = \frac{[\xi.\gamma^{q_i}]}{e_O(\gamma^{q_i})} = \sum_{j=1}^{k} \frac{[\xi_j.\gamma^{q_i}]}{e_O(\gamma^{q_i})} = \sum_{j=1}^{k} e_O(\xi_j) \frac{[\xi_j.\gamma^{q_i}]}{e_O(\xi_j)e_O(\gamma^{q_i})}. \tag{2.5}$$

Now for each $j = 1, \ldots, k$ we have one of the following situations:

- $p \notin \mathcal{N}_O(\xi_j)$. In this case the Noether's formula 1.1.3 implies that

$$\frac{[\xi_j.\gamma^{q_i}]}{e_O(\xi_j)e_O(\gamma^{q_i})} = \sum_{p' \in \mathcal{N}_O(\xi_j) \cap \mathcal{N}_O(\gamma^{q_i})} e'_{p'}(\xi_j) e'_{p'}(\gamma^{q_i}).$$

  Now note that $e'_{p'}(\gamma^{p_1}) = e'_{p'}(\gamma^{p_2})$ for every $p' < p$, and since $p \notin \mathcal{N}_O(\xi_j) \cap \mathcal{N}_O(\gamma^{q_i})$ we have the equality

$$\frac{[\xi_j.\gamma^{q_2}]}{e_O(\xi_j)e_O(\gamma^{q_2})} = \frac{[\xi_j.\gamma^{q_2}]}{e_O(\xi_j)e_O(\gamma^{q_2})}.$$

- $p \in \mathcal{N}_O(\xi_j)$. In this case we can apply Proposition 2.2.14 to obtain

$$\frac{[\xi_j.\gamma^{q_i}]}{e_O(\xi_j)e_O(\gamma^{q_i})} = \sum_{p'<p} e'_{p'}(\xi_j)^2 + b'_p(\xi_j) \min\{e'_p(\xi_j), e'_p(\gamma^{q_i})\}. \tag{2.6}$$

  In the case $e'_p(\xi_j) \leqslant e'_p(\gamma^{q_1})$ the minimum in the last summand of (2.6) is the same for $i = 1$ and $i = 2$ (namely, $e'_p(\xi_j)$), so we also have equality

$$\frac{[\xi_j.\gamma^{q_1}]}{e_O(\xi_j)e_O(\gamma^{q_1})} = \frac{[\xi_j.\gamma^{q_2}]}{e_O(\xi_j)e_O(\gamma^{q_2})}.$$

  But in the case $e'_p(\xi_j) > e'_p(\gamma^{q_1})$, which is precisely the case $q_1 \prec \xi_j$, the minimum in (2.6) for $i = 1$ is $e'_p(\gamma^{q_1})$, which is strictly smaller than $\min\{e'_p(\xi_j), e'_p(\gamma^{q_2})\}$, and then we obtain the strict inequality

$$\frac{[\xi_j.\gamma^{q_1}]}{e_O(\xi_j)e_O(\gamma^{q_1})} < \frac{[\xi_j.\gamma^{q_2}]}{e_O(\xi_j)e_O(\gamma^{q_2})}.$$

If we add up all these equalities or inequalities (depending on the case) according to (2.5), we get

$$I_\xi(q_1) = \frac{[\xi.\gamma^{q_1}]}{e_O(\gamma^{q_1})} \leqslant \frac{[\xi.\gamma^{q_2}]}{e_O(\gamma^{q_2})} = I_\xi(q_2)$$

with strict inequality if and only if there exists some ξ_j branch of ξ such that $q_1 \prec \xi_j$, as wanted. □

We can apply the previous theorem to the case $q_2 = p$, and we obtain refinements of the first and second statements of Proposition 7.6.8 of [3]:

Corollary 2.2.19. *Let $p \in \mathcal{N}_O$ be any free point different from O. Then*

$$I_\xi(q) \leqslant I_\xi(p)$$

for any p-satellite point q, with strict inequality if and only q is smaller than some branch of ξ. In particular, if there is no free point on ξ in the first neighbourhood of p, we have

$$I_\xi(q) \leqslant I_\xi(p)$$

with equality if and only if q is bigger than or equal to the biggest rupture point of ξ which is satellite of p.

To close this section, we need to treat carefully some special situation which will appear in the next chapter.

Theorem 2.2.20. *Let ξ be a germ of curve at O, let $p \in \mathcal{N}_O$ be any free point different from O, and assume there is only one branch γ of ξ going through p and whose point in the first neighbourhood of p is free and non-singular (on ξ). Assume also that there is at least another branch of ξ going through p (whose point in the first neighbourhood of p must be satellite) and let $q \in \mathcal{N}_O(\xi)$ be the biggest p-satellite rupture point on ξ. Then*

$$\frac{[\xi.\gamma^p]-1}{e_O(\gamma^p)} = I_\xi(p) - \frac{1}{e_O(\gamma^p)} < I_\xi(q) < I_\xi(p).$$

Proof. First of all, note that the second inequality is given by Corollary 2.2.19, so we only have to proof the first one.

As in the previous proof, let $\xi_1, \ldots, \xi_k$ be the branches of ξ, where ξ_1 is *the* branch going through p and having a free point in its first neighbourhood, and $\xi_2, \ldots, \xi_l$ are the rest of the branches going through p. We again have decompositions

$$I_\xi(q_i) = \frac{[\xi.\gamma^p]}{e_O(\gamma^p)} = \sum_{j=1}^{k} \frac{[\xi_j.\gamma^p]}{e_O(\gamma^p)} = \sum_{j=1}^{k} e_O(\xi_j)\frac{[\xi_j.\gamma^p]}{e_O(\xi_j)e_O(\gamma^p)} \tag{2.7}$$

and

$$I_\xi(q_i) = \frac{[\xi.\gamma^q]}{e_O(\gamma^q)} = \sum_{j=1}^{k} \frac{[\xi_j.\gamma^q]}{e_O(\gamma^q)} = \sum_{j=1}^{k} e_O(\xi_j)\frac{[\xi_j.\gamma^q]}{e_O(\xi_j)e_O(\gamma^q)}. \tag{2.8}$$

Now, for each $j \neq 1$ we consider the same cases as in the proof of Theorem 2.2.17:

- $p \notin \mathcal{N}_O(\xi_j)$. In this case we can prove exactly as in the preceding theorem that we have the equality
$$\frac{[\xi_j.\gamma^p]}{e_O(\xi_j)e_O(\gamma^p)} = \frac{[\xi_j.\gamma^q]}{e_O(\xi_j)e_O(\gamma^q)}.$$
- $p \in \mathcal{N}_O(\xi_j)$. In this case, since q is the biggest p-satellite rupture point on ξ and we are assuming $j \neq 1$, the same reasoning as above provides also equalities
$$\frac{[\xi_j.\gamma^p]}{e_O(\xi_j)e_O(\gamma^p)} = \frac{[\xi_j.\gamma^q]}{e_O(\xi_j)e_O(\gamma^q)}.$$

So all the summands in (2.7) and (2.8) are equal but for $j = 1$. Thus we only need to check the inequality

$$\frac{[\xi_1.\gamma^p]}{e_O(\gamma^p)} - \frac{1}{e_O(\gamma^p)} < \frac{[\xi_1.\gamma^q]}{e_O(\gamma^q)},$$

which is equivalent to

$$\frac{[\xi_1.\gamma^p]}{e_O(\xi_1)e_O(\gamma^p)} - \frac{[\xi_1.\gamma^q]}{e_O(\xi_1)e_O(\gamma^q)} < \frac{1}{e_O(\xi_1)e_O(\gamma^p)}. \tag{2.9}$$

By Proposition 2.2.14, the difference in the left hand side of (2.9) is just

$$b'_p(\xi_1)(e'_p(\gamma^p) - e'_p(\gamma^q)).$$

Now note that since the point on ξ_1 in the first neighbourhood of p is free and non-singular, we have $e_p(\xi_1) = b_p(\xi_1) = 1$, and then $e'_p(\xi_1) = b'_p(\xi_1) = \frac{1}{e_O(\xi_1)}$. Moreover, since γ^p has the same properties of ξ_1, we also have $e_O(\xi_1) = e_O(\gamma^p)$, and then we only have to check that

$$\frac{1}{e_O(\xi_1)}\left(\frac{1}{e_O(\xi_1)} - e'_p(\gamma^q)\right) < \frac{1}{e_O(\xi_1)^2}$$

which is obvious since $e'_p(\gamma^q) > 0$. □

Remark 2.2.21. *The statement of Theorem 2.2.20 is quite similar to the third statement of Proposition 7.6.8 in [3], with the difference that here it is stated precisely which are the integers b and c appearing there.*

2.3 Values versus heights of the trunks.

Remember that, as we explained in section 2.1, we have taken $\eta : g = 0$ to be any smooth germ at O such that $\zeta : P_g(f) = 0$ is a generic transverse polar, and so goes sharply through $BP(\mathcal{J}(\xi))$. Furthermore, we can consider ζ as the jacobian germ of the morphism $\varphi = (f, g) : S \longrightarrow \mathbb{C}^2$.

In this section we will study the relation between the value $v_p(\xi)$ of ξ at p and the height m_p of the trunk $\mathcal{T}_p$ of $\varphi_p = \varphi \circ \pi_p$ for the points $p \in \mathcal{N}_O$ such that γ^p is not tangent to η (and in particular for the satellite points of any point in $BP(\mathcal{J}(\xi))$).

The first result is immediate from the definitions, and it holds for any point infinitely near O, not necessarily fundamental of φ:

Lemma 2.3.1. *For any point p infinitely near to O, we have the inequality*

$$v_p(\xi) \leqslant m_p.$$

Proof. The proof is quite straightforward. On the one hand

$$v_p(\xi) = e_p(\varphi_p^*(\{u = 0\})) = [\mathcal{T}_p.\{u = 0\}]$$

by Theorem 1.3.4, and on the other hand, the Puiseux series of $\mathcal{T}_p$ is of the form

$$\mathcal{S}(v, \theta) = \sum_{n_p \leqslant i < m_p} a_i v^{\frac{i}{n_p}} + \theta a_{m_p} v^{\frac{m_p}{n_p}}, \qquad a_{m_p} \neq 0$$

(notice that since $o(f) = e_O(\xi) \geqslant 2 > 1 = e_O(\eta) = o(g)$, we have interchanged the role of the variables u and v with respect to the notation of section 1.3).

Now, the intersection product $[\mathcal{T}_p.\{u = 0\}]$ may be computed as

$$\sum_{\epsilon^{n_p}=1} o_v(\mathcal{S}(\epsilon v, \theta)),$$

(where o_v denotes the vanishing order with respect to v) and thus

$$v_p(\xi) = [\mathcal{T}_p.\{u = 0\}] = n_p \frac{\min\{i \,|\, a_i \neq 0\}}{n_p} \leqslant n_p \frac{m_p}{n_p} = m_p$$

and we are done. □

But this previous result is too general and weak to our purposes. The next proposition characterizes whether $v_p(\xi) = m_p$. Recall that γ^p is any irreducible curve going through p and such that its point in the first neighbourhood of p is free and does not lie on ξ.

Proposition 2.3.2. *Let p be any point infinitely near to O such that γ^p is not tangent to η. Then the equality $v_p(\xi) = m_p$ holds if and only if the total transforms $\bar{\xi}_p$ and $\bar{\eta}_p$ at p have non-homothetical tangent cones (counting multiplicities, or equivalently, considered as divisors on E_p, the first neighbourhood of p).*

Proof. First of all, notice that the algorithm in section 1.3 which computes the trunk $\mathcal{T}_p$ stops at the first step if and only if the initial forms of $\varphi_p^*(u)$ and $\varphi_p^*(v)$ are non-homothetical, and this is equivalent to the total transforms $\bar{\xi}_p$ and $\bar{\eta}_p$ at p having non-homothetical tangent cones.

In order to complete the proof it is enough to show that the cluster $\widetilde{\mathcal{K}}_{0,1}$ of the first step of the algorithm (see Remark 1.3.5) has height $v_p(\xi)$. Thus, if the algorithm stops at the first step then $m_p = v_p(\xi)$, and if it performs another step then $m_p > v_p(\xi)$ (again by Remark 1.3.5).

So let us consider the first step of the algorithm (i.e, $i = 0$ and $j = 1$). Since $o(\varphi_p^*(v)) = e_p(\bar{\eta}_p) = n_p \leqslant v_p(\xi) = e_p(\bar{\xi}_p) = o(\varphi_p^*(u))$, we must take $P_0 = v$ and $h_{0,1} = u$, and thus $\Gamma_o = \langle \check{m}_0 \rangle = \langle n_p \rangle$. Now we must distinguish two cases:

- $v_p(\xi) \in \langle n_p \rangle$. Let $c_0 = \frac{v_p(\xi)}{n_p}$. Then $Q_{0,1} = P_0^{c_0} = v^{\frac{v_p(\xi)}{n_p}}$ and the first pencil is

  $$\mathcal{Q}_{0,1} = \{\alpha u + \beta v^{\frac{v_p(\xi)}{n_p}} = 0\}.$$

  Since the general germ in this pencil has Puiseux series $-\frac{\beta}{\alpha} v^{\frac{v_p(\xi)}{n_p}}$, the Puiseux series of $\mathcal{Q}_{0,1}$ is

  $$\mathcal{S}_{0,1}(v, \theta) = \theta v^{\frac{v_p(\xi)}{n_p}},$$

  and hence $\mathcal{K}_{0,1}$ has height $c_0 = \frac{v_p(\xi)}{n_p}$. But according to 1.3.5, $\widetilde{\mathcal{K}}_{0,1}$ is n_p times $\mathcal{K}_{0,1}$, and thus $\widetilde{\mathcal{K}}_{0,1}$ has height $n_p c_0 = v_p(\xi)$, which proves the claim in this case.

- $v_p(\xi) \notin \langle n_p \rangle$. This case is similar to the first one, with the difference that we have to consider more parameters. We take $P_1 = h_{1,0} = u$, $\check{m}_1 = v_p(\xi)$, $\Gamma_1 = \langle n_p, v_p(\xi)\rangle$ and $n' = \frac{n_p}{\gcd(n_p, v_p(\xi))}$. Now c_0 must be

$$c_0 = \frac{n' v_p(\xi)}{n_p} = \frac{v_p(\xi)}{\gcd(n_p, v_p(\xi))},$$

and then

$$Q_1 = P_0^{c_0} = v^{\frac{v_p(\xi)}{\gcd(n_p, v_p(\xi))}}.$$

Thus the first pencil is

$$\mathcal{Q}_{0,1} = \{\alpha u^{\frac{n_p}{\gcd(n_p, v_p(\xi))}} + \beta v^{\frac{v_p(\xi)}{\gcd(n_p, v_p(\xi))}} = 0\},$$

whose general germ has Puiseux series

$$\sqrt[n']{-\frac{\beta}{\alpha}} v^{\frac{v_p(\xi)/\gcd(n_p, v_p(\xi))}{n_p/\gcd(n_p, v_p(\xi))}},$$

and hence $\mathcal{K}_{0,1}$ has height $\frac{v_p(\xi)}{\gcd(n_p, v_p(\xi))}$. But again by 1.3.5, to obtain the height of $\widetilde{\mathcal{K}}_{0,1}$ we have to multiply $\frac{v_p(\xi)}{\gcd(n_p, v_p(\xi))}$ by $n_1 = \gcd(n_p, v_p(\xi))$, and hence $\widetilde{\mathcal{K}}_{0,1}$ has height $v_p(\xi)$. The claim is proved also in this case, completing the proof.

□

Theorem 2.3.3. *Let $p \in \mathcal{N}_O$ be any point such that γ^p is not tangent to η. Then $v_p(\xi) = m_p$ if and only if*

- *either p is free and there is a free point proximate to p lying on ξ (which in particular implies that p lies on ξ),*
- *or p is satellite and there exists a branch of ξ which goes through the point of which p is satellite, and it is not smaller than p.*

Equivalently, $v_p(\xi) = m_p$ if and only if there exists a branch of ξ going through the point of which p is satellite and not being smaller than p.

Proof. Let us first consider the case p free. By Proposition 2.3.2 we know that $v_p(\xi) = m_p$ if and only if the total transforms $\bar{\xi}_p$ and $\bar{\eta}_p$ have non-homothetical tangent cones. Since p is free, it is proximate to a single point, namely q. Let E_q be the germ (at p) of the exceptional divisor of the composition of blowing-ups $\pi_p : S_p \longrightarrow S$ giving rise to p. By definition, we have $\bar{\xi}_p = v_q(\xi)E_q + \tilde{\xi}_p$, and by the hypothesis on p we can write $\bar{\eta}_p = n_q E_q$. So, $\bar{\xi}_p$ and $\bar{\eta}_p$ have homothetical tangent cones if and only if every branch of $\tilde{\xi}_p$ is also tangent to E_q, which means that there is no free point in the first neighbourhood of p lying on ξ.

So, $v_p(\xi) = m_p$ if and only if there is some free point in the first neighbourhood of p lying on ξ, as wanted.

Now let us deal with the second case. Since p is satellite, it is proximate to two points q and q'. Interchanging them if necessary, we may assume that $q \prec q'$. Hence, by 2.2.11

we have $q \prec p \prec q'$. Note that q' and p are both satellite of p', while q might not (it might be the point preceding p', if p was in the first neighbourhood of p', for instance).

By definition and the hypothesis on p we have $\bar{\xi}_p = v_q(\xi)E_q + v_{q'}(\xi)E_{q'} + \tilde{\xi}_p$ and $\bar{\eta}_p = n_q E_q + n_{q'} E_{q'}$. Let a_q (resp. $a_{q'}$) denote the multiplicity of E_q (resp. $E_{q'}$) in the tangent cone of $\tilde{\xi}_p$. Then $\bar{\xi}_p$ and $\bar{\eta}_p$ have homothetical tangent cones if and only if every branch of $\tilde{\xi}_p$ is tangent either to E_q or $E_{q'}$ (which is equivalent to the equality $a_q + a_{q'} = e_p(\xi)$) and we have the equality

$$\frac{v_q(\xi) + a_q}{n_q} = \frac{v_{q'}(\xi) + a_{q'}}{n_{q'}}.$$

So assume $\bar{\xi}_p$ and $\bar{\eta}_p$ have homothetical tangent cones, which by 2.3.2 means that $v_p(\xi) < m_p$, and take $\alpha = \frac{v_q(\xi)+a_q}{n_q} = \frac{v_{q'}(\xi)+a_{q'}}{n_{q'}}$. Then on the one hand we have (remind that $a_q + a_{q'} = e_p(\xi)$, $n_p = n_q + n_{q'}$ and $v_p(\xi) = v_q(\xi) + v_{q'}(\xi) + e_p(\xi)$)

$$\alpha = \frac{v_q(\xi) + a_q + v_{q'}(\xi) + a_{q'}}{n_q + n_{q'}} = \frac{v_q(\xi) + v_{q'}(\xi) + e_p(\xi)}{n_p} = \frac{v_p(\xi)}{n_p} = I_\xi(p).$$

And on the other hand

$$\alpha = I_\xi(q) + \frac{a_q}{n_q} \geqslant I_\xi(q) \text{ and } \alpha = I_\xi(q') + \frac{a_{q'}}{n_{q'}} \geqslant I_\xi(q').$$

But we have assumed $q \prec p \prec q'$, and then by 2.2.7 we have

$$I_\xi(q) \leqslant I_\xi(p) \leqslant I_\xi(q'),$$

which combined with the above equalities implies that $I_\xi(p) = I_\xi(q')(= \alpha)$ and $a_{q'} = 0$. This in turn implies that $a_q = e_p(\xi)$. Now, by 2.2.18, $I_\xi(p) = I_\xi(q')$ holds if and only if every branch of ξ through p' is smaller than p, as wanted.

It remains to prove that if $\bar{\xi}_p$ and $\bar{\eta}_p$ have non-homothetical tangent cones (or equivalently, if $v_p(\xi) = m_p$), then there is some branch of ξ going through p' which is bigger than p. But this case only may occur either if $a_q + a_{q'} < e_p(\xi)$ or if

$$\frac{v_q(\xi) + a_q}{n_q} \neq \frac{v_{q'}(\xi) + a_{q'}}{n_{q'}}.$$

In the former case there is a branch of ξ through p whose point in its first neighbourhood is free, and such a branch is not smaller than p. In the latter case we can assume that $a_q + a_{q'} = e_p(\xi)$ (for if not we are in the previous case) and then we have that the quotient

$$\frac{v_q(\xi) + a_q + v_{q'}(\xi) + a_{q'}}{n_q + n_{q'}} = \frac{v_q(\xi) + v_{q'}(\xi) + e_p(\xi)}{n_p} = \frac{v_p(\xi)}{n_p} = I_\xi(p)$$

fits between $\frac{v_q(\xi)+a_q}{n_q} = I_\xi(q) + \frac{a_q}{n_q}$ and $\frac{v_{q'}(\xi)+a_{q'}}{n_{q'}} = I_\xi(q') + \frac{a_{q'}}{n_{q'}}$. Since $p \prec q'$ implies $I_\xi(p) \leqslant I_\xi(q')$, we are in fact in the situation

$$I_\xi(q) + \frac{a_q}{n_q} < I_\xi(p) < I_\xi(q') + \frac{a_{q'}}{n_{q'}}.$$

Now we have to consider in which cases the second inequality holds. If we already have $I_\xi(p) < I_\xi(q')$, then by 2.2.18 there exists a branch of ξ going through p' and bigger than p, as we want. If otherwise $I_\xi(p) = I_\xi(q')$, then $a_{q'}$ must be strictly positive and there is at least one branch of ξ whose strict transform at p is tangent to $E_{q'}$. But this branch is by definition bigger than p, and we are done. □

Corollary 2.3.4. *If p is a rupture point of ξ, then*

$$v_p(\xi) = m_p.$$

Proof. Since p is a rupture point of ξ, there is at least one branch of ξ going through it and whose point in the first neighbourhood if free. Such a branch clearly goes through the point of which p is satellite and it is not smaller than p. Thus, we have $v_p(\xi) = m_p$ in virtue of Theorem 2.3.3. □

Chapter 3

Solving the problem

3.1 Recovering rupture points.

This section is devoted to explain how the set of singular points $\mathcal{S}(\xi)$ (of a singular reduced germ of curve ξ) can be recovered from the weighted cluster $BP(\mathcal{J}(\xi))$ of base points of the jacobian system of ξ.

We will use the same notations as in the previous chapter. Let us recall them:

- ξ denotes a singular reduced germ of curve at O, with equation f.
- η is a smooth germ at O given by $g = 0$, non tangent to ξ and such that $\zeta : \frac{\partial(f,g)}{\partial(x,y)} = 0$ is a generic transverse polar of ξ.
- φ is the morphism from a neighbourhood U of O to $\mathbb{C}^2$ given by the equations (f, g). Therefore ζ can be viewed as $\mathbf{J}(\varphi)$, the jacobian germ of φ.
- For each $p \in \mathcal{N}_O$, n_p denotes the multiplicity of the composite morphism $\varphi_p = \varphi \circ \pi_p$, and m_p stands for the height of the p-trunk $\mathcal{T}_p = \mathcal{T}(\varphi_p)$.
- Also for each point $p \in \mathcal{N}_O$, γ^p is any irreducible curve going through p and such that its point in the first neighbourhood of p is free and does not belong to ξ.

Recall also from section 2.1 that for every $p \in \mathcal{N}_O$ such that γ^p is not tangent to η, n_p can be computed by means of the recursive formula

$$n_p = \sum_{p \to q} n_q,$$

starting with $n_O = 1$. In addition, if γ^p is not tangent to η, then p is non-fundamental, and Corollary 1.3.9 applies to give $m_O = e_O(\zeta)$ and

$$m_p = \sum_{p \to q} m_q + e_p(\zeta)$$

if $p \neq O$. Thus, if γ^p is not tangent to η, m_p can be computed just from the generic polar ζ. In particular, if p is in $BP(\mathcal{J}(\xi))$, or more generally, it is satellite of a point in $BP(\mathcal{J}(\xi))$, m_p can be computed just from the virtual multiplicities of the cluster of base points of the jacobian system.

Back to the problem, recall that the maximal singular points of ξ are rupture points (see Remark 1.2.2). So, in order to recover $\mathcal{S}(\xi)$, it will be enough to recover $\mathcal{R}(\xi)$, and this is what we will do.

We already know (Theorem 1.2.3) that for any rupture point q there exists a branch γ (not necessarily unique) of ζ such that $I_\xi(q) = \frac{[\gamma.\xi]}{e_O(\gamma)}$. Moreover, Theorem 1.2.3 also says that if p is the point of which q is satellite, then p is the last free point lying both on ξ and γ.

It is natural to wonder if the same branch γ could work for several rupture points. The answer is negative, as the following proposition shows.

Proposition 3.1.1. *Let γ be a branch of ζ, and let p_γ be the last free point lying both on ξ and γ. Then there exists a unique rupture point $q_\gamma \in \mathcal{R}^p(\xi)$ such that $I_\xi(q_\gamma) = \frac{[\gamma.\xi]}{e_O(\gamma)}$. Furthermore, $q_\gamma \preceq \gamma$.*

Proof. The existence of such a q_γ is guaranteed by Theorem 1.2.3, so it only remains to prove uniqueness.

The case $p_\gamma = O$ is quite easy, since O is the only O-satellite point, and thus $q_\gamma = O$ is the only possibility.

For the rest of the proof assume $p_\gamma \neq O$, and suppose that $q_1 \prec q_2$ are two rupture points of ξ satellite of p_γ and such that

$$I_\xi(q_1) = I_\xi(q_2) = \frac{[\gamma.\xi]}{e_O(\gamma)}.$$

By Theorem 2.2.18, no branch of ξ can be bigger than q_1. But since q_2 is a rupture point, there exists a branch of ξ going through q_2 and having a free point in its first neighbourhood, and such a branch is clearly bigger than q_1, which leads to a contradiction. Therefore, there exists a unique p_γ-satellite rupture point q_γ satisfying $I_\xi(q_\gamma) = \frac{[\gamma.\xi]}{e_O(\gamma)}$.

In order to prove that $q_\gamma \preceq \gamma$, note that we can consider $\frac{[\gamma.\xi]}{e_O(\gamma)}$ as $I_\xi(q')$, where q' is the last p_γ-satellite point on γ. Then $I_\xi(q_\gamma) = I_\xi(q')$, and again by Theorem 2.2.18 we obtain that $q_\gamma \preceq q'$, which implies $q_\gamma \preceq \gamma$ by definition. □

From now on, if γ is a branch of the polar ζ, p_γ will denote the last free point lying both in γ and ζ, and q_γ will stand for the p_γ-satellite rupture point associated to γ (which is uniquely determined because of proposition 3.1.1).

So, in order to recover the set of rupture points $\mathcal{R}(\xi)$ just from $BP(\mathcal{J}(\xi))$ (and hence recover the set of singular points $\mathcal{S}(\xi)$), we only have to find the rupture point q_γ associated to each branch γ of ζ by the previous result. We will say that $I_\xi(q_\gamma) = \frac{[\xi.\gamma]}{e_O(\gamma)}$ is the *polar invariant associated to* the branch γ.

The first step to determine q_γ is to compute the polar invariant $I_\xi(q_\gamma) = \frac{[\xi.\gamma]}{e_O(\gamma)}$ from $BP(\mathcal{J}(\xi))$, and we can do it thanks to the following

Lemma 3.1.2. *If ξ is any germ of curve and γ is any branch of a generic transverse polar ζ of ξ, we have*

$$I_\xi(q_\gamma) = \frac{[\xi.\gamma]}{e_O(\gamma)} = \frac{[BP(\mathcal{J}(\xi)).\gamma]}{e_O(\gamma)} + 1.$$

Proof. By definition, there exist some equation f of ξ and some smooth germ $g = 0$ such that ζ is given by the equation $\frac{\partial(f,g)}{\partial(x,y)} = 0$. Up to change of coordinates, we may assume $g = x$, and thus $\zeta : \frac{\partial f}{\partial y} = 0$.

By Theorem 1.1.12 and Lemma 1.2.5, generic germs ζ' of the pencil $\left\{\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} = 0\right\}$ go sharply through $BP(\mathcal{J}(\xi))$ and miss the first point lying on γ and not in $BP(\mathcal{J}(\xi))$. Then, for any such ζ', we have

$$[BP(\mathcal{J}(\xi)).\gamma] = [\zeta'.\gamma].$$

Moreover, up to a linear change of the coordinate y, we may assume that $\zeta' : \frac{\partial f}{\partial x} = 0$ is the y-polar of ξ.

Now, let $n = e_O(\gamma)$ and let $s(x)$ be a Puiseux series of γ. Thus, we have

$$[\xi.\gamma] = \sum_{\epsilon^n=1} o_x(f(x, s(\epsilon x))) \text{ and } [\zeta'.\gamma] = \sum_{\epsilon^n=1} o_x\left(\frac{\partial f}{\partial x}(x, s(\epsilon x))\right).$$

We may relate the summands in the two formulas as follows:

$$o_x(f(x, s(\epsilon x))) = 1 + o_x\left(\frac{d}{dx}f(x, s(\epsilon x))\right) = 1 + o_x\left(\frac{\partial f}{\partial x}(x, s(\epsilon x)) + \epsilon\frac{\partial f}{\partial y}(x, s(\epsilon x))s'(x)\right)$$

and since γ is a branch of $\zeta : \frac{\partial f}{\partial y} = 0$, the summand $\epsilon\frac{\partial f}{\partial y}(x, s(\epsilon x))s'(x)$ vanishes identically. Now adding-up all this equalities for every n-th root of the unity ϵ, we finally obtain

$$[\xi.\gamma] = n + [\zeta'.\gamma] = e_O(\gamma) + [BP(\mathcal{J}(\xi)).\gamma]$$

and the claim follows. □

The second step in order to determine q_γ is to determine p_γ, the free point of which q_γ is satellite, or equivalently, the last free point lying both on γ and ξ. To achieve this we will use a property that relates p_γ with the polar invariant $I_\xi(q_\gamma)$, and which is stated in forthcoming Lemma 3.1.5.

Remark 3.1.3. *Note that p_γ is a singular point of ξ because it precedes a rupture point. Thus, by Lemma 1.2.6, $p_\gamma \in BP(\mathcal{J}(\xi))$, and so in order to solve the problem there is no need to know the points lying on a specific generic polar, and knowing the base points of the jacobian system suffices*

We first study the very easy (and exceptional) case $p_\gamma = O$, and then deal with the general one.

Lemma 3.1.4. *Keeping the notations as above, we have $p_\gamma = q_\gamma = O$ if and only if $\frac{[BP(\mathcal{J}(\xi)).\gamma]}{e_O(\gamma)} = e_O(\zeta)$, or equivalently, if and only if γ is a branch whose only point in $BP(\mathcal{J}(\xi))$ is the origin O.*

Proof. Suppose first that $p_\gamma = q_\gamma = O$. Then it obviously holds that $I_\xi(O) = e_O(\xi)$. But by Lemma 3.1.2, the left hand side is equal to $\frac{[BP(\mathcal{J}(\xi)).\gamma]}{e_O(\gamma)} + 1$, and since we are assuming ζ to be a generic polar, the right hand side is precisely $e_O(\zeta) + 1$. Thus we have proved that if $p_\gamma = q_\gamma = O$, then $\frac{[BP(\mathcal{J}(\xi)).\gamma]}{e_O(\gamma)} = e_O(\zeta)$.

Conversely, suppose that $\frac{[BP(\mathcal{J}(\xi)).\gamma]}{e_O(\gamma)} = e_O(\zeta)$. This is equivalent to the equality $I_\xi(q_\gamma) = e_O(\xi)$. But by the Noether's Formula 1.1.3 this means that $[\xi.\mathcal{K}(q_\gamma)] = e_O(\xi)$, which implies that any germ going through q_γ shares with ξ just the origin, and then $p_\gamma = O$. Finally, since O has no satellite points besides O itself, we also have $q_\gamma = O$. □

Lemma 3.1.5. *Suppose that $p_\gamma \neq O$ is proximate to p'_γ. Then we have $v_{p'_\gamma}(\xi) = m_{p'_\gamma}$, and thus*

$$I_\xi(p'_\gamma) = \frac{m_{p'_\gamma}}{n_{p'_\gamma}} < I_\xi(q_\gamma) \leqslant I_\xi(p_\gamma) \leqslant \frac{m_{p_\gamma}}{n_{p_\gamma}}.$$

Proof. Since p_γ is a free point lying on ξ in the first neighbourhood of p'_γ, the equality $v_{p'_\gamma}(\xi) = m_{p'_\gamma}$ is a consequence of Theorem 2.3.3. Hence we have $I_\xi(p'_\gamma) = \frac{m_{p'_\gamma}}{n_{p'_\gamma}}$.

The inequalities $I_\xi(p'_\gamma) < I_\xi(q_\gamma) \leqslant I_\xi(p_\gamma)$ are consequence of Theorems 2.2.17 and 2.2.18, and the final inequality follows from Lemma 2.3.1. □

So, among all free points p lying on γ, we only need to consider those ones satisfying $\frac{m_{p'}}{n_{p'}} < I_\xi(q_\gamma) \leqslant \frac{m_p}{n_p}$ (where p' is the point to which p is proximate). At first sight, there might be many free points p lying on γ and satisfying this condition. Next proposition will allow us to determine which of those is exactly p_γ.

Proposition 3.1.6. *Assume again that $p_\gamma \neq O$. Let $p' \in \mathcal{N}_O(\gamma)$ be any point such that the point p in its first neighbourhood lying on γ is free. Then $I_\xi(p') < I_\xi(q_\gamma)$ if and only if $p \leqslant p_\gamma$ (or equivalently $p' \leqslant p'_\gamma$).*

Proof. Let $\{O < p_1 < p_2 < \ldots < p_k < \ldots\}$ be the free points lying on γ, and for any $i \geqslant 1$, let p'_i be the point to which p_i is proximate. Then p'_{i+1} is satellite of p_i (in fact, it may be equal to p_i), and by Theorems 2.2.17 and 2.2.18 we have

$$I_\xi(p'_i) \leqslant I_\xi(p'_{i+1}) \leqslant I_\xi(p_i),$$

and the first inequality is strict if and only if $p'_i \leqslant p'_\gamma$, since this is equivalent to p_i lying on ξ. In particular $\{I_\xi(p'_i)\}_{i\geqslant 1}$ is a strictly increasing sequence up to p'_γ, and afterwards it becomes constant. So if $p_\gamma = p_k$, we have

$$I_\xi(p'_k) = I_\xi(p'_\gamma) < I_\xi(q_\gamma) \leqslant I_\xi(p_k) = I_\xi(p_\gamma).$$

Now let us consider $I_\xi(p'_{k+1})$. Since $\gamma \succeq q_\gamma$ (Proposition 3.1.1), it results that

$$I_\xi(q_\gamma) \leqslant \frac{[\xi.\gamma]}{e_O(\gamma)} = I_\xi(p'_{k+1}) = I_\xi(p'_{k+2}) = \cdots$$

and we are done. □

Corollary 3.1.7. *In the case $I_\xi(q_\gamma) \neq \nu_O(BP(\mathcal{J}(\xi))) + 1$ (i.e., $p_\gamma \neq O$), the point p'_γ to which p_γ is proximate is the greatest point $p' \in \mathcal{N}_O(\gamma)$ such that $\frac{m_{p'}}{n_{p'}} < I_\xi(q_\gamma)$ and the point p lying on γ in the first neighbourhood of p' is free.*

Proof. With the notations of the proof of the previous proposition, for $i \leqslant k$ Theorem 2.3.3 implies that $v_{p'_i}(\xi) = m_{p'_i}$, and thus $I_\xi(p'_i) = \frac{m_{p'_i}}{n_{p'_i}} < I_\xi(q_\gamma)$ by the previous proposition.

However, for $i > k$, Lemma 2.3.1 and Proposition 3.1.6 give

$$\frac{m_{p'_i}}{n_{p'_i}} \geqslant I_\xi(p'_i) = I_\xi(p'_{k+1}) \geqslant I_\xi(q_\gamma),$$

completing the proof. □

Now that we have determined p_γ, it only remains to know which of its satellite points is q_γ. We will first deal with some cases in which this task is quite straightforward, and then treat the general case.

Proposition 3.1.8. *Let γ be a branch of ζ with associated polar invariant $I = \frac{[\xi.\gamma]}{e_O(\gamma)}$ and whose last free point on ξ is p_γ. Assume that there exists another branch γ' of ζ whose last free point on ξ is also p_γ but whose associated invariant I' is greater than I. Then q_γ is the last p_γ-satellite point on γ.*

Proof. Suppose the claim is false and let $\bar{q}_\gamma$ be the last p_γ-satellite point on γ. Proposition 3.1.1 implies that $\bar{q}_\gamma \succeq q_\gamma$, and hence $\bar{q}_\gamma \succ q_\gamma$. Moreover, since p_γ is the last free point lying both on γ and ξ, we can take γ as $\gamma^{\bar{q}_\gamma}$ and then

$$I_\xi(\bar{q}_\gamma) = \frac{[\xi.\gamma^{\bar{q}_\gamma}]}{e_O(\gamma^{\bar{q}_\gamma})} = \frac{[\xi.\gamma]}{e_O(\gamma)} = I.$$

If $q_{\gamma'}$ is the rupture point associated to γ', we claim that $q_{\gamma'} \succ q_\gamma$. Indeed, if it is not the case, Theorem 2.2.18 would imply that

$$I' = I_\xi(q_{\gamma'}) \leqslant I_\xi(\bar{q}_\gamma) = I$$

contradicting our hypothesis.

Therefore, there exists some branch of ξ bigger than q_γ, and then Theorem 2.2.18 again will give

$$I_\xi(q_\gamma) < I_\xi(\bar{q}_\gamma) = I,$$

which contradicts that q_γ is the rupture point associated to γ. □

This proposition leaves only untreated the case of the biggest p_γ-satellite rupture point, which we shall study now.

The problem with this case is that there are many p_γ-satellite points p with the same invariant $I_\xi(p) = I_\xi(q_\gamma)$. Moreover, although it can be proved that q_γ is the smallest such point, there is no way to determine it explicitly from the last p_γ-satellite point on γ, since, for the moment, we have no way to kno2 $I_\xi(p)$ but for the last p_γ satellite point on γ.

Fortunately, the p_γ-satellite points p bigger than q_γ are exactly the points for which $v_p(\xi) < m_p$ (Theorem 2.3.3), and this fact enables us to solve this case. In other words, the heights m_p can distinguish between the p_γ-satellite points when the invariants $I_\xi(p)$ cannot.

Although this fact is not enough to give a result analogous to Proposition 3.1.8, it allows us to develop an algorithm which computes q_γ just from the polar invariant $I_\xi(q_\gamma)$

and the already determined p_γ, by seeking the unique p_γ satellite point p for which the equality $\frac{m_p}{n_p} = I_\xi(q_\gamma)$ holds. In fact, it computes step by step all the intermediate points $p_\gamma = q_0 < q_1 < \cdots < q_{k-1} < q_k = q_\gamma$ (where q_i is in the first neighbourhood of q_{i-1}). It is worth noting that it computes q_γ for any γ, even those considered in Proposition 3.1.8.

The procedure works as follows:

- Start with $i = 0$ and $q_0 = p_\gamma$.
- While $\frac{m_{q_i}}{n_{q_i}} \neq I_\xi(q_\gamma)$ do
  - If $\frac{m_{q_i}}{n_{q_i}} > I_\xi(q_\gamma)$ take q_{i+1} to be the first satellite of q_i.
  - If $\frac{m_{q_i}}{n_{q_i}} < I_\xi(q_\gamma)$ take q_{i+1} to be the second satellite of q_i.

  Increase i to $i + 1$.
- If $\frac{m_{q_i}}{n_{q_i}} = I_\xi(q_\gamma)$, end by taking $k = i$ and $q_\gamma = q_k$.

Theorem 3.1.9. *The above procedure ends after a finite number of steps, and actually computes the rupture point q_γ.*

Proof. First of all, note that since q_γ is a rupture point, Corollary 2.3.4 implies that

$$I_\xi(q_\gamma) = \frac{m_{q_\gamma}}{n_{q_\gamma}}.$$

Therefore, since there are finitely many points between p_γ and q_γ, it is enough to check that each q_i actually precedes q_γ and that if $\frac{m_{q_i}}{n_{q_i}} = I_\xi(q_\gamma)$ then $q_i = q_\gamma$.

To see that $q_i \leqslant q_\gamma$ for each i we use induction on i.

For $i = 0$, we have $q_0 = p_\gamma$, and hence $q_0 = p_\gamma \leqslant q_\gamma$ by definition of p_γ.

Now suppose we have reached the step i of the algorithm and we have to perform another step. This means that $q_i \leqslant q_\gamma$ and $\frac{m_{q_i}}{n_{q_i}} \neq I_\xi(q_\gamma)$. We know that in this case $q_i < q_\gamma$, and we claim that the point q_{i+1} computed by the algorithm still precedes q_γ.

Indeed, since both q_i and q_γ are p_γ-satellite points, the point in the first neighbourhood of q_i preceding q_γ must be satellite. Hence, it only remains to check that the choice made by the algorithm is the correct one.

- If $\frac{m_{q_i}}{n_{q_i}} < I_\xi(q_\gamma)$, then
$$I_\xi(q_i) = \frac{v_{q_i}(\xi)}{n_{q_i}} \leqslant \frac{m_{q_i}}{n_{q_i}} < I_\xi(q_\gamma)$$
by 2.3.1. Therefore, by Theorem 2.2.9, the next point q_{i+1} must be the second satellite point, for if it was the first the invariants $I_\xi(q)$ would be strictly smaller than $I_\xi(q_\gamma)$ for every satellite $q \geqslant q_{i+1}$.
- If $\frac{m_{q_i}}{n_{q_i}} > I_\xi(q_\gamma)$ then either $I_\xi(q_\gamma) < I_\xi(q_i) \leqslant \frac{m_{q_i}}{n_{q_i}}$ or $I_\xi(q_i) \leqslant I_\xi(q_\gamma) < \frac{m_{q_i}}{n_{q_i}}$. In the former case we apply Theorem 2.2.9 and see that q_{i+1} must be the first satellite point of q_i. In the latter case we have that $v_{q_i}(\xi) < m_{q_i}$, and hence by Theorem 2.3.3 it holds that every branch of ξ through p_γ is smaller than q_i. This implies in particular that $q_\gamma \prec q_i$, and thus by Theorem 2.2.9 q_γ must be infinitely near to the first satellite of q_i.

In any case, the algorithm is correct.

In order to complete the proof, we must check that the algorithm does not stop before reaching the point q_γ. That is, we have to show that if q is any p_γ-satellite point strictly preceding q_γ, then $\frac{m_q}{n_q} \neq I_\xi(q_\gamma)$.

- If $q \prec q_\gamma$, any branch of ξ going through q_γ is bigger than q. Then Theorem 2.2.18 implies that $I_\xi(q) < I_\xi(q_\gamma)$, and by Theorem 2.3.3 we also have that $v_q(\xi) = m_q$. So

$$I_\xi(q) = \frac{m_q}{n_q} < I_\xi(q_\gamma)$$

  and in particular $\frac{m_q}{n_q} \neq I_\xi(q_\gamma)$.

- Consider now the case $q \succ q_\gamma$. Then, on the one hand Theorem 2.2.18 implies that $I_\xi(q_\gamma) \leqslant I_\xi(q)$, with equality if and only if every branch of ξ going through p_γ is not bigger than q_γ. On the other hand, Theorem 2.3.3 says that $v_q(\xi) \leqslant m_q$, and equality holds if and only if there is some branch of ξ not smaller than q. Summarizing, we have

$$I_\xi(q_\gamma) \leqslant I_\xi(q) \leqslant \frac{m_q}{n_q},$$

  and having equality $I_\xi(q_\gamma) = \frac{m_q}{n_q}$ would imply (by Theorem 2.2.18) that there is some branch of ξ through p_γ which is not smaller than q. But such a branch would be bigger than q_γ, implying that $I_\xi(q_\gamma) < I_\xi(q) \leqslant \frac{m_q}{n_q}$ and thus contradicting the equality $I_\xi(q_\gamma) = \frac{m_q}{n_q}$.

□

Remark 3.1.10. *As a final remark, note that two branches of ζ having the same last point in $BP(\mathcal{J}(\xi))$ in fact define the same rupture point q. Therefore, we can in fact define the rupture point q associated to a dicritical point of $BP(\mathcal{J}(\xi))$, and apply the results in this section to each dicritical point of $BP(\mathcal{J}(\xi))$ instead of to each branch of ζ.*

3.2 Recovering values.

This section is devoted to explain how the values of a curve ξ at its singular points can be recovered from the quotients $\frac{m_p}{n_p}$, provided the set of rupture points $\mathcal{R}(\xi)$ (and hence the cluster of singular points $\mathcal{S}(\xi)$) is already known (for example, if it has been determined as explained in the previous section). Recall that from Lemma 2.2.1 we already know that $v_p(\xi) = n_p I_\xi(p)$ at any $p \in \mathcal{S}(\xi)$.

We will consider the different cases ordered by increasing difficulty: we will start with the rupture points (which are the easiest ones), followed by the free singular points, and we will finish by considering the satellite points which are not rupture points (the most complicated case).

The easiest cases are rupture points, because Corollary 2.3.4 implies that $v_p(\xi) = m_p$ for any $p \in \mathcal{R}(\xi)$.

Let us now consider $p \in \mathcal{S}(\xi)$ a free singular point which is not a rupture point. By Theorem 2.3.3, we have the equality $v_p(\xi) = m_p$ if and only if there is a free point in the

first neighbourhood of p lying on ξ. In particular, if there is a free singular point in the first neighbourhood of p, we can also assert that $v_p(\xi) = m_p$. But what can we say if there are no free singular points in the first neighbourhood of p and p is not a rupture point? This situation is in which Theorem 2.2.20 plays an important role. So suppose that p is a free singular point of ξ, but that it is not a rupture point and that there is no free singular point on ξ in its first neighbourhood. This means that there is at most one free point lying on ξ in the first neighbourhood of p, and if it exists, it is non-singular.

If there is no such a point, then Corollary 2.2.19 implies that

$$v_p(\xi) = n_p I_\xi(p) = n_p I_\xi(q) = \frac{n_p}{n_q} v_q(\xi) = \frac{n_p}{n_q} m_q,$$

where q is the biggest p-satellite point in $\mathcal{R}(\xi)$. On the contrary, if there is one such point, Theorem 2.2.20 gives the inequalities

$$\frac{[\xi.\gamma^p] - 1}{e_O(\gamma^p)} = \frac{v_p(\xi) - 1}{n_p} < I_\xi(q) = \frac{v_q(\xi)}{n_q} < I_\xi(p) = \frac{v_p(\xi)}{n_p},$$

which are equivalent to

$$\frac{n_p}{n_q} m_q < v_p(\xi) < \frac{n_p}{n_q} m_q + 1,$$

where as above q is the biggest p-satellite point in $\mathcal{R}(\xi)$.

Hence, in any case, $v_p(\xi)$ belongs to the real interval $\left[\frac{n_p}{n_q} m_q, \frac{n_p}{n_q} m_q + 1\right)$. But the width of this interval is one, so there is exactly one integer in it, and thus the value $v_p(\xi)$ is uniquely determined.

So far we have proved the following

Proposition 3.2.1. *Let $p \in \mathcal{S}(\xi)$ be a free singular point which is not a rupture point.*

- *If there is a free singular point in the first neighbourhood of p, then $v_p(\xi) = m_p$.*
- *Otherwise, let q be the biggest point in $\mathcal{R}^p(\xi)$ (which must be non-empty). Then $v_p(\xi)$ is the only integer in the interval*

$$\left[\frac{n_p}{n_q} m_q, \frac{n_p}{n_q} m_q + 1\right).$$

Moreover, the equality $v_p(\xi) = \frac{n_p}{n_q} m_q$ holds if and only if there is no branch of ξ going through p and whose point in the first neighbourhood of p is free.

It only remains to consider the satellite points $p \in \mathcal{S}(\xi)$ which are not rupture points, and it is solved by the next

Proposition 3.2.2. *Let $p \in \mathcal{S}(\xi)$ be a satellite point of ξ which is not a rupture point. Suppose moreover that p is satellite of $p' \in \mathcal{S}(\xi)$ and let q be the biggest p'-satellite rupture point of ξ. Then*

$$v_p(\xi) = \begin{cases} \frac{n_p}{n_{p'}} v_{p'}(\xi) & \text{if } p \succ q \text{ and } v_{p'}(\xi) = \frac{n_{p'}}{n_q} m_q \text{ both hold,} \\ m_p & \text{otherwise.} \end{cases}$$

Proof. If $p' = q$ is itself a rupture point, there exists a branch of ξ going through p' and having a free point in its first neighbourhood, and the same holds if $v_{p'}(\xi) \neq \frac{n_{p'}}{n_q} m_q$ (by Proposition 3.2.1). Thus, in any case Theorem 2.3.3 implies that $v_p(\xi) = m_p$.

Suppose now that p' is not a rupture point and $v_{p'}(\xi) = \frac{n_{p'}}{n_q} m_q$. Then there is no branch of ξ going through p' and having a free point in its first neighbourhood. If furthermore $p \prec q$, Theorem 2.3.3 applies to give $v_p(\xi) = m_p$ again, but if otherwise $p \succ q$, Corollary 2.2.19 gives that

$$v_p(\xi) = n_p I_\xi(p) = n_p I_\xi(p') = \frac{n_p}{n_{p'}} v_{p'}(\xi).$$

□

As a consequence of the proof of Proposition 3.2.2 we infer the following result, which determines for which free points $p \in \mathcal{S}(\xi)$ (besides the rupture points) there exists some branch of ξ which goes through p and is non-singular after it.

Corollary 3.2.3. *Let $p \in \mathcal{S}(\xi)$ be a free singular point. Then there is some branch of ξ non-smaller than p if and only if either p is itself a rupture point or $v_p(\xi) \neq \frac{n_p}{n_q} m_q$ (where q is the biggest p-satellite rupture point of ξ).*

3.3 The algorithm.

In this section, we present explicitly the algorithm which computes the cluster $\mathcal{S}(\xi)$ and the values $v_p(\xi)$ for $p \in \mathcal{S}(\xi)$ from the weighted cluster $BP(\mathcal{J}(\xi))$ of base points of the polar germs of ξ.

The algorithm:

- *Recovering the rupture points and the singular points.*

  1. Start with $\mathcal{R} = \mathcal{S} = \emptyset$, and let $\mathcal{D}$ be the set of dicritical points of $BP(\mathcal{J}(\xi))$.
  2. If $O \in \mathcal{D}$, then set $\mathcal{R} = \mathcal{S} = \{O\}$
  3. For each $p^* \in \mathcal{D}$ compute $I_{p^*} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p^*)]}{n_{p^*}} + 1$, and order $\mathcal{D} - \{O\} = \{p_1^*, \ldots, p_k^*\}$ by descending order of I_{p^*} (i.e., $I_{p_1^*} \geqslant I_{p_2^*} \geqslant \ldots \geqslant I_{p_k^*}$).
  4. For each $j = 1, \ldots k$ do:
     (a) Find the last point $p' < p_j^*$ such that $\frac{m_{p'}}{n_{p'}} < I_{p_j^*}$ and its next point p in $\mathcal{K}(p_j^*)$ is free (by Remark 3.1.3, we know that p is in $BP(\mathcal{J}(\xi))$, and so $p \leqslant p_j^*$).
     (b) If p has already appeared at this step, let q_j be the last p-satellite point in $\mathcal{K}(p_j^*)$ and set $\mathcal{R} = \mathcal{R} \cup \{q_j\}$ and $\mathcal{S} = \mathcal{S} \cup \{q \,|\, q \leqslant q_j\}$. Then skip to the next j.
     (c) Otherwise, take $i = 0$ and $q_0 = p$.
     (d) While $\frac{m_{q_i}}{n_{q_i}} \neq I_{p_j^*}$ do
        – If $\frac{m_{q_i}}{n_{q_i}} > I_{p_j^*}$, take q_{i+1} to be the first satellite of q_i.

– If $\frac{m_{q_i}}{n_{q_i}} < I_{p_j^*}$, take q_{i+1} to be the second satellite of q_i.

Increase i to $i+1$.

(e) If $\frac{m_{q_i}}{n_{q_i}} = I_{p_j^*}$, set $\mathcal{R} = \mathcal{R} \cup \{q_i\}$ and $\mathcal{S} = \mathcal{S} \cup \{q \,|\, q \leqslant q_i\}$.

- *Recovering values.*

1. For each $p \in \mathcal{R}$ set $v_p = m_p$.
2. For each free point $p \in \mathcal{S} - \mathcal{R}$
   - If there is a free point both in $\mathcal{S}$ and in the first neighbourhood of p, set $v_p = m_p$.
   - Otherwise, let q be the biggest p-satellite point in $\mathcal{R}$ and set v_p the only integer in the interval $\left[\frac{n_p}{n_q} m_q, \frac{n_p}{n_q} m_q + 1\right)$.
3. For each satellite point $p \in \mathcal{S} - \mathcal{R}$, let p' be the free point of which p is satellite, and let q be the biggest point in $\mathcal{R}$ which is satellite of p'.
   - If $p \succ q$ and $v_{p'}(\xi) = \frac{n_{p'}}{n_q} m_q$ both hold, set $v_p = \frac{n_p}{n_{p'}} v_{p'}(\xi)$.
   - Otherwise, set $v_p = m_p$.

From the previous results in this chapter, it is immediate to check the following

Theorem 3.3.1. *The previous algorithm actually computes the set of rupture points and the cluster of singular points of ξ, together with the values of ξ at these points. More precisely, at the end of the algorithm we have $\mathcal{R}(\xi) = \mathcal{R}$, $\mathcal{S}(\xi) = \mathcal{S}$ and $v_p(\xi) = v_p$ for any $p \in \mathcal{S}(\xi)$.*

Remark 3.3.2. *It is possible to shorten the presentation of the algorithm replacing the first part by*

1. *Start with $\mathcal{R} = \mathcal{S} = \emptyset$, and let $\mathcal{D}$ be the set of dicritical points of $BP(\mathcal{J}(\xi))$.*
2. *If $O \in \mathcal{D}$, then set $\mathcal{R} = \mathcal{S} = \{O\}$*
3. *For each $p^* \in \mathcal{D}$ do:*

   (a) Compute $I = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p^*)]}{n_{p^*}} + 1$.

   (b) Find the last point $p' < p^*$ such that $\frac{m_{p'}}{n_{p'}} < I$ and its next point p in $\mathcal{K}(p_j^*)$ is free.

   (c) Take $i = 0$ and $q_0 = p$.

   (d) While $\frac{m_{q_i}}{n_{q_i}} \neq I$ do

   - *If $\frac{m_{q_i}}{n_{q_i}} > I$, take q_{i+1} to be the first satellite of q_i.*
   - *If $\frac{m_{q_i}}{n_{q_i}} < I$, take q_{i+1} to be the second satellite of q_i.*

   Increase i to $i+1$.

   (e) If $\frac{m_{q_i}}{n_{q_i}} = I$, set $\mathcal{R} = \mathcal{R} \cup \{q_i\}$ and $\mathcal{S} = \mathcal{S} \cup \{q \,|\, q \leqslant q_i\}$.

However, we have chosen the longer version because it gives a more precise and geometrical description of some of the rupture points q_γ, for which also avoids the tedious task of performing the iterations in step (d).

Remark 3.3.3. *This algorithm gives a proof of the first statement in Theorem 1.2.7. Furthermore, it is obvious that the algorithm yields similar clusters if it is applied to similar clusters, so in fact it also proves the second statement in Theorem 1.2.7, as we wanted.*

Appendix A

Examples.

A.1 Some little examples.

In this section we expose some explicit applications of the algorithm to some "simple" curves ξ (i.e., curves with a little number of branches and just a few singular points). Some of them will be worked in detail (the first ones and some others which illustrate special cases), and the other will just be sketched.

In each example we will start from an equation f of ξ and then we will compute the cluster of base points $BP(\mathcal{J}(\xi)) = BP\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ using the algorithm given in [2] (this part will not be explained in any case). Then we will apply the algorithm of section 3.3 to recover the cluster $\mathcal{S}(\xi)$ with the corresponding values, showing the invariants $\frac{m_p}{n_p}$ computed and explaining how the algorithm works (the explicitness of the explanation will vary in each case).

Each example will be illustrated with some Enriques' diagrams. For the sake of clarity, the singular points of ξ will be represented with black filled circles, while the points in $BP(\mathcal{J}(\xi))$ not lying on ξ will be represented with white filled circles.

When reading each example, it is advisable to look at the corresponding figure in order to fix some notation, paying attention to the labels of the points of the clusters.

Example 1: (See figure A.1.)

ξ is the curve given by $f = (y^2 - \alpha x^3)(y^2 + \alpha x^3) = 0$, where α is any non-zero complex number. It consists of two branches, both of them with characteristic exponent $\frac{3}{2}$, and sharing no points besides the singular ones.

The cluster $BP(\mathcal{J}(\xi))$ is the irreducible cluster consisting of the singular points of ξ plus the second satellite of the last point in $\mathcal{S}(\xi)$ (which is labeled with p_3).

The first part of the algorithm is quite short, because p_3 is the only one dicritical point. Thus we start with $\mathcal{R} = \mathcal{S} = \emptyset$ and compute

$$I = I_{p_3} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_3)]}{n_{p_3}} + 1 = \frac{BP(\mathcal{J}(\xi))^2}{3} + 1 = \frac{3^2 + 2^2 + 1^2 + 1^2}{3} + 1 = 6.$$

The point p' is the origin O, and hence the rupture point q associated to p_3 is satellite of $q_0 = p_1$. The step 4(d) in the algorithm consists just of one iteration:

- $\frac{m_{q_0}}{n_{q_0}} = 7 > 6 = I$, so we take $q_1 = p_2$, the first satellite of p_1.

Since $\frac{m_{q_1}}{n_{q_1}} = 6 = I$, we end by taking $\mathcal{R} = \{p_2\}$ and $\mathcal{S} = \{O, p_1, p_2\}$.

The second part of the algorithm starts setting $v_{p_2} = m_{p_2} = 12$, and the second step is:

- Since $p_1 \in \mathcal{S}$ is a free point in the first neighbourhood of O, we set $v_O = m_O = 4$.
- Since there is no free singular point in the first neighbourhood of p_1, we apply the second instance. The biggest p_1-satellite point in $\mathcal{R}$ is p_2, and so we take v_{p_1} to be the only integer in the interval $\left[\frac{1}{2}12, \frac{1}{2}12 + 1\right) = [6, 7)$, that is, $v_{p_1} = 6$.

There is no third step because the only satellite point in $\mathcal{S}$ is also in $\mathcal{R}$, so we have finished and the cluster $\mathcal{S}$ and the values v_p obtained clearly coincide with $\mathcal{S}(\xi)$ and $v_p(\xi)$.

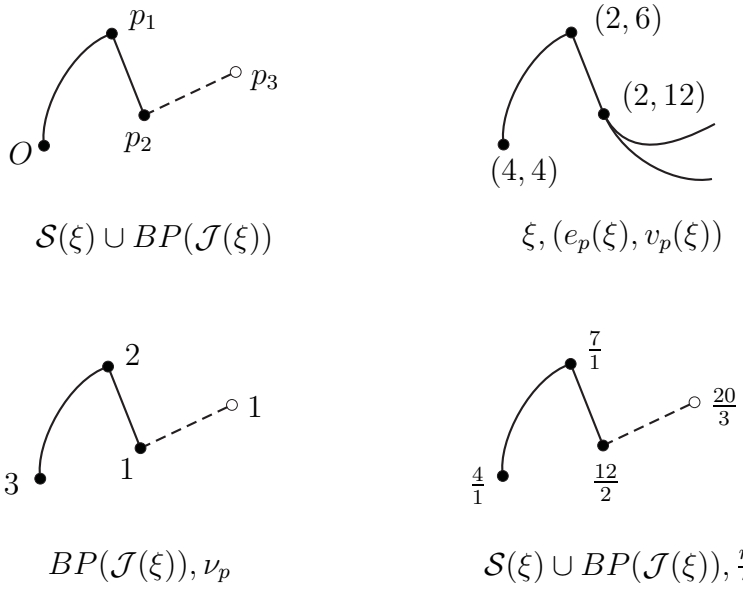


Figure A.1: $\xi : (y^2 - \alpha x^3)(y^2 + \alpha x^3) = 0$.

Example 2: (See figure A.2.)

ξ is the curve given by $(y^2 - \alpha x^3)(y^2 - \beta x^3) = 0$, where α and β are non-zero complex numbers such that $\alpha + \beta \neq 0$.

The cluster $BP(\mathcal{J}(\xi))$ consists of the singular points of ξ and one further free point p_3 on the polar $\zeta_x = P_x(f) : \frac{\partial f}{\partial y} = 0$ in the first neighbourhood of the last singular point of ξ. The virtual multiplicities are displayed in figure A.2.

Note that ξ consists of two branches, both of them with characteristic exponent $\frac{3}{2}$ and sharing no non-singular point. Hence, ξ is equisingular to the curve in the previous example, but its generic polars are not. Indeed, generic polars of ξ have one smooth branch tangent to ξ and one singular branch sharing four points with ζ_x, while generic polars in example 1 were irreducible.

The set of dicritical points of $BP(\mathcal{J}(\xi))$ is $\mathcal{D} = \{p_1, p_3\}$, and since $O \notin \mathcal{D}$ the algorithm starts with $\mathcal{R} = \mathcal{S} = \emptyset$. The polar invariants are

$$I_1 = I_{p_1} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_1)]}{n_{p_1}} + 1 = \frac{3 \cdot 1 + 2 \cdot 1}{1} + 1 = 6,$$

$$I_3 = I_{p_3} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_3)]}{n_{p_3}} + 1 = \frac{3\cdot 2 + 2\cdot 1 + 1^2 + 1^2}{2} + 1 = 6.$$

Thus any order of $\mathcal{D}$ is valid, and we choose to start with p_1. The corresponding polar invariant is $I_1 = 6$ and thus p' is the origin. The rest of the computations are the same as in Example 1, obtaining $\mathcal{R} = \{p_2\}$ and $\mathcal{S} = \{O, p_1, p_2\}$.

Take now p_3, whose polar invariant is also $I_3 = 6$. Therefore p' is again O, and the procedure yields the same result as for p_1. Hence it is not necessary to add any point to $\mathcal{R}$ nor $\mathcal{S}$.

So far we have determined $\mathcal{R}(\xi) = \{p_2\}$ (and thus $\mathcal{S}(\xi) = \{O, p_1, p_2\}$), as in example 1. Since the invariants m_p and n_p for $p \in \mathcal{S}$ are the same as in that example, the second part of the algorithm is exactly the same and thus we obtain the same cluster of singular points (as we already knew).

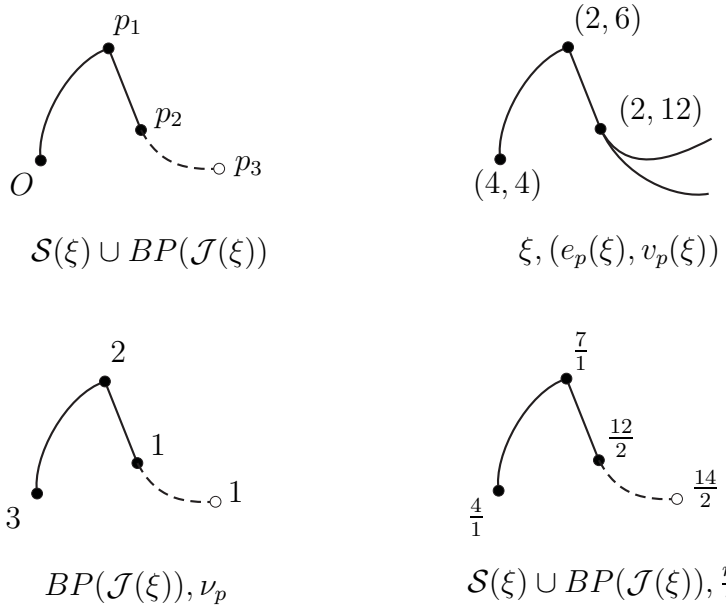


Figure A.2: $\xi : (y^2 - \alpha x^3)(y^2 - \beta x^3) = 0$, $\alpha, \beta, \alpha + \beta \neq 0$.

Example 3: (See figure A.3.)

Now consider ξ the curve given by $(y^2 - \beta_1 x^3)((y - \alpha x)^2 - \beta_2 x^3) = 0$, where $\alpha, \beta_1, \beta_2 \neq 0$. It consists of two non-tangent branches, both of them with characteristic exponent $\frac{3}{2}$.

The cluster $BP(\mathcal{J}(\xi))$ consists of the origin O taken with virtual multiplicity 3, and the two points on ξ in the first neighbourhood of P taken with multiplicity 1. Note that the points in $BP(\mathcal{J}(\xi))$ are exactly of the free singular points of ξ_3, which by Lemma 1.2.6 must belong to the base points of the polar germs. Thus, in this case, $BP(\mathcal{J}(\xi))$ is as small as possible.

In this case all points in $BP(\mathcal{J}(\xi))$ are dicritical. In particular, the origin O is dicritical and the algorithm starts with $\mathcal{R} = \mathcal{S} = \{O\}$. The polar invariants associated to the rest of the points in $BP(\mathcal{J}(\xi))$ are

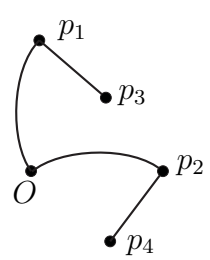
$$I = I_{p_1} = I_{p_2} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_1)]}{n_{p_1}} + 1 = \frac{3\cdot 1 + 1^2}{1} + 1 = 5,$$

The situation is obviously symmetrical, and thus we can start with p_1. The corresponding polar invariant is 5, and hence $p' = O$ is the origin. Therefore, the associated rupture point is satellite of $q_0 = p_1$. As in the previous examples, step 4(d) in the algorithm is performed just one time:

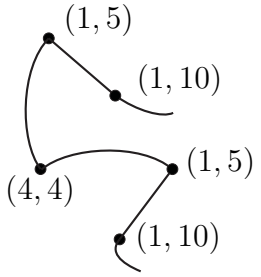
- $\frac{m_{q_0}}{n_{q_0}} = 6 > 5 = I$, so we take $q_1 = p_3$, the first satellite of p_1.

Since $\frac{m_{q_1}}{n_{q_1}} = 5 = I$, we end by taking $\mathcal{R} = \{O, p_3\}$ and $\mathcal{S} = \{O, p_1, p_3\}$.

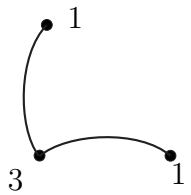
The case of p_2 is analogous, leading to $\mathcal{R} = \{O, p_3, p_4\}, \mathcal{S} = \{O, p_1, p_2, p_3, p_4\}$.



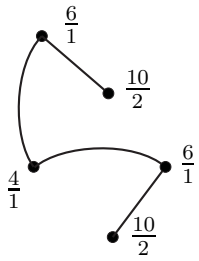
$\mathcal{S}(\xi) \cup BP(\mathcal{J}(\xi))$



$\xi, (e_p(\xi), v_p(\xi))$



$BP(\mathcal{J}(\xi)), \nu_p$



$\mathcal{S}(\xi) \cup BP(\mathcal{J}(\xi)), \frac{m_p}{n_p}$

Figure A.3: $\xi : (y^2 - \beta_1 x^3)((y - \alpha x)^2 - \beta_2 x^3) = 0$ $(\alpha, \beta_1, \beta_2 \neq 0)$.

The second part starts setting $v_O = m_O = 4$ and $v_{p_3} = v_{p_4} = m_{p_3} = m_{p_4} = 10$. The second step is the same for p_1 and p_2: since there are no free points in $\mathcal{S}$ in its first neighbourhoods, we must apply the second instance to obtain $v_{p_1} = v_{p_2} = 5$, the only integer in the interval

$$\left[\frac{n_{p_1}}{n_{p_3}} m_{p_3}, \frac{n_{p_1}}{n_{p_3}} m_{p_3} + 1\right) = \left[\frac{n_{p_2}}{n_{p_4}} m_{p_4}, \frac{n_{p_2}}{n_{p_4}} m_{p_4} + 1\right) = [5, 6).$$

As in the previous examples, there is no third step because there are no satellite points in $\mathcal{S}$ and not in $\mathcal{R}$. Therefore the process has finished and we have certainly recovered $\mathcal{R}(\xi)$ and $\mathcal{S}(\xi)$ with its values.

Example 4: (See figure A.4.)

Take ξ to be given by $y^3 - x^{11} + \alpha x^8 y = 0$, with $\alpha \neq 0$. It is irreducible and has only one characteristic exponent: $\frac{11}{3}$.

The cluster $BP(\mathcal{J}(\xi))$ is shown in figure A.4, and hence generic polars of ξ consist of two smooth branches sharing the points on ξ up to p_3, the point on ξ in the third neighbourhood of O. Moreover, generic polars of ξ share four further fixed free points after p_3, two on each branch.

Since $O \notin \mathcal{D} = \{p_8, p_9\}$, we start with $\mathcal{R} = \mathcal{S} = \emptyset$. The polar invariants are

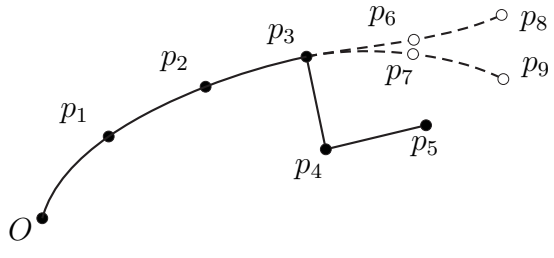
$$I = I_{p_8} = I_{p_9} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_8)]}{n_{p_8}} + 1 = \frac{2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 1^2 + 1^2}{1} + 1 = 11,$$

and therefore we can start with p_8. The point p' is p_2, and thus the rupture point associated to p_8 is satellite of $q_0 = p_3$. Step 4(d) consists of the next two iterations:

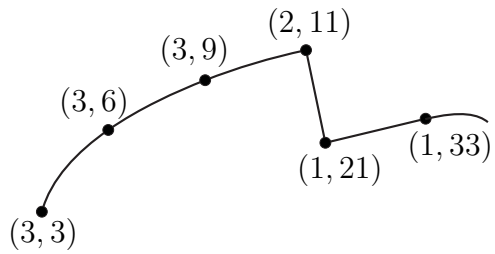
- $\frac{m_{q_0}}{n_{q_0}} = 12 > 11 = I$, so we take $q_1 = p_4$, the first satellite of p_3.
- $\frac{m_{q_1}}{n_{q_1}} = \frac{21}{2} < 11 = I$, so we take $q_2 = p_5$, the second satellite of p_4.

Since $\frac{m_{q_2}}{n_{q_2}} = 11 = I$, we end by taking $\mathcal{R} = \{p_5\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_5\}$.

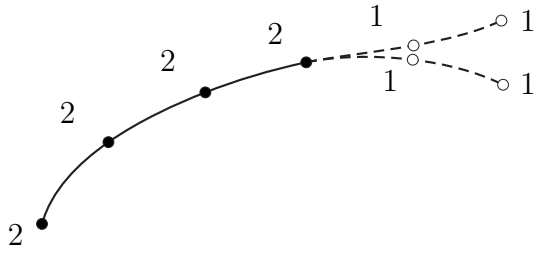
Taking p_9 we have $I_{p_9} = I = 11$ and again $p' = p_2$. Hence we obtain the same results as for p_8 and it is not necessary to add any point to $\mathcal{R}$ or $\mathcal{S}$.



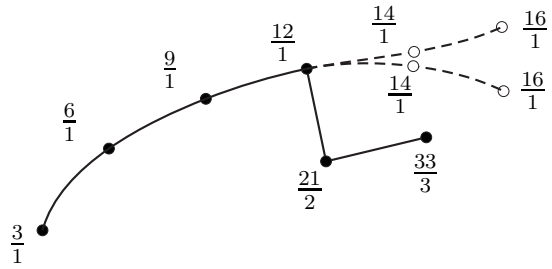
$\mathcal{S}(\xi) \cup BP(\mathcal{J}(\xi))$



$\xi, (e_p(\xi), v_p(\xi))$



$BP(\mathcal{J}(\xi)), \nu_p$



$\mathcal{S}(\xi) \cup BP(\mathcal{J}(\xi)), \frac{m_p}{n_p}$

Figure A.4: $\xi : y^3 - x^{11} + \alpha x^8 y = 0$ ($\alpha \neq 0$).

The second part of the algorithm starts setting $v_{p_5} = m_{p_5} = 33$. On the one hand, since there are free singular points in the first neighbourhood of O, p_1 and p_2, Step 2 yields $v_O = 3, v_{p_1} = 6$ and $v_{p_2} = 9$. On the other hand, since there are no free singular points in the first neighbourhood of p_3, the second instance of step 2 gives $v_{p_3} = 11$, the only integer in the interval

$$\left[\frac{n_{p_3}}{n_{p_5}} m_{p_5}, \frac{n_{p_3}}{n_{p_5}} m_{p_5} + 1\right) = [11, 12).$$

Finally, the third step of the second part applies to recover v_{p_4}. p' is p_3 and q is p_5. Since $p_4 \prec p_5$, we must follow the second instance of step 3 and set $v_{p_4} = m_{p_4} = 21$.

Example 5: (See figure A.5.)

Now consider the curve ξ given by $y^3 - x^{11} = 0$. It is again irreducible with single characteristic exponent $\frac{11}{3}$, and hence it is equisingular to the curves in the previous example (in fact, it corresponds to take $\alpha = 0$ in the equation of Example 4).

However, $BP(\mathcal{J}(\xi))$ is not similar to the base points in Example 4 (showing again that the reciprocal of Theorem 1.2.7 does not hold). In this case, generic polars also consist of two smooth branches, but they share five points (instead of four, as happened in the previous example) and there are no more base points.

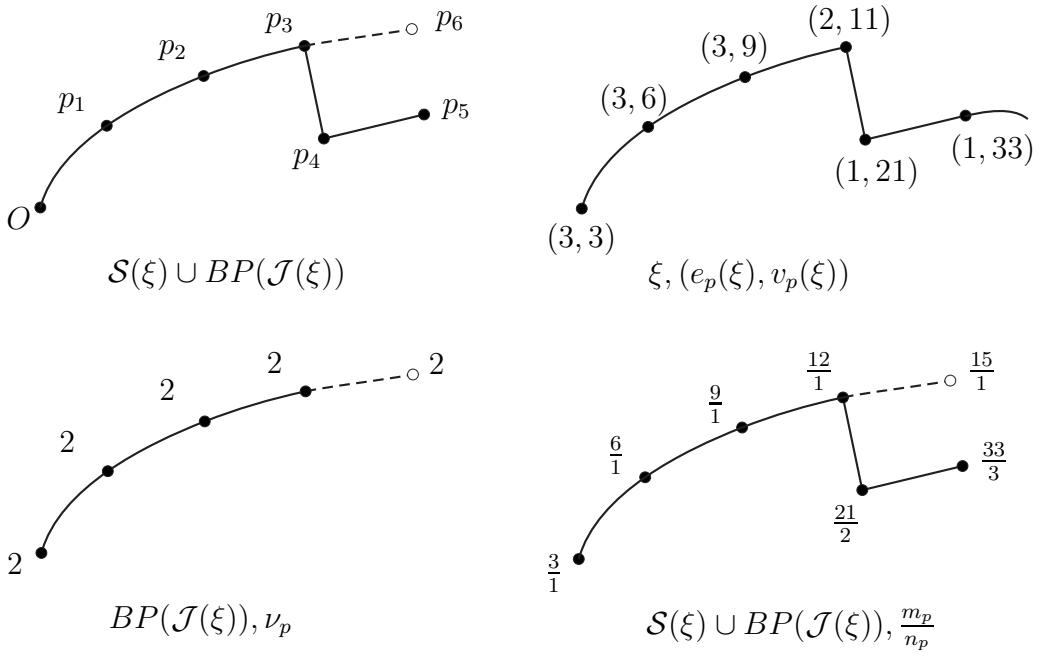


Figure A.5: $\xi : y^3 - x^{11} = 0$.

In this case there is only one dicritical point in $BP(\mathcal{J}(\xi))$: p_6, and its corresponding polar invariant is again

$$I = I_{p_6} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_6)]}{n_{p_6}} + 1 = \frac{2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1}{1} + 1 = 11.$$

Moreover, the point p is again p_3, and hence the algorithm works as it does in example 4 (recovering both the rupture points and the values).

Example 6: (See figure A.6.)

Let ξ be the curve given by $(y^5 - x^3)(y^3 - x^5) = 0$. Hence ξ consists of two non-tangent branches with characteristic exponent $\frac{5}{3}$. This example is quite similar to Example 3.

The cluster $BP(\mathcal{J}(\xi))$ has been represented in figure A.6. According to it, generic polars of ξ consist of five smooth branches, two of them tangent to one branch of ξ, another

two tangent to the other branch of ξ, and the remaining one with variable tangent. As in Example 3, the points in $BP(\mathcal{J}(\xi))$ are just the singular free points of ξ (obviously, with different multiplicities).

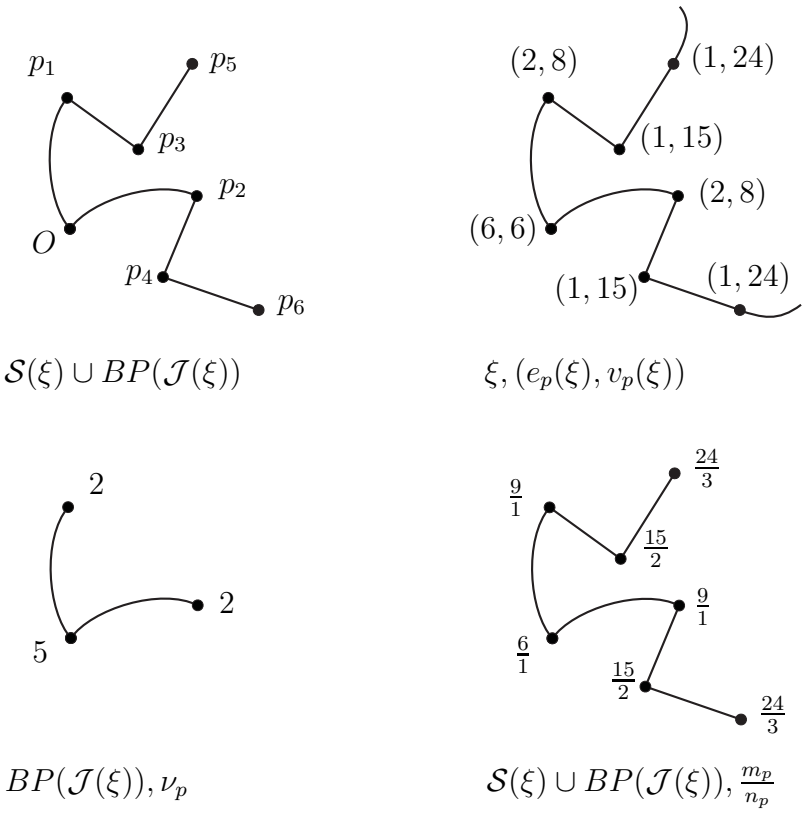


Figure A.6: $\xi : (y^5 - x^3)(y^3 - x^5) = 0$.

Again as in Example 3, all points in $BP(\mathcal{J}(\xi))$ are dicritical, and in particular so is O. Thus, the algorithm starts with $\mathcal{R} = \mathcal{S} = \{O\}$. The polar invariant associated both to p_1 and p_2 is

$$I = I_{p_1} = I_{p_2} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_1)]}{n_{p_1}} + 1 = \frac{5 \cdot 1 + 2 \cdot 1}{1} + 1 = 8.$$

The situation is again symmetrical, so we start with p_1. Since $I_{p_1} = 8$, the point p' is the origin O, and hence the rupture point associated to p_1 is satellite of $q_0 = p_1$. Step 4(d) consists of two iterations:

- $\frac{m_{q_0}}{n_{q_0}} = 9 > 8 = I$, so we take $q_1 = p_3$, the first satellite of p_1.
- $\frac{m_{q_1}}{n_{q_1}} = \frac{15}{2} < 8 = I$, so we take $q_2 = p_5$, the second satellite of p_3.

Since $\frac{m_{q_2}}{n_{q_2}} = 8 = I$, we end by taking $\mathcal{R} = \{O, p_5\}$ and $\mathcal{S} = \{O, p_1, p_3, p_5\}$.

The case p_2 is analogous, so we obtain $\mathcal{R} = \{O, p_5, p_6\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_6\}$.

The second part of the algorithm works as follows. First of all, we set $v_O = m_O = 6$ and $v_{p_5} = v_{p_6} = 24$. Step 2 applies to p_1 and p_2, giving $v_{p_1} = v_{p_2} = 8$, the only integer in

the interval

$$\left[\frac{n_{p_1}}{n_{p_5}}m_{p_5}, \frac{n_{p_1}}{n_{p_5}}m_{p_5}+1\right) = \left[\frac{n_{p_2}}{n_{p_6}}m_{p_6}, \frac{n_{p_2}}{n_{p_6}}m_{p_6}+1\right) = [8,9).$$

To conclude, we set $v_{p_3} = v_{p_4} = 15$ because $p_3 \prec p_5$ and $p_4 \prec p_6$, and hence the second instance of step 3 applies.

Example 7: (See figure A.7.)

Now consider the curve $\xi : (x^2-y)(x^2+y)(x-y^2)(x+y^2)(x^2-(x-y))(x^2+(x-y)) = 0$. It consists of three pairs of tangent smooth branches having different points in the second neighbourhood of O.

The cluster $BP(\mathcal{J}(\xi))$ consists of the singular points of ξ and three more simple free points not lying on ξ, as is shown in figure A.7. Therefore generic polars of ξ consist of five smooth branches with different tangent directions, three of them are tangent to ξ and the remaining two have variable tangents.

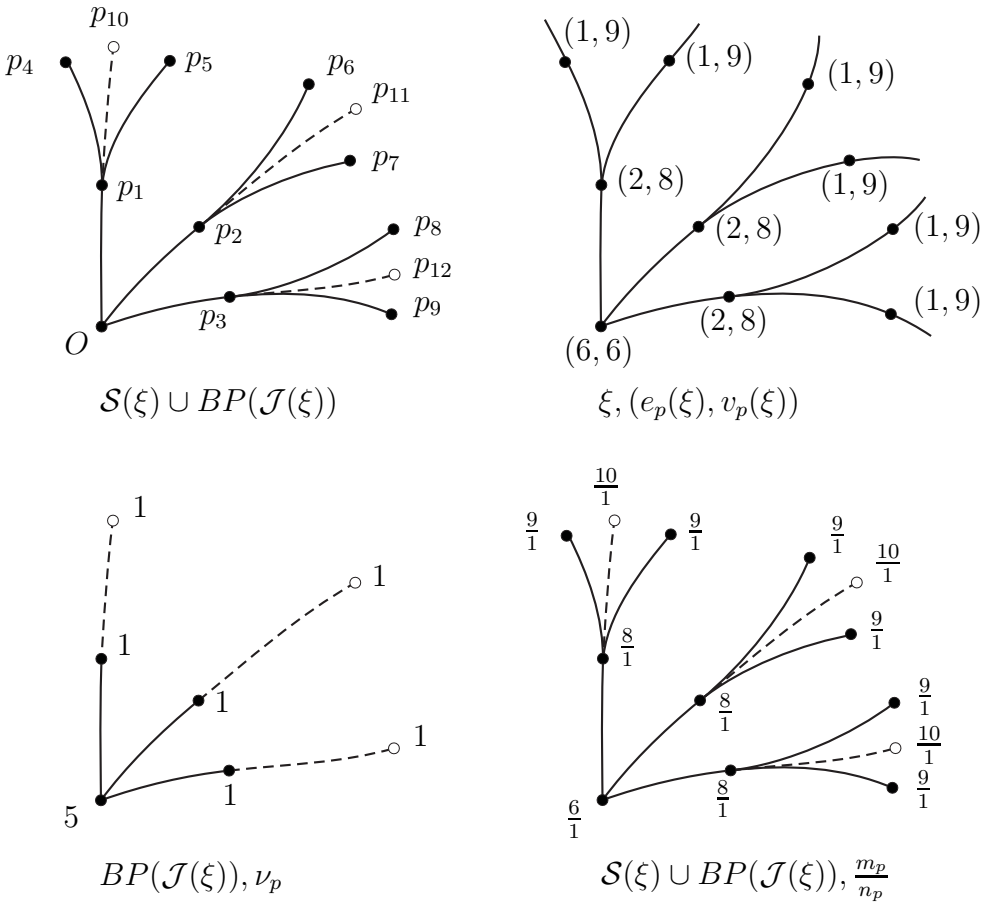


Figure A.7: $\xi : (x^2-y)(x^2+y)(x-y^2)(x+y^2)(x^2-(x-y))(x^2+(x-y)) = 0$.

The set of dicritical points of $BP(\mathcal{J}(\xi))$ is $\mathcal{D} = \{O, p_{10}, p_{11}, p_{12}\}$. In particular $O \in \mathcal{D}$, and thus the algorithm starts with $\mathcal{R} = \mathcal{S} = \{O\}$. The polar invariant associated to

p_{10}, p_{11} and p_{12} is

$$I = I_{p_{10}} = I_{p_{11}} = I_{p_{12}} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{10})]}{n_{p_{10}}} + 1 = \frac{5 \cdot 1 + 1^2 + 1^2}{1} + 1 = 8.$$

If we start, for example, with p_{10}, we find that p' is the origin O. Hence the rupture point associated to p_{10} is satellite of p_1. Indeed, since $\frac{m_{p_1}}{n_{p_1}} = 8 = I$, this rupture point is precisely p_1.

The procedure works analogously for p_{11} and p_{12}, and the result of the first part of the algorithm is $\mathcal{R} = \mathcal{S} = \{O, p_1, p_2, p_3\}$.

Since $\mathcal{R} = \mathcal{S}$, the second part of the algorithm finishes at the first step, setting $v_O = m_O = 6$ and $v_{p_1} = v_{p_2} = v_{p_3} = m_{p_1} = m_{p_2} = m_{p_3} = 8$, which actually are the values of the singular points on ξ.

Example 8: (See figure A.8.)

Now let ξ be the curve given by $y^3 - \alpha x^{10} = 0$, with $\alpha \neq 0$. It is irreducible with only one characteristic exponent: $\frac{10}{3}$.

Its generic polars are irreducible with characteristic exponent $\frac{9}{2}$, and they have no common points besides the singular ones. Hence the cluster of base points $BP(\mathcal{J}(\xi))$ is irreducible and consists precisely of these singular points.

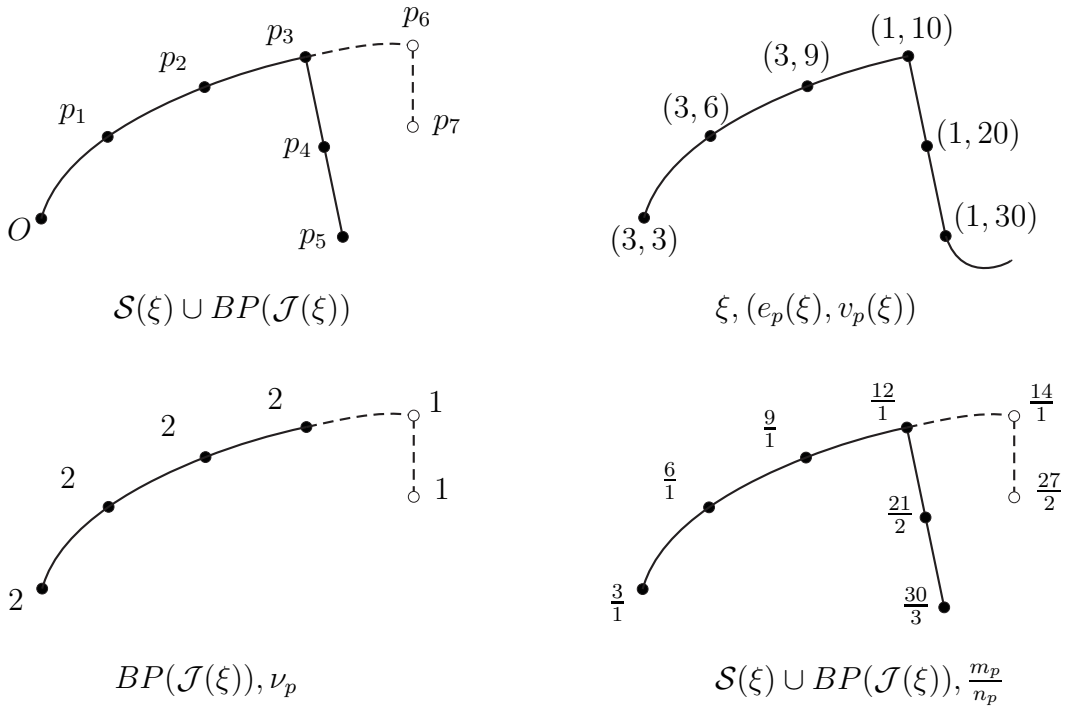


Figure A.8: $\xi : y^3 - \alpha x^{10} = 0$, $\alpha \neq 0$.

Since $BP(\mathcal{J}(\xi))$ is irreducible, $\mathcal{D} = \{p_7\}$, and the algorithm starts with $\mathcal{R} = \mathcal{S} = \emptyset$. The only polar invariant is

$$I = I_{p_7} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_7)]}{n_{p_7}} + 1 = \frac{BP(\mathcal{J}(\xi))^2}{n_{p_7}} + 1 = \frac{2^2 + 2^2 + 2^2 + 2^2 + 1^2 + 1^2}{2} + 1 = 10,$$

and the point p' is p_2. Thus, the only rupture point of ξ must be satellite of $q_0 = p_3$. Step 4(d) works as follows:

- $\frac{m_{q_0}}{n_{q_0}} = 12 > 10 = I$, so we take $q_1 = p_4$, the first satellite of p_3.
- $\frac{m_{q_1}}{n_{q_1}} = \frac{21}{2} > 10 = I$, so we take $q_2 = p_5$, the first satellite of p_4.

Since $\frac{m_{q_2}}{n_{q_2}} = 10 = I$, we end taking $\mathcal{R} = \{p_5\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_5\}$.

The second part of the algorithm starts setting $v_{p_5} = m_{p_5} = 30$. The second step computes $v_O = m_O = 3$, $v_{p_1} = m_{p_1} = 6$ and $v_{p_2} = m_{p_2} = 9$ because O, p_1 and p_2 have free singular points in their respective first neighbourhoods. However, since there are no free points in $\mathcal{S}$ in the first neighbourhood of p_3, we must apply the second instance of step 2 to obtain $v_{p_3} = 10$, the only integer in the interval

$$\left[\frac{n_{p_3}}{n_{p_5}} m_{p_5}, \frac{n_{p_3}}{n_{p_5}} m_{p_5} + 1\right) = [10, 11).$$

It remains to recover the value of p_4, which is done in step 3. We take $p' = p_3$ and $q = p_5$, and since $p_4 \succ p_5$ and $v_{p_3} = \frac{n_{p_3}}{n_{p_5}} m_{p_5} = 10$, we finish just setting $v_{p_4} = \frac{n_{p_4}}{n_{p_5}} m_{p_5} = 20$.

Example 9: (See figure A.9.)

Now consider the curve ξ given by $y(y^2 - x^3) = 0$. It consists of two tangent branches, one of them is smooth and the other one has $\frac{3}{2}$ as the only characteristic exponent.

Generic polars of ξ are irreducible also with $\frac{3}{2}$ as the only characteristic exponent. They share all the singular points of ξ and one further point (p_3) which does not lie on ξ.

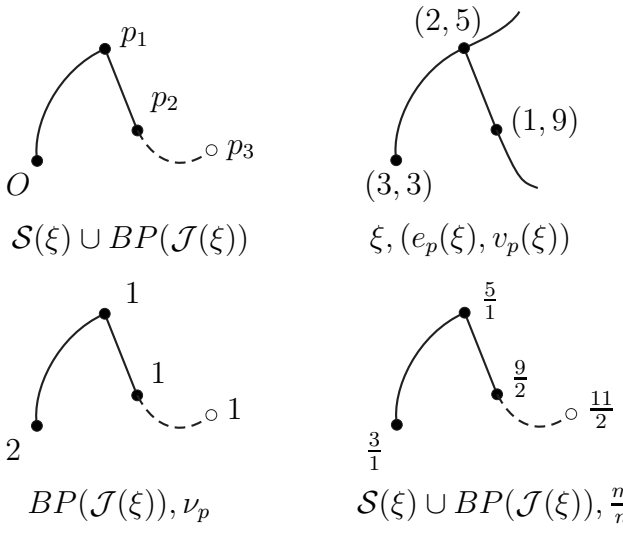


Figure A.9: $\xi : y(y^2 - x^3) = 0$.

Since $BP(\mathcal{J}(\xi))$ is irreducible, the set of dicritical points is $\mathcal{D} = \{p_3\}$ and thus the algorithm starts with $\mathcal{R} = \mathcal{S} = \emptyset$. The only polar invariant is

$$I = I_{p_3} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_3)]}{n_{p_3}} + 1 = \frac{BP(\mathcal{J}(\xi))^2}{n_{p_3}} + 1 = \frac{2^2 + 1^2 + 1^2 + 1^2}{2} + 1 = \frac{9}{2},$$

and the point p' is the origin. Thus, the only rupture point of ξ must be satellite of $q_0 = p_1$. Step 4(d) consists just of the iteration

- $\frac{m_{q_0}}{n_{q_0}} = 5 > \frac{9}{2} = I$, so we take $q_1 = p_2$, the first satellite of p_3.

Since $\frac{m_{q_1}}{n_{q_1}} = \frac{9}{2} = I$, we end by taking $\mathcal{R} = \{p_2\}$ and $\mathcal{S} = \{O, p_1, p_2\}$.

The second part of the algorithm starts setting $v_{p_2} = m_{p_2} = 9$. On the one hand, since p_1 is a free singular point in the first neighbourhood of O, step 2 computes $v_O = m_O = 3$. On the other hand, since there are no free singular points in the first neighbourhood of p_1, we must apply the second instance of step 2 to obtain $v_{p_1} = 5$, the only integer in the interval

$$\left[\frac{n_{p_1}}{n_{p_2}} m_{p_2}, \frac{n_{p_1}}{n_{p_2}} m_{p_2} + 1\right) = \left[\frac{9}{2}, \frac{11}{2}\right).$$

Since there are no satellite points but the rupture point p_2, step 3 is not performed and the algorithm finishes.

Example 10: (See figure A.10.)

Let ξ be the curve given by $y(y - x^2)(y^2 - x^3) = 0$. It is the curve in the previous example with an extra tangent smooth branch. This new branch shares no point with the other ones but the point in the first neighbourhood of the origin.

The cluster $BP(\mathcal{J}(\xi))$ consists of the singular points of ξ and two more free points, as is shown in the figure. Thus, generic polars of ξ have two tangent branches. One of them is smooth and always go through p_3, a fixed free point in the second neighbourhood of O, and the other one is singular of characteristic exponent $\frac{3}{2}$ and also goes through one further fixed free point (p_4) after the singular ones.

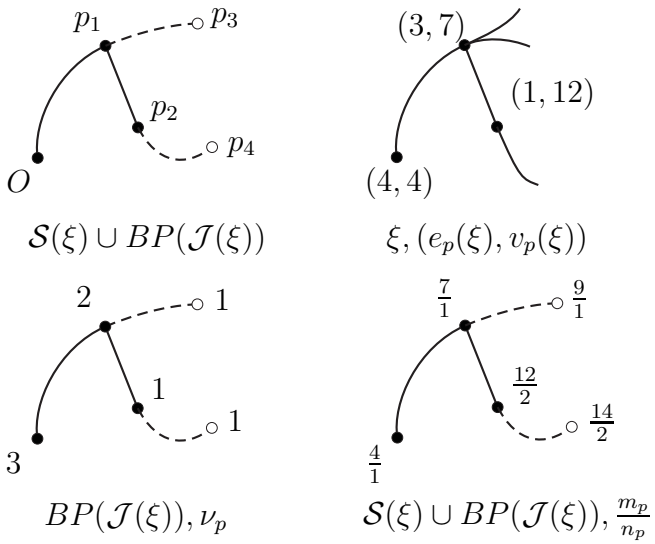


Figure A.10: $\xi : y(y - x^2)(y^2 - x^3) = 0$.

In this case $\mathcal{D} = \{p_3, p_4\}$, and since $O \notin \mathcal{D}$, the algorithm starts with $\mathcal{R} = \mathcal{S} = \emptyset$. The polar invariants are

$$I_3 = I_{p_3} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_3)]}{n_{p_3}} + 1 = \frac{3 \cdot 1 + 2 \cdot 1 + 1^2}{1} + 1 = 7,$$

$$I_4 = I_{p_4} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_4)]}{n_{p_4}} + 1 = \frac{3\cdot 2 + 2\cdot 1 + 1^2 + 1^2}{2} + 1 = 6.$$

Since $I_3 > I_4$, we must start with p_3. The point p' is the origin, and hence the rupture point associated to p_3 is satellite of $p = p_1$. Furthermore, since $\frac{m_{p_1}}{n_{p_1}} = 7 = I_1$, the rupture point is exactly p_1, and we temporarily set $\mathcal{R} = \{p_1\}$ and $\mathcal{S} = \{O, p_1\}$.

We now take p_4. The point p' is again the origin, and $p = p_1$. Since p_1 has already appeared, we know that the rupture point associated to p_4 is p_2, the last p_1-satellite point in $\mathcal{K}(p_4)$. Thus we end the first part of the algorithm by setting $\mathcal{R} = \{p_1, p_2\}$ and $\mathcal{S} = \{O, p_1, p_2\}$.

The recovering of the values if quite straightforward. First of all, we start with the rupture points and set $v_{p_1} = m_{p_1} = 7$ and $v_{p_2} = m_{p_2} = 12$. Then, since p_1 is a free singular point in the first neighbourhood of O, we finish applying step 2 to obtain $v_O = m_O = 5$.

Example 11: (See figure A.11.)

Now consider the curve ξ given by $y(y-x^2)(y-x^3)(y^2-x^3) = 0$. It is the curve in Example 10 with an extra smooth branch sharing three points with one of the smooth branches.

The cluster $BP(\mathcal{J}(\xi))$ is shown in the figure. In particular, generic polars have tree branches, two of which are smooth and the other one has also characteristic exponent $\frac{3}{2}$.

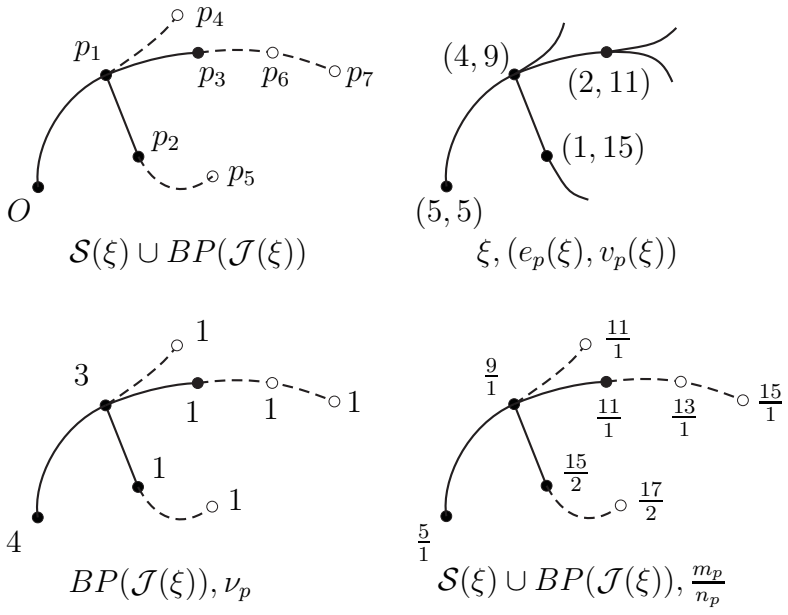


Figure A.11: $\xi : y(y-x^2)(y-x^3)(y^2-x^3) = 0$.

We now have three dicritical points (p_4, p_5 and p_7) which lead to three different polar invariants:

$$I_4 = I_{p_4} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_4)]}{n_{p_4}} + 1 = \frac{4\cdot 1 + 3\cdot 1 + 1^2}{1} + 1 = 9,$$

$$I_5 = I_{p_5} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_5)]}{n_{p_5}} + 1 = \frac{4\cdot 2 + 3\cdot 1 + 1^2 + 1^2}{2} + 1 = \frac{15}{2},$$

$$I_7 = I_{p_7} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_7)]}{n_{p_7}} + 1 = \frac{4\cdot 1 + 3\cdot 1 + 1^2 + 1^2 + 1^2}{1} + 1 = 11.$$

Since O is not a dicritical point, we start with $\mathcal{R} = \mathcal{S} = \emptyset$. In addition, we must consider the dicritical points ordered as (p_7, p_4, p_5) because of the polar invariants.

So let us start with p_7. The point p' is p_1, and hence the rupture point is satellite of $p = p_3$. Moreover, since $\frac{m_{p_3}}{n_{p_3}} = 11 = I_7$, we end this step with $\mathcal{R} = \{p_3\}$ and $\mathcal{S} = \{O, p_1, p_3\}$.

Now take p_4. The point p' is the origin, and thus the rupture point associated to p_4 is satellite of $p = p_1$. Again, since $\frac{m_{p_1}}{n_{p_1}} = 9 = I_4$, this step finishes with $\mathcal{R} = \{p_1, p_3\}$ and $\mathcal{S} = \{O, p_1, p_3\}$.

Finally, take p_5. The point p' is the origin, and the point p is again p_1. Since it has already appeared, the rupture point associated to p_5 is directly p_2, the last p_1-satellite point in $\mathcal{K}(p_5)$. Thus, the first part of the algorithm ends with $\mathcal{R} = \{p_1, p_2, p_3\}$ and $\mathcal{S} = \{O, p_1, p_2, p_3\}$.

As for the second part, it is quite straightforward. The first step gives immediately $v_{p_1} = 9$, $v_{p_2} = 15$ and $v_{p_3} = 11$, and step 2 gives $v_O = m_O = 5$, which are the correct values of ξ at its singular points.

Example 12: (See figure A.12.)

Now let ξ be the curve given by $(y^2 - x^3)(y^2 - x^5) = 0$. It clearly consists of two tangent branches with characteristic exponents $\frac{3}{2}$ and $\frac{5}{2}$, respectively.

As it is shown in figure A.12, $BP(\mathcal{J}(\xi))$ consists of the singular points of ξ but p_4, and two further points (p_5 and p_6) which do not lie on ξ. Thus, generic polars of ξ have two branches, one of which is smooth and the other one is singular with characteristic exponent $\frac{3}{2}$. Furthermore, both of them go through one fixed free point not lying on ξ.

In this case $O \notin \mathcal{D} = \{p_5, p_6\}$, and thus we start with $\mathcal{R} = \mathcal{S} = \emptyset$. The polar invariants are

$$I_5 = I_{p_5} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_5)]}{n_{p_5}} + 1 = \frac{3\cdot 2 + 2\cdot 1 + 1^2 + 1^2}{2} + 1 = 6,$$

$$I_6 = I_{p_6} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_6)]}{n_{p_6}} + 1 = \frac{3\cdot 1 + 2\cdot 1 + 1^2 + 1^2}{1} + 1 = 8.$$

Thus we must start with p_6. The point p' is p_1, and hence the rupture point associated to p_6 is satellite of $p = q_0 = p_3$. Step 4(d) consists just of the iteration

- $\frac{m_{q_0}}{n_{q_0}} = 9 > 8 = I_6$, so we take $q_1 = p_4$, the first satellite of p_3,

and since $\frac{m_{q_1}}{n_{q_1}} = 8 = I$, this step ends with $\mathcal{R} = \{p_4\}$ and $\mathcal{S} = \{O, p_1, p_3, p_4\}$.

Now take p_5. p' is the origin and thus the rupture point is satellite of p_1. Step 4(d) also consists of one iteration which yields p_2 as the last rupture point. Therefore the first part of the algorithm ends with $\mathcal{R} = \{p_2, p_4\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_4\}$.

The second part of the algorithm starts by setting $v_{p_2} = 12$ and $v_{p_4} = 16$. Step 2 sets $v_O = 4$, $v_{p_1} = 7$ and $v_{p_3} = 8$, the last one because it is the only integer in the interval

$$\left[\frac{n_{p_3}}{n_{p_4}} m_{p_4}, \frac{n_{p_3}}{n_{p_4}} m_{p_4} + 1\right) = [8, 9).$$

There is no third step because all satellite points are rupture points.

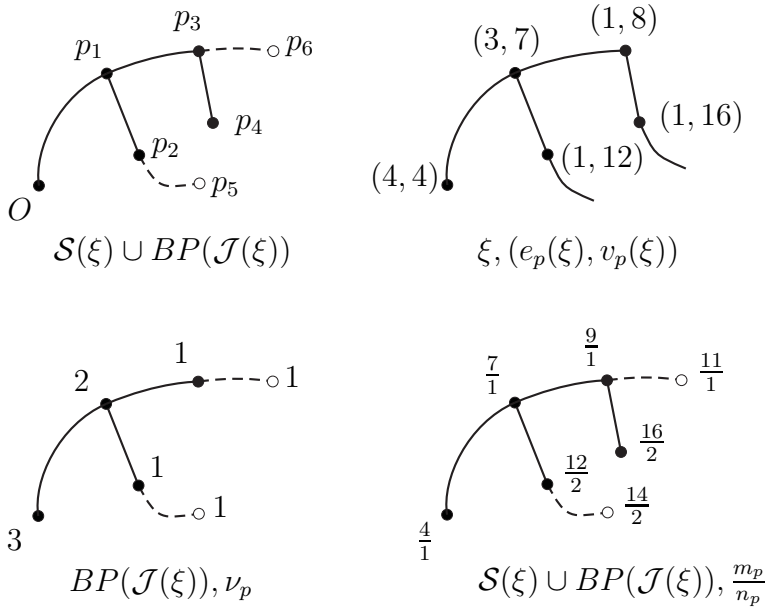


Figure A.12: $\xi : (y^2 - x^3)(y^2 - x^5) = 0$.

Example 13: (See figure A.13.)

Now consider the curve ξ given by the equation $y(y^2-x^3)(y^2-x^5) = 0$. It is the curve in the previous example with an extra smooth branch going through all the free singular points.

Generic polars of ξ consist of two singular branches of characteristic exponents $\frac{3}{2}$ and $\frac{5}{2}$. Moreover, the first branch always goes through one fixed free point not on ξ (p_5), and the second one goes through tree fixed free points not on ξ (p_6, p_7 and p_8).

The dicritical points of $BP(\mathcal{J}(\xi))$ are $\mathcal{D} = \{p_5, p_8\}$, and the polar invariants are

$$I_5 = I_{p_5} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_5)]}{n_{p_5}} + 1 = \frac{4 \cdot 2 + 3 \cdot 1 + 1^2 + 1^2}{2} + 1 = \frac{15}{2},$$

$$I_8 = I_{p_8} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_8)]}{n_{p_8}} + 1 = \frac{4 \cdot 2 + 3 \cdot 2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2}{2} + 1 = \frac{21}{2}.$$

Thus, we start with p_8. The point p' is p_1, and hence the rupture point associated to p_8 is satellite of $p = q_0 = p_3$. Step 4(d) consists of just one iteration:

- $\frac{m_{q_0}}{n_{q_0}} = 11 > \frac{21}{2} = I_8$, so we take $q_1 = p_4$, the first satellite of p_3.

Since $\frac{m_{q_0}}{n_{q_0}} = I_8$, we have find that the rupture point associated to p_8 is p_4, so for the moment we have $\mathcal{R} = \{p_4\}$ and $\mathcal{S} = \{O, p_1, p_3, p_4\}$.

Applying the procedure to p_5, the first part of the algorithm ends with $\mathcal{R} = \{p_2, p_4\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_4\}$.

The second part of the algorithm works as follows. In first place, we recover the values of the rupture points as $v_{p_2} = 12$ and $v_{p_4} = 21$. Then we set $v_O = 5$ and $v_{p_1} = 9$ because

both O and p_1 have free singular points in their first neighbourhoods. Finally, $v_{p_3} = 11$ because it is the only integer in the interval

$$\left[\frac{n_{p_3}}{n_{p_4}}m_{p_4}, \frac{n_{p_3}}{n_{p_4}}m_{p_4} + 1\right) = \left[\frac{21}{2}, \frac{23}{2}\right).$$

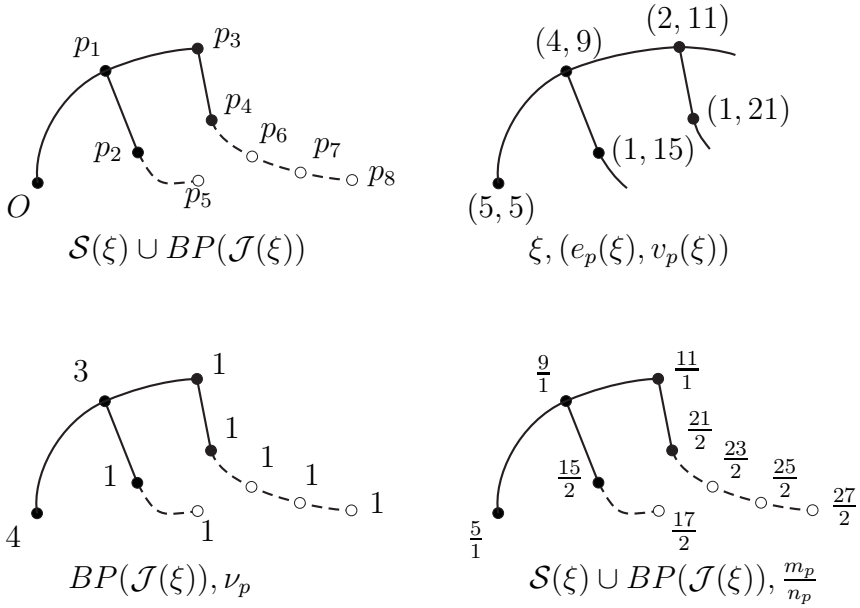


Figure A.13: $\xi : y(y^2 - x^3)(y^2 - x^5) = 0$.

Example 14: (See figure A.14.)

Let ξ be the curve given by $(y^3 - x^5)((y - x^2)^3 - x^5) = 0$. It consists of two tangent branches with characteristic exponent $\frac{5}{3}$ sharing only its singular points.

The generic polars of ξ have three branches. Two of them are smooth and share no points but the one on ξ in the first neighbourhood of the origin. The other branch has characteristic exponent $\frac{5}{3}$ (as each branch of ξ) and goes through two fixed free points not on ξ after the last satellite point of ξ.

Therefore $BP(\mathcal{J}(\xi))$ has two dicritical points: p_1 with excess 2 and p_5 with excess one. The associated polar invariants are

$$I_1 = I_{p_1} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_1)]}{n_{p_1}} + 1 = \frac{5 \cdot 1 + 4 \cdot 1}{1} + 1 = 10,$$

$$I_5 = I_{p_5} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_5)]}{n_{p_5}} + 1 = \frac{5 \cdot 3 + 4 \cdot 2 + 1^2 + 1^2 + 1^2 + 1^2}{3} + 1 = 10.$$

Hence we can proceed in any order, and in fact both p_1 and p_5 have p_3 as associated rupture point. So the first part of the algorithm finishes with $\mathcal{R} = \{p_3\}$ and $\mathcal{S} = \{O, p_1, p_2, p_3\}$.

The second part starts with $v_{p_3} = m_{p_3} = 30$. Step 2 gives $v_O = m_O = 6$ (because p_1 is a free singular point in the first neighbourhood of O), and $v_{p_1} = 10$, the only integer in the interval

$$\left[\frac{n_{p_1}}{n_{p_3}}m_{p_3}, \frac{n_{p_1}}{n_{p_3}}m_{p_3} + 1\right) = [10, 11).$$

Finally, since $p_2 \prec p_3$, step 3 gives $v_{p_2} = m_{p_2} = 18$, and we have in fact recovered $\mathcal{S}(\xi)$ with the correct values.

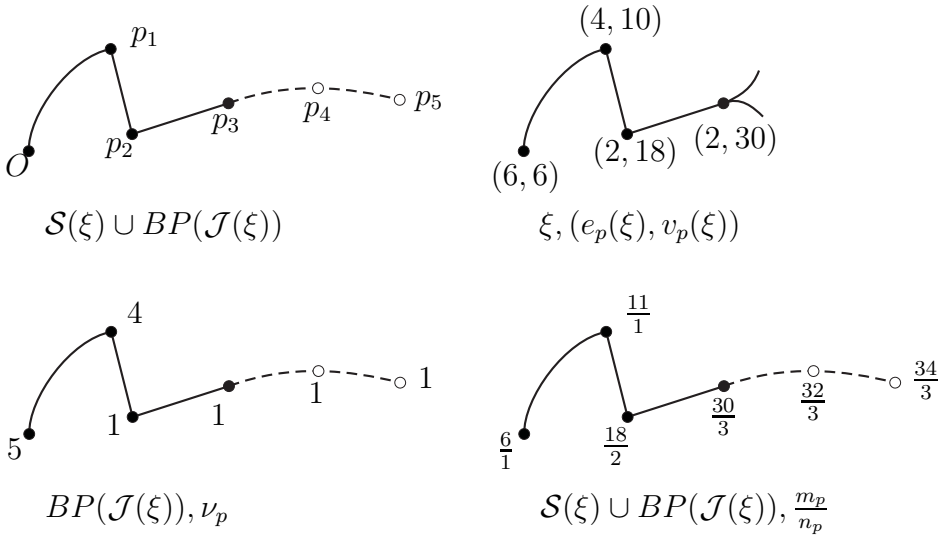


Figure A.14: $\xi : (y^3 - x^5)((y - x^2)^3 - x^5) = 0$.

Example 15: (See figure A.15.)

Now consider the curve ξ given by $y(y - x^2)(y + x^2) = 0$. It consists of three smooth tangent branches which have different points in the second neighbourhood of the origin.

The cluster $BP(\mathcal{J}(\xi))$ consists of the two singular points of ξ with multiplicity 2 and two further free points in the first neighbourhood of p_1 with multiplicity 1. Hence generic polars of ξ have two smooth tangent branches which go through two fixed points in the second neighbourhood of O.

Thus $BP(\mathcal{J}(\xi))$ has two dicritical points (p_2 and p_3) which play a symmetrical role. The associated polar invariant is

$$I = I_{p_2} = I_{p_3} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_2)]}{n_{p_2}} + 1 = \frac{2 \cdot 1 + 2 \cdot 1 + 1^2}{1} + 1 = 6.$$

The point p' (both for p_2 and p_3) is the origin, and therefore the rupture points associated to p_2 and p_3 are p_1-satellite. Note that since the polar invariant is the same for p_2 and p_3, the associated rupture point is the same for p_2 and p_3. In fact, it is p_1 because $\frac{m_{p_1}}{n_{p_1}} = 6 = I$. Hence the first part of the algorithm ends with $\mathcal{R} = \{p_1\}$ and $\mathcal{S} = \{O, p_1\}$.

As for the second part of the algorithm, the first step gives $v_{p_1} = m_{p_1} = 6$, and step 2 gives $v_O = m_O = 3$ (because p_1 is a free singular point in the first neighbourhood of O), which are the actual values of ξ at its singular points.

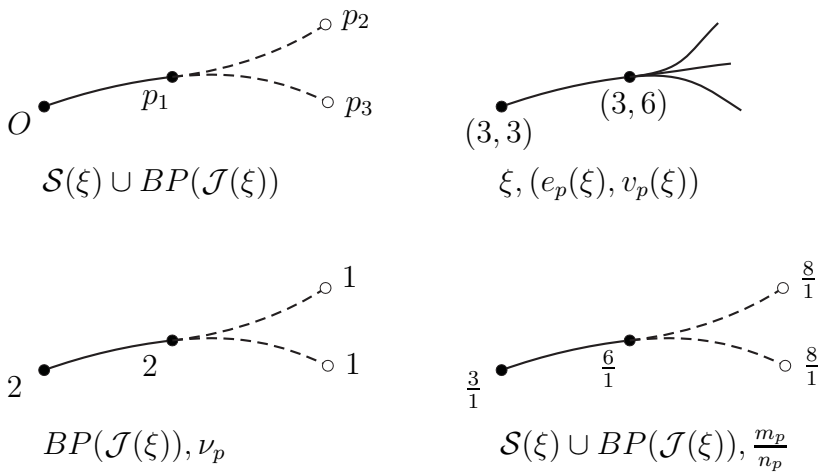


Figure A.15: $\xi : y(y-x^2)(y+x^2) = 0$.

Example 16: (See figure A.16.)

Let ξ be the curve given by $y(y-x^2)(y+x^2)(y^3-x^5)((y-x^2)^3-x^5) = 0$. It is the sum of the curves in Example 14 and Example 15.

The cluster $BP(\mathcal{J}(\xi))$ is shown in figure A.16. Generic polars of ξ have four branches. Two of them are smooth and tangent to ξ, and they go through two fixed free points in the second neighbourhood of the origin which are not on ξ. The other two branches are singular and go through the last singular point of ξ and four free fixed points (two for each branch) not on ξ.

The dicritical points of $BP(\mathcal{J}(\xi))$ are p_4, p_5 (which play the same role), p_8 and p_9 (which also play analogous roles). The corresponding polar invariants are

$$I_4 = I_{p_4} = I_{p_5} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_4)]}{n_{p_4}} + 1 = \frac{8 \cdot 1 + 6 \cdot 1 + 1^2}{1} + 1 = 16,$$

$$I_8 = I_{p_8} = I_{p_9} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_8)]}{n_{p_8}} + 1 = \frac{8 \cdot 3 + 6 \cdot 2 + 2 \cdot 1 + 2 \cdot 1 + 1^2 + 1^2}{3} + 1 = 15.$$

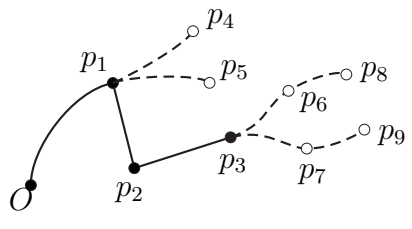
Hence we must consider first p_4 and p_5. The corresponding p' is the origin, both for p_4 and p_5. Therefore the rupture points associated to p_4 and p_5 are satellite of p_1, and in fact are the same point since the two points p_4 and p_5 have the same associated polar invariant. Moreover, this rupture point is directly p_1, because $\frac{m_{p_1}}{n_{p_1}} = 16 = I_4$. Thus we have for the moment that $\mathcal{R} = \{p_1\}$ and $\mathcal{S} = \{O, p_1\}$.

To complete the recovering of rupture points, it remains to consider p_8 and p_9. The point p' is again the origin, and hence the rupture points associated to p_8 and p_9 are p_1-satellite. Furthermore, they coincide because p_8 and p_9 have the same associated polar invariant. Since p_1 has already appeared, this rupture point is the last p_1-satellite point in $\mathcal{K}(p_8)$ (or $\mathcal{K}(p_9)$), which is p_3.

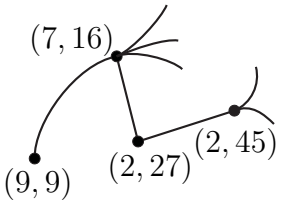
So the first part of the algorithm has finished with $\mathcal{R} = \{p_1, p_3\}$ and $\mathcal{S} = \{O, p_1, p_2, p_3\}$.

The second part of the algorithm starts by setting $v_{p_1} = m_{p_1} = 16$ and $v_{p_3} = m_{p_3} = 45$. The second step applies to O and gives $v_O = m_O = 9$ (because p_1 is a free singular point

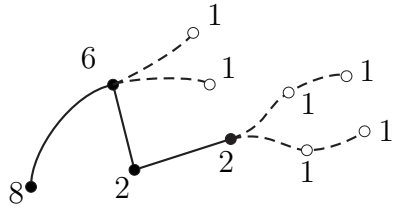
in the first neighbourhood of the origin). Finally, step 3 applies to p_2 giving $v_{p_2} = 27$ because $p_2 \prec p_3$.



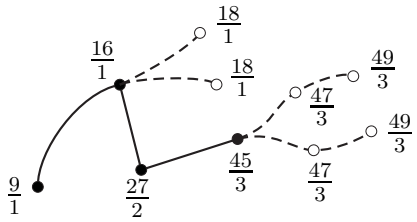
$\mathcal{S}(\xi) \cup BP(\mathcal{J}(\xi))$



$\xi, (e_p(\xi), v_p(\xi))$



$BP(\mathcal{J}(\xi)), \nu_p$



$\mathcal{S}(\xi) \cup BP(\mathcal{J}(\xi)), \frac{m_p}{n_p}$

Figure A.16: $\xi : y(y - x^2)(y + x^2)(y^3 - x^5)((y - x^2)^3 - x^5) = 0$.

Example 17: (See figure A.17.)

Now consider the curve ξ given by $\prod_{k=1}^{7}(y^4 - kx^7) = 0$. It consists of seven singular branches with characteristic exponent $\frac{7}{4}$, which share only their singular points $O, p_1, \ldots, p_4$.

The cluster $BP(\mathcal{J}(\xi))$ is shown in figure A.17. Thus generic polars of ξ consist of nine branches. Three of them are smooth, tangent to ξ and have different points in the second neighbourhood of O, while each of the other 6 branches go through the singular points of ξ and three more fixed free points not lying on ξ in successive neighbourhoods of p_4.

Thus, $BP(\mathcal{J}(\xi))$ has seven dicritical points. The first one is p_1 and the other six are $p_{17}, \ldots, p_{22}$ and play exactly the same role. The associated polar invariants are

$$I_1 = I_{p_1} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_1)]}{n_{p_1}} + 1 = \frac{27 \cdot 1 + 21 \cdot 1}{1} + 1 = 49,$$

$$I_{17} = I_{p_{17}} = \cdots = I_{p_{22}} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{17})]}{n_{p_{17}}} + 1 =$$
$$= \frac{27 \cdot 4 + 21 \cdot 3 + 6 \cdot 1 + 6 \cdot 1 + 6 \cdot 1 + 1^2 + 1^2 + 1^2}{4} + 1 = 49.$$

So in fact all the dicritical points of $BP(\mathcal{J}(\xi))$ have the same associated polar invariant, and thus they play analogous roles.

Take for example p_1. The point p' is the origin, and thus the associated rupture point is satellite of $q_0 = p_1$. Step 4(b) consists of three iterations:

- $\frac{m_{q_0}}{n_{q_0}} = 50 > 49 = I_1$, so we take $q_1 = p_2$, the first satellite of p_1.
- $\frac{m_{q_1}}{n_{q_1}} = \frac{84}{2} = 42 < 49 = I_1$, so we take $q_1 = p_3$, the second satellite of p_2.
- $\frac{m_{q_2}}{n_{q_2}} = \frac{140}{3} < 49 = I_1$, so we take $q_1 = p_4$, the second satellite of p_3.

Since $\frac{m_{q_3}}{n_{q_3}} = \frac{196}{4} = 49 = I_1$, we stop and take $\mathcal{R} = \{p_4\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_4\}$.

It is straightforward to check that any of the points $p_{17}, \ldots, p_{22}$ gives rise to the same rupture point, so the first part of the algorithm is finished.

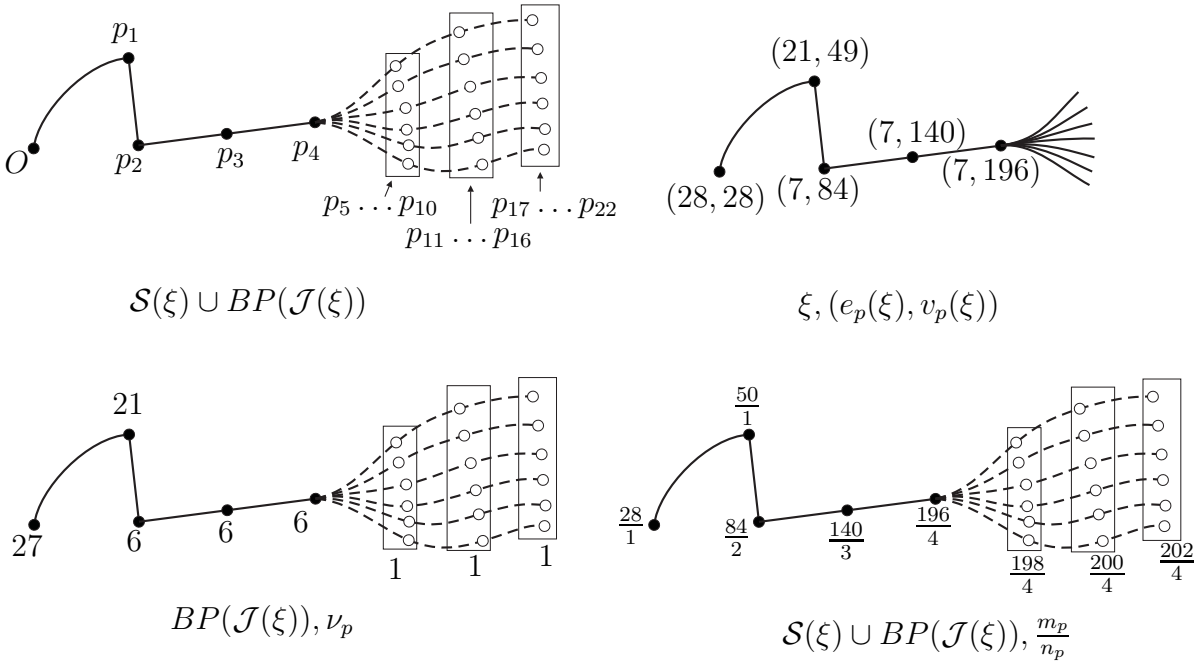


Figure A.17: $\xi : \prod_{k=1}^{7}(y^4 - kx^7) = 0$.

To recover the values of ξ at the points of $\mathcal{S}$, the algorithm starts with $v_{p_4} = m_{p_4} = 196$. Step two give $v_O = m_O = 28$, because p_1 is a free singular point in the first neighbourhood of the origin, and $v_{p_1} = 49$ because it is the only integer in the interval

$$\left[\frac{n_{p_1}}{n_{p_4}} m_{p_4}, \frac{n_{p_1}}{n_{p_4}} m_{p_4} + 1\right) = [49, 50).$$

Finally, step 3 gives $v_{p_2} = m_{p_2} = 84$ and $v_{p_3} = m_{p_3} = 140$ because $p_2 \prec p_3 \prec p_4$.

Example 18: (See figure A.18.)

Let ξ be the curve given by $y^7 - x^{16} = 0$. It is irreducible with only one characteristic exponent: $\frac{16}{7}$.

Generic polars of ξ have three branches which go through the first satellite point on ξ (p_3) and afterwards they go through three different and variable free points. Therefore, $BP(\mathcal{J}(\xi))$ is three times the irreducible cluster $\mathcal{K}(p_3)$.

Thus $\mathcal{D} = \{p_3\}$ consists of just one dicritical point, whose associated polar invariant is

$$I = I_{p_3} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_3)]}{n_{p_3}} + 1 = \frac{6 \cdot 2 + 6 \cdot 2 + 3 \cdot 1 + 3 \cdot 1}{2} + 1 = 16.$$

The point p' is thus p_1, and hence the rupture point associated to p_3 is satellite of $q_0 = p_2$. Step 4(d) of the first part of the algorithm consists of four iterations:

- $\frac{m_{q_0}}{n_{q_0}} = 18 > 16 = I$, so we take $q_1 = p_3$, the first satellite of p_2.
- $\frac{m_{q_1}}{n_{q_1}} = \frac{35}{2} > 16 = I$, so we take $q_2 = p_4$, the first satellite of p_3.
- $\frac{m_{q_2}}{n_{q_2}} = \frac{49}{3} > 16 = I$, so we take $q_3 = p_5$, the first satellite of p_4.
- $\frac{m_{q_3}}{n_{q_3}} = \frac{63}{4} < 16 = I$, so we take $q_4 = p_6$, the second satellite of p_5.

Now $\frac{m_{q_4}}{n_{q_4}} = 16 = I$, so the first part of the algorithm finishes with $\mathcal{R} = \{p_6\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_6\}$.

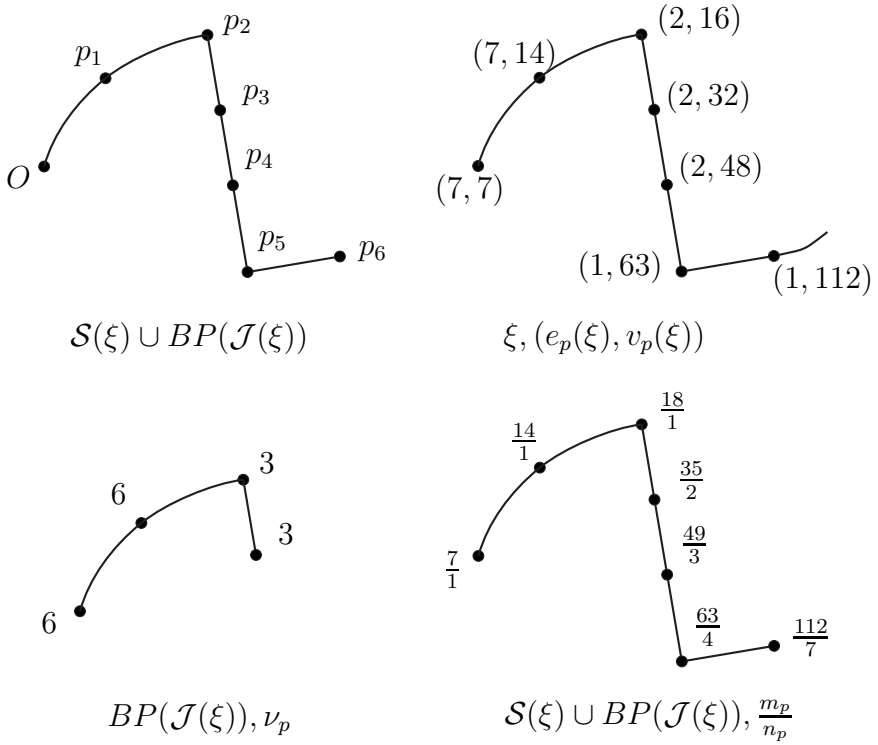


Figure A.18: $\xi : y^7 - x^{16} = 0$.

The second part of the algorithm starts with $v_{p_6} = m_{p_6} = 112$. The second step sets $v_O = m_O = 7$, $v_{p_1} = m_{p_1} = 14$ and $v_{p_2} = 16$, where the last value is the only integer in the interval

$$\left[\frac{n_{p_2}}{n_{p_6}} m_{p_6}, \frac{n_{p_2}}{n_{p_6}} m_{p_6} + 1\right) = [16, 17).$$

Finally, step 3 sets

- $v_{p_3} = \frac{n_{p_3}}{n_{p_2}} v_{p_2} = \frac{2}{1}16 = 32$, because $p_3 \succ p_6$ and $v_{p_2} = \frac{n_{p_2}}{n_{p_6}} m_{p_6}$,
- $v_{p_4} = \frac{n_{p_4}}{n_{p_2}} v_{p_2} = \frac{3}{1}16 = 48$, because $p_4 \succ p_6$ and $v_{p_2} = \frac{n_{p_2}}{n_{p_6}} m_{p_6}$, and
- $v_{p_5} = m_{p_5} = 63$ because $p_5 \prec p_6$.

Example 19: (See figure A.19.)

Now consider the curve ξ given by $y^{10} - x^{23} = 0$. It is irreducible with $\frac{23}{10}$ as its only characteristic exponent.

Its generic polars are also irreducible with characteristic exponent $\frac{22}{9}$, and thus $BP(\mathcal{J}(\xi))$ consists of the singular points of ξ up to p_4 and three further points p_8, p_9, p_{10} proximate to p_3.

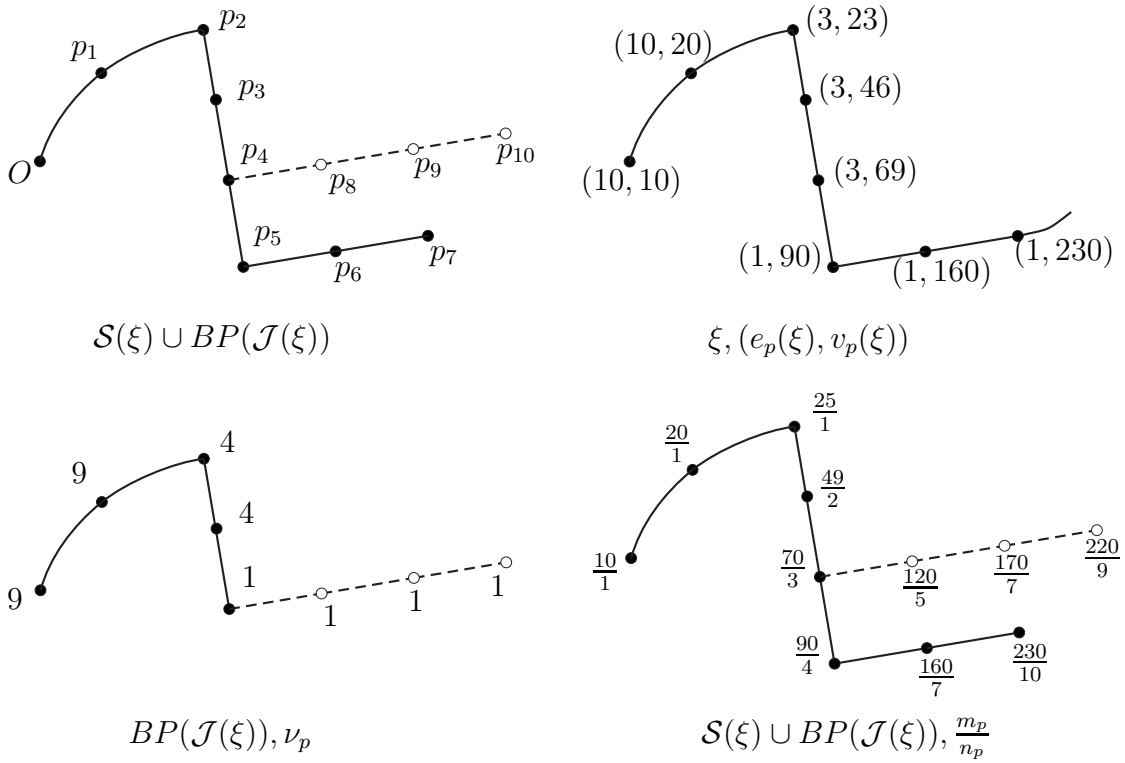


Figure A.19: $\xi : y^{10} - x^{23} = 0$.

Hence there is only one dicritical point p_{10} with associated polar invariant

$$I = I_{p_{10}} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{10})]}{n_{p_{10}}} + 1 = \frac{BP(\mathcal{J}(\xi))^2}{n_{p_{10}}} + 1 = \\ = \frac{9^2 + 9^2 + 4^2 + 4^2 + 1^2 + 1^2 + 1^2 + 1^2}{9} + 1 = 23.$$

Thus the point p' is p_1, and hence the rupture point associated to p_{10} must be satellite of $q_0 = p_2$. Step 4(c) consists of five iterations:

- $\frac{m_{q_0}}{n_{q_0}} = 25 > 23 = I$, so we take $q_1 = p_3$, the first satellite of p_2.

- $\frac{m_{q_1}}{n_{q_1}} = \frac{49}{2} > 23 = I$, so we take $q_2 = p_4$, the first satellite of p_3.
- $\frac{m_{q_2}}{n_{q_2}} = \frac{70}{3} > 23 = I$, so we take $q_3 = p_5$, the first satellite of p_4.
- $\frac{m_{q_3}}{n_{q_3}} = \frac{90}{4} < 23 = I$, so we take $q_4 = p_6$, the second satellite of p_5.
- $\frac{m_{q_4}}{n_{q_4}} = \frac{160}{7} < 23 = I$, so we take $q_5 = p_7$, the second satellite of p_6.

Since $\frac{m_{q_4}}{n_{q_4}} = 23 = I$, we have finished the first part of the algorithm with $\mathcal{R} = \{p_7\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_7\}$.

The second part of the algorithm starts setting $v_{p_7} = m_{p_7} = 230$. Afterwards, step 2 gives $v_O = m_O = 10$, $v_{p_1} = m_{p_1} = 20$ and $v_{p_2} = 23$, where the last value is the only integer in the interval

$$\left[\frac{n_{p_2}}{n_{p_7}} m_{p_7}, \frac{n_{p_2}}{n_{p_7}} m_{p_7} + 1\right) = [23, 24).$$

The algorithm finishes with step 3, which applies to $p_3, \ldots, p_6$ and gives:

- $v_{p_3} = \frac{n_{p_3}}{n_{p_2}} v_{p_2} = \frac{2}{1} 23 = 46$, because $p_3 \succ p_7$ and $v_{p_2} = \frac{n_{p_2}}{n_{p_7}} m_{p_7}$,
- $v_{p_4} = \frac{n_{p_4}}{n_{p_2}} v_{p_2} = \frac{3}{1} 23 = 69$, because $p_4 \succ p_7$ and $v_{p_2} = \frac{n_{p_2}}{n_{p_7}} m_7$,
- $v_{p_5} = m_{p_5} = 90$ because $p_5 \prec p_7$, and
- $v_{p_6} = m_{p_6} = 160$ because $p_6 \prec p_7$.

Example 20: (See figure A.20.)

Finally, let ξ be the curve given by $(y - x)(y^2 - x^3)(x^2 - y^3) = 0$. It consists of two transverse branches with characteristic exponent $\frac{3}{2}$ (as in Example 3) and one smooth branch non-tangent to the singular ones.

Generic polars of ξ have four smooth branches. Two of them are always tangent to the singular branches of ξ, and the other two have variable tangent directions. Hence the cluster $BP(\mathcal{J}(\xi))$ consists of the origin with multiplicity 4 and the free singular points of ξ in the first neighbourhood of O taken with multiplicity 1, as it is shown in figure A.20.

Now every base point is dicritical, and in particular so is the origin. Therefore, the algorithm starts with $\mathcal{R} = \mathcal{S} = \{O\}$, and it remains to consider p_1 and p_2. These points have the same associated polar invariant

$$I = I_{p_1} = I_{p_2} \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_1)]}{n_{p_1}} + 1 = \frac{4 \cdot 1 + 1 \cdot 1}{1} + 1 = 6,$$

so we can start with any of them.

Take for example p_1. The point p' is thus the origin, and hence the rupture point associated to p_1 is satellite of $q_0 = p_1$. Step 4(d) consists of just one iteration:

- $\frac{m_{q_0}}{n_{q_0}} = 7 > 6 = I$, so we take $q_1 = p_3$, the first satellite of p_1.

And since $\frac{m_{q_1}}{n_{q_1}} = \frac{12}{2} = 6 = I$, the procedure stops to give $\mathcal{R} = \{p_3\}$ and $\mathcal{S} = \{O, p_1, p_3\}$.

The case of p_2 is analogous, and we finally obtain $\mathcal{R} = \{p_2, p_4\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_4\}$.

The second part of the algorithm works as follows. Step 1 gives $v_O = m_O = 5$ and $v_{p_2} = v_{p_4} = 12$, and step 2 sets $v_{p_1} = v_{p_2} = 6$, which is the only integer in the interval

$$\left[\frac{n_{p_1}}{n_{p_3}} m_{p_3}, \frac{n_{p_1}}{n_{p_3}} m_{p_3} + 1\right) = \left[\frac{n_{p_2}}{n_{p_4}} m_{p_4}, \frac{n_{p_2}}{n_{p_4}} m_{p_4} + 1\right) = [6, 7).$$

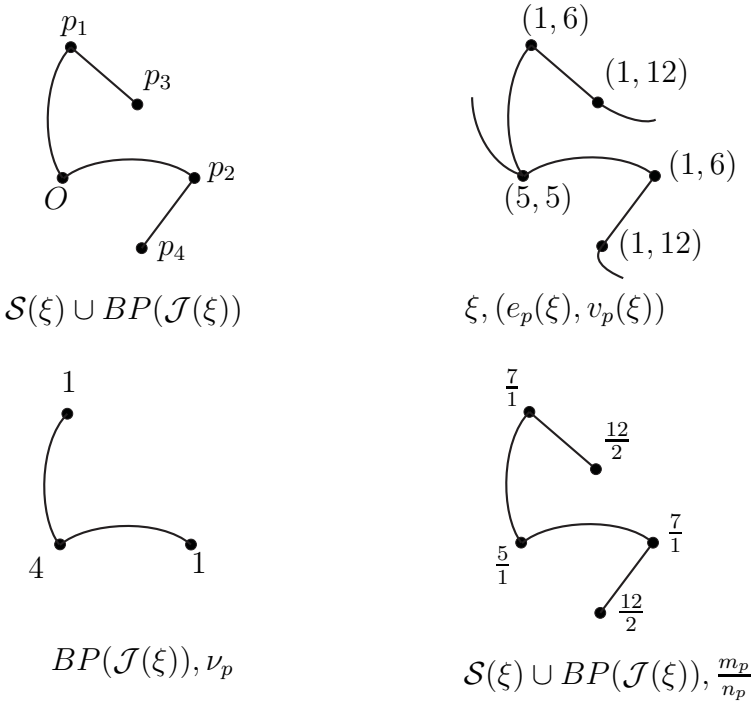


Figure A.20: $\xi : (y - x)(y^2 - x^3)(x^2 - y^3) = 0$.

A.2 Three bigger examples.

The second section of the appendix is devoted to expose in detail three "big" examples of application of the algorithm in section 3.3. After these examples it will be clear that the computation of $\mathcal{S}(\xi)$ is much faster using the definitive version of the algorithm than the simplest version in Remark 3.3.2.

As in the previous section, there are four Enriques' diagrams for each example. The first one shows the equisingularity type of the curve ξ. The second one contains the names of the singular points of ξ and the base points of $\mathcal{J}(\xi)$, where the dots in each square mean that there are as many free points as the number in the same square. The third diagram represents the cluster $BP(\mathcal{J}(\xi))$ with its virtual multiplicities, and the fourth one shows the heights of the trunks m_p and the multiplicities n_p of the morphism φ_p for each $p \in \mathcal{S}(\xi) \cup BP(\mathcal{J}(\xi))$ (which are computed with Corollary 1.3.9 and formula 2.1). Recall that points lying on ξ are represented with black filled circles, while the circles representing points not lying on ξ are filled in white.

Example A: (See figures A.21, A.22, A.23 and A.24)
As a first example, consider the curve ξ given by

$$(y^4 - \alpha_1 x^{11})(y^3 - \alpha_2 x^8)(y^9 - \alpha_3 x^{22})(y^{12} - \alpha_4 x^{29})(y^4 - \alpha_5 x^9) = 0,$$

where $\alpha_i \neq 0$ for $i = 1, \ldots 5$. Thus ξ consists of five branches $\gamma_1, \ldots, \gamma_5$ of characteristic exponents $\frac{11}{4}, \frac{8}{3}, \frac{22}{9}, \frac{29}{12}$ and $\frac{9}{4}$, respectively. From the Enriques' diagram of figure A.21 it is immediate that the set of rupture points of ξ is $\mathcal{R}(\xi) = \{p_4, p_5, p_9, p_{10}, p_{11}\}$.

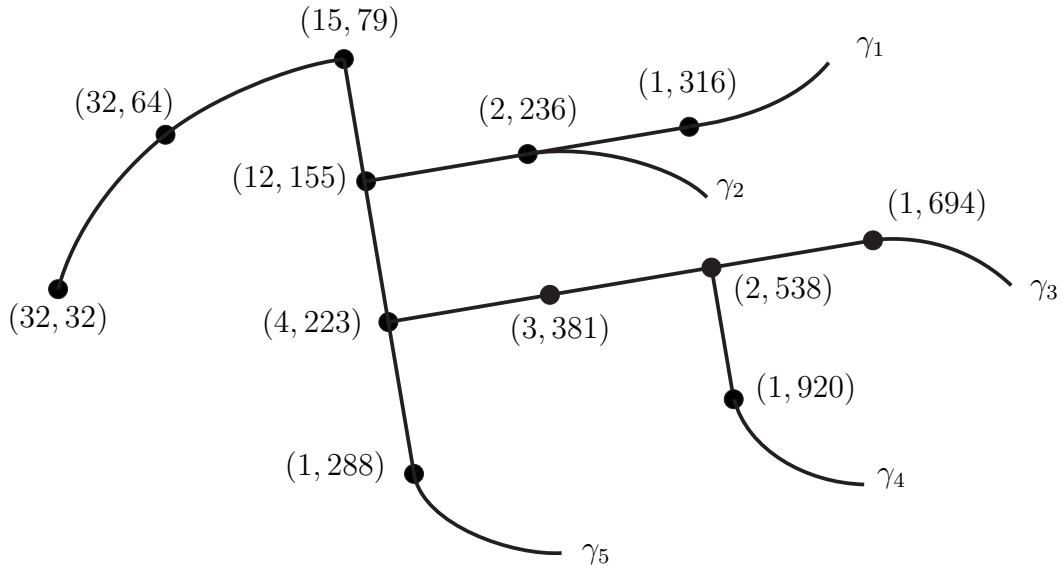


Figure A.21: Singular points of ξ with its multiplicities and values $(e_p(\xi), v_p(\xi))$.

According to figure A.23, generic polars of ξ have six branches with the following descriptions:

- One branch is smooth and goes through the last free singular point p_2 on ξ and one further free point p_{12}.
- Another branch is singular with characteristic exponent $\frac{7}{2}$. Its free singular point in the first neighbourhood of p_2 is fixed (p_{13}), and hence so is the satellite point p_{14}.
- Another branch has characteristic exponent $\frac{8}{3}$, and goes through p_4 and five fixed free points (up to p_{19}) in the successive neighbourhoods of p_4.
- Another branch has characteristic exponent $\frac{22}{9}$, and goes through p_9 and thirteen fixed free points (up to p_{20}) in the successive neighbourhoods of p_9.
- Another branch has characteristic exponent $\frac{29}{12}$, and goes through p_{11} and seventeen fixed free points (up to p_{21}) in the successive neighbourhoods of p_{11}.
- The last branch has characteristic exponent $\frac{9}{4}$, and goes through p_{10} and five fixed free points (up to p_{22}) in the successive neighbourhoods of p_{10}.

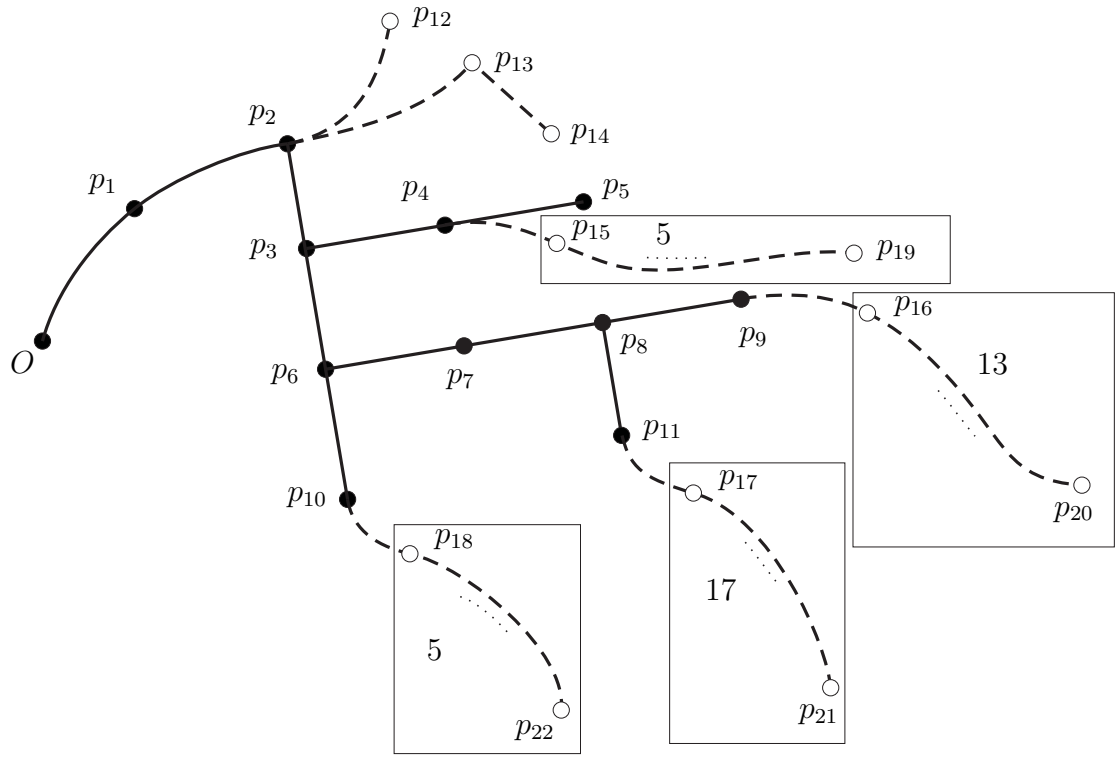


Figure A.22: Singular points of ξ and base points of $\mathcal{J}(\xi)$.

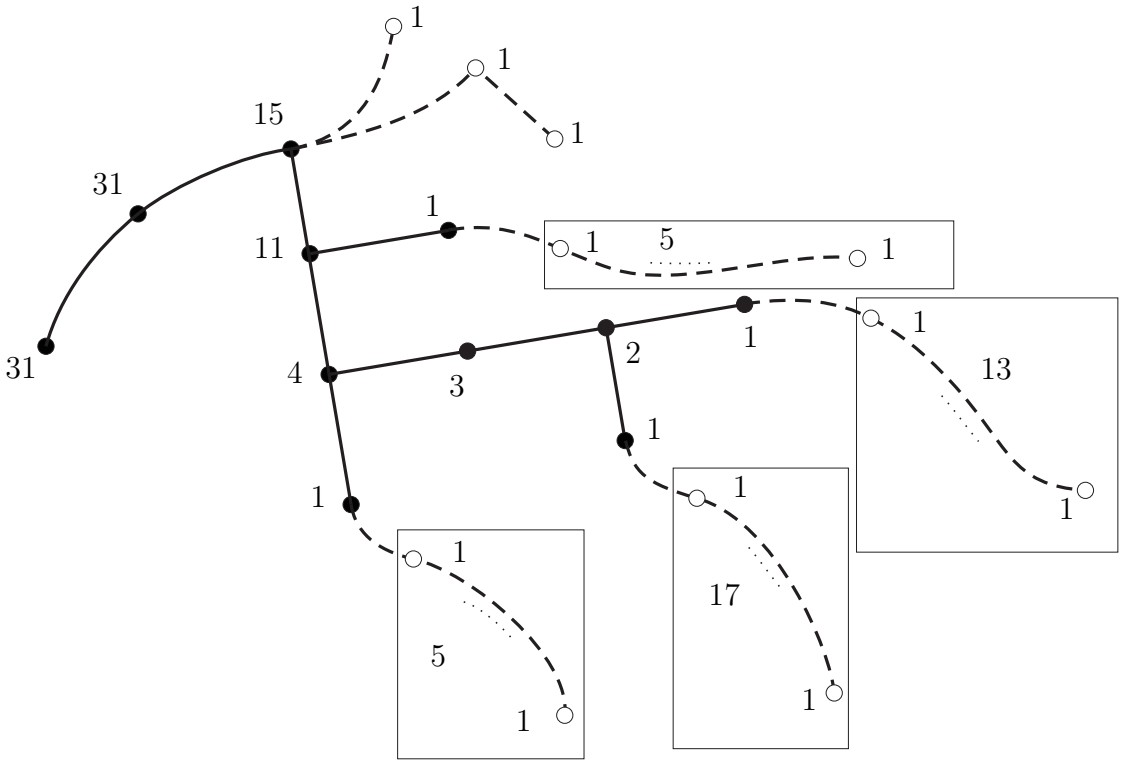


Figure A.23: Base points of $\mathcal{J}(\xi)$ with its virtual multiplicities ν_p.

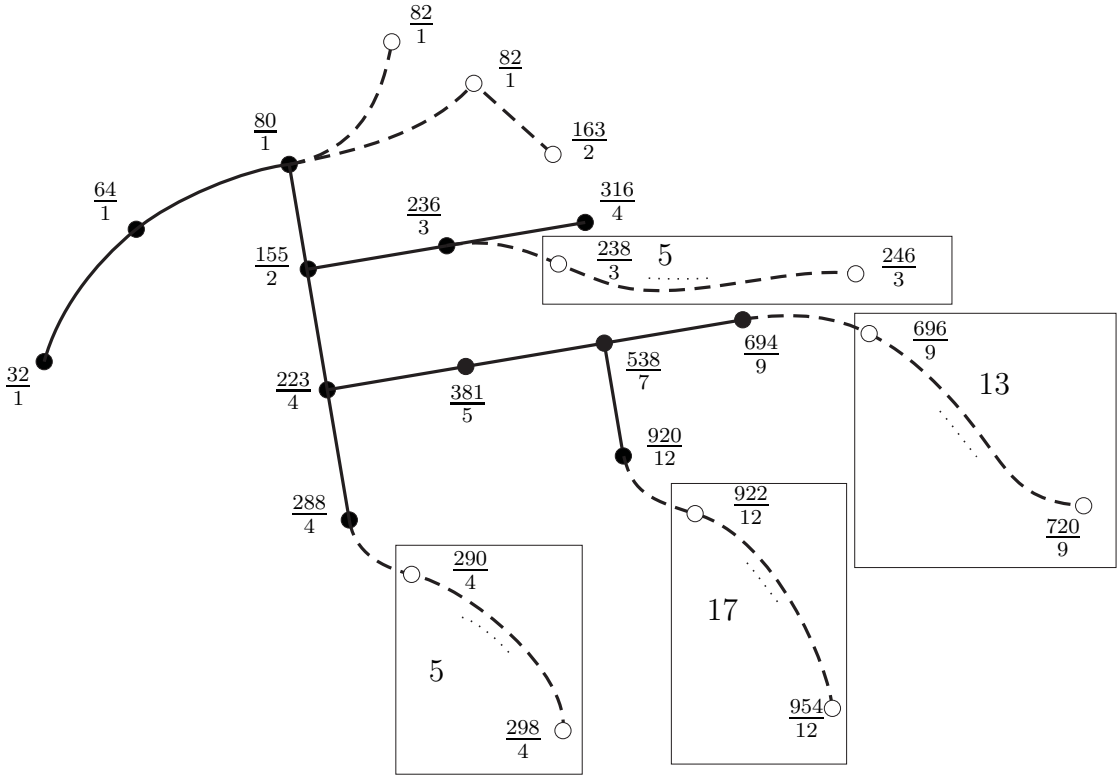


Figure A.24: $\mathcal{S}(\xi) \cup BP(\mathcal{J}(\xi))$ with heights of the trunks and multiplicities of φ_p, $\left(\frac{m_p}{n_p}\right)$.

Let us apply the algorithm to this case step by step. It starts with $\mathcal{R} = \mathcal{S} = \emptyset$ and $\mathcal{D} = \{p_{12}, p_{14}, p_{19}, p_{20}, p_{21}, p_{22}\}$. The polar invariants are

$$I_{12} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{12})]}{n_{p_{12}}} + 1 = \frac{31 \cdot 1 + 31 \cdot 1 + 15 \cdot 1 + 1^2}{1} + 1 = 79,$$

$$I_{14} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{14})]}{n_{p_{14}}} + 1 = \frac{31 \cdot 2 + 31 \cdot 2 + 15 \cdot 2 + 1^2 + 1^2}{3} + 1 = 79,$$

$$I_{19} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{19})]}{n_{p_{19}}} + 1 = \frac{31 \cdot 3 + 31 \cdot 3 + 15 \cdot 2 + 11 \cdot 1 + 1^2 + 5 \cdot 1^2}{3} + 1 = \frac{236}{3},$$

$$I_{20} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{20})]}{n_{p_{20}}} + 1 =$$

$$= \frac{31 \cdot 9 + 31 \cdot 9 + 15 \cdot 4 + 11 \cdot 4 + 4 \cdot 1 + 3 \cdot 1 + 2 \cdot 1 + 1^2 + 13 \cdot 1^2}{9} + 1 = \frac{694}{9},$$

$$I_{21} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{21})]}{n_{p_{21}}} + 1 =$$

$$= \frac{31 \cdot 12 + 31 \cdot 12 + 15 \cdot 5 + 11 \cdot 5 + 4 \cdot 2 + 3 \cdot 2 + 2 \cdot 1 + 1^2 + 17 \cdot 1^2}{12} + 1 = \frac{230}{3},$$

$$I_{22} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{22})]}{n_{p_{22}}} + 1 =$$

$$= \frac{31 \cdot 4 + 31 \cdot 4 + 15 \cdot 1 + 11 \cdot 1 + 4 \cdot 1 + 1^2 + 5 \cdot 1^2}{4} + 1 = 72.$$

So the order we have already chosen is valid and we can start step 4 of the first part.

- Start with p_{12}. We have $p' = p_1$ because $\frac{m_{p_1}}{n_{p_1}} = 64 < I_{12} = 79 \leqslant \frac{m_{p_2}}{n_{p_2}} = 80$, and then $p = p_2$. Since it is the first step, we must take $q_0 = p_2$ and perform 4(d).
  - $\frac{m_{q_0}}{n_{q_0}} = 80 > 79 = I_{12}$, and hence we take $q_1 = p_3$, the first satellite of p_2.
  - $\frac{m_{q_1}}{n_{q_1}} = \frac{155}{2} < 79 = I_{12}$, and hence we take $q_2 = p_4$, the second satellite of p_3.
  - $\frac{m_{q_2}}{n_{q_2}} = \frac{263}{3} < 79 = I_{12}$, and hence we take $q_3 = p_5$, the second satellite of p_4.

  Since $\frac{m_{q_3}}{n_{q_3}} = 79 = I_{12}$, we stop and set $\mathcal{K} = \{p_5\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_5\}$.

- Take now p_{14}. By the same reason as in the previous step, we have $p' = p_1$ and $p = p_2$. Moreover, the equality $I_{14} = I_{12} = 79$ implies that the procedure yields the same result as p_{12}, so we will omit it.

- Take now the point p_{19}. Since $\frac{m_{p_1}}{n_{p_1}} = 64 < I_{19} = \frac{236}{3} \leqslant \frac{m_{p_4}}{n_{p_4}} = \frac{236}{3}$, we again have $p' = p_1$, and hence $p = p_2$. But p_2 has already appeared as p, so the rupture point associated to p_{19} is p_4, the last p_2-satellite point in $\mathcal{K}(p_{19})$. Thus, we must set $\mathcal{R} = \{p_4, p_5\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_5\}$.

- Consider now p_{20}. p' is again p_1 because $\frac{m_{p_1}}{n_{p_1}} = 64 < I_{20} = \frac{694}{9} \leqslant \frac{m_{p_9}}{n_{p_9}} = \frac{694}{9}$. Therefore $p = p_2$, which has already appeared, and thus the rupture point associated to p_{20} is the last p_2-satellite in $\mathcal{K}(p_{20})$: p_9. This step finishes with $\mathcal{R} = \{p_4, p_5, p_9\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_9\}$.

- Take now the point p_{21}. Since $\frac{m_{p_1}}{n_{p_1}} = 64 < I_{21} = \frac{230}{3} \leqslant \frac{m_{p_{11}}}{n_{p_{11}}} = \frac{920}{12}$, we have $p' = p_1$ and $p = p_2$. But since p_2 has already appeared, the rupture point associated to p_{20} is p_{11}, the last p_2-satellite in $\mathcal{K}(p_{22})$. We finish this step with $\mathcal{R} = \{p_4, p_5, p_9, p_{11}\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_9, p_{11}\}$.

- We finally take p_{22}, the last dicritical point. We have again that $p' = p_1$ because $\frac{m_{p_1}}{n_{p_1}} = 64 < I_{22} = 72 \leqslant \frac{m_{p_{10}}}{n_{p_{10}}} = \frac{288}{4}$, and hence $p = p_2$. But it has already appeared, and therefore the rupture point associated to p_{22} is p_{10}.

Thus, the first part of the algorithm is completed with $\mathcal{R} = \{p_4, p_5, p_9, p_{10}, p_{11}\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_{11}\}$, which actually coincide with $\mathcal{R}(\xi)$ and $\mathcal{S}(\xi)$ respectively.

The second part begins recovering of the values of the rupture points:

$$v_{p_4} = m_{p_4} = 236, \quad v_{p_5} = 316, \quad v_{p_9} = 694, \quad v_{p_{10}} = 288, \quad \text{and} \quad v_{p_{11}} = 920.$$

Then it is the turn of the free singular non-rupture points. Firstly, $v_O = m_O = 32$ and $v_{p_1} = m_{p_1} = 64$ because p_1 and p_2 are free points in the first neighbourhoods of O and p_1 respectively. Secondly, $v_{p_2} = 79$ because it is the only integer in the interval

$$\left[\frac{n_{p_2}}{n_{p_5}} m_{p_5}, \frac{n_{p_2}}{n_{p_5}} m_{p_5} + 1\right) = [79, 80)$$

(where p_5 is the biggest p_2-satellite rupture point).

Finally, we must consider the satellite non-rupture points, which are p_3, p_6, p_7 and p_8. But all these points are smaller than p_5, which is the biggest p_2-satellite rupture point. Therefore we must always apply the second instance of step 3, obtaining the equality $v_p = m_p$ for all these points. More explicitly, we have

$$v_{p_3} = 155, \quad v_{p_6} = 223, \quad v_{p_7} = 381, \quad \text{and} \quad v_{p_8} = 538.$$

It is immediate to check that these values are the values of ξ at its singular points, as claimed.

Example B: (See figures A.25, A.26, A.27 and A.28)
Let ξ be the curve whose four branches $\gamma_1, \ldots, \gamma_4$ are given by the Puiseux series

$$\begin{aligned} s_1(x) &= x^{\frac{7}{4}} + x^2, \\ s_2(x) &= x^{\frac{7}{4}} + x^{\frac{35}{16}}, \\ s_3(x) &= x^{\frac{7}{4}} + x^{\frac{21}{10}}, \\ s_4(x) &= x^{\frac{8}{5}}. \end{aligned}$$

One possible equation for ξ is

$$\begin{aligned} f =& (y^5 - x^8)(y^4 - 4x^2y^3 + 6x^4y^2 - 4x^6y - x^7 + x^8) \\ & (y^{16} - 4x^7y^{12} + 6x^{14}y^8 - 80x^{14}y^9 - 4x^{21}y^4 - \\ & \quad 160x^{21}y^5 - 72x^{21}y^6 + x^{28} - 16x^{28}y + 56x^{28}y^2 - 16x^{28}y^3 - x^{35}) \\ & (y^{20} - 5x^7y^{16} + 10x^{14}y^{12} - 60x^{14}y^{13} - 10x^{21}y^8 - 540x^{21}y^9 - 2x^{21}y^{10} + \\ & \quad 5x^{28}y^4 - 404x^{28}y^5 + 380x^{28}y^6 - x^{35} - 20x^{35}y - 90x^{35}y^2 40x^{35}y^3 + x^{42}) \end{aligned}$$

It is straightforward to check from the Puiseux series of the branches of ξ that the corresponding Enriques' diagram is this of figure A.25.

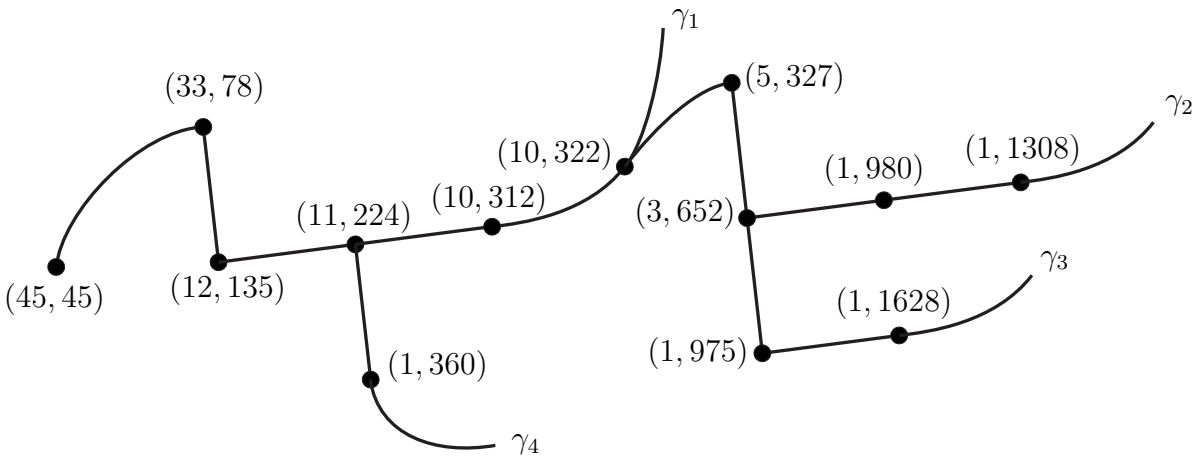


Figure A.25: Singular points of ξ with its multiplicities and values $(e_p(\xi), v_p(\xi))$.

It follows from the Enriques' diagram of figure A.21 that the set of rupture points of ξ is $\mathcal{R}(\xi) = \{p_4, p_5, p_6, p_{10}, p_{12}\}$.

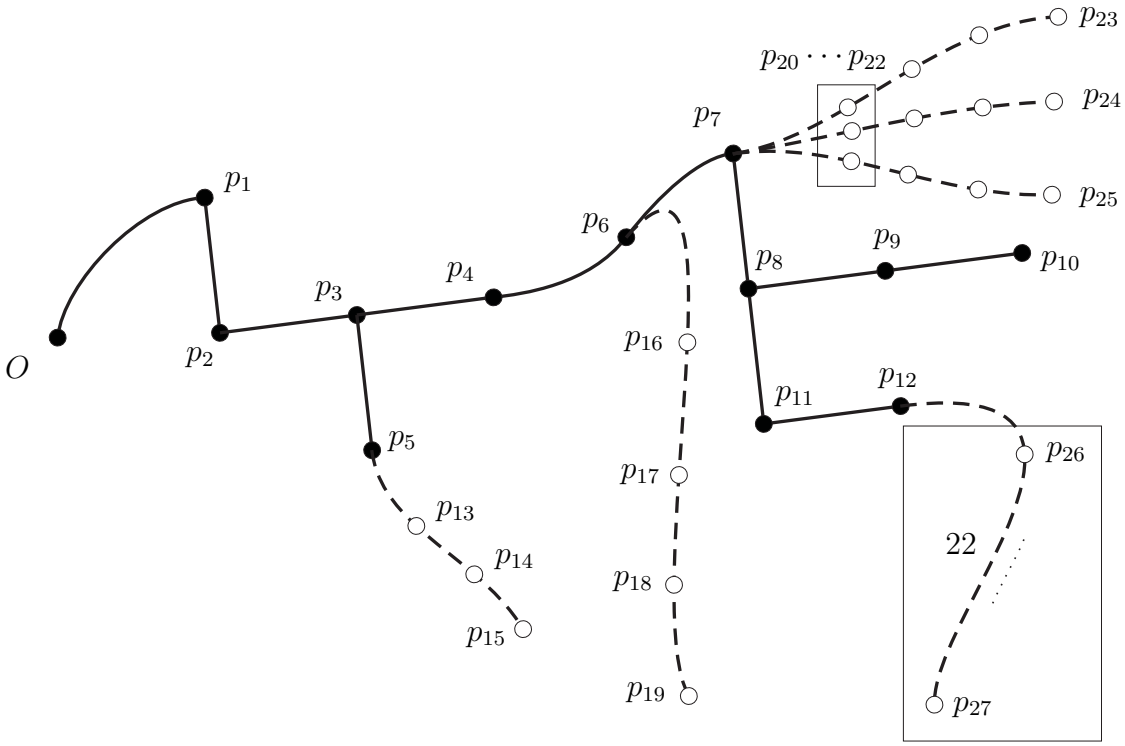


Figure A.26: Singular points of ξ and base points of $\mathcal{J}(\xi)$.

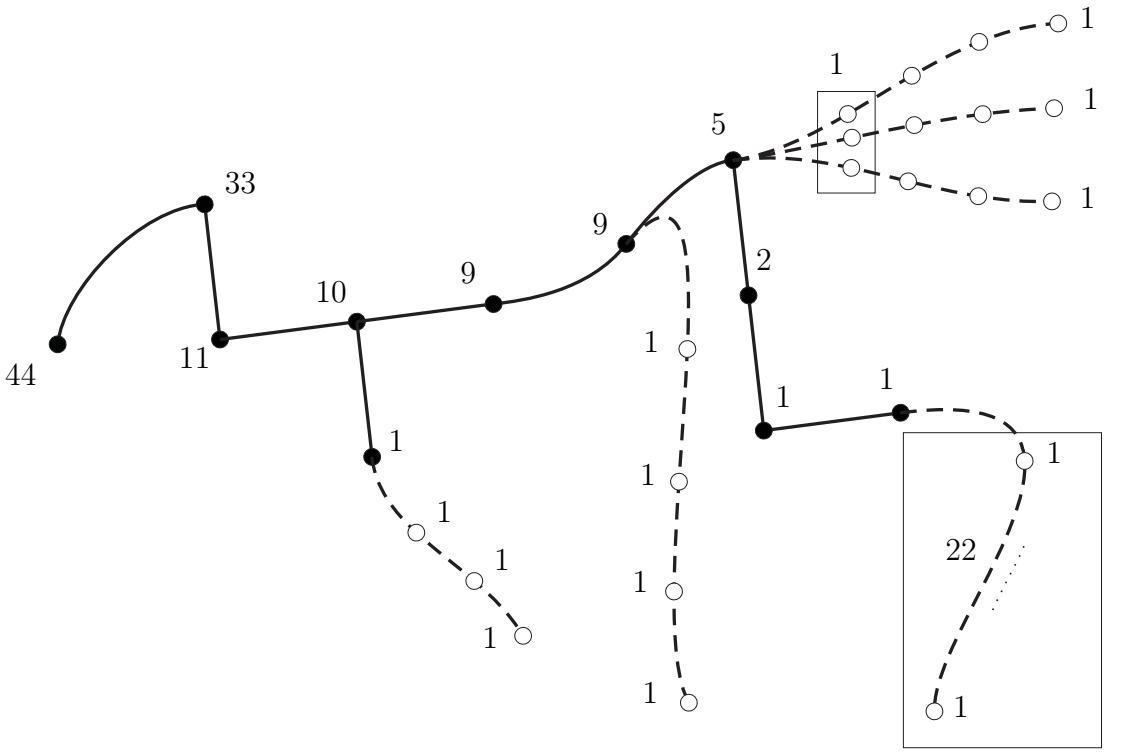


Figure A.27: Base points of $\mathcal{J}(\xi)$ with its virtual multiplicities ν_p.

According to figure A.27, generic polars of ξ have nine branches with the following descriptions:

- Three branches are smooth and go through p_1 (i.e, are tangent to ξ). Furthermore, they have no more fixed points.

- Three more branches go through p_7. They also go through four fixed free points (different for each branch) in successive neighbourhoods of p_7 (up to p_{23}, p_{24} and p_{25}.

- Another branch goes through p_{12} and 22 fixed free points (not lying on ξ) in successive neighbourhoods (up to p_{27}).

- Another branch has characteristic exponent $\frac{7}{4}$, as γ_1. Moreover, this branch goes through p_6 (also as γ_1 does) and four further free fixed points which do not lie on ξ (up to p_{19}).

- The last branch has characteristic exponent $\frac{8}{5}$, and goes through p_5 and three fixed free points (up to p_{15}) in the successive neighbourhoods of p_5.

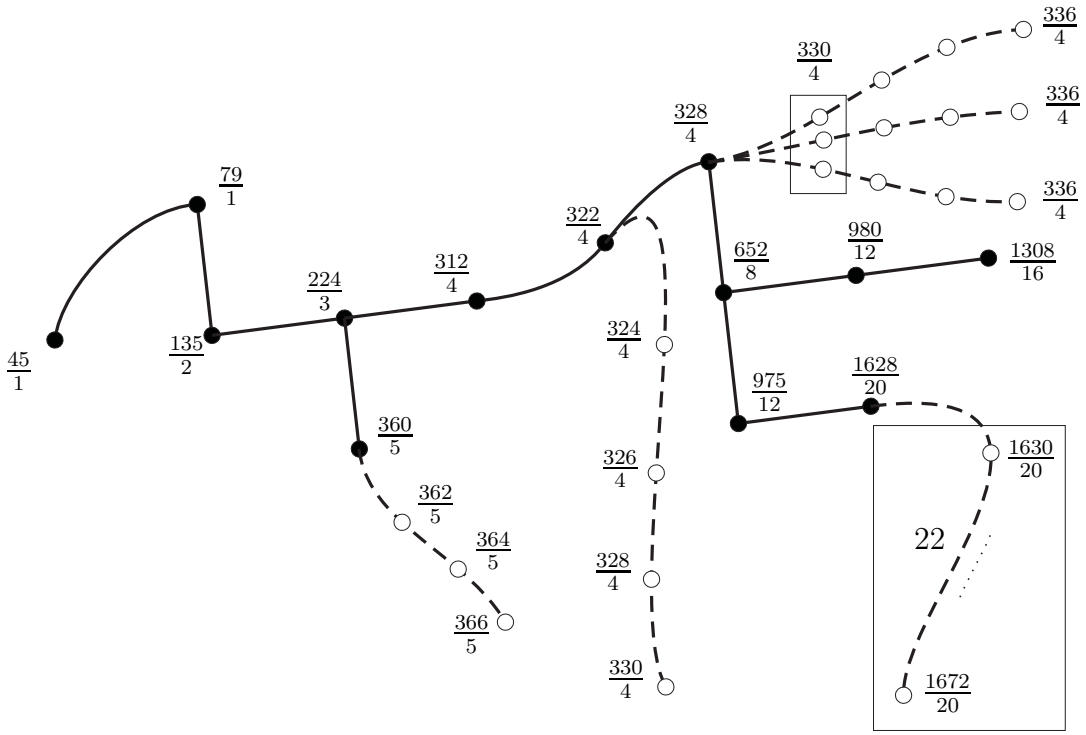


Figure A.28: $\mathcal{S}(\xi) \cup BP(\mathcal{J}(\xi))$ with heights of the trunks and multiplicities of φ_p, $\left(\frac{m_p}{n_p}\right)$.

Now we run the algorithm. Step 1 sets $\mathcal{R} = \mathcal{S} = \emptyset$, $\mathcal{D} = \{p_1, p_{15}, p_{19}, p_{23}, p_{24}, p_{25}, p_{27}\}$, and since $O \notin \mathcal{D}$ we go to step 3.

The polar invariants are

$$I_1 = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_1)]}{n_{p_1}} + 1 = \frac{44\cdot 1 + 33\cdot 1}{1} + 1 = 78,$$

$$I_{15} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{15})]}{n_{p_{15}}} + 1 = \frac{44\cdot 5 + 33\cdot 3 + 11\cdot 2 + 10\cdot 1 + 1^2 + 3\cdot 1^2}{5} + 1 = 72,$$

$$I_{19} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{19})]}{n_{p_{19}}} + 1 =$$

$$= \frac{44\cdot 4 + 33\cdot 3 + 11\cdot 1 + 10\cdot 1 + 9\cdot 1 + 9\cdot 1 + 4\cdot 1^2}{4} + 1 = \frac{161}{2},$$

$$I_{23} = I_{24} = I_{25} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{23})]}{n_{p_{23}}} + 1 =$$

$$= \frac{44\cdot 4 + 33\cdot 3 + 11\cdot 1 + 10\cdot 1 + 9\cdot 1 + 9\cdot 1 + 5\cdot 1 + 4\cdot 1^2}{4} + 1 = \frac{327}{4},$$

$$I_{27} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{27})]}{n_{p_{27}}} + 1 =$$

$$= \frac{44\cdot 20 + 33\cdot 15 + 11\cdot 5 + 10\cdot 5 + 9\cdot 5 + 9\cdot 5 + 5^2 + 2^2 + 1^2 + 1^2 + 22\cdot 1^2}{20} + 1 =$$

$$= \frac{407}{5}.$$

Hence, we must process the dicritical points in the order $p_{23}, p_{24}, p_{25}, p_{27}, p_{19}, p_1, p_{15}$.

- Start with p_{23}. We have $p' = p_6$ because $\frac{m_{p_6}}{n_{p_6}} = \frac{322}{4} < I_{23} = \frac{327}{4} \leqslant \frac{m_{p_7}}{n_{p_7}} = 82$, and then $p = p_7$. Since it is the first step, we must take $q_0 = p_7$ and perform 4(d).
  - $\frac{m_{q_0}}{n_{q_0}} = 82 > \frac{327}{4} = I_{23}$, and hence we take $q_1 = p_8$, the first satellite of p_7.
  - $\frac{m_{q_1}}{n_{q_1}} = \frac{652}{8} < \frac{327}{4} = I_{23}$, and hence we take $q_2 = p_9$, the second satellite of p_8.
  - $\frac{m_{q_2}}{n_{q_2}} = \frac{980}{12} < \frac{327}{4} = I_{23}$, and hence we take $q_3 = p_{10}$, the second satellite of p_9.

  Since $\frac{m_{q_3}}{n_{q_3}} = \frac{1308}{16} = I_{23}$, we stop and set $\mathcal{K} = \{p_{10}\}$, $\mathcal{S} = \{O, p_1, \ldots, p_4, p_6, \ldots, p_{10}\}$.

- Take now p_{24}. By the same reason as in the previous step, we have $p' = p_6$ and $p = p_7$. Moreover, the equality $I_{23} = I_{24}$ implies that the procedure yields the same result as p_{23}, so we will omit it. The same happens with p_{25}.

- Take now the point p_{27}. Since $\frac{m_{p_6}}{n_{p_6}} = \frac{322}{4} < I_{27} = \frac{407}{5} \leqslant \frac{m_{p_{12}}}{n_{p_{12}}} = \frac{1628}{20}$, we again have $p' = p_6$, and hence $p = p_7$. But p_7 has already appeared as p, so the rupture point associated to p_{27} is p_{12}, the last p_7-satellite point in $\mathcal{K}(p_{27})$. Thus, we must set $\mathcal{R} = \{p_{10}, p_{12}\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_4, p_6, \ldots, p_{12}\}$.

- Consider now p_{19}. The point p' is now p_4 because $\frac{m_{p_4}}{n_{p_4}} = \frac{312}{4} < I_{19} = \frac{161}{2} \leqslant \frac{m_{p_6}}{n_{p_6}} = \frac{322}{4}$. Therefore $p = p_6$, which has not appeared before as p. Hence we must perform 4(d). But fortunately $\frac{m_{p_6}}{n_{p_6}} = \frac{322}{4} = \frac{161}{2} = I_{19}$, which implies that this step finishes by setting $\mathcal{R} = \{p_6, p_{10}, p_{12}\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_4, p_6, \ldots, p_{12}\}$.

- Take now the point p_1. Since $\frac{m_O}{n_O} = 45 < I_1 = 78 \leqslant \frac{m_{p_1}}{n_{p_1}} = 79$, we have $p' = O$ and $p = p_1$. Since p_1 has not appeared in a previous step, we must iterate 4(d) from $q_0 = p = p_1$.
  - $\frac{m_{q_0}}{n_{q_0}} = 79 > 78 = I_1$, and hence we take $q_1 = p_2$, the first satellite of p_1.
  - $\frac{m_{q_1}}{n_{q_1}} = \frac{135}{2} < 78 = I_1$, and hence we take $q_2 = p_3$, the second satellite of p_2.
  - $\frac{m_{q_2}}{n_{q_2}} = \frac{224}{3} < 78 = I_1$, and hence we take $q_3 = p_4$, the second satellite of p_3.

  Now we have arrived to $\frac{m_{q_3}}{n_{q_3}} = \frac{312}{4} = I_1$. Hence we stop with $\mathcal{R} = \{p_4, p_6, p_{10}, p_{12}\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_4, p_6, \ldots, p_{12}\}$.

- We finally take p_{15}, the last dicritical point. We have again that $p' = O$ because $\frac{m_O}{n_O} = 45 < I_{15} = 72 \leqslant \frac{m_{p_1}}{n_{p_1}} = 79$, and hence $p = p_1$. But it has already appeared, and therefore the rupture point associated to p_{15} is p_5.

Thus, the first part of the algorithm is completed with $\mathcal{R} = \{p_4, p_5, p_6, p_{10}, p_{12}\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_{12}\}$, which actually coincide with $\mathcal{R}(\xi)$ and $\mathcal{S}(\xi)$ respectively.

The second part begins recovering the values of the rupture points:

$$v_{p_4} = m_{p_4} = 312, \quad v_{p_5} = 360, \quad v_{p_6} = 322, \quad v_{p_{10}} = 1308, \quad \text{and} \quad v_{p_{12}} = 1628.$$

Then it is the turn of the free singular non-rupture points. Firstly, $v_O = m_O = 45$ because p_1 is a free point in the first neighbourhood of O. Secondly,

$$v_{p_1} = 78 \quad \text{and} \quad v_{p_7} = 327$$

because they are the only integers in the intervals

$$\left[\frac{n_{p_1}}{n_{p_4}} m_{p_4}, \frac{n_{p_1}}{n_{p_4}} m_{p_4} + 1\right) = [78, 79) \quad \text{and} \quad \left[\frac{n_{p_7}}{n_{p_{10}}} m_{p_{10}}, \frac{n_{p_7}}{n_{p_{10}}} m_{p_{10}} + 1\right) = [327, 328)$$

respectively, where p_4 (resp. p_{10}) is the biggest p_1-satellite (resp. p_7-satellite) rupture point.

Finally, we must consider the satellite non-rupture points, which are p_2, p_3, p_8, p_9 and p_{11}. In first place, both p_2 and p_3 are smaller than p_4, the biggest p_1-satellite rupture point, and hence we have

$$v_{p_2} = m_{p_2} = 135 \quad \text{and} \quad v_{p_3} = m_{p_3} = 224$$

because the second instance of step 3 applies. In second place, all three p_8, p_9 and p_{11} are smaller than p_{10}, which is the biggest p_7-satellite rupture point. Therefore we get

$$v_{p_8} = m_{p_8} = 652, \quad v_{p_9} = m_{p_9} = 980 \quad \text{and} \quad v_{p_{11}} = m_{p_{11}} = 975$$

by the same reason as above.

As in the previous example, it is immediate to check that these values are the values of ξ at its singular points, as claimed.

Example C: (See figures A.29, A.30, A.31 and A.32)

For the last example, let ξ be the curve whose five branches $\gamma_1, \ldots, \gamma_5$ are given by the Puiseux series

$$\begin{aligned} s_1(x) &= x^{\frac{11}{4}} + x^{\frac{51}{16}}, \\ s_2(x) &= x^{\frac{11}{4}} + x^{\frac{63}{20}}, \\ s_3(x) &= x^{\frac{8}{3}}, \\ s_4(x) &= x^{\frac{16}{7}}, \\ s_5(x) &= x^{\frac{9}{4}}. \end{aligned}$$

One possible equation for ξ is

$$\begin{aligned} f =& (y^3 - x^8)(y^4 - x^9)(y^7 - x^{16}) \\ & (y^{16} - 4x^{11}y^{12} - 80x^{21}y^9 + 6x^{22}y^8 - 72x^{31}y^6 - \\ & \quad 160x^{32}y^5 - 4x^{33}y^4 - 16x^{41}y^3 + 56x^{42}y^2 - 16x^{43}y + x^{44} - x^{51}) \\ & (y^{20} - 5x^{11}y^{16} + 10x^{22}y^{12} - 140x^{24}y^{12} - 10x^{33}y^8 - 620x^{35}y^8 - 110x^{37}y^8 + \\ & \quad 5x^{44}y^4 - 260x^{46}y^4 + 340x^{48}y^4 - 20x^{50}y^4 - x^{55} - 4x^{57} - 6x^{59} - 4x^{61} - x^{63}) \end{aligned}$$

As in the previous example, it is straightforward to check from the Puiseux series of the branches of ξ that the corresponding Enriques' diagram is this of figure A.29.

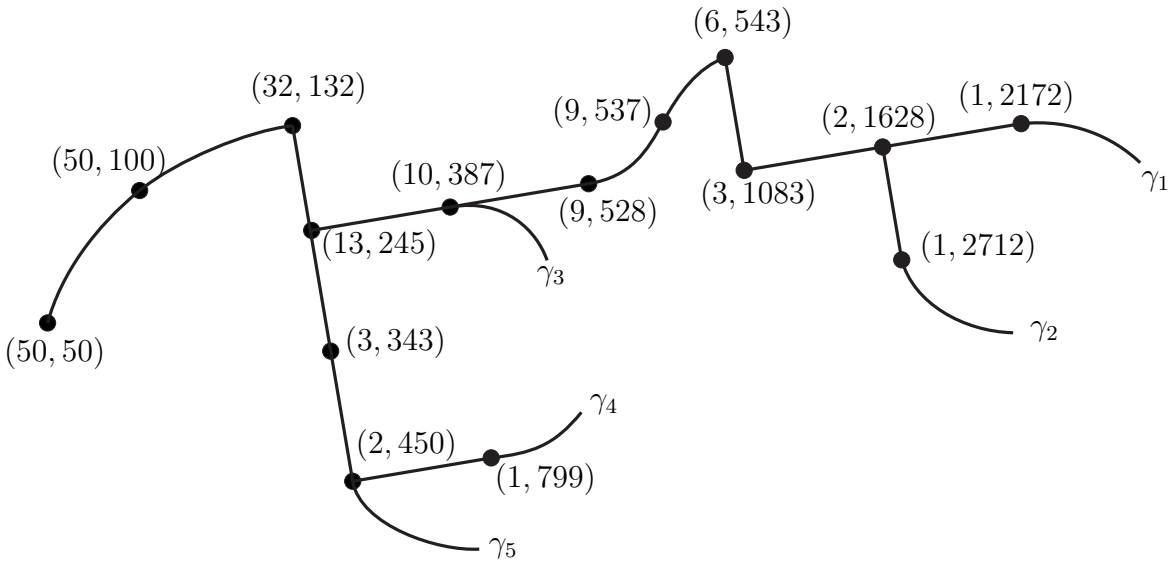


Figure A.29: Singular points of ξ with its multiplicities and values $(e_p(\xi), v_p(\xi))$.

It is also immediate that the set of rupture points of ξ is $\mathcal{R}(\xi) = \{p_4, p_5, p_7, p_8, p_{13}, p_{14}\}$.

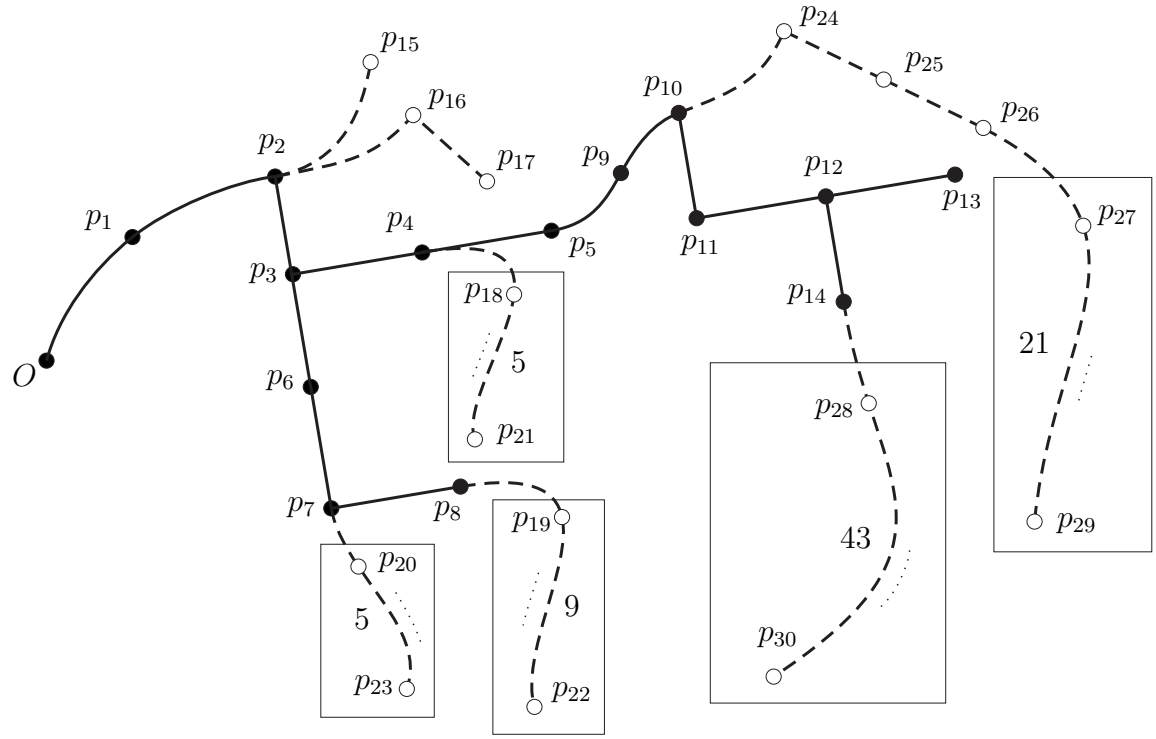


Figure A.30: Singular points of ξ and base points of $\mathcal{J}(\xi)$.

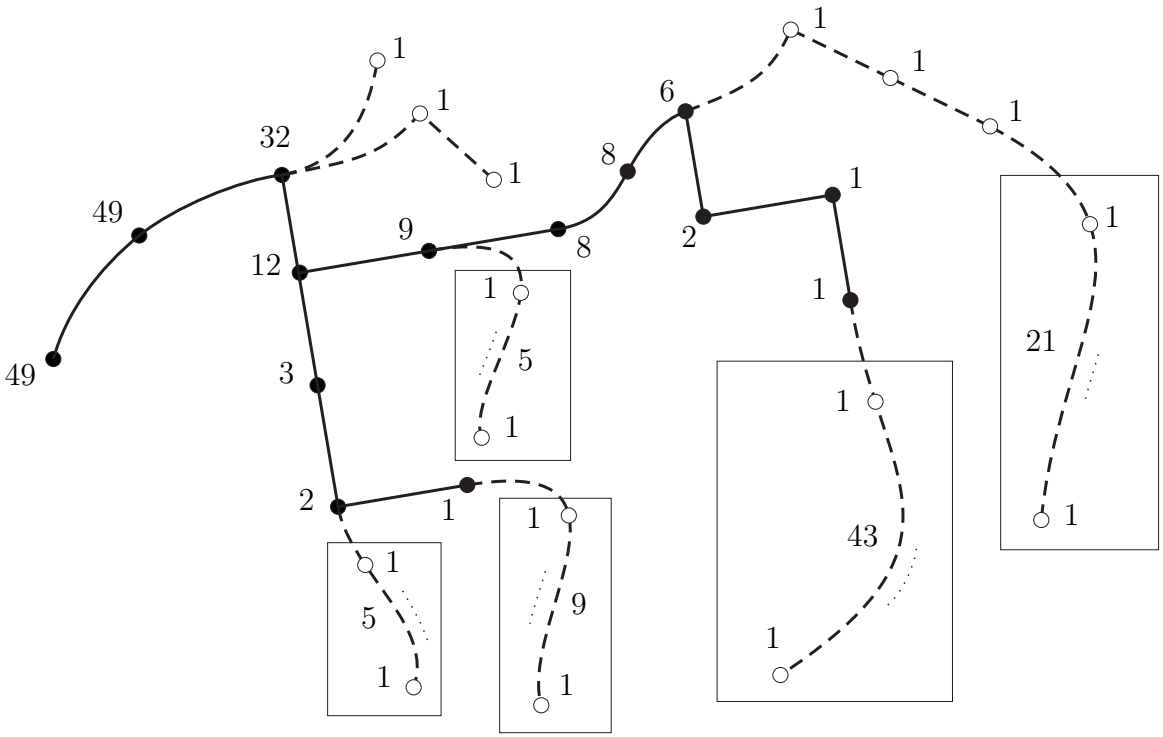


Figure A.31: Base points of $\mathcal{J}(\xi)$ with its virtual multiplicities ν_p.

According to figure A.31, generic polars of ξ have seven branches with the following descriptions:

- One branch is smooth and goes through the last free singular point p_2 on ξ and one further free point p_{15}.
- Another branch is singular with characteristic exponent $\frac{7}{2}$. Its free singular point in the first neighbourhood of p_2 is fixed (p_{16}), and hence so is the satellite point p_{17}.
- Another branch has characteristic exponent $\frac{8}{3}$, and goes through p_4 and five fixed free points (up to p_{21}) in the successive neighbourhoods of p_4.
- Another branch has characteristic exponent $\frac{16}{7}$, and goes through p_8 and nine fixed free points (up to p_{22}) in the successive neighbourhoods of p_8.
- Another branch has characteristic exponent $\frac{9}{4}$, and goes through p_7 and five fixed free points (up to p_{23}) in the successive neighbourhoods of p_7.
- Another branch has characteristic exponents $\frac{11}{4}$ and $\frac{63}{20}$, as γ_2. In fact, the branch of the polar goes through p_{14} and 43 fixed free points (up to p_{30}) in the successive neighbourhoods of p_{14}.
- The last branch goes through p_{10}, one free point, two satellite points and 21 further fixed free points (up to p_{29}) in the successive neighbourhoods of p_{10}.

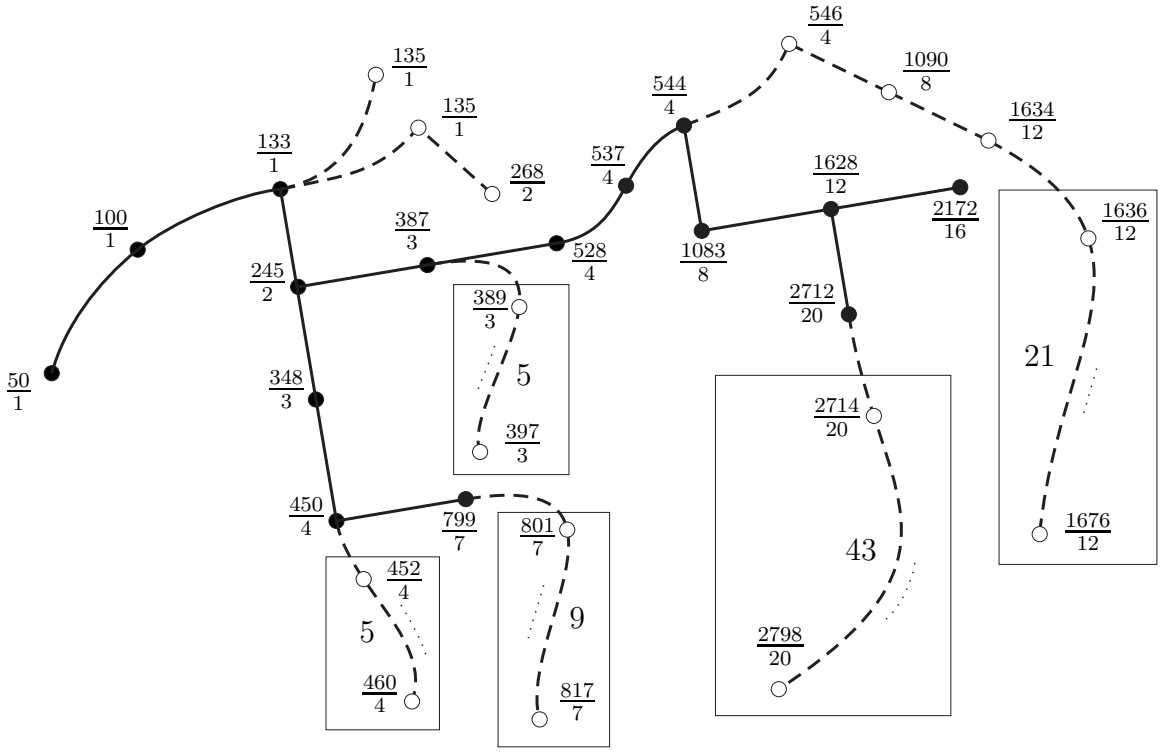


Figure A.32: $\mathcal{S}(\xi) \cup BP(\mathcal{J}(\xi))$ with heights of the trunks and multiplicities of φ_p, $\left(\frac{m_p}{n_p}\right)$.

Now we run the algorithm. Step 1 sets $\mathcal{R} = \mathcal{S} = \emptyset$, $\mathcal{D} = \{p_{15}, p_{17}, p_{21}, p_{22}, p_{23}, p_{29}, p_{30}\}$, and since $O \notin \mathcal{D}$ we go to step 3.

The polar invariants are

$$I_{15} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{15})]}{n_{p_{15}}} + 1 = \frac{49 \cdot 1 + 49 \cdot 1 + 32 \cdot 1 + 1^2}{1} + 1 = 132,$$

$$I_{17} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{17})]}{n_{p_{17}}} + 1 = \frac{49 \cdot 2 + 49 \cdot 2 + 32 \cdot 2 + 1^2 + 1^2}{2} + 1 = 132,$$

$$I_{21} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{21})]}{n_{p_{21}}} + 1 = \frac{49 \cdot 3 + 49 \cdot 3 + 32 \cdot 2 + 12 \cdot 1 + 9 \cdot 1 + 5 \cdot 1^2}{3} + 1 = 129,$$

$$I_{22} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{22})]}{n_{p_{22}}} + 1 =$$
$$= \frac{49 \cdot 7 + 49 \cdot 7 + 32 \cdot 2 + 12 \cdot 2 + 3 \cdot 2 + 2 \cdot 1 + 1^2 + 9 \cdot 1^2}{7} + 1 = \frac{799}{7},$$

$$I_{23} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{23})]}{n_{p_{23}}} + 1 =$$
$$= \frac{49 \cdot 4 + 49 \cdot 4 + 32 \cdot 1 + 12 \cdot 1 + 3 \cdot 1 + 2 \cdot 1 + 5 \cdot 1^2}{4} + 1 = \frac{225}{2},$$

$$I_{29} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{29})]}{n_{p_{29}}} + 1 =$$
$$= \frac{49 \cdot 12 + 49 \cdot 12 + 32 \cdot 9 + 12 \cdot 3 + 9 \cdot 3 + 8 \cdot 3 + 8 \cdot 3 + 6 \cdot 3 + 24 \cdot 1^2}{12} + 1 = \frac{543}{4},$$

$$I_{30} = \frac{[BP(\mathcal{J}(\xi)).\mathcal{K}(p_{30})]}{n_{p_{30}}} + 1 =$$
$$= \frac{49 \cdot 20 + 49 \cdot 20 + 32 \cdot 15 + 12 \cdot 5 + 9 \cdot 5 + 8 \cdot 5 + 8 \cdot 5 + 6 \cdot 3 + 2^2 + 45 \cdot 1^2}{20} + 1 =$$
$$= \frac{678}{5}.$$

Hence, we must process the dicritical points in the order $p_{29}, p_{30}, p_{15}, p_{17}, p_{21}, p_{22}, p_{23}$.

- Start with p_{29}. We have $p' = p_9$ because $\frac{m_{p_9}}{n_{p_9}} = \frac{537}{4} < I_{29} = \frac{543}{4} \leqslant \frac{m_{p_{10}}}{n_{p_{10}}} = \frac{544}{4} = 136$, and then $p = p_{10}$. Since it is the first step, we must take $q_0 = p_{10}$ and perform 4(d).
  - $\frac{m_{q_0}}{n_{q_0}} = 136 > \frac{543}{4} = I_{29}$, and hence we take $q_1 = p_{11}$, the first satellite of p_{10}.
  - $\frac{m_{q_1}}{n_{q_1}} = \frac{1083}{8} < \frac{543}{4} = I_{29}$, and hence we take $q_2 = p_{12}$, the second satellite of p_{11}.
  - $\frac{m_{q_2}}{n_{q_2}} = \frac{407}{3} < \frac{543}{4} = I_{29}$, and hence we take $q_3 = p_{13}$, the second satellite of p_{12}.

  Since $\frac{m_{q_3}}{n_{q_3}} = \frac{2172}{16} = I_{29}$, we stop and set $\mathcal{R} = \{p_{13}\}$, $\mathcal{S} = \{O, p_1, \ldots, p_5, p_9, \ldots, p_{13}\}$.
- Take now p_{30}. Since $\frac{m_{p_9}}{n_{p_9}} = \frac{537}{4} < I_{30} = \frac{678}{5} \leqslant \frac{m_{p_{14}}}{n_{p_{14}}} = \frac{2712}{20}$, we have $p' = p_9$ and $p = p_{10}$. But p_{10} has already appeared as p, and so the rupture point associated to p_{30} is p_{14}, the last p_{10}-satellite point in $\mathcal{K}(p_{30})$. Up to now we have $\mathcal{R} = \{p_{13}, p_{14}\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_5, p_9, \ldots, p_{14}\}$.

- Take now the point p_{15}. Since $\frac{m_{p_1}}{n_{p_1}} = 100 < I_{15} = 132 \leqslant \frac{m_{p_2}}{n_{p_2}} = 133$, we have $p' = p_1$, and hence $p = p_2$. It is the first time p_2 appears, so we must perform the iterations of 4(d) starting from $q_0 = p = p_2$:
  - $\frac{m_{q_0}}{n_{q_0}} = 133 > 132 = I_{15}$, and hence we take $q_1 = p_3$, the first satellite of p_2.
  - $\frac{m_{q_1}}{n_{q_1}} = \frac{245}{2} < 132 = I_{15}$, and hence we take $q_2 = p_4$, the second satellite of p_3.
  - $\frac{m_{q_2}}{n_{q_2}} = \frac{387}{3} < 132 = I_{15}$, and hence we take $q_3 = p_5$, the second satellite of p_4.

  And now we stop because $\frac{m_{q_3}}{n_{q_3}} = \frac{528}{4} = 132 = I_{15}$. We finish this step by setting $\mathcal{R} = \{p_5, p_{13}, p_{14}\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_5, p_9, \ldots, p_{14}\}$.

- The case of p_{17} is exactly the same of p_{15}, so we will omit it.

- Take now the point p_{21}. Since $\frac{m_{p_1}}{n_{p_1}} = 100 < I_{21} = \frac{387}{3} \leqslant \frac{m_{p_2}}{n_{p_2}} = 133$, we have $p' = p_1$ and $p = p_2$. But p_2 has already appeared, and hence we obtain that the rupture point associated to p_{21} is p_4, the last p_2-satellite point in $\mathcal{K}(p_{21})$. Therefore we have by the moment $\mathcal{R} = \{p_4, p_5, p_{13}, p_{14}\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_5, p_9, \ldots, p_{14}\}$.

- Consider now the point p_{22}. We have again that $p' = p_1$ and $p = p_2$ because $\frac{m_{p_1}}{n_{p_1}} = 100 < I_{22} = \frac{799}{7} \leqslant \frac{m_{p_2}}{n_{p_2}} = 133$, and since p_2 has already appeared as the point p, the rupture point associated to p_{22} is the last p_2-satellite point in $\mathcal{K}(p_{22})$: p_8. We finish this step by setting $\mathcal{R} = \{p_4, p_5, p_8, p_{13}, p_{14}\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_{14}\}$.

- We finally take p_{23}, the last dicritical point. We have again that $p' = p_1$ because $\frac{m_{p_1}}{n_{p_1}} = 100 < I_{23} = \frac{225}{2} \leqslant \frac{m_{p_2}}{n_{p_2}} = 133$, and hence $p = p_2$. But it has already appeared (three times), and therefore the rupture point associated to p_{23} is p_7.

Thus, the first part of the algorithm finishes with $\mathcal{R} = \{p_4, p_5, p_7, p_8, p_{13}, p_{14}\}$ and $\mathcal{S} = \{O, p_1, \ldots, p_{14}\}$, which actually coincide with $\mathcal{R}(\xi)$ and $\mathcal{S}(\xi)$ respectively.

The second part begins recovering the values of the rupture points:

$$v_{p_4} = 387, \quad v_{p_5} = 528, \quad v_{p_7} = 450, \quad v_{p_8} = 799, \quad v_{p_{13}} = 2172, \quad \text{and} \quad v_{p_{14}} = 2712.$$

Then it is the turn of the free singular non-rupture points. Firstly,

$$v_O = m_O = 50, \quad v_{p_1} = m_{p_1} = 50 \quad \text{and} \quad v_{p_9} = m_{p_9} = 537$$

because p_1, p_2 and p_{10} are a free singular points in the first neighbourhoods of O, p_1 and p_9 respectively. Secondly,

$$v_{p_2} = 132 \quad \text{and} \quad v_{p_{10}} = 543$$

because they are the only integers in the intervals

$$\left[\frac{n_{p_2}}{n_{p_5}} m_{p_5}, \frac{n_{p_2}}{n_{p_5}} m_{p_5} + 1\right) = [132, 133) \quad \text{and} \quad \left[\frac{n_{p_{10}}}{n_{p_{13}}} m_{p_{13}}, \frac{n_{p_{10}}}{n_{p_{13}}} m_{p_{13}} + 1\right) = [543, 544)$$

respectively, where p_5 (resp. p_{13}) is the biggest p_2-satellite (resp. p_{10}-satellite) rupture point.

Finally, we must consider the satellite non-rupture points, which are p_3, p_6, p_{11} and p_{12}. In first place, both p_3 and p_6 are smaller than p_5, the biggest p_2-satellite rupture point, and hence we have

$$v_{p_3} = m_{p_3} = 245 \quad \text{and} \quad v_{p_6} = m_{p_6} = 348$$

because the second instance of step 3 applies. In second place, both p_{11} and p_{12} are smaller than p_{13}, which is the biggest p_{10}-satellite rupture point. Therefore we get

$$v_{p_{11}} = m_{p_{11}} = 1083 \quad \text{and} \quad v_{p_{12}} = m_{p_{12}} = 1628$$

by the same reason as above.

As in all the other examples, it is immediate to check that these values are the values of ξ at its singular points, as claimed.

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