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Introduction to Algebraic K-Theory

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Introduction

Algebraic K-theory is a subject of mathematics that serves, in a way, as a kind of cohomology theory for rings and other algebraic objects. It brings some of the tools of algebraic topology to the study of more abstract structures.

The "K" comes from the German word for class, *klasse*. K-theory has many formulations, and it is itself separated in different fields of mathematics, thus the word "algebraic" distinguishes algebraic K-theory, for example, from its topological counterpart, topological K-theory. All variants of K-theory are of course connected, being some of them generalizations of previous formulations that allow an abstract formulation of some of their properties, while others are focused on a concrete ambient that might add special tools or properties to the theory. Algebraic K-theory itself is used in different contexts, mostly the one regarding algebraic varieties and the one that studies rings.

We give here an introduction to algebraic K-theory focusing on the study of rings, although we will give some geometric interpretations and some relations to topological K-theory.

The field of K-theory, and algebraic K-theory in particular, was first introduced by Alexander Grothendieck in 1957, in the context of algebraic geometry, with the aim of formulating a generalization of the Riemann-Roch Theorem. Grothendieck himself realized that what he had defined, what we call the Grothendieck group K_0, was part of a much bigger theory. He intentionally indexed the K_0 group, since he believed it was part of a long sequence of groups K_n.

Shortly after Grothendieck introduced the K_0 group and proved the Grothendieck-Riemann-Roch Theorem, Michael Atiyah and Friedrich Hirzebruch used a similar construction applied to the category of vector bundles over a topological space X, starting with $K^0(X)$, and afterwards extending it to a complete cohomology theory.

The progress in topological K-theory motivated the further study of algebraic K-theory regarding rings, and the groups $K_1(R)$ and $K_2(R)$ were defined for a ring R. By that time, what we now understand as classical algebraic K-theory, i.e., the study of the properties of the lower K groups K_0, K_1 and

K_2, had become a consolidated area. Alongside many other results, a relation between the K^0 group of topological K-theory and the K_0 group of algebraic K-theory was proved by Swan, which was a topological formulation of a previous result due to Serre. Swan's Theorem gives an equivalence of the category of vector bundles over a (compact Hausdorff) topological space X and the category of finitely generated projective modules over the ring of continuous functions over X.

The study of K_0 and projective modules led Serre to the formulation of what we know as "Serre's problem" or "Serre's conjecture". Projective modules are those that are a direct summand of a free module, and the results on K_0 for rings showed that, if $R = \Bbbk[t_1, \ldots, t_n]$ is a polynomial ring over a field with n variables, then finitely generated projective R-modules P were stably free, i.e., $P \oplus R^n \cong R^m$ for some m, n, which is a consequence of the fact that $K_0(R) \cong \mathbb{Z}$ for this ring.

There are many cases of rings R for which $K_0(R) \cong \mathbb{Z}$, as we will see. In particular, for any ring R such that every finitely generated projective R-module is free, it holds $K_0(R) \cong \mathbb{Z}$. The converse is not true in general. Instead, we can only know that they are stably free as we already said. What Serre stated is that he did not know if finitely generated projective modules over a polynomial ring $\Bbbk[t_1, \ldots, t_n]$, with $\Bbbk$ a field, were actually free. The result was proved many years later outside the field of K-theory, but it has been an important problem that motivated the progress of algebraic K-theory, its formalization and an accurate analysis of the known subject.

It was not until 1972, when Daniel Quillen gave his formulation of higher algebraic K-theory. He gave a general definition of $K_n(R)$, which agreed with the previous definitions and extended the exact sequences given by the lower K groups.

We present here an introduction to algebraic K-theory, in particular we will study the Grothendieck group K_0. The background in algebraic geometry of the author is still insufficient, so we will study the K_0 group of a ring. This thesis consists of 5 chapters, which are summarized below.

Chapter 1 This chapter is a brief introduction to rings and modules, and some results in commutative algebra that we will need in the other chapters. We will see basic definitions and properties of rings and modules, including Nakayama's Lemma and flat modules; we will introduce localization and some of its properties, the definition and results regarding Noetherian rings, including Hilbert's Basis Theorem; and we will finish by defining the prime spectrum of a ring $\operatorname{Spec} R$ and seeing some basic properties.

Chapter 2 We give here some basic notions of category theory, focusing in the concept of abelian category. This chapter is mostly descriptive, in the sense that it consists essentially of definitions, since the aim of this part is to introduce the language of category theory, in order to use it later in the following chapters, especially in the last chapter, where the concept of quotient abelian category defined here will be used to formulate one of the main abstract theorems.

Chapter 3 In this chapter we introduce the concept of projective module and its basic properties and characterize them in terms of idempotent matrices. We will study the exactness of the functors $\mathrm{Hom}(P, -)$ and $P \otimes -$ when P is projective. Afterwards, we will see some specific cases: principal ideal domains and local rings; we will see that in both cases, finitely generated projective modules are actually free. Some basic concepts on projective resolutions and local rings are also included in this chapter. At the end of the chapter we will introduce the notion of vector bundle over a compact Hausdorff topological space X, and some results, including Swan's Theorem.

Chapter 4 The Grothendieck group $K_0(R)$ of a ring is introduced in this chapter. We will see some examples, including the cases of PIDs and local rings, and we will see the definition of $K^0(X)$ of topological K-theory, which is linked to algebraic K-theory by the Swan's Theorem. We give a proof of the additivity of the K_0 functor, and we use it to give a definition of $K_0(I)$ for a nonunital ring I, that agrees with the definition of K_0 of rings when I is also a ring. The last section of this chapter will give some definitions and results regarding the notion of a relative K_0 group of a ring R and an ideal I of R, the group $K_0(R, I)$. This section includes the proof of the Excision Theorem, which shows that $K_0(R, I)$ is isomorphic to $K_0(I)$.

Chapter 5 The last chapter brings the category theory back to our focus, by defining the K_0 group of a category with exact sequences, within the context of abelian categories. We will also see in this section the definition of the G_0 group of a ring. The second part of this chapter consists of the three main abstract theorems of K_0: we give in this section a proof of the Devissage Theorem, the Resolution Theorem and the Localization Theorem. We finish by giving Grothendieck's proof of the Fundamental Theorem of G_0 for a commutative ring and some of its consequences.

At the end of Chapter 5, we see the relation of K_0, in particular the Fundamental Theorem of G_0 for a field, with Serre's problem.

This thesis is intended to be as self-contained as possible. There is one remarkable exception: in Chapter 5, we will need some basic concepts and properties of homology, which are briefly explained, but they are not proved here. It is assumed the reader has enough knowledge on this matter, since every concept used regarding homology can be seen, for example, in an undergraduate algebraic topology course.

1

Rings and Modules

This chapter is a brief introduction to rings and modules and some of their properties that will be used in subsequent chapters. Some of the concepts of this chapter can be found in any graduate introductory commutative algebra course, and so some basic proofs are skipped in this section. Most concepts, results and proofs from this chapter can be found in [AtiMacD].

1.1 Rings and ideals

Basic concepts

Definition 1.1.1. A *ring* R is a set with two binary operations, usually written as $+$ for addition and $\cdot$ (or juxtaposition) for multiplication, such that:

i) R is an abelian group with respect to addition (with zero element 0).

ii) R is a monoid with respect to multiplication (with identity element 1).

iii) Multiplication is distributive over addition (i.e., $x(y+z) = xy + xz$, $(x+y)z = xz + yz$, $\forall x, y, z \in R$).

Note. A ring is called *commutative* if $xy = yx$, $\forall x, y \in R$. We shall consider every ring is commutative unless indicated otherwise. This is done to avoid tedious proofs, some of which can be inferred easily from the proofs in the commutative case.

Remark. It is possible that $1 = 0$ in a ring. In this case the ring must be 0. From now on, it is assumed that every ring R is not 0 unless stated otherwise.

Definition 1.1.2. A *ring homomorphism* between two rings R and R' is a map $f : R \longrightarrow R'$ such that ($\forall x, y \in R$):

i) $f(x+y) = f(x) + f(y)$.

ii) $f(xy) = f(x)f(y)$.

iii) $f(1) = 1$.

Notice that the composition of two ring homomorphism is trivially a ring homomorphism.

A subring S of a ring R is a subset of R that is also a ring. For most scenarios in ring theory, subrings are not useful. Instead, the notion of ideal as a subset of a ring is of important value.

Definition 1.1.3. An *ideal* I of a ring R is an additive subgroup of R such that $RI \subseteq I$ (i.e., $xy \in I$, $\forall x \in R$ and $\forall y \in I$).

Ideals allow the construction of quotient rings R/I in the sense that $\bar{x} = \bar{y} \iff x - y \in I$ (being $\bar{x} = \phi(x)$ given by the quotient map $\phi : R \longrightarrow R/I$). In other words, $\bar{x} = x + I$. This induces a ring structure in the quotient with $\bar{x} + \bar{y} = \overline{x+y}$ and $\bar{x}\bar{y} = \overline{xy}$ (this is well-defined by the properties of ideals).

The following proposition is a basic and important relation between the concepts of ideals and quotient rings.

Proposition 1.1.4. *There is a one-to-one correspondence between the ideals of R containing I and the ideals J of R/I given by $\phi^{-1}(J)$.* □

Some common concepts regarding rings and ideals are stated below.

Definition 1.1.5. i) An element $x \in R$ for which $\exists y \neq 0$ such that $xy = 0$ is called a *zero-divisor*.

ii) An element $x \in R$ such that $x^n = 0$ for some $n > 0$ is called *nilpotent*.

iii) An element $x \in R$ for which $\exists y \in R$ such that $xy = 1$ is called a *unit*. The element y is unique and is the inverse of x, noted x^{-1}. A unit cannot be a zero-divisor.

iv) A ring without zero-divisors (besides 0) is an *integral domain*.

v) A ring where every $x \neq 0$ is a unit, is called a *field*. Fields are integral domains.

vi) The set of all multiples ax ($a \in R$) of an element $x \in R$, denoted by (x) is a *principal ideal*.

vii) A *principal ideal domain*, or PID, is a ring that is an integral domain and such that all its ideals are principal.

Example 1.1.6. i) The ring $\mathbb{Z}$ is an integral domain. Moreover, it is a principal ideal domain.

ii) If R is an integral domain, the polynomial ring $R[t]$ is an integral domain (since the leading coefficient of the product of two polynomials is the product of their leading coefficients). For example, $\mathbb{Z}[t]$ is an integral domain.

iii) The ring $\mathbb{Z}/4\mathbb{Z}$ is not an integral domain, since $2 \cdot 2 \equiv 0 \pmod 4$.

iv) The ring $C^0([-1,1])$ of continuous $\mathbb{R}$-valued functions over the interval $[-1,1]$ is not an integral domain. For example, if $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$, then $f(x) \cdot f(-x) = 0$.

v) Every field $\mathbb{K}$ is a PID, since its only ideals are 0 and $\mathbb{K}$. Moreover the polynomial ring $\mathbb{K}[t]$ is a PID:

We know that $\mathbb{K}[t]$ is an integral domain. Let I be an ideal of $\mathbb{K}[t]$. If $I = 0$ or $I = \mathbb{K}[t]$, the ideal is finitely generated. Let d be a nonzero polynomial of I with minimum degree. If it has degree 0, then it is a unit and therefore $I = \mathbb{K}[t]$. Assume it has degree $n > 0$. Then, for every polynomial $f \in I$, we can divide f by d to obtain $f = d \cdot q + r$ where r has degree smaller than n. Since $r = f - d \cdot q \in I$, then r is necessarily 0 and therefore $I = (d)$.

The following definitions describe the most important notions of ideals:

Definition 1.1.7. i) An ideal $\mathfrak{p} \neq R$ is a *prime ideal* if for each pair $x, y \in R$ such that $xy \in \mathfrak{p}$ then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

ii) An ideal $\mathfrak{m} \neq R$ is *maximal* if there is no ideal I such that $\mathfrak{m} \subsetneq I \subsetneq R$.

Equivalently:

Proposition 1.1.8. *i)* $\mathfrak{p}$ *is prime* $\iff R/\mathfrak{p}$ *is an integral domain.*

ii) $\mathfrak{m}$ *is maximal* $\iff R/\mathfrak{m}$ *is a field.* □

Hence, maximal ideals are prime ideals.

Theorem 1.1.9. *Every ring R has at least one maximal ideal.*

Proof. Let $\{I_i\}$ be a chain in the set Σ of ideals different from R (not empty because $0 \in \Sigma$). Then $\bigcup I_i$ is an ideal different from R and it is an upper bound of the chain $\{I_i\}$. By Zorn's Lemma, there is a maximal element in Σ, i.e., a maximal ideal. □

As consequences of this theorem, we know that every ideal of a ring R is contained in a maximal ideal of R and also that every non-unit of R belongs to a maximal ideal of R.

Example 1.1.10. i) A nonzero ideal (p) of $\mathbb{Z}$ is prime if and only if p is prime. Moreover, it is a maximal ideal.

ii) In an integral domain that it is not a field, the zero ideal is prime, but it is not maximal.

Local rings

Definition 1.1.11. A ring R is a *local ring* if it has exactly one maximal ideal.

Proposition 1.1.12. *R is a local ring $\iff$ R has an ideal $\mathfrak{m}$ such that every element in $R - \mathfrak{m}$ is a unit.*

Proof. $\Longrightarrow$: Take its maximal ideal $\mathfrak{m}$, then the ideal generated by an element of $R - \mathfrak{m}$ is not included in $\mathfrak{m}$ and hence is R, so the element is a unit.

$\Longleftarrow$: If every element of $R - \mathfrak{m}$ is a unit, then the ideal it generates is R, so all proper ideals are contained in $\mathfrak{m}$ and hence is the only maximal ideal. □

Example 1.1.13. i) Every field is a local ring, with maximal ideal 0.

ii) If $\Bbbk$ is a field, then $\Bbbk[t]$ is not a local ring: suppose $\mathfrak{m}$ is its maximal ideal, then $t - a, t - b \in \mathfrak{m}$ with $a \neq b$, because they are not units, but then $(t - a) - (t - b) = b - a \in \mathfrak{m}$, which cannot be, since $b - a$ is a unit.

iii) If R is a local ring, the ring of formal power series $R[[t]]$ is local. Recall that an element of $R[[t]]$ is a unit if and only if its constant coefficient is a unit, hence, the ideal formed by the elements with constant coefficient in the maximal ideal of R is the only maximal ideal in $R[[t]]$.

Nilradical and Jacobson radical

The nilradical and Jacobson radical of a ring are two ideals that have special properties.

Definition 1.1.14. The *nilradical* $\mathcal{R}$ of a ring R is the set of all nilpotent elements of R.

Proposition 1.1.15. *i) The nilradical $\mathcal{R}$ of R is an ideal, and $R/\mathcal{R}$ has no nontrivial nilpotent elements.*

ii) $\mathcal{R}$ *is the intersection of all the prime ideals of* R.

Proof. i) If $x \in \mathcal{R}$, clearly $ax \in \mathcal{R}$. If $x, y \in \mathcal{R}$, then $(x+y)^n = 0$ for n large enough, by the binomial theorem, and thus $x + y \in \mathcal{R}$ and $\mathcal{R}$ is an ideal. Also, if $\bar{x} \in R/\mathcal{R}$, then $0 = \bar{x}^n = \overline{x^n} \implies x^n \in \mathcal{R} \implies x \in \mathcal{R} \implies \bar{x} = 0$.

ii) Let $\mathcal{R}'$ be the intersection of all prime ideals. If $x^n = 0 \in \mathfrak{p}$, then $x \in \mathfrak{p}$, for all $\mathfrak{p}$ prime, so $x \in \mathcal{R}'$.

Now, let $z \notin \mathcal{R}$. Let $\Sigma = \{I \subseteq R \text{ ideal} \mid z^n \notin I, \forall n > 0\}$. Zorn's Lemma can be applied to Σ once again, thus there is a maximal ideal $\mathfrak{p} \in \Sigma$. If $x, y \notin \mathfrak{p}$ then $\mathfrak{p} + (x), \mathfrak{p} + (y) \notin \Sigma \implies z^n \in \mathfrak{p} + (x)$ and $z^m \in \mathfrak{p} + (y)$ for some $m, n \implies z^{n+m} \in \mathfrak{p} + (xy) \implies xy \notin \mathfrak{p} \implies \mathfrak{p}$ is prime. Therefore, $z \notin \mathfrak{p} \implies z \notin \mathcal{R}'$.

□

Definition 1.1.16. The *Jacobson radical* $\mathfrak{R}$ of R is the intersection of all the maximal ideals of R.

Here is a useful characterization of the Jacobson radical:

Proposition 1.1.17. $x \in \mathfrak{R} \iff 1 - xy$ *is a unit of* R, $\forall y \in R$.

Proof. Suppose $x \in \mathfrak{R}$. If $1 - xy$ is not a unit then it belongs to a maximal ideal $\mathfrak{m}$ (consequence of Theorem 1.1.9), but $x \in \mathfrak{R} \subseteq \mathfrak{m} \implies xy \in \mathfrak{m} \implies 1 \in \mathfrak{m}$, which cannot be ($\mathfrak{m} \neq R$).

Now suppose $x \notin \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Then $\mathfrak{m} + (x) = R \implies z + xy = 1$ for some $z \in \mathfrak{m}$ and some $y \in R \implies 1 - xy \in \mathfrak{m} \implies 1 - xy$ is not a unit. □

1.2 Modules

Basic concepts and Nakayama's Lemma

Let R be a (commutative) ring.

Definition 1.2.1. An *R-module* is an abelian group $(M, +)$ and an action $R \times M \longrightarrow M$ such that $\forall r, s \in R$ and $\forall x, y \in M$:

i) $r(x+y) = rx + ry$

ii) $(r+s)x = rx + sx$

iii) $(rs)x = r(sx)$

iv) $1x = x$

A *set of generators* of an R-module M is a set $\{x_i\}_{i\in\mathcal{I}}$ such that every element of M can be expressed as $\sum_{i\in\mathcal{I}} r_i x_i$ for some $r_i \in R$. A module is *finitely generated* if it has a finite set of generators.

Definition 1.2.2. An *R-module homomorphism* between two R-modules M and N is a map $f : M \longrightarrow N$ such that $\forall x, y \in M$ and $\forall r \in R$:

i) $f(x + y) = f(x) + f(y)$

ii) $f(rx) = r \cdot f(x)$

Definition 1.2.3. A *submodule* M' of an R-module M is a subgroup of M that is closed under multiplication by elements of R.

Given an R-module M and a submodule M', the quotient M/M' of M by M' as abelian groups inherits an R-module structure given by $r\bar{x} = \overline{rx}$ for $x \in M$, $r \in R$ and being $\bar{x} \in M/M'$ the class of x.

An analogue result to Proposition 1.1.4 can be stated about this construction: there is a one-to-one correspondence between submodules of M/M' and submodules of M that contain M'. Observe that Proposition 1.1.4 is actually a special case of this statement, since ideals of a ring R are submodules of the R-module R.

Given an R-module homomorphism $f : M \longrightarrow N$, its kernel $\ker f$ is a submodule of M, its image $\operatorname{Im} f$ is a submodule of N, and its cokernel $\operatorname{coker} f := N/\operatorname{Im} f$ is a quotient module.

The usual isomorphism theorems are true for modules:

Proposition 1.2.4. *i) If $f : M \longrightarrow N$ is an R-module homomorphism, then $M/\ker f \cong \operatorname{Im} f$.*

ii) If $N \subseteq M \subseteq L$ are R-modules, then $(L/N)/(M/N) \cong L/M$.

iii) If M_1, M_2 are submodules of M, then $(M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2)$.

□

Example 1.2.5. i) If $\Bbbk$ is a field, then $\Bbbk$-modules are $\Bbbk$-vector spaces.

ii) Abelian groups are naturally $\mathbb{Z}$-modules.

Definition 1.2.6. The direct sum of a family of R-modules $(M_i)_{i\in\mathcal{I}}$, denoted by $\bigoplus_{i\in\mathcal{I}} M_i$, is the set of families $(x_i)_{i\in\mathcal{I}}$ with $x_i \in M_i$ such that all but a finite number of the x_i are 0 with the obvious induced addition and multiplication by scalar $((x_i)_{i\in\mathcal{I}} + (y_i)_{i\in\mathcal{I}} = (x_i + y_i)_{i\in\mathcal{I}}$ and $r \cdot (x_i)_{i\in\mathcal{I}} = (rx_i)_{i\in\mathcal{I}})$.

Note. If the condition over the x_i is not met, it defines a direct product, denoted by $\prod_{i\in\mathcal{I}} M_i$.

Definition 1.2.7. A free R-module is an R-module that is isomorphic to $\bigoplus_{i\in\mathcal{I}} M_i$, in which $M_i \cong R$, $\forall i \in \mathcal{I}$. It is finitely generated if $\mathcal{I}$ is finite. In this case, it is written as R^n where $|\mathcal{I}| = n$.

Proposition 1.2.8. *M is a finitely generated R-module $\Longleftrightarrow$ M is isomorphic to a quotient of R^n for some n.*

Proof. $\Longrightarrow$: If $x_1, \dots, x_n$ generate M, then $\phi : R^n \longrightarrow M$ defined by $\phi(r_1, ...r_n) = r_1x_1 + \cdots + r_nx_n$ is onto and therefore $M \cong R^n / \ker\phi$.

$\Longleftarrow$: If $M = R^n/M'$, and $e_1, \dots, e_n$ generate R^n, then $\phi(e_1), \dots, \phi(e_n)$ generate M, where $\phi : R^n \longrightarrow M$ is the quotient map. □

Theorem 1.2.9 (Nakayama's Lemma)**.** *Let M be a finitely generated R-module and I an ideal of R contained in its Jacobson radical $\mathfrak{R}$. Then $IM = M$ $\Longrightarrow M = 0$.*

Proof. Suppose $M \neq 0$, and let $x_1, \dots, x_n$ be a minimal set of generators. Then $x_n \in M = IM \Longrightarrow$ there are $r_1, \dots, r_n \in I$ such that:

$$x_n = r_1x_1 + \cdots + r_nx_n$$

Hence

$$(1 - r_n)x_n = r_1x_1 + \cdots + r_{n-1}x_{n-1}$$

But $r_n \in I \subseteq \mathfrak{R} \Longrightarrow 1 - r_n$ is a unit by Proposition 1.1.17 and therefore x_n is a linear combination of $x_1, \dots, x_{n-1}$, which is a contradiction. □

Corollary 1.2.10. *Let M be an R-module and $I \subseteq \mathfrak{R}$ an ideal. Let $\bar{x_1}, \dots, \bar{x_n} \in M/IM$ be the classes of $x_1, \dots, x_n \in M$. Then $\bar{x_1}, \dots, \bar{x_n}$ generate $M/IM \Longrightarrow$ $x_1, \dots, x_n$ generate M.*

Proof. Let N be the submodule of M generated by $x_1, \dots, x_n$. The composition map $N \hookrightarrow M \to M/IM$ is onto, so $N + IM = M$. Hence $M/N = (N+IN)/N \cong I(M/N)$ by Proposition 1.2.4 *iii)* and, by Nakayama's Lemma, $M/N = 0$, which implies $M = N$. □

This result is useful for a local ring R with maximal ideal $\mathfrak{m}$, since in this case $\mathfrak{m} = \mathfrak{R}$ and $M/\mathfrak{m}M$ is an $R/\mathfrak{m}$-module and $R/\mathfrak{m}$ is a field, hence the module structure becomes a vector space structure and so the previous result is reduced to the case of checking if a set of vectors form a base of a vector space.

Exactness and tensor product. Flat modules

The notion of exact sequence is important to study some properties of rings and modules.

Definition 1.2.11. A sequence of R-modules and homomorphisms

$$\cdots \longrightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots$$

such that $d_k d_{k+1} = 0$, $\forall k \in \mathbb{Z}$, is *exact at* M_i if $\ker d_i = \operatorname{Im} d_{i+1}$. The sequence is *exact* if it is exact at each M_i.

Remark. A sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is exact if f is injective, g is surjective, and $\ker g = \operatorname{Im} f$. This is called a *short exact sequence.*

One important construction regarding modules is the tensor product.

Definition 1.2.12. Let M, N be R-modules. The *tensor product* of M and N is an R-module T and an R-bilinear map $g : M \times N \longrightarrow T$ such that for every R-bilinear map $f : M \times N \longrightarrow P$ (with P an R-module) there exists a unique R-module homomorphism $f' : T \longrightarrow P$ such that $f = f' \circ g$.

This R-module T is usually denoted by $M \otimes_R N$. It can be constructed by imposing R-bilinear relations to the free R-module over $M \times N$. The pairs given by the map g are written as $x \otimes y$ with $x \in M$ and $y \in N$. If there is another R-module T' with a map g' satisfying the definition, then there is a unique isomorphism $j : T \longrightarrow T'$ such that $j \circ g = g'$.

If $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ are R-module homomorphisms, then they induce a homomorphism $f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N'$ defined by $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$. Also, if we have $f' : M' \longrightarrow M''$ and $g' : N' \longrightarrow N''$, then $((f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g))$.

Remark. This definition of $M \otimes_R N$ can be extended to define a tensor product of any finite number of R-modules by imposing R-multilinear relations. It can also be constructed by making consecutive tensor products, and both definitions give the same construction up to isomorphism.

Definition 1.2.13. Let $f : R \longrightarrow S$ be a ring homomorphism and M an R-module. Then there is a module $M_S = S \otimes_R M$, where S is seen as an R-module with the action $r \cdot s = f(r)s$ (for $r \in R$ and $s \in S$). This module can be seen as an S-module with action $s \cdot (s' \otimes r) = ss' \otimes r$ (for $r \in r$ and $s, s' \in S$), and it is said to be obtained by *extension of scalars*.

The following proposition will be proved in Chapter 3, since the concepts of next chapters will help us write a clearer proof.

Proposition 1.2.14. *Let*

$$0 \longrightarrow M' \xrightarrow{\ f\ } M \xrightarrow{\ g\ } M'' \longrightarrow 0$$

be a short exact sequence of R-modules, and let N be an R-module. Then

$$M' \otimes_R N \xrightarrow{f \otimes \mathrm{id}} M \otimes_R N \xrightarrow{g \otimes \mathrm{id}} M'' \otimes_R N \longrightarrow 0$$

is exact.

The following example shows how an injective map $f : M' \longrightarrow M$ can induce a map that is not injective.

Example 1.2.15. Let $R = \mathbb{Z}$, $N = \mathbb{Z}/2\mathbb{Z}$, and $f : 2\mathbb{Z} \hookrightarrow \mathbb{Z}$. Then, $f \otimes \mathrm{id} : 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is not injective since the element $2 \otimes x \in 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ maps to $2 \otimes x = 1 \otimes 2x = 1 \otimes 0 = 0$.

There are modules N that do induce injective homomorphisms from injective homomorphisms:

Definition 1.2.16. An R-module N is said to be *flat* if for all short exact sequences

$$0 \longrightarrow M' \xrightarrow{\ f\ } M \xrightarrow{\ g\ } M'' \longrightarrow 0$$

the sequence

$$0 \longrightarrow M' \otimes_R N \xrightarrow{f \otimes \mathrm{id}} M \otimes_R N \xrightarrow{g \otimes \mathrm{id}} M'' \otimes_R N \longrightarrow 0$$

is exact.

Remark. In the language of category theory, it is said that the functor T_N, defined by $T_N : M \mapsto M \otimes_R N$ and $T_N : f \mapsto f \otimes \mathrm{id}$, is in general right exact, and it is exact when N is flat.

Example 1.2.17. R^n is a flat R-module. To see this, notice that for any R-module M, we have $R^n \otimes_R M \cong (R \oplus \overset{n}{\cdots} \oplus R) \otimes_R M \cong (R \otimes_R M) \oplus \overset{n}{\cdots} \oplus (R \otimes_R M)$ and every element of $R \otimes_R M$ is of the form $\sum_i r_i \otimes m_i = \sum_i 1 \otimes r_i m_i = 1 \otimes (\sum r_i m_i)$, so $R \otimes_R M \cong M$ and therefore $R^n \otimes_R M \cong M^n$. If $f : M \longrightarrow N$ is an R-module homomorphism, the induced morphism when tensoring by R^n is $f \oplus \overset{n}{\cdots} \oplus f$. Therefore R^n is flat.

Example 1.2.18. $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. In particular, $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/d\mathbb{Z}) = 0$, and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$. These facts are a consequence of $\mathbb{Q}$ being the field of fractions of $\mathbb{Z}$, as seen in the next section.

1.3 Rings and modules of fractions

Let R be a (commutative) ring. Let S be a multiplicatively closed subset of R (i.e., it is closed under multiplication and $1 \in S$).

Proposition 1.3.1. *The relation $\equiv$ defined in $R \times S$ by*

$$(r, s) \equiv (r', s') \Longleftrightarrow \exists u \in S : (rs' - r's)u = 0$$

is an equivalence relation.

Proof. The relation is clearly reflexive and symmetric. To see that it is transitive, suppose $(r, s) \equiv (r', s')$ and $(r', s') \equiv (r'', s'')$, so $(rs' - r's)u = 0$ and $(r's'' - r''s')u' = 0$ for some $u, u' \in S$. Multiplying the first expression by $u's''$, the second by us and adding them, we get $uu's'(rs'' - r''s) = 0$, so $(r, s) \equiv (r'', s'')$. □

Definition 1.3.2. The set $S^{-1}R$ of equivalence classes (each denoted by r/s for a class with representative (r, s)), defined by the relation $\equiv$, is called the *ring of fractions* of R with respect to S. It has a ring structure given by $r/s + r'/s' = (rs' + r's)/ss'$ and $r/s \cdot r'/s' = rr'/ss'$.

Remark. If R is an integral domain, then for $S = R - \{0\}$, $S^{-1}R$ is a field, called the *field of fractions* of R.

Notice that if $0 \in S$, then $S^{-1}R = 0$, so most times we will use multiplicatively closed subsets S not containing 0.

Proposition 1.3.3. *Let $f : R \longrightarrow S^{-1}R$ be the ring homomorphism defined by $f(r) = r/1$ and let $g : R \longrightarrow R'$ be a ring homomorphism such that $g(s)$ is a unit $\forall s \in S$. Then there is a unique homomorphism $h : S^{-1}R \longrightarrow R'$ such that $g = h \circ f$.*

Proof. This result is proved by checking that $h(r/s) = g(r)g(s)^{-1}$ is well-defined and that it is the only homomorphism satisfying the stated condition. □

Example 1.3.4. An important case of a ring of fractions is when $S = R - \mathfrak{p}$ with $\mathfrak{p}$ a prime ideal of R. In this case, $S^{-1}R$ is denoted by $R_{\mathfrak{p}}$. Note that the elements r/s with $r \in \mathfrak{p}$ form an ideal $\mathfrak{m}$ and if $r'/s' \notin \mathfrak{m}$ then $r' \notin \mathfrak{p} \Longrightarrow r' \in S \Longrightarrow r'/s'$ is a unit and so, any ideal containing r'/s' must be $R_{\mathfrak{p}}$. Therefore, all proper ideals are contained in $\mathfrak{m}$, i.e, $R_{\mathfrak{p}}$ has only one maximal ideal ($R_{\mathfrak{p}}$ is a local ring). The construction of this ring of fractions is called *localization at* $\mathfrak{p}$.

This construction can easily be extended to R-modules:

Definition 1.3.5. Let M be an R-module. The relation $\equiv$ defined in $M \times S$ by

$$(m,s) \equiv (m',s') \iff \exists u \in S : (ms' - m's)u = 0$$

is an equivalence relation. The set $S^{-1}M$ of equivalence classes (m/s) is the *module of fractions* of M with respect to S. It has an $S^{-1}R$-module structure given by $m/s + m'/s' = (ms' + m's)/ss'$ and $(r/s') \cdot (m/s) = rm/ss'$.

If $f : M \longrightarrow N$ is an R-module homomorphism, then there is an induced homomorphism $S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$ defined by $(S^{-1}f)(m/s) = f(m)/s$, and if we have $g : N \longrightarrow P$ then $s^{-1}(g \circ f) = (S^{-1}g) \circ (S^{-1}f)$. This defines an operation S^{-1}, which is actually exact:

Proposition 1.3.6. *For all short exact sequences*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

the sequence

$$0 \longrightarrow S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M'' \longrightarrow 0$$

is exact. This is, the operation S^{-1} is exact.

Proof. Exactness at $S^{-1}M'$: $m_1'/s_1 \neq m_2'/s_2 \iff u(m_1's_2 - m_2's_1) \neq 0 \ \forall u \in S$. Since f is injective, $0 \neq f(u(m_1's_2 - m_2's_1)) = u(f(m_1')s_2 - f(m_2')s_1) \ \forall u \in S$ $\Longrightarrow (S^{-1}f)(m_1'/s_1) = f(m_1')/s_1 \neq f(m_2')/s_2 = (S^{-1}f)(m_2'/s_2) \Longrightarrow S^{-1}f$ is injective.

Exactness at $S^{-1}M$: $g \circ f = 0 \Longrightarrow (S^{-1}g) \circ (S^{-1}f) = S^{-1}(g \circ f) = S^{-1}(0) = 0$ $\Longrightarrow \operatorname{Im} S^{-1}f \subseteq \ker S^{-1}g$. Now, take $m/s \in \ker S^{-1}g$, then $(S^{-1}g)(m/s) = g(m)/s = 0$, which, by using the definition, is equivalent to $t\,g(m) = 0$ for some $t \in S$. Hence, $0 = t\,g(m) = g(tm) \Longrightarrow \exists m' \in M'$ such that $f(m') = tm$. Therefore, $(S^{-1}f)(m'/ts) = f(m')/ts = tm/ts = m/s \Longrightarrow \operatorname{Im} S^{-1}f \supseteq \ker S^{-1}g \Longrightarrow \operatorname{Im} S^{-1}f = \ker S^{-1}g$.

Exactness at $S^{-1}M''$: for $m''/s \in S^{-1}M''$, $\exists m \in M$ such that $g(m) = m''$, now $(S^{-1}g)(m/s) = g(m)/s = m''/s$. □

Proposition 1.3.7. *Let M be an R-module. Then $S^{-1}M \cong S^{-1}R \otimes_R M$.*

Proof. Let $f : S^{-1}R \otimes_R M \longrightarrow S^{-1}M$ be a homomorphism defined by $f(r/s \otimes m) = rm/s$. Clearly f is onto.

Any element of $S^{-1}R \otimes_R M$ can be written as $\sum_i r_i/s_i \otimes m_i$. Let $s = \prod_i s_i$ and $t_i = \prod_{j \neq i} s_j$, then

$$\sum_i r_i/s_i \otimes m_i = \sum_i t_i r_i/s \otimes m_i = \sum_i 1/s \otimes t_i r_i m_i = 1/s \otimes \sum_i t_i r_i m_i = 1/s \otimes m$$

with $m = \sum_i t_i r_i m_i \in M$. So, every element of $S^{-1}R \otimes_R M$ can be expressed as $1/s \otimes m$ for some $s \in S$ and $m \in M$.

Then $f(1/s \otimes m) = 0 \implies m/s = 0 \implies tm = 0$ for some $t \in S \implies 1/s \otimes m = t/st \otimes m = 1/st \otimes tm = 0$, hence f is injective and thus an isomorphism. $\square$

By Proposition 1.3.6 and Proposition 1.3.7:

Corollary 1.3.8. *$S^{-1}R$ is flat as an R-module.*

Example 1.3.9. As stated above, $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}$, hence it is a flat $\mathbb{Z}$-module by Proposition 1.3.6. Also, notice that every element of $\mathbb{Z}/d\mathbb{Z}$ is annihilated by d, so $\mathbb{Q}\otimes_{\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}) = (\frac{1}{d}\cdot d\cdot\mathbb{Q})\otimes_{\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}) = (\frac{1}{d}\cdot\mathbb{Q})\otimes_{\mathbb{Z}}(d\cdot\mathbb{Z}/d\mathbb{Z}) = 0$.

Let $f : R \longrightarrow S^{-1}R$ be the ring homomorphism defined by $f(r) = r/1$. Notice that if $I \subseteq R$ is an ideal, then the ideal generated by $f(I)$ is actually $S^{-1}I$, since every element in the ideal generated by $f(I)$ is of the form $\sum_i r_i/s_i$ with $r_i \in I$, which can be rewritten with a common denominator.

Proposition 1.3.10. *i) Every ideal in $S^{-1}R$ is generated by $f(I)$ for some ideal I of R.*

ii) The prime ideals of $S^{-1}R$ are in one-to-one correspondence with the prime ideals of R that don't meet S.

Proof. i) Let I' be an ideal of $S^{-1}R$ and $I = f^{-1}(I')$, which is an ideal of R. Notice that the ideal generated by $f(I)$ is included in I'. Let $x/s \in I'$, then $x/1 = (s/1)(x/s) \in I'$ and so $x \in I$. Hence, x/s belongs to the ideal generated by $f(I)$, and therefore I' is the ideal generated by $f(I)$.

ii) If $\mathfrak{p}'$ is a maximal ideal of $S^{-1}R$ then $f^{-1}(\mathfrak{p}')$ is a prime ideal in R. If $f^{-1}(\mathfrak{p}') \cap S \neq \varnothing$, take $s \in f^{-1}(\mathfrak{p}') \cap S$, then $f(s) = s/1 \in \mathfrak{p}' \implies 1 = (1/s)\cdot(s/1) \in \mathfrak{p}'$ which is absurd.

If $\mathfrak{p}$ is a prime ideal of R, notice that the sequence of R-modules

$$0 \longrightarrow \mathfrak{p} \hookrightarrow R \longrightarrow R/\mathfrak{p} \longrightarrow 0$$

is exact, and by Proposition 1.3.6,

$$0 \longrightarrow S^{-1}\mathfrak{p} \hookrightarrow S^{-1}R \longrightarrow S^{-1}(R/\mathfrak{p}) \longrightarrow 0$$

is also exact. Then, $S^{-1}R/S^{-1}\mathfrak{p} \cong S^{-1}(R/\mathfrak{p}) \cong \bar{S}^{-1}(R/\mathfrak{p})$, where $\bar{S}$ is the image of S in $R/\mathfrak{p}$. The last isomorphism is true because $S\cap\mathfrak{p} = \varnothing$ (which also implies that $\bar{S}^{-1}(R/\mathfrak{p})$ is not 0). Then, $\bar{S}^{-1}(R/\mathfrak{p})$ is contained in the field of fractions of $R/\mathfrak{p}$ and thus it is an integral domain. Hence, $S^{-1}\mathfrak{p}$ is prime.

□

The notion of localizing at a prime ideal $\mathfrak{p}$ can also be done with R-modules. If M is an R-module, then localizing at $\mathfrak{p}$ is denoted by $M_{\mathfrak{p}}$. Localizing at a maximal ideal is denoted in the same way.

Proposition 1.3.11. *Let M be an R-module. Then the following are equivalent:*

i) $M = 0$

ii) $M_{\mathfrak{p}} = 0$ *for every prime ideal* $\mathfrak{p}$.

iii) $M_{\mathfrak{m}} = 0$ *for every maximal ideal* $\mathfrak{m}$.

Proof. It is clear that *i)* $\Longrightarrow$ *ii)* $\Longrightarrow$ *iii)*. Now suppose *iii)* holds. Let $m \in M$ be a nonzero element, and $\mathfrak{a} = \{a \in R \mid am = 0\}$, which is a proper ideal, and hence is contained in a maximal ideal $\mathfrak{m}$. Let $m/1 \in M_{\mathfrak{m}} = 0$, so $m/1 = 0$, and therefore $sm = 0$ for some $s \in R - \mathfrak{m}$, but $s \in \mathfrak{a} \subseteq \mathfrak{m}$, which is a contradiction. Hence, nonzero elements of M do not exist and $M = 0$. □

1.4 Noetherian rings

Let R be a commutative ring.

The following proposition characterizes a special kind of modules, and will be used to introduce the concept of Noetherian ring.

Proposition 1.4.1. *Let M be an R-module. Then the following conditions are equivalent:*

i) Every submodule N of M is finitely generated.

ii) For every chain

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_i \subseteq \cdots$$

of submodules of M, $\exists n$ such that $N_n = N_{n+1} = \cdots$. This is, every ascending chain is stationary. This property is called the ascending chain condition.

iii) Every nonempty set Σ of submodules of M, ordered by inclusion, has a maximal element.

Proof. i) $\Longrightarrow$ *ii)*: Let

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_i \subseteq \cdots$$

be an ascending chain of submodules of N. Then $N = \bigcup_{i\geq 1} N_i$ is a submodule of M, and so it is finitely generated. Let $x_1, \ldots, x_r$ be a set of generators of N. Then each x_i is in N_{n_i} for some n_i. Take $n = \max_{i=1}^r n_i$, so $N_n = N$, and therefore $N_n = N_{n+1} = \cdots$.

ii) $\Longrightarrow$ *iii)*: If *iii)* was false, then there is a set of submodules Σ that has no maximal element. Take $N_1 \in \Sigma$, then $\exists N_2 \in \Sigma$ such that $N_1 \subsetneq N_2$. Applying the same argument to N_2 and so on, we get an ascending chain that is not stationary.

iii) $\Longrightarrow$ *i)*: Let N be a submodule of N and Σ the set of all finitely generated submodules of N. Σ is not empty since $0 \in \Sigma$, so it has a maximal element N_0. If $N_0 \neq N$, take $x \in N \setminus N_0$, then $N_0 + (x)$ is finitely generated and $N_0 \subsetneq N_0 + (x)$, which is a contradiction. Therefore, $N_0 = N$, so N is finitely generated. □

Definition 1.4.2. An R-module M is *Noetherian* if it satisfies the equivalent conditions of Proposition 1.4.1.

Definition 1.4.3. A ring R is said to be *Noetherian* if it is a Noetherian R-module. Equivalently, a ring R is Noetherian if any ascending chain of ideals stabilizes, if every ideal of R is finitely generated or if any nonempty set of ideals, ordered by inclusion, has a maximal element.

The Noetherian property has good stability properties, as the succeeding results show.

Proposition 1.4.4. *Let*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence of R-modules. Then M is Noetherian $\Longleftrightarrow$ M' and M'' are Noetherian.

Proof. $\Longrightarrow$: If

$$N'_1 \subseteq N'_2 \subseteq \cdots \subseteq N'_i \subseteq \cdots$$

is an ascending chain of submodules of M', then

$$\alpha(N'_1) \subseteq \alpha(N'_2) \subseteq \cdots \subseteq \alpha(N'_i) \subseteq \cdots$$

is an ascending chain of submodules of M, hence $\alpha(N'_n) = \alpha(N'_{n+1}) = \cdots$, but α is injective, hence $N'_n = N'_{n+1} = \cdots$.

If

$$N_1'' \subseteq N_2'' \subseteq \cdots \subseteq N_i'' \subseteq \cdots$$

is an ascending chain of submodules of M'', then

$$\beta^{-1}(N_1'') \subseteq \beta^{-1}(N_2'') \subseteq \cdots \subseteq \beta^{-1}(N_i'') \subseteq \cdots$$

is an ascending chain of submodules of M, hence $\beta^{-1}(N_n'') = \beta^{-1}(N_{n+1)=\cdots}''$, but then $N_n'' = N_{n+1}'' = \cdots$.

$\Longleftarrow$: Suppose

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_i \subseteq \cdots$$

is an ascending chain of submodules of M that does not stabilize. Since M'' is Noetherian,

$$\beta(N_1) \subseteq \beta(N_2) \subseteq \cdots \subseteq \beta(N_i) \subseteq \cdots$$

stabilizes, this is, $\beta(N_n) = \beta(N_{n+1}) = \cdots$. But then, the elements of $N_i \setminus N_n$ for $i > n$ are in $\ker \beta$, hence in the image of α, and taking the preimage, they generate an ascending chain of submodules of M' that does not stabilize, which is a contradiction. □

Corollary 1.4.5. *If $M_1, \ldots, M_n$ are Noetherian R-modules, then $\bigoplus_{i=1}^n M_i$ is Noetherian.*

Proof. By induction on n:

M_1 is Noetherian. Suppose $\bigoplus_{i=1}^{n-1} M_i$ is Noetherian. By Proposition 1.4.4 applied to

$$0 \longrightarrow M_n \hookrightarrow \bigoplus_{i=1}^{n} M_i \longrightarrow \bigoplus_{i=1}^{n-1} M_i \longrightarrow 0,$$

we have that $\bigoplus_{i=1}^n M_i$ is Noetherian. □

Corollary 1.4.6. *Let R be a Noetherian ring and M a finitely generated R-module. Then, M is Noetherian. This is, submodules of finitely generated R-modules are finitely generated if R is Noetherian.*

Proof. We have a short exact sequence

$$0 \longrightarrow \ker \pi \hookrightarrow R^n \xrightarrow{\pi} M \longrightarrow 0$$

for some n, because M is finitely generated. By Corollary 1.4.5, R^n is Noetherian, therefore by Proposition 1.4.4, M is Noetherian. □

Corollary 1.4.7. *Let R be a Noetherian ring and $I \subseteq R$ an ideal. Then R/I is a Noetherian ring.*

Proof. The short exact sequence

$$0 \longrightarrow I \hookrightarrow R \longrightarrow R/I \longrightarrow 0$$

shows that R/I is Noetherian as an R-module, but submodules of R/I are R/I-modules, hence R/I is Noetherian as a ring. □

The following properties show that if R is Noetherian, localization preserves the Noetherian property; and the polynomial ring $R[t]$ is Noetherian as well. These results will be used in subsequent chapters.

Proposition 1.4.8. *If R is a Noetherian ring and S is a multiplicatively closed subset of R, then $S^{-1}R$ is Noetherian.*

Proof. By Proposition 1.3.10 i), the ideals of $S^{-1}R$ are $S^{-1}I$ where I is an ideal of R. But then I is finitely generated, say, by elements $x_1, \dots, x_r$. Therefore, $S^{-1}I$ is generated by $x_1/1, \dots, x_r/1$. □

Theorem 1.4.9 (Hilbert's Basis Theorem)**.** *Let R be a Noetherian ring. Then, the polynomial ring $R[t]$ is Noetherian.*

Proof. Let I be an ideal of $R[t]$. Let L be the ideal formed by the leading coefficients of the polynomials in I. Since R is Noetherian, L is finitely generated, say, by elements $a_1, \dots, a_n$. Let f_i be polynomials in I with leading coefficient a_i and degree r_i and let r be the maximum of the r_i. The f_i generate an ideal $I' \subseteq I$.

Let $f \in I$ be a polynomial of degree $m \geq r$, and a its leading coefficient. Then $a = \sum u_i a_i$ with $u_i \in R$. Therefore $f - \sum u_i f_i x^{m-r_i}$ is in I and has degree $m_0 < m$. This procedure can be repeated until we have a polynomial f' of degree $m' < r$. Therefore $f = f' + g$, with $g \in I'$.

Let M be the R-module generated by $1, x, \dots, x^{r-1}$, which is Noetherian because R is Noetherian. Then $I \cap M \subseteq M$ is finitely generated, say, by elements $f_1', \dots, f_k'$. Then $I = (I \cap M) + I'$ is generated by $f_1, \dots, f_n, f_1', \dots, f_k'$, i.e., it is finitely generated, and so $R[t]$ is Noetherian. □

Applying the result multiple times, it follows that polynomial rings with a finite number of variables are Noetherian if the ring is Noetherian.

Corollary 1.4.10. *Let R be a Noetherian ring. Then, the polynomial ring $R[t_1, \dots, t_n]$ is Noetherian.* □

Example 1.4.11. i) Any field is a Noetherian ring, since its only ideals are 0 and itself.

ii) Moreover, any PID is a Noetherian ring, since every ideal is generated by one element.

iii) The polynomial ring $R[t_1, t_2, \dots]$ with infinite variables is not Noetherian, since
$$(t_1) \subseteq (t_1, t_2) \subseteq (t_1, t_2, t_3) \subseteq \cdots$$
is an ascending chain that does not stabilize.

1.5 Prime spectrum of a ring

Let R be a (commutative) ring.

Definition 1.5.1. The *prime spectrum of* R, denoted by $\operatorname{Spec} R$ is the set of all prime ideals of R.

Proposition 1.5.2. $\operatorname{Spec} R$ *can be equipped with a topology, in which its closed sets are defined by* $V(E) = \{\mathfrak{p} \in \operatorname{Spec} R \mid E \subseteq \mathfrak{p}\}$ *for any subset* $E \subseteq R$. *In particular:*

i) $V(E) = V(I)$, *where* I *is the ideal generated by* E.

ii) $V(0) = \operatorname{Spec} R$, $V(1) = \varnothing$.

iii) $V\left(\bigcup_{i\in\mathcal{I}} E_i\right) = \bigcap_{i\in\mathcal{I}} V(E_i)$.

iv) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$ *for any* $I, J \in R$ *ideals.*

This topology is called the Zariski topology. □

Proposition 1.5.3. *Let* $r \in R$ *and* $X_r = \operatorname{Spec} R \setminus V(r)$. *The sets* X_r *form a basis of open sets for the Zariski topology, and:*

i) $X_r \cap X_s = X_{rs}$.

ii) $X_r = \varnothing \iff r$ *is nilpotent.*

iii) $X_r = \operatorname{Spec} R \iff r$ *is a unit.*

iv) $\operatorname{Spec} R$ *is quasi-compact.*

Proof. i), *ii)* and *iii)* are easy.

For *iv)*, notice that $\bigcup_{i\in\mathcal{I}} X_{r_i} = \operatorname{Spec} R \setminus V(\{r_i\}_{i\in\mathcal{I}})$ (by Proposition 1.5.2 *iii)*), then every open set $\operatorname{Spec} R \setminus V(E)$ can be thought as the union of all X_r with $r \in E$ (this also proves that the X_r form a basis of open sets), so each open covering of $\operatorname{Spec} R$ can be expressed as $\bigcup_{i\in\mathcal{I}} X_{r_i}$. Now, $\bigcup_{i\in\mathcal{I}} X_{r_i} = \operatorname{Spec} R \iff \operatorname{Spec} R \setminus V(\{r_i\}_{i\in\mathcal{I}}) = \operatorname{Spec} R \iff V(\{r_i\}_{i\in\mathcal{I}}) = \varnothing$ so the ideal generated by $\{r_i\}_{i\in\mathcal{I}}$ is R (otherwise it would be contained in a maximal ideal, that is prime). Hence, $1 = \sum_{i\in\mathcal{J}} s_i r_i$ with $s_i \in R$ and $\mathcal{J}$ a finite subset of $\mathcal{I}$. Therefore $\operatorname{Spec} R = \operatorname{Spec} R \setminus V(\{r_i\}_{i\in\mathcal{J}}) = \bigcup_{i\in\mathcal{J}} X_{r_i}$, which is a finite subcover. □

To better explain the geometric interpretation of $\operatorname{Spec} R$, let us introduce first a motivating example.

Example 1.5.4. Let X be a topological space, and $x \in X$. Let $C^0(X)$ be the set of $\mathbb{R}$-valued continuous functions over X.

We want to focus attention on x. To do this, we define the following ring:

$$C_x := \{f : U \longrightarrow \mathbb{R} \mid U \text{ neighbourhood of } x,\ f \text{ continuous}\}/\sim,$$

where $\sim$ is the relation defined by: $f \sim g \iff \exists\, W \subseteq U \cap V$ such that $f|_W = g|_W$, with $f : U \longrightarrow \mathbb{R}$ and $g : V \longrightarrow \mathbb{R}$. Addition and multiplication are defined pointwise in a suitable neighbourhood, and they are well-defined since all elements of an equivalence class are equal locally, near x.

We claim that C_x is a local ring. To see this, let $\mathfrak{m}_x = \{f \in C_x \mid f(x) = 0\} \subseteq C_x$. Notice that if $g \notin \mathfrak{m}_x$, then $g(x) \neq 0$, and by continuity, it is not zero in a neighbourhood U of x. Thus, $\exists \frac{1}{g} \in C_x$, since it is defined in U. By Proposition 1.1.12, C_x is a local ring and $\mathfrak{m}_x$ its maximal ideal.

Consider the quotient map $C_x \longrightarrow C_x/\mathfrak{m}_x$. Notice that if $f, g \in C_x$ are such that $f = g$ in $C_x/\mathfrak{m}_x$, then $(f-g)(x) = 0$, this is, $f(x) = g(x)$. Therefore the quotient map can be thought as evaluating a function f at x, and $C_x/\mathfrak{m}_x \cong \mathbb{R}$:

$$\begin{array}{ccc} C_x & \longrightarrow & C_x/\mathfrak{m}_x \cong \mathbb{R} \\ f & \longmapsto & f(x) \end{array}$$

To see the geometric notion of $\operatorname{Spec} R$, we must see R as a ring of "functions" over a topological space $\operatorname{Spec} R$. Now, for $\mathfrak{p} \in \operatorname{Spec} R$, we focus on the point $\mathfrak{p}$ precisely by localizing at $\mathfrak{p}$, and the quotient map is now

$$\begin{array}{ccc} R_{\mathfrak{p}} & \longrightarrow & R_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} =: \Bbbk(\mathfrak{p}) \\ f & \longmapsto & f(\mathfrak{p}) \end{array}$$

What makes it different from the given example is that now the field $\Bbbk(\mathfrak{p})$ is different at each point $\mathfrak{p}$, so the "functions" of R are evaluated on different fields at each point.

2

Categories

In this chapter, we introduce some definitions and properties of category theory, with a special focus on abelian categories. Most parts of this chapter can be seen in detail in [MacL]. Basic concepts on categories are also found in [Awod] and in Appendix A of [WeibHA]. For abelian categories and quotient abelian categories, [Groth] and [WeibK] are suitable sources. Nevertheless, for a detailed definition of abelian quotient categories, one should read [Swan].

2.1 Categories, functors and natural transformations

Categories

Definition 2.1.1. A *category* $\mathcal{C}$ consists of:

i) A class $\operatorname{Obj}\mathcal{C}$ of *objects*. It is usual to write $C \in \mathcal{C}$ instead of $C \in \operatorname{Obj} C$.

ii) A set $\operatorname{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* for every ordered pair (A, B) of objects. The objects A and B are called *domain* and *codomain* respectively.

iii) An *identity morphism* $\mathrm{id}_A \in \operatorname{Hom}_{\mathcal{C}}(A, A)$, $\forall A \in \operatorname{Obj}\mathcal{C}$.

iv) A *composition* function $\operatorname{Hom}_{\mathcal{C}}(A, B) \times \operatorname{Hom}_{\mathcal{C}}(B, C) \to \operatorname{Hom}_{\mathcal{C}}(A, C)$ for every ordered triple (A, B, C) of objects.

Subject to the following axioms:

a) *Associativity axiom*: $(hg)f = h(gf)$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$.

b) *Unit axiom*: $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Remark. A category $\mathcal{C}$ is called *small* if $\mathrm{Obj}\,\mathcal{C}$ is a set.

Example 2.1.2. The following are examples of categories.

i) The category **Set** that has sets as objects and functions as morphisms is the most common example.

ii) **Ab**, the category of abelian groups and group homomorphisms.

iii) R-**mod**, the category of R-modules and R-module homomorphisms.

iv) The category **Top** of topological spaces and continuous maps.

v) A partially ordered set, or poset, I with order relation $\leq$, can be thought as a category, with a unique morphism between i and j whenever $i \leq j$. The identity morphisms are given by the reflexive property, and compositions are well-defined by the transitive property.

vi) The open sets of a topological space, ordered by inclusion can also be thought as a category.

vii) *Discrete categories*, which are categories with no morphisms other than identities.

Let $\mathcal{C}$ be a category.

Definition 2.1.3. i) A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ is an *isomorphism* if $\exists g \in \mathrm{Hom}_{\mathcal{C}}(C, B)$ such that $fg = \mathrm{id}_C$ and $gf = \mathrm{id}_B$. In this case, g is unique and it is denoted by f^{-1}.

ii) A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ is called *monic* if $\forall e_1, e_2 \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $e_1 \neq e_2 \implies fe_1 \neq fe_2$.

iii) A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ is called *epi* if $\forall g_1, g_2 \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, $g_1 \neq g_2 \implies g_1 f \neq g_2 f$.

Remark. Notice that by this definitions, a morphism is monic if it can be cancelled on the right, and it is epi if it can be cancelled on the left. Since isomorphisms can be cancelled on both sides by composing or precomposing with their inverses, all isomorphisms are monic and epi. The converse is not true in general.

Example 2.1.4. i) In the category **Set**, since maps are defined by the images of the elements of the domain, the definitions of monic and epi coincide with the notion of injective and surjective. In this case, isomorphisms are bijections.

ii) The correspondence of the previous example is true in many categories. The following example shows a case where the correspondence fails:

In the category **Mon** of monoids and monoid homomorphisms, the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ is an epi. To show this fact, we will see that whenever $f, g : \mathbb{Z} \longrightarrow M$ are two monoid homomorphisms such that they are equal when they are restricted to $\mathbb{N}$, then they are equal. Notice that

$$f(-1) = f(-1) + g(1-1) = f(-1) + g(1) + g(-1) =$$

$$= f(-1) + f(1) + g(-1) = f(1-1) + g(-1) = g(-1)$$

And therefore $f(-n) = f(-1) + \overset{n}{\cdots} + f(-1) = g(-1) + \overset{n}{\cdots} + g(-1) = g(-n)$, so $f = g$.

Definition 2.1.5. i) An *initial object* in $\mathcal{C}$ is an object I such that $\forall C \in \mathcal{C}$, $\exists! f \in \mathrm{Hom}_{\mathcal{C}}(I, C)$.

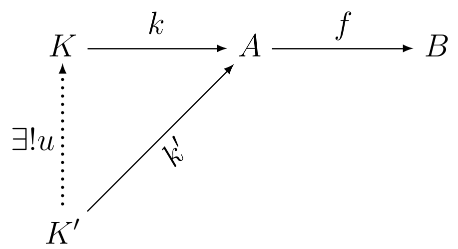
ii) A *terminal object* in $\mathcal{C}$ is an object T such that $\forall C \in \mathcal{C}$, $\exists! f \in \mathrm{Hom}_{\mathcal{C}}(C, T)$.

iii) A *zero object* 0 is an object that is both initial and terminal.

If $\mathcal{C}$ has a zero object 0, then each set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has a morphism defined by $A \to 0 \to B$ which is written 0 as well. It is unique by the definition of the zero object.

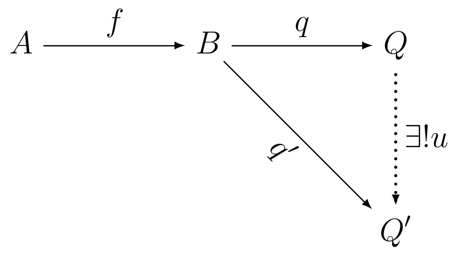
Definition 2.1.6. Let $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, where $\mathcal{C}$ has a zero object 0.

i) A *kernel* of f is a morphism $k \in \mathrm{Hom}_{\mathcal{C}}(K, A)$ such that $fk = 0$ and that satisfies the universal property: for every $k' \in \mathrm{Hom}_{\mathcal{C}}(K', A)$ such that $fk' = 0$, there is a unique $u \in \mathrm{Hom}_{\mathcal{C}}(K', K)$ such that $k' = ku$. This is, $fk = 0$ and the following diagram commutes for all k' such that $fk' = 0$:



It is unique up to a unique isomorphism between domains.

ii) A *cokernel* of f is a morphism $q \in \mathrm{Hom}_{\mathcal{C}}(B, Q)$ such that $qf = 0$ and that satisfies the universal property: for every $q' \in \mathrm{Hom}_{\mathcal{C}}(B, Q')$ such that $q'f = 0$, there is a unique $u \in \mathrm{Hom}_{\mathcal{C}}(Q, Q')$ such that $q' = uq$. This is, $qf = 0$ and the following diagram commutes for all q' such that $q'f = 0$:



It is unique up to a unique isomorphism between codomains.

Remark. Not all morphisms have kernels or cokernels. When they do, in some cases, the kernel and cokernel are identified with their domain and codomain respectively. In other cases, they represent both the morphism and the domain/codomain.

Notice that kernels are monics, since if $e_1, e_2 : C \longrightarrow K$ are such that $ke_1 = ke_2$, then $fke_1 = fke_2 = 0$ and therefore $\exists! u : C \longrightarrow K$ such that $ku = ke_1 = ke_2$, so $e_1 = e_2 = u$. Similarly, cokernels are epis, since if $g_1, g_2 : Q \longrightarrow D$ are such that $g_1 q = g_2 q$, then $g_1 qf = g_2 qf$ and $\exists! u : Q \longrightarrow D$ such that $g_1 q = g_2 q = uq$, so $g_1 = g_2 = u$.

Definition 2.1.7. The category $\mathcal{C}^{\mathrm{op}}$, that has the same objects as $\mathcal{C}$ but the morphisms and composition are reversed (i.e., $f^{\mathrm{op}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(B, A)$ if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$; and $f^{\mathrm{op}} g^{\mathrm{op}} = h^{\mathrm{op}}$ if $gf = h$), is called the *opposite category*.

Taking the opposite category interchanges epis and monics, kernels and cokernels, initial objects and terminal objects, and many other definitions, because they only differ in the orientation of the morphisms. For this reason, sometimes $\mathcal{C}^{\mathrm{op}}$ is called the *dual category* of $\mathcal{C}$.

Functors

Definition 2.1.8. A *covariant functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is a map that associates an object $F(C) \in \mathcal{D}$ (also written FC or F_C) to each $C \in \mathcal{C}$ and a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ (also Ff or F_f) to each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and such that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(gf) = F(g)F(f)$. These two conditions are referred to as *functoriality*.

A *contravariant functor* between two categories $\mathcal{C}$ and $\mathcal{D}$ is a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$, this is, $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(gf) = F(f)F(g)$. It is usually written $F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$.

Remark. Note that functors send isomorphisms to isomorphisms, since if $f : A \longrightarrow B$ and $g : B \longrightarrow A$ are such that $gf = \mathrm{id}_A$ and $fg = \mathrm{id}_B$, then $F(g)F(f) = F(gf) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(f)F(g) = F(fg) = F(\mathrm{id}_B) = \mathrm{id}_{F(B)}$. Nevertheless, it is not true in general that epis are sent to epis or that monics are sent to monics.

Example 2.1.9. The following are examples of functors.

i) $\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \longrightarrow \mathbf{Set}$ is a (covariant) functor sending the objects $C \in \mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(A, C)$ and morphisms $f : C \longrightarrow D$ to $\mathrm{Hom}_{\mathcal{C}}(A, f) : \mathrm{Hom}_{\mathcal{C}}(A, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, D)$ defined by $\mathrm{Hom}_{\mathcal{C}}(A, f) : g \mapsto fg$.

ii) $\mathrm{Hom}_{\mathcal{C}}(-, B) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a contravariant functor: objects $C \in \mathcal{C}$ are sent to $\mathrm{Hom}_{\mathcal{C}}(C, B)$, but morphisms $f : C \longrightarrow D$ are sent to $\mathrm{Hom}_{\mathcal{C}}(f, B) : \mathrm{Hom}_{\mathcal{C}}(D, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, B)$, defined by $\mathrm{Hom}_{\mathcal{C}}(f, B) : g \mapsto gf$.

iii) A functor $F : I \longrightarrow \mathcal{C}$ with I a poset, can be thought as a diagram in $\mathcal{C}$, since it chooses some of the objects and gives a unique morphism between two objects if they come from two elements $i, j \in I$ such that $i \leq j$. For example, if $I = \{a, b, c\}$ with the only relations $a, b \leq c$, then F gives diagrams $F_a \longrightarrow F_c \longleftarrow F_b$ in $\mathcal{C}$.

Definition 2.1.10. i) A *forgetful functor* is a functor that forgets some of the structure of a category (e.g. from **Ab** to **Set**, forget the group structure). It is usually denoted by U.

ii) A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called *faithful* if the map $\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FA, FB)$ given by $f \mapsto F(f)$ is an injection $\forall A, B \in \mathcal{C}$.

iii) A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called *full* if the map $\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FA, FB)$ is a surjection $\forall A, B \in \mathcal{C}$.

iv) A functor that is both faithful and full is called *fully faithful.*

Definition 2.1.11. A functor is called an *embedding* if it is fully faithful and it is injective on objects.

Definition 2.1.12. A *subcategory* $\mathcal{B}$ of $\mathcal{C}$ consists of some of the objects of $\mathcal{C}$ and some of the morphisms, and it is a category itself. If $\mathrm{Hom}_{\mathcal{B}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$, $\forall A, B \in \mathcal{B}$, it is called a *full subcategory.* The inclusion functor of a full subcategory is an embedding.

Natural transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

Definition 2.1.13. Let F and G be two functors between the categories $\mathcal{C}$ and $\mathcal{D}$. A *natural transformation* $\eta : F \Longrightarrow G$ is a map that associates a morphism $\eta_C : F(C) \longrightarrow G(C)$ in $\mathcal{D}$ to every $C \in \mathcal{C}$ such that $\forall f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$, the following diagram commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(C') \\ {\scriptstyle \eta_C}\big\downarrow & & \big\downarrow{\scriptstyle \eta_{C'}} \\ G(C) & \xrightarrow{G(f)} & G(C') \end{array}$$

If η_C is an isomorphism $\forall C \in \mathcal{C}$, η is a *natural isomorphism*, and we write $F \cong G$.

Definition 2.1.14. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there is a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ (called *equivalence of categories*) such that $\exists G : \mathcal{D} \longrightarrow \mathcal{C}$ and we have natural isomorphisms $\mathrm{id}_{\mathcal{C}} \cong GF$ and $\mathrm{id}_{\mathcal{D}} \cong FG$.

Proposition 2.1.15. *Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if there is a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ that is fully faithful and, $\forall D \in \mathcal{D}$, there is an object $C \in \mathcal{C}$ such that $D \cong F(C)$.*

Proof. $\Longrightarrow$: Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be the equivalence of categories and $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that there are natural isomorphisms $\alpha : \mathrm{id}_{\mathcal{C}} \longrightarrow GF$ and $\beta : \mathrm{id}_{\mathcal{D}} \longrightarrow GF$. Then for $f, f' : C \longrightarrow C'$ in $\mathcal{C}$ we have a diagram

$$\begin{array}{ccc} C & \xrightarrow{\alpha_C} & GF(C) \\ {\scriptstyle f}\big\downarrow\big\downarrow{\scriptstyle f'} & & {\scriptstyle GF(f)}\big\downarrow\big\downarrow{\scriptstyle GF(f')} \\ C' & \xrightarrow{\alpha_{C'}} & GF(C') \end{array}$$

such that $f = \alpha_{C'}^{-1} \circ GF(f) \circ \alpha_C$ and $f' = \alpha_{C'}^{-1} \circ GF(f') \circ \alpha_C$. This implies that if $F(f) = F(f')$, then $f = f'$, which proves that F is faithful. Using the symmetric argument, we conclude that G is also faithful.

Let $h : F(C) \longrightarrow F(C')$ be a morphism in $\mathcal{D}$. Let $f = \alpha_{C'}^{-1} \circ G(h) \circ \alpha_C$ and consider the following diagram:

$$\begin{array}{ccc} C & \xrightarrow{\alpha_C} & GF(C) \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle GF(f)}\big\downarrow\big\downarrow{\scriptstyle G(h)} \\ C' & \xrightarrow{\alpha_{C'}} & GF(C') \end{array}$$

Then, $GF(f) = \alpha_{C'} \circ f \circ \alpha_C^{-1} = G(h)$ and since G is faithful, $F(f) = h$, and hence F is fully faithful. Moreover, it is clear that $D \in \mathcal{D}$ is isomorphic to $F(GD)$ with $GD \in \mathcal{C}$, since $\beta_D : D \longrightarrow FG(D)$ is an isomorphism.

$\Longleftarrow$: We will define a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ and natural transformations $\alpha : \mathrm{id}_{\mathcal{C}} \longrightarrow GF$, $\beta : \mathrm{id}_{\mathcal{D}} \longrightarrow FG$.

For each $D \in \mathcal{D}$, let $C \in \mathcal{C}$ be an object such that there is an isomorphism $\beta_D : D \longrightarrow F(C)$. Define $G(D) = C$. This defines the functor G on objects and the natural isomorphism β. We will define G on morphisms and see that everything is well-defined. Let $h' = \beta_{D'} \circ h \circ \beta_D^{-1} : FG(D) \longrightarrow FG(D')$, so that

$$\begin{array}{ccc} D & \xrightarrow{\beta_D} & FG(D) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ D' & \xrightarrow{\beta_{D'}} & FG(D') \end{array}$$

commutes. Since F is fully faithful, there is a unique morphism $f : G(D) \longrightarrow G(D')$ such that $F(f) = h'$. Let $G(h) = f$, so $h' = FG(h)$. This makes G into a functor and the commutative diagram proves that $\beta : \mathrm{id}_{\mathcal{D}} \longrightarrow FG$ is well-defined. To define α, let $C \in \mathcal{C}$ and consider $\beta_{FC} : F(C) \longrightarrow FGF(C)$ and let $\alpha_C = F^{-1}(\beta_{FC})$, which is well-defined because F is fully faithful. □

Definition 2.1.16. If I and $\mathcal{A}$ are categories, the *functor category* $\mathcal{A}^I$ is a category such that its objects are covariant functors $F : I \longrightarrow \mathcal{A}$ and its morphisms are natural transformations $\eta : F \Longrightarrow G$. Composition is given by $(\zeta\eta)_i = \zeta_i\eta_i$, $\forall i \in I$, and identities satisfy $(\mathrm{id}_F)_i = \mathrm{id}_{F(i)}$, $\forall i \in I$.

Example 2.1.17. i) The functor category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is called the *category of presheaves*. The topological sense of *presheaf* is obtained when $\mathcal{C}$ is the poset of open sets of a topological space, ordered by inclusion.

ii) When I is a poset, the category $\mathcal{C}^I$ is a functor category formed by the *I-diagrams* in $\mathcal{C}$.

2.2 Constructions in categories. Limits and colimits

Definition 2.2.1. i) The *limit* of a functor $F : I \longrightarrow \mathcal{A}$, if it exists, is an object L of $\mathcal{A}$, and maps $\pi_i : L \longrightarrow F_i$, $\forall i \in I$ such that for every morphism $\alpha : j \longrightarrow i$ $(i, j \in I)$, $F_\alpha \pi_j = \pi_i$, and such that $\forall A \in \mathcal{A}$ and every collection of maps $f_i : A \longrightarrow F_i$ with $F_\alpha f_j = f_i$, there is a unique $u : A \longrightarrow L$ satisfying $f_i = \pi_i u$. It is written $\lim_{i \in I} F_i$ and it is unique up to isomorphism.

ii) The *colimit* of a functor $F : I \longrightarrow \mathcal{A}$, if it exists, is an object C of $\mathcal{A}$, and maps $\iota_i : F_i \longrightarrow C$, $\forall i \in I$ such that for every morphism $\alpha : j \longrightarrow i$ ($i, j \in I$), $\iota_i F_\alpha = \iota_j$, and such that $\forall A \in \mathcal{A}$ and every collection of maps $f_i : F_i \longrightarrow A$ with $f_i F_\alpha = f_j$, there is a unique $u : C \longrightarrow A$ satisfying $f_i = u\iota_i$. It is written $\operatorname{colim}_{i \in I} F_i$ and it is unique up to isomorphism.

The most common example of limits and colimits are products and coproducts. If $\mathcal{I}$ is a discrete category we obtain the usual definition of product and coproduct:

Definition 2.2.2. i) Given a set of objects $\{C_i\}_{i \in \mathcal{I}}$ of $\mathcal{C}$, a *product* $\prod_{i \in \mathcal{I}} C_i$, if it exists, is an object of $\mathcal{C}$ and morphisms $\pi_i : \prod C_j \longrightarrow C_i$, $\forall i \in \mathcal{I}$ such that $\forall A \in \mathcal{C}$ and every family of morphisms $\alpha_i : A \longrightarrow C_i$, $\forall i \in \mathcal{I}$, there is a unique morphism $\alpha : A \longrightarrow \prod C_i$ such that $\pi_i \alpha = \alpha_i$, $\forall i \in \mathcal{I}$. Finite products are written $C_1 \times \cdots \times C_n$.

ii) Given a set of objects $\{C_i\}_{i \in \mathcal{I}}$ of $\mathcal{C}$, a *coproduct* $\coprod_{i \in \mathcal{I}} C_i$, if it exists, is an object of $\mathcal{C}$ and morphisms $\iota_i : C_i \longrightarrow \coprod C_j$, $\forall i \in \mathcal{I}$ such that $\forall A \in \mathcal{C}$ and every family of morphisms $\alpha_i : C_i \longrightarrow A$, $\forall i \in \mathcal{I}$, there is a unique morphism $\alpha : \coprod C_i \longrightarrow A$ such that $\alpha \iota_i = \alpha_i$, $\forall i \in \mathcal{I}$. Finite coproducts are written $C_1 \amalg \cdots \amalg C_n$. Alternatively, a coproduct in $\mathcal{C}$ is a product in $\mathcal{C}^{\text{op}}$.

Example 2.2.3. In the category **Set**, the product of two sets A and B corresponds to the cartesian product $A \times B = \{(a, b) \mid a \in A, b \in B\}$, and the morphisms π_A and π_B are the projections onto A and B respectively.

The coproduct of two objects A and B in **Set** is the disjoint union $A \amalg B = (A \times \{0\}) \cup (B \times \{1\})$, and the morphisms ι_A and ι_B are the injections of A and B into $A \amalg B$ respectively.

Definition 2.2.4. A poset I is called *filtered* if for every $i, j \in I$, $\exists k \in I$ such that $i \leq k$ and $j \leq k$.

Definition 2.2.5. i) A *direct limit* is the colimit of a functor $A : I \longrightarrow \mathcal{A}$ over a poset I, and it is written $\varinjlim A_i$.

ii) An *inverse limit* is the limit of a functor $A : I \longrightarrow \mathcal{A}$ over a poset I, and it is written $\varprojlim A_i$.

Example 2.2.6. The following are some examples of constructions that can be made with inverse and direct limits.

i) The inverse limit over a poset $a \leq c \geq b$ is called *pullback*. We shall see this definition in more detail later.

ii) Its dual, the colimit over a poset $a \geq c \leq b$ (notice that this is not filtered) is called *pushout*.

Example 2.2.7. i) If a filtered poset I has an element m such that $m \geq i$ for all $i \in I$, then $\varinjlim A_i = A_m$. To see this, notice that if α_{ij} are the morphisms between A_i and A_j, then the morphisms $\iota_i : A_i \longrightarrow A_m$ can be taken to be the α_{im}, and they are compatible with the α_{jk} by hypothesis. Also, if there is an object A and morphisms $f_i : A_i \longrightarrow A$ that are compatible with the α_{jk}, then there is a unique morphism $u = f_m$ such that the f_i factor through A_m because $m \geq i$.

ii) Limits do not always exist. For example, let $(x_i)_{i\in\mathbb{N}}$ be a sequence of elements, then in $\mathbf{Set_{fin}}$, the category of finite sets. Consider the diagram

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq \cdots,$$

where $A_i = \{x_1, \ldots, x_i\}$ and the inclusions are the compatible morphisms. Now suppose $A = \{y_1, \ldots, y_n\} \in \mathbf{Set_{fin}}$ is the direct limit of this diagram, with morphisms $\iota_i : A_i \longrightarrow A$. Now take $B = A_{n+1}$ and morphisms $f_i : A_i \longrightarrow B$ which are inclusions for $i \leq n+1$, and when $i > n+1$, the elements $x_1, \ldots, x_{n+1}$ are mapped to themselves and the $x_{n+2}, \ldots, x_i$ are mapped to x_1. Then, the f_i are compatible with the diagram of inclusions, but then there should be a morphism $u : A \longrightarrow B$ such that, in particular $u\iota_{n+1} = f_{n+1}$, but the image of f_{n+1} is B and the image of u is strictly included in B because B has one more element than A, hence A cannot be the direct limit.

The failure of this example is better understood in **Set**, since the direct limit in this case would be $\{x_i\}_{i\in\mathbb{N}}$, which is not finite.

Direct limits in R-mod

Let R be a commutative ring. Let I be a partially ordered set. It can be seen as a category with objects its elements and a unique morphism between i and j whenever $i \leq j$. Let $\alpha : I \longrightarrow R$-**mod** be a functor, with $\alpha(i) = M_i$ and $\alpha_{ij} : M_i \longrightarrow M_j$ with $i \leq j$.

Recall that the direct sum $\bigoplus_i M_i$ is the submodule of $\prod_i M_i$ formed by elements $(x_i)_{i\in I}$ with $x_i = 0$ except for a finite number of $i \in I$. Let $C = \bigoplus_i M_i$ and $D = \langle x_i - \alpha_{ij}(x_i) \mid x_i \in M_i, i \leq j\rangle \subseteq C$ (we think the M_i as submodules of C). Let $M = C/D$ and $\alpha_i : M_i \hookrightarrow C \xrightarrow{\pi} M$.

Proposition 2.2.8. *With the above notation,* $\varinjlim M_i = M$.

Proof. Notice that $\alpha_j\alpha_{ij} = \alpha_i$ with $i \leq j$ because $(\alpha_j\alpha_{ij} - \alpha_i)(x_i) = \pi(\alpha_{ij}(x_i) - x_i)$ and $\alpha_{ij}(x_i) - x_i \in D$.

Let N be an R-module and $\beta_i : M_i \longrightarrow N$ such that $\beta_i = \beta_j\alpha_{ij}$ with $i \leq j$. Define $\tilde{\beta} : C \longrightarrow N$ by $\tilde{\beta} : (x_i)_i \mapsto \sum_i \beta_i(x_i)$, which is well-defined because only a finite set of the x_i are nonzero. Notice that

$$x_i - \alpha_{ij}(x_i) \mapsto \beta_i(x_i) - \beta_j\alpha_{ij}(x_i) = (\beta_i - \beta_j\alpha_{ij})(x_i) = 0.$$

Therefore, there is a morphism $\beta : M \longrightarrow N$ defined by $\beta(\overline{(x_i)_i}) = \tilde{\beta}((x_i)_i)$ (with $\overline{(x_i)_i} = \pi((x_i)_i)$), which is well-defined because the elements of D are mapped to zero and it is the unique morphism satisfying $\beta\pi = \tilde{\beta}$ by construction. Hence, $M = \varinjlim M_i$. □

If I is filtered, this is, if $\exists k$ such that $k \geq i, j$ for all $i, j \in I$, then the direct limit has some interesting properties.

Proposition 2.2.9. *With the above notation, if I is filtered, then:*

i) Every element of M is the image of an element of M_i for some $i \in I$.

ii) If $\alpha_i(x_i) = 0$, then $\exists j \geq i$ such that $\alpha_{ij}(x_i) = 0$.

Proof. For every $\overline{(x_i)_i} \in M$, there is a finite number of elements of $(x_i)_i$ that are nonzero, say, $x_{i_1}, \ldots, x_{i_r}$. Take $k \geq i_1, \ldots i_r$, then $\sum_j \alpha_{i_jk}(x_{i_j}) \in M_k$ and it maps to $\overline{(x_i)_i} \in M$ because $\alpha_{i_jk}(x_{i_j}) - x_{i_j} = 0$ in M, which proves *i)*.

To see *ii)*, notice that $\alpha_i(x_i) = 0 \Longrightarrow x_i \in D$, so $x_i = \sum_{k=1}^n (x_{i_k} - \alpha_{i_kj_k}(x_{i_k}))$ in C, and observe that if $l \geq i_1, \ldots, i_n$ then

$$x_{i_k} - \alpha_{i_kj_k}(x_{i_k}) = x_{i_k} - \alpha_{i_kl}(x_{i_k}) + \alpha_{i_kl}(x_{i_k}) - \alpha_{i_kj_k}(x_{i_k}) =$$

$$= x_{i_k} - \alpha_{i_kl}(x_{i_k}) + \alpha_{j_kl}(\alpha_{i_kj_k}(x_{i_k})) - \alpha_{i_kj_k}(x_{i_k}),$$

so x_i can be written as $x_i = \sum_{k=1}^n (x_{i_k} - \alpha_{i_kl}(x_{i_k}))$ where all indexes are taken to be different. Since we are in $C = \bigoplus_i M_i$, we have two cases:

- If $i = i_k$ for some k, then $x_{i_{k'}} = 0$ for $i_{k'} \neq i_k$ and therefore $\alpha_{i_{k'}l}(x_{i_{k'}}) = 0$, but $\sum_{j=1}^n \alpha_{i_jl}(x_{i_j})$ must be 0, hence $\alpha_{i_kl}(x_{i_k}) = 0$, this is, $\alpha_{il}(x_i) = 0$.
- If $i = l$, then all x_{i_k} are zero, and $x_i = \sum_{k=1}^n \alpha i_k l(x_{i_k}) = 0$.

□

If we take $R = \mathbb{Z}$, we have the particular result for abelian groups. Since every ring is an abelian group, the construction can be made for a directed system of rings in the same way. It has to be checked that the resulting abelian group is actually a ring.

Proposition 2.2.10. *Take $R = \mathbb{Z}$ in the above construction and apply it to a directed system of rings R_i. Then, the direct limit $R_0 = \varinjlim R_i$ is a ring.*

Proof. Let $a, b \in R_0$. Then $a = \alpha_i(x)$ and $b = \alpha_j(y)$. Take $k \geq i, j$ and define the product of a and b in R_0 as $a \cdot b := \alpha_k(\alpha_{ik}(x)\alpha_{jk}(y))$.

To see that it is well-defined, first let $k, l \geq i, j$ and let $m \geq k, l$. Then $\alpha_k(\alpha_{ik}(x)\alpha_{jk}(y)) = \alpha_m(\alpha_{km}(\alpha_{ik}(x)\alpha_{jk}(y))) = \alpha_m(\alpha_{im}(x)\alpha_{jm}(y))$, and changing k by l we see that the product does not depend on k.

Now suppose $\alpha_{i'}(x') = \alpha_i(x)$ and $\alpha_{j'}(y') = \alpha_j(y)$ and take $k \geq i, i', j, j'$. Then $\alpha_k(\alpha_{i'k}(x')) = \alpha_k(\alpha_{ik}(x))$ and $\alpha_k(\alpha_{j'k}(y')) = \alpha_k(\alpha_{jk}(y))$, so there is an $l \geq k$ such that $\alpha_{kl}(\alpha_{i'k}(x')) = \alpha_{kl}(\alpha_{ik}(x))$ and $\alpha_{kl}(\alpha_{j'k}(y')) = \alpha_{kl}(\alpha_{jk}(y))$, i.e., $\alpha_{i'l}(x') = \alpha_{il}(x)$ and $\alpha_{j'l}(y') = \alpha_{jl}(y)$, so the product does not depend on i and j either:

$$\alpha_l(\alpha_{il}(x)\alpha_{jl}(y)) = \alpha_l(\alpha_{i'l}(x')\alpha_{j'l}(y')).$$

The rest of properties can easily be checked from this definition of the product, and it makes the maps α_i into ring homomorphisms by construction. □

The assignment of direct limits in R-**mod** is actually a functor, as the following proposition states.

Proposition 2.2.11. *Let (M_i, α_{ij}) and (M'_i, α'_{ij}) be two directed systems of R-modules (this is, an I-diagram in R-**mod** with I a filtered poset) and $M = \varinjlim M_i$, $M' = \varinjlim M'_i$. Let $\{f_i\}$ be a collection of homomorphisms such that $\alpha'_{ij} f_i = f_j \alpha_{ij}$ with $i \leq j$. Then there is a homomorphism $f : M \longrightarrow M'$ that makes $\varinjlim$ into a functor.*

Proof. Let $\mu_i : M_i \longrightarrow M'$ be defined by $\mu_i = \alpha'_i f_i$. Notice that $\mu_j \alpha_{ij} = \alpha'_j f_j \alpha_{ij} = \alpha'_j \alpha'_{ij} f_i = \alpha'_i f_i = \mu_i$. Therefore, by the universal property of the limit, there is a homomorphism $f : M \longrightarrow M'$ such that $\mu_i = f \alpha_i$.

Observe that this implies that $f\alpha_i = \alpha'_i f_i$. From this fact and the uniqueness of the homomorphism satisfying this relation follows the functoriality of this assignment. □

2.3 Abelian Categories

Definition

The notion of abelian category is crucial for homological algebra. It was introduced by Alexander Grothendieck's article *Sur quelques points d'algèbre homologique* ([Groth]), and generalizes the structure and properties of many nice categories, like **Ab** or R-**mod**. There is no better manner of explaining the motivation for the definition of abelian category than the following quote from [Bass]:

> "The intention of these axioms [those that define an abelian category] is to make available, in any Abelian category, all of the elementary arguments and constructions (involving only a finite amount of data) which one performs in categories of modules. The achievement of this aim is testified to by the "Embedding Theorem", [...]. In view of that theorem one might protest that the notion of Abelian category is superfluous; why not speak of subcategories of categories of modules instead. This is roughly analogous to asking that we only speak of vector spaces with fixed coordinate systems, or that we speak only of groups of permutations (after all, every group is one). There are many reasons beyond linguistic simplification that make the notion of Abelian category natural and useful. The most obvious one derives from the fact that the axioms are self-dual, so that the dual of a theorem about Abelian categories is again one. Only rarely does the dual of a category of modules have a natural representation as a category of modules. Furthermore, there is the important notion of quotient category [...], which would be awkward, to say the least, to formalize using only categories of modules. Of greatest importance, perhaps, is the fact that, with respect to certain infinite constructions (e.g. limits) categories of modules betray certain definite idiosyncrasies."
>
> *Hyman Bass,* Algebraic K-Theory*, p. 21.*

We begin with some previous concepts.

Definition 2.3.1. An *additive category* is a category $\mathcal{A}$ such that:

i) It has a zero object 0.

ii) Every set $\operatorname{Hom}_{\mathcal{A}}(B, C)$ is an abelian group with respect to addition, and composition distributes over addition, i.e. for the following diagram:

$$A \xrightarrow{\;f\;} B \underset{g'}{\overset{g}{\rightrightarrows}} C \xrightarrow{\;h\;} D$$

We have $h(g+g')f = hgf + hg'f$.

iii) For every pair of objects $A, B \in \mathcal{A}$ their product $A \times B$ exists. It can be seen that this implies that their coproduct also exists and they are isomorphic (see [MacL] p. 194, Theorem 2). In this case it is sometimes written $A \oplus B$ instead.

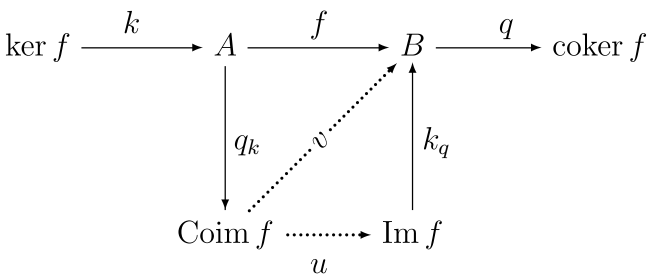
Remark. In an additive category, a morphism f is monic if $fe_1 - fe_2 = f(e_1 - e_2) = fe = 0 \Longrightarrow e_1 - e_2 = e = 0$. Also notice that in this case, the zero object 0 satisfies the definition of $\ker f$. Dually, a morphism f is epi if $ef = 0 \Longrightarrow e = 0$, and then $\operatorname{coker} f = 0$.

Definition 2.3.2. Let $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$. The *image* $\operatorname{Im} f$ of f, if it exists, is the kernel of the cokernel of f. Dually, the *coimage* $\operatorname{Coim} f$ of f, if it exists, is the cokernel of the kernel of f.

The image and coimage are related by a morphism, which will be the center of attention when defining the concept of abelian category.

Proposition 2.3.3. *In an additive category, if the image and coimage of a morphism $f : A \longrightarrow B$ exist, then there is a unique canonical morphism $u : \operatorname{Coim} f \longrightarrow \operatorname{Im} f$ such that $f = k_q u q_k$, where k_q and q_k are the image and coimage morphisms respectively.*

Proof. We have the following commutative diagram:



The morphism v exists because $fk = 0$, but q_k is a cokernel, and by the universal property, there is a v such that $vq_k = f$. Now, observe that $qvq_k = 0 = 0q_k$, but q_k is a cokernel and so it is an epi $\Longrightarrow qv = 0$, but k_q is a kernel, hence there is a u such that $k_q u = v \Longrightarrow f = vq_k = k_q u q_k$.

It is unique because if there is another morphism u' such that $f = k_q u' q_k = k_q u q_k$, then using that q_k is epi we have $k_q u' = k_q u$ and using that k_q is monic we have $u' = u$. $\square$

Definition 2.3.4. An *abelian category* is an additive category $\mathcal{A}$ satisfying:

i) Every morphism admits a kernel and a cokernel.

ii) For every morphism f, the canonical morphism between $\operatorname{Coim} f$ and $\operatorname{Im} f$ is an isomorphism.

Remark. The last condition is equivalent to the condition that every monic is the kernel of its cokernel and every epi is the cokernel if its kernel.

Notice that in an abelian category, if a morphism $f : A \longrightarrow B$ is monic and epi, then it is an isomorphism, because in this case $\ker f = \operatorname{coker} f = 0$, so the image of f is B and the coimage is A, so $A \cong B$ by the isomorphism u.

Observe that the axioms that define an abelian category are self-dual, so that if $\mathcal{A}$ is an abelian category, then $\mathcal{A}^{\mathrm{op}}$ is also an abelian category.

Example 2.3.5. i) The category **Ab** of abelian groups is an abelian category. It is additive because it has a zero object, the trivial group 0, group homomorphisms between two abelian groups behave as elements of an abelian group, and it has all products of two abelian groups. It is abelian because the kernel and cokernel of any abelian group homomorphism are abelian groups and the first isomorphism theorem is exactly the second property of an abelian category.

ii) With the same arguments, the category R-**mod** of R-modules is an abelian category. In fact, **Ab** is just R-**mod** with $R = \mathbb{Z}$.

iii) The category R-$\mathbf{mod_{fg}}$ of finitely generated R-modules is not abelian in general because submodules of a finitely generated module in R-**mod** may not be finitely generated. However, if R is Noetherian, then R-$\mathbf{mod_{fg}}$ is abelian.

iv) The category $\mathbf{TVect}_{\mathbb{K}}$ of topological vector spaces, which are vector spaces equipped with a topological structure[1], is not an abelian category because for a map $f : X \longrightarrow Y$, the quotient topology of $\operatorname{Coim} f = X/\ker f$ differs from the one of $\operatorname{Im} f$, this is, both spaces are isomorphic as vector spaces but not as topological vector spaces.

In abelian categories, the notion of exact sequence can be defined in the usual way.

Definition 2.3.6. Let $\mathcal{A}$ be an abelian category. A sequence of objects and morphisms of $\mathcal{A}$

$$\cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots$$

such that $d_k d_{k+1} = 0$, $\forall k \in \mathbb{Z}$, is *exact at* A_i if $\ker d_i = \operatorname{Im} d_{i+1}$. The sequence is *exact* if it is exact at each A_i.

[1]A topological vector space is a vector space over a field $\mathbb{K}$ that is also a topological space, satisfying that addition and multiplication are continuous maps, and such that in the vector space, addition and multiplication by a scalar are also continuous.

Remark. Note that $\ker d_i = \operatorname{Im} d_{i+1} \Longrightarrow \operatorname{Coim} d_i = \operatorname{coker}(\ker d_i) = \operatorname{coker}(\operatorname{Im} d_{i+1}) = \operatorname{coker}(\ker(\operatorname{coker} d_{i+1}))$, and by the universal properties of kernels and cokernels, it can be seen that $\ker(\operatorname{coker}(\ker f)) = \ker f$ and $\operatorname{coker}(\ker(\operatorname{coker} f)) = \operatorname{coker} f$, so $\operatorname{Coim} d_i = \operatorname{coker} d_{i+1}$. Now taking kernels we obtain again the original condition, so $\operatorname{Coim} d_i = \operatorname{coker} d_{i+1}$ is an equivalent condition for exactness.

A sequence

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

is exact if f is monic, g is epi, and $\ker g = \operatorname{Im} f$. This is called a *short exact sequence.*

Exact functors

Definition 2.3.7. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between two additive categories $\mathcal{A}$ and $\mathcal{B}$ is additive if $F(f+g) = Ff + Fg$, $\forall f, g \in \operatorname{Hom}_{\mathcal{C}}(A, A')$. In particular, $F(0) = 0$.

Definition 2.3.8. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a covariant functor between two abelian categories $\mathcal{A}$ and $\mathcal{B}$.

i) F is *left exact* if it is additive and for all short exact sequences

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

the sequence

$$0 \longrightarrow FA' \xrightarrow{Ff} FA \xrightarrow{Fg} FA''$$

is exact.

ii) F is *right exact* if it is additive and for all short exact sequences

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

the sequence

$$FA' \xrightarrow{Ff} FA \xrightarrow{Fg} FA'' \longrightarrow 0$$

is exact.

Definition 2.3.9. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a contravariant functor between two abelian categories $\mathcal{A}$ and $\mathcal{B}$.

i) F is *left exact* if it is additive and for all short exact sequences

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

the sequence

$$0 \longrightarrow FA'' \xrightarrow{Fg} FA \xrightarrow{Ff} FA'$$

is exact.

ii) F is *right exact* if it is additive and for all short exact sequences

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

the sequence

$$FA'' \xrightarrow{Fg} FA \xrightarrow{Ff} FA' \longrightarrow 0$$

is exact.

Definition 2.3.10. An functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between two abelian categories $\mathcal{A}$ and $\mathcal{B}$ is *exact* if it is both left and right exact.

Proposition 2.3.11. *A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is exact if and only if it preserves all exact sequences.*

Proof. If it preserves all exact sequences, in particular it preserves short exact sequences.

Now, if it preserves short exact sequences, notice that any exact sequence

$$\cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots$$

in $\mathcal{A}$ can be split up into short exact sequences

$$0 \longrightarrow \ker d_i \longrightarrow A_i \longrightarrow \ker d_{i-1} \longrightarrow 0$$

with $\ker d_i = \operatorname{Im} d_{i+1}$ by exactness. Hence, in $\mathcal{B}$ the short exact sequences are preserved and they can build up a sequence

$$\cdots \longrightarrow FA_{i+1} \xrightarrow{Fd_{i+1}} FA_i \xrightarrow{Fd_i} FA_{i-1} \longrightarrow \cdots$$

which is exact by the exactness of the short exact sequences. □

Example 2.3.12. i) The functor $\operatorname{Hom}(A, -) : R\text{-}\mathbf{mod} \longrightarrow \mathbf{Ab}$ is a covariant left exact functor; and similarly $\operatorname{Hom}(-, B)$ is a contravariant left exact functor.

ii) The functor $M \otimes_R - : R\textbf{-mod} \longrightarrow R\textbf{-mod}$ is a covariant right exact functor. If M is a flat R-module, then the functor is exact. For example, $S^{-1}R \otimes_R - : R\textbf{-mod} \longrightarrow S^{-1}R\textbf{-mod}$ is an exact functor.

The proof of these facts is given in the next chapter. The exactness of $S^{-1}R \otimes_R -$ was proved in the previous chapter.

Recall by Proposition 2.2.11 the functor behaviour of the direct limit in R-**mod**. This is an example of an exact functor.

Proposition 2.3.13. *The functor* $\varinjlim : R\textbf{-mod}^I \longrightarrow R\textbf{-mod}$ *that assigns the direct limit to an* I*-diagram in* R-**mod**, *with* I *a poset, is an exact functor. This is:*

Let (M_i', α_{ij}'), (M_i, α_{ij}) *and* (M_i'', α_{ij}'') *be directed systems in* R-**mod** *with direct limits* M', M *and* M'' *respectively, and* $\{f_i\}$, $\{g_i\}$ *collections of homomorphisms such that* $\alpha_{ij} f_i = f_j \alpha_{ij}'$ *and* $\alpha_{ij}'' g_i = g_j \alpha_{ij}$. *If the sequences*

$$0 \longrightarrow M_i' \xrightarrow{f_i} M_i \xrightarrow{g_i} M_i'' \longrightarrow 0$$

are exact $\forall i \in I$, *then*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is exact, where f *and* g *are the homomorphisms induced by the* f_i *and* g_i *on the direct limits.*

Proof. We will actually show that if the sequences

$$M_i' \xrightarrow{f_i} M_i \xrightarrow{g_i} M_i''$$

are exact, then

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

is exact. If we apply this result to $0 \longrightarrow M_i' \longrightarrow M_i$ and $M_i \longrightarrow M_i'' \longrightarrow 0$ as well, we get the original statement

Notice that, for $i \leq j$, the following diagram commutes by the construction in Proposition 2.2.11:

$$\begin{array}{ccccc}
M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' \\
{\scriptstyle \alpha'_j}\uparrow & & {\scriptstyle \alpha_j}\uparrow & & {\scriptstyle \alpha''_j}\uparrow \\
M'_j & \xrightarrow{f_j} & M_j & \xrightarrow{g_j} & M''_j \\
{\scriptstyle \alpha'_{ij}}\uparrow & & {\scriptstyle \alpha_{ij}}\uparrow & & {\scriptstyle \alpha''_{ij}}\uparrow \\
M'_i & \xrightarrow{f_i} & M_i & \xrightarrow{g_i} & M''_i
\end{array}$$

Let $x \in M'$. By Proposition 2.2.9 i), there is an element $x_j \in M'_j$ such that $x = \alpha'_j(x_j)$. Hence,

$$gf(x) = gf\alpha'_j(x_j) = g\alpha_j f_j(x_j) = \alpha''_j g_j f_j(x_j) = 0$$

because $f_j g_j = 0$, so $\operatorname{Im} f \subseteq \ker g$.

Now, let $y \in M$ such that $g(y) = 0$, and $y_i \in M_i$ such that $y = \alpha_i(y_i)$. Then, $0 = g(y) = g\alpha_i(y_i) = \alpha''_i g_i(y_i)$. By Proposition 2.2.9 ii), $\exists j \geq i$ such that $\alpha''_{ij} g_i(y_i) = 0$, hence $g_j \alpha_{ij}(y_i) = 0$, and by exactness of the sequence in i, $\exists x_j \in M'_j$ such that $\alpha_{ij}(y_i) = f_j(x_j)$. Let $x = \alpha'_j(x_j) \in M'$, then

$$y = \alpha_i(y_i) = \alpha_j \alpha_{ij}(y_i) = \alpha_j f_j(x_j) = f\alpha'_j(x_j) = f(x)$$

so $\ker g \subseteq \operatorname{Im} f$ and therefore $\operatorname{Im} f = \ker g$ and we are done. □

Remark. It is not true, though, that $\varprojlim : R\text{-}\mathbf{mod}^I \longrightarrow R\text{-}\mathbf{mod}$ is an exact functor.

This shows that, direct limits in abelian categories are not exact in general: the category $\mathcal{A} = R\text{-}\mathbf{mod}^{\mathrm{op}}$ is abelian, but its direct limits are precisely the inverse limits of R-**mod**, and therefore the functor $\varinjlim$ is not exact in $\mathcal{A}$.

Definition 2.3.14. An *abelian subcategory* $\mathcal{B}$ of $\mathcal{A}$ is a subcategory of $\mathcal{A}$ such that $\mathcal{B}$ is also abelian and the inclusion functor is exact.

Theorem 2.3.15 (Freyd-Mitchell Embedding Theorem)**.** *Let $\mathcal{A}$ be a small abelian category. There is a ring R (in general, noncommutative) and an exact, fully faithful functor $\mathcal{A} \longrightarrow R$-**mod** which embeds $\mathcal{A}$ as a full subcategory of R-**mod**.*

Remark. Observe that since the functor $\mathcal{A} \longrightarrow R$-**mod** is exact, it preserves kernels, cokernels and images, so exact sequences in $\mathcal{A}$ are exact in R-**mod**. Moreover, since the functor is fully faithful, it reflects exactness, this is, the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact $\iff$ the sequence

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC \longrightarrow 0$$

is exact. Hence, to prove that a sequence is exact in $\mathcal{A}$ it is sufficient to prove its exactness in R-**mod**, i.e., we can use techniques that work in R-**mod** such as diagram chasing to prove statements about $\mathcal{A}$. We will use this observation to simplify some proofs.

This theorem is actually more powerful than one could think at first sight: notice that any statement regarding, for example, a finite set of objects in an abelian category $\mathcal{A}$, can be seen in the smallest full abelian subcategory $\mathcal{A}'$ of $\mathcal{A}$, which will be small, and hence the Freyd-Mitchell Embedding Theorem can be applied to $\mathcal{A}'$ and obtain exactness properties in $\mathcal{A}$, because the inclusion of abelian subcategories is also exact by definition.

2.4 Quotient abelian categories

Let $\mathcal{A}$ be an abelian category.

Definition 2.4.1. A *Serre subcategory* $\mathcal{B}$ of $\mathcal{A}$ is an abelian subcategory such that if

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$, then $A \in \mathcal{B} \iff A', A'' \in \mathcal{B}$.

Example 2.4.2. Let $\mathcal{A} = \mathbf{Ab}$. Then, the following are examples of Serre subcategories. Note that they are abelian subcategories since they are closed under subgroups and quotients. For all examples, consider a short exact sequence in **Ab**:

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

i) The subcategory $\mathcal{B}_t$ of torsion groups:

Indeed, if $B \in \mathcal{B}_t$ (i.e, all elements of B have finite order), then, since g is surjective, $\forall c \in C$ we have $c = g(b)$ for some $b \in B$ and $nb = 0$ for some $n \in \mathbb{N}^*$, hence $nc = n\,g(b) = g(nb) = 0$; and since f is injective and $f(ma) = m\,f(a) = 0$ for some $m \in \mathbb{N}^*$, then $ma = 0$. Hence, $A, C \in \mathcal{B}_t$.

If $A, C \in \mathcal{B}_t$, then $\forall b \in B$ either $g(b) = 0$ or $g(b) \neq 0$. In the first case, $\exists a \in A$ such that $f(a) = b$ and $na = 0$ for some $n \in \mathbb{N}^*$, hence $nb = n\,f(a) = f(na) = 0$. If $g(b) \neq 0$, then $g(mb) = m\,g(b) = 0$ for some $m \in \mathbb{N}^*$, and now we are in the first case, so there is a $k \in \mathbb{N}^*$ such that $kmb = 0$. Hence, $B \in \mathcal{B}_t$.

ii) The subcategory $\mathcal{B}_{fg}$ of finitely generated groups:

Indeed, if $B \in \mathcal{B}_{fg}$, then $C = g(B)$ is finitely generated and $A \cong f(A) \subseteq B$ is finitely generated $\Longrightarrow A, C \in \mathcal{B}_{fg}$.

If $A, C \in \mathcal{B}_{fg}$, suppose B is not finitely generated. Then $C \cong B/\ker g$ finitely generated $\Longrightarrow \ker g = f(A) \cong A$ is not finitely generated, which contradicts the hypothesis. Hence, $B \in \mathcal{B}_{fg}$.

iii) The subcategory $\mathcal{B}_{fin}$ of finite groups. Finite groups are finitely generated torsion groups, so this is a consequence of i) and ii).

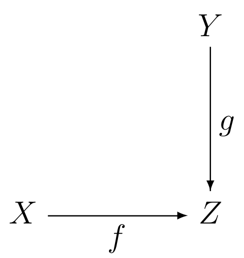
iv) The subcategory $\mathcal{B}_p$ of p-groups. This can be shown proceeding in the same way as in i), because p-groups are torsion groups in which their elements have order p^n for some $n \in \mathbb{N}^*$ and $p^n x = 0 \Longrightarrow x$ has order p^m for some $m \leq n$.

Assume $\mathcal{A}$ is small and let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$.

Definition 2.4.3. A morphism f in $\mathcal{A}$ is a $\mathcal{B}$-*iso* if $\ker f$ and $\operatorname{coker} f$ are in $\mathcal{B}$.

Recall that a pullback is a limit over a diagram like so: $\cdot \to \cdot \leftarrow \cdot$. The definition of limit applied to this diagram results in the following definition of pullback.

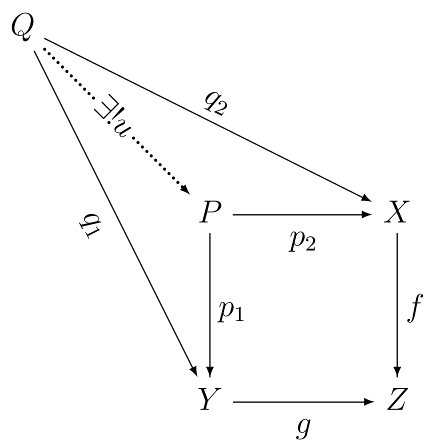
Definition 2.4.4. Let



be a diagram in any category. The *pullback* of X and Y over Z, if it exists, is an object P and morphisms $p_1 : P \longrightarrow X$ and $p_2 : P \longrightarrow Y$ such that

$$\begin{array}{ccc} P & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

commutes and satisfies the following universal property: given an object Q and morphisms $q_1 : Q \longrightarrow X$ and $q_2 : Q \longrightarrow Y$ such that $fq_1 = gq_2$, there is a unique morphism $u : Q \longrightarrow P$ such that $q_1 = p_1u$ and $q_2 = p_2u$. This is, the following diagram commutes:



Remark. In an abelian category, the pullback always exists. To see this, notice that we have the product $X \times Y$ and the kernel of the morphism $(f, -g) : X \times Y \longrightarrow Z$, and this kernel is precisely the pullback. Composing the kernel morphism with the maps π_X and π_Y of the product, we obtain the morphisms p_2 and p_1 respectively. By construction $fp_2 = gp_1$, and the universal property is given by the universal property of the kernel.

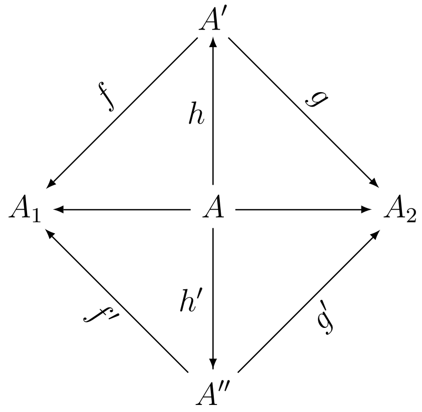
Definition 2.4.5. The *quotient abelian category* $\mathcal{A}/\mathcal{B}$ is a category that has the same objects as $\mathcal{A}$, and a morphism between A_1 and A_2 is defined as the equivalence class of diagrams in $\mathcal{A}$

$$A_1 \xleftarrow{f} A' \xrightarrow{g} A_2$$

with f a $\mathcal{B}$-iso and the equivalence with another diagram

$$A_1 \xleftarrow{f'} A'' \xrightarrow{g'} A_2$$

with f' a $\mathcal{B}$-iso is given by



where h and h' are $\mathcal{B}$-isos.

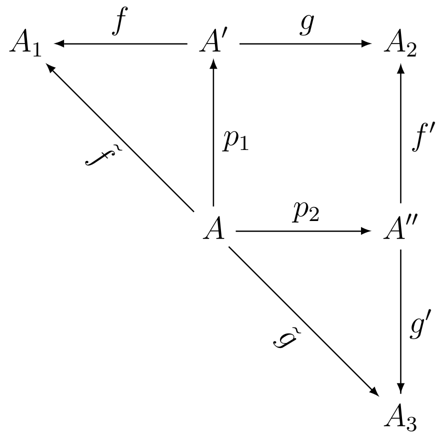
Given two morphisms

$$A_1 \xleftarrow{\ f\ } A' \xrightarrow{\ g\ } A_2$$

and

$$A_2 \xleftarrow{\ f'\ } A'' \xrightarrow{\ g'\ } A_3$$

(f and f' $\mathcal{B}$-isos), the composition is constructed in the following way:



(∗)

Where A is the pullback of A' and A'' over A_2, and $\tilde{f} = fp_1$, $\tilde{g} = g'p_2$.

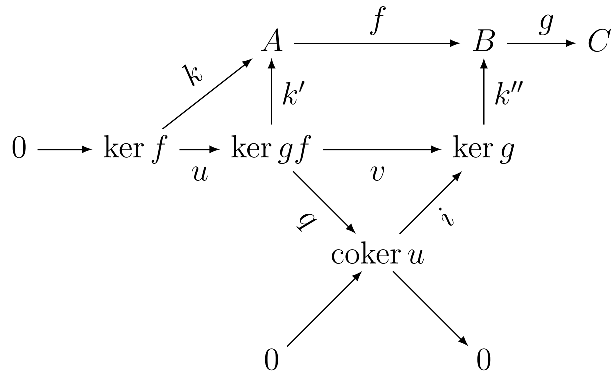
It can be checked that the composition is well-defined, that $\mathcal{A}/\mathcal{B}$ is abelian and that the quotient functor $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{B}$ is exact. The proof of the following result can be found in [Swan].

Theorem 2.4.6. *The quotient category $\mathcal{A}/\mathcal{B}$ exists, it is abelian, and there is a functor $T : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{B}$ that is exact, $T(\mathcal{B}) \cong 0$ (in the sense that $T(B) \cong 0$ in $\mathcal{A}/\mathcal{B}$, $\forall B \in \mathcal{B}$) and for every exact functor $F : \mathcal{A} \longrightarrow \mathcal{C}$ (with $\mathcal{C}$ abelian) such that $F(\mathcal{B}) \cong 0$, there is a unique exact functor $G : \mathcal{A}/\mathcal{B} \longrightarrow \mathcal{C}$ such that $F = GT$.*

We will only show that the composition is well-defined, so we need to see in $(*)$ that $\tilde{f}$ is a $\mathcal{B}$-iso. To do so, we will see that p_1 is a $\mathcal{B}$-iso and that fp_1 is a $\mathcal{B}$-iso. This is a consequence of the following propositions.

Proposition 2.4.7. *Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be $\mathcal{B}$-isos. Then $gf : A \longrightarrow C$ is a $\mathcal{B}$-iso.*

Proof. Consider the following diagram:



The morphism u exists because $fk = 0 \Longrightarrow (gf)k = 0$ and by the universal property of $\ker gf$, there is a unique morphism u such that $k'u = k$. It is monic because $ue = 0 \Longrightarrow ke = k'ue = 0$, but k is a kernel and hence it is monic, so $e = 0$.

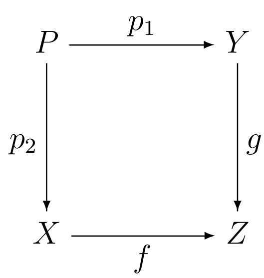
The morphism v exists because $gfk' = g(fk') = 0$, and by the universal property of $\ker g$, there is a unique morphism v such that $k''v = fk'$.

Notice that the morphism u is actually the inclusion, and v is the restriction of f, so if an element in $\ker gf \subseteq A$ is sent to 0 in $\ker g$, then it belongs to $\ker f$; and clearly $vu = 0$, therefore the sequence of kernels is exact at $\ker gf$. Thus, we have $\operatorname{coker} u = \operatorname{Coim} v \cong \operatorname{Im} v$, hence $i = \ker(\operatorname{coker} v)$ is monic.

If g is a $\mathcal{B}$-iso, then $\ker g \in \mathcal{B}$ which implies that $\operatorname{coker} u \in \mathcal{B}$ (because $\mathcal{B}$ is a Serre subcategory), and $\ker f, \operatorname{coker} u \in \mathcal{B}$ implies that $\ker gf \in \mathcal{B}$.

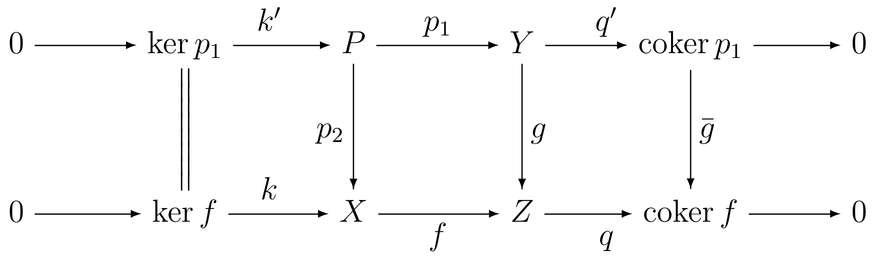
Proceeding dually, we obtain $\operatorname{coker} gf \in \mathcal{B}$, so gf is a $\mathcal{B}$-iso. □

Proposition 2.4.8. *Let P be the pullback of X and Y over Z, and $f : X \longrightarrow Z$ be a $\mathcal{B}$-iso.*

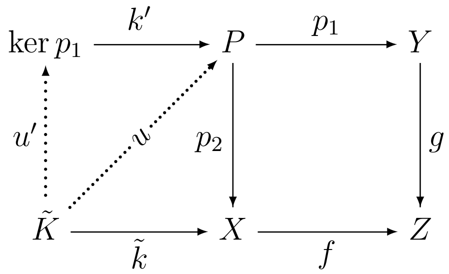


Then p_1 is a $\mathcal{B}$-iso.

Proof. Consider the following diagram:



To see that $\ker p_1 = \ker f$, see the following diagram:



If $\tilde{k}$ is such that $f\tilde{k} = 0$, then take the 0 morphism from $\tilde{K}$ to Y. By the universal property of the pullback P, since $g0 = f\tilde{k} = 0$, there is a unique morphism u such that $p_2 u = \tilde{k}$ and $p_1 u = 0$. The latter implies that there is a unique morphism u' such that $k'u' = u$ (universal property of $\ker p_1$). Hence, $\ker p_1$ satisfies the definition of $\ker f$ and therefore they are equal.

The morphism $\bar{g}$ exists because of the universal property of $\operatorname{coker} p_1$, since $qgp_1 = qfp_2 = 0$.

If we take elements, the pullback is actually the kernel of $(f, -g) : X \times Y \longrightarrow Z$. This is, $P = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$. Also, we can write $\operatorname{coker} f$ as $Z/\operatorname{Im} f$ and $\operatorname{coker} p_1$ as $Y/\operatorname{Im} p_1$. If $\bar{g}(a) = 0$, then $q(gq'^{-1}(a)) = 0 \Longrightarrow g(q'^{-1}(a)) \in \operatorname{Im} f$, so $g(q'^{-1}(a)) = f(x)$ for some x. Therefore, $(x, q'^{-1}(a)) \in P$ and so $q'^{-1}(a) \in \operatorname{Im} p_1$, hence $a = 0$ and we have that $\bar{g}$ is monic.

Since $\ker p_1 = \ker f \in \mathcal{B}$ and $\bar{g}$ monic $\Longrightarrow \operatorname{coker} p_1 \in \mathcal{B}$, p_1 is a $\mathcal{B}$-iso. □

Example 2.4.9. Let R be a Noetherian ring and S a multiplicatively closed subset of R. Let R-$\mathbf{mod}_S$ be the category of finitely generated S-torsion R-modules, this is, there is an $s \in S$ such that $sM = 0$ for $M \in R$-$\mathbf{mod}_S$. Using similar arguments as in Example 2.4.2, it can be shown that this is a Serre subcategory of R-$\mathbf{mod}_{\mathbf{fg}}$. Notice that $S^{-1}M = 0 \Longleftrightarrow M$ is an S-torsion R-module. In this case, it can be seen that $S^{-1}R$-$\mathbf{mod}_{\mathbf{fg}}$ satisfies the definition of quotient abelian category so Theorem 2.4.6 can be applied to see that actually $S^{-1}R\text{-}\mathbf{mod}_{\mathbf{fg}} \cong R\text{-}\mathbf{mod}_{\mathbf{fg}}/R\text{-}\mathbf{mod}_S$ (see [Swan] pp. 114-115).

3

Projective Modules

In this chapter, we introduce the concept of projective module and study some basic properties. We will also introduce projective resolutions and regular rings, although we will not enter into much detail, leaving out the homological part of the topic. In the last section we will study vector bundles over a compact Hausdorff space and we will give a proof of Swan's Theorem, that relates them to projective modules.

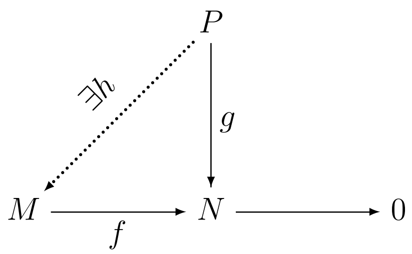
Most parts on projective modules and projective resolutions are found in [WeibHA]. Some concepts of regular rings can be found in [Mat] or [Ros], the latter containing the proof of Swan's Theorem as well. Some previous concepts and examples of vector bundles can be found in more detail in [Hatch], and we will use some results in topology that can be found in [Rud].

Let R be a commutative ring.

3.1 Definition and characterization by idempotents

Definition

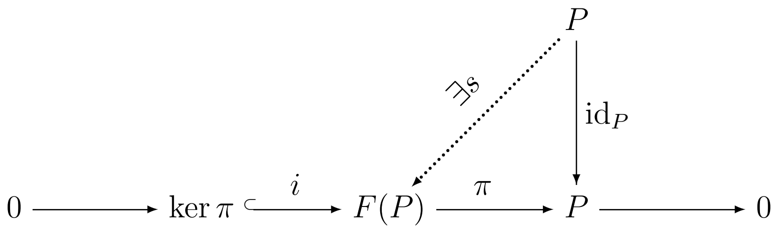
Definition 3.1.1. An R-module P is *projective* if, given a surjective R-module homomorphism $f : M \longrightarrow N$ and a homomorphism $g : P \longrightarrow N$, there is a homomorphism $h : P \longrightarrow M$ such that $g = fh$. This is, the following diagram commutes:



Remark. This definition can be extended to the case of abelian categories in the obvious way.

Proposition 3.1.2. *An R-module P is projective $\Longleftrightarrow$ $P \oplus Q \cong F$ for some R-module Q, and F a free R-module.*

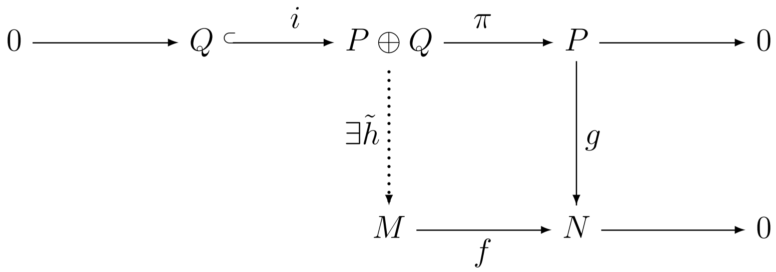
Proof. $\Longrightarrow$: Let $F(P)$ be the free R-module over a set of generators of P and $\pi : F(P) \longrightarrow P$ the projection to P (i.e., it sends each generator of $F(P)$ to the corresponding generator of P and extend by linearity), which is surjective. A section of π can be constructed using the lifting property of projectives as follows:



Therefore, by the splitting lemma[1], $P \oplus \ker \pi \cong F(P)$. Note that, if P is finitely generated, then $F(P)$ is finitely generated and $P \oplus \ker \pi \cong R^n$ for some n.

$\Longleftarrow$: Observe that free R-modules are projective, since we can take $h(e_i) = g^{-1}(f(e_i))$ in the definition, for the generators e_i, and extend by linearity.

Now, if $P \oplus Q$ is free, it is projective and for any surjective R-module homomorphism $f : M \longrightarrow N$, and any homomorphism $g : P \longrightarrow N$, we have:



Taking $\tilde{g} = g\pi$, by the definition of projective module applied to $P \oplus Q$, we have $\tilde{g} = f\tilde{h}$. We have a section s of π given by $s = (\mathrm{id}, 0)$ (i.e., the injection of P into $P \oplus Q$), and thus $g = g\pi s = \tilde{g}s = f\tilde{h}s$. If we take $h = \tilde{h}s$, we have $g = fh$ and hence P is projective. If $P \oplus Q$ is finitely generated, the image of a set of generators by π generates P, and so P is finitely generated. $\square$

Example 3.1.3. i) As we just said, free modules are projective.

[1] A short exact sequence $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$ is split (i.e., $A \cong A' \oplus A''$) $\Longleftrightarrow$ g has a section $\Longleftrightarrow$ f has a retraction.

ii) Let $R = \mathbb{Z}/6\mathbb{Z}$. Then, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong R$ so $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are projective R-modules.

Proposition 3.1.4. *The set of isomorphism classes of finitely generated projective R-modules is denoted by* $\operatorname{Proj} R$ *and it is an abelian monoid with direct sum $\oplus$.*

Proof. From the definition, it is obvious that if P is projective and $P' \cong P$, then P' is projective.

Using Proposition 3.1.2, if P and P' are projective modules, then $P \oplus Q$ and $P' \oplus Q'$ are free for some Q and Q', hence $(P \oplus Q) \oplus (P' \oplus Q') \cong (P \oplus P') \oplus (Q \oplus Q')$ is free, and therefore $P \oplus P'$ is projective. Moreover, the direct sum is well-defined: if $P \cong P'$ and $Q \cong Q'$, then $P \oplus Q \cong P' \oplus Q'$.

For commutativity and associativity, it is clear that $P \oplus P' \cong P' \oplus P$ and $(P \oplus P') \oplus P'' \cong P \oplus (P' \oplus P'')$. The zero element is given by the zero R-module. □

Remark. Note that if $P \oplus P'$ is projective, then P and P' are also projective, since $(P \oplus P') \oplus Q'' \cong P \oplus (P' \oplus Q'') \cong P' \oplus (P \oplus Q'')$ free $\Longrightarrow$ P and P' projective.

Proposition 3.1.5. *Projective R-modules are torsion-free.*

Proof. Let P be a projective R-module. Then $P \oplus Q \cong F$ is free for some Q. Take $p : F \longrightarrow P$ the projection onto P. Using the definition of projective over this surjection, we have a section s of p. For any nonzero element $m \in P$ and every nonzero $r \in R$, since s is a section, $s(m) \neq 0$ and hence $r\,s(m) = s(rm) \neq 0$, so $rm \neq 0$. This is, P has no torsion elements. □

Characterization by idempotents

If P is a finitely generated projective R-module, we have $P \oplus Q = R^n$ for some Q and some n, maybe replacing P by an isomorphic module. We consider a morphism $p : R^n \longrightarrow R^n$ such that $p|_P = \mathrm{id}_P$ and $p|_Q = 0$. This is an idempotent, i.e. $p^2 = p$, and it can be seen as an $n \times n$ matrix, since a homomorphism from R^n to R^n is determined by the n coordinates of the images of the n canonical generators. Equivalently, P is given by an idempotent p, by $P = pR^n$. It is projective since p idempotent $\Longrightarrow pR^n \oplus \ker p = pR^n \oplus (1 - p)R^n = R^n$. We will write P_p for the projective determined by the idempotent p.

Proposition 3.1.6. *Let $p \in \mathrm{Idem}(n, R)$ (idempotent $n \times n$ matrices over R) and $q \in \mathrm{Idem}(m, R)$. Then $P_p \cong P_q \Longleftrightarrow \exists u \in GL(N, R)$ such that $upu^{-1} = q$ (understanding p and q in $\mathrm{Idem}(N, R)$ by adding the needed zeroes) for some N.*

Proof. If $upu^{-1} = q$, then $P_p \cong P_q$ because u is an isomorphism.

If $P_p \cong P_q$, take the isomorphism $\alpha : pR^n \longrightarrow qR^m$. We extend it to an R-module homomorphism $a : R^n = pR^n \oplus (1-p)R^n \longrightarrow R^m$ with $a|_{(1-p)R^n} = 0$ and embedding qR^m into R^m. Define $b : R^m = qR^m \oplus (1-q)R^m \longrightarrow R^n$ extending α^{-1} in the same way. We can see a as an $m \times n$ matrix and b as an $n \times m$ matrix, satisfying $ab = q$, $ba = p$, $a = aba = ap = qa$ and $b = bab = bq = pb$. Take $N = n + m$, and notice that

$$\begin{pmatrix} 1-p & b \\ a & 1-q \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1-p & b \\ a & 1-q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-p & b \\ a & 1-q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \sim \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

where the last step is done by conjugation by a permutation matrix. □

We can define $M(R) = \bigcup M(n,R)$ with embeddings $M(n,R) \longrightarrow M(n+1,R)$ given by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. In a similar way, we can define $GL(R) = \bigcup GL(n,R)$ with embeddings $GL(n,R) \longrightarrow GL(n+1,R)$ given by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. Let $\mathrm{Idem}(R)$ be the set of idempotent matrices in $M(R)$. With this definitions, $GL(R)$ acts on $\mathrm{Idem}(R)$ by conjugation and Proposition 3.1.6 can be written as follows.

Proposition 3.1.7. $\mathrm{Proj}\, R \cong \mathrm{Idem}(R)_{GL(R)}$. *This is, the monoid* $\mathrm{Proj}\, R$ *is isomorphic to the conjugation classes of* $\mathrm{Idem}(R)$ *by elements of* $GL(R)$. *The monoid operation in the second case is defined by* $a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, *which is well-defined up to conjugation.* □

3.2 Exactness of Hom $(P, -)$ and $P \otimes -$

In this section we will characterize a projective P by the exactness of the functor $\mathrm{Hom}(P, -) : R$-**mod** $\longrightarrow$ **Set** and we will see that $P \otimes - : R$-**mod** $\longrightarrow R$-**mod** is also exact when P is projective. We will start by proving that $\mathrm{Hom}(M, -)$ is in general left exact and afterwards we will study the case $\mathrm{Hom}(P, -)$ for P projective.

The following is true in any abelian category:

Proposition 3.2.1. *The functor* $\mathrm{Hom}(M,-)$ *is left exact. Recall that the functor* $\mathrm{Hom}(M,-)$ *takes an object* A *to* $\mathrm{Hom}(M,A)$ *and a morphism* $f : A \longrightarrow B$ *to* $f_* := \mathrm{Hom}(M,f) : \mathrm{Hom}(M,A) \longrightarrow \mathrm{Hom}(M,B)$ *defined by* $f_* : g \mapsto fg$.

Proof. For a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

we must show that

$$0 \longrightarrow \mathrm{Hom}(M,A) \xrightarrow{f_*} \mathrm{Hom}(M,B) \xrightarrow{g_*} \mathrm{Hom}(M,C)$$

is exact. If $\alpha \in \mathrm{Hom}(M,A)$, then $f_*\alpha = f\alpha = 0 \Longrightarrow \alpha = 0$ because f is monic, hence f_* is monic. Since $gf = 0$, $g_*f_* = (gf)_* = 0$. Now notice that f monic $\Longrightarrow A \cong \ker g$ and therefore A has the universal kernel property: if $g_*\beta = g\beta = 0$ for $\beta \in \mathrm{Hom}(M,B)$, then $\exists \alpha' \in \mathrm{Hom}(M,A)$ such that $\beta = f\alpha' = f_*\alpha'$. Therefore $\mathrm{Im}\, f_* = \ker g_*$. □

Proposition 3.2.2. *P is projective $\Longleftrightarrow$ the functor* $\mathrm{Hom}(P,-)$ *is exact.*

Proof. We only have to show that if $g : B \longrightarrow C$ is surjective, then P is projective $\Longleftrightarrow g_* : \mathrm{Hom}(P,B) \longrightarrow \mathrm{Hom}(P,C)$ is surjective. But, given $\gamma \in \mathrm{Hom}(P,C)$, by definition P is projective $\Longleftrightarrow \exists \beta \in \mathrm{Hom}(P,B)$ such that $\alpha = g\beta = g_*\beta \Longleftrightarrow g_*$ is surjective. □

Now we want to see that $P \otimes -$ is exact. We will first see some results that are needed to prove that $M \otimes -$ is right exact in general, and then we will see some results regarding flat modules to finally see that $P \otimes -$ is exact when P is projective.

Proposition 3.2.3. *The sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if

$$0 \longrightarrow \mathrm{Hom}(M,A) \xrightarrow{f_*} \mathrm{Hom}(M,B) \xrightarrow{g_*} \mathrm{Hom}(M,C) \longrightarrow 0$$

is exact for all M.

Proof. Exactness at A: if we take $M = \ker f$ then the kernel morphism $k_f \in \mathrm{Hom}(M,A)$ satisfies $0 = fk_f = f_*k_f$ and f_* is monic, so $k_f = 0 \Longrightarrow f$ is monic.

Exactness at B: if $M = A$, $0 = g_* f_* \operatorname{id}_A = gf$, so $\operatorname{Im} f \subseteq \ker g$. If $M = \ker g$, then the kernel morphism $k_g \in \operatorname{Hom}(M, B)$ satisfies $0 = gk_g = g_* k_g \implies \exists \alpha \in \operatorname{Hom}(M, A)$ such that $k_g = f_* \alpha = f\alpha \implies \ker g \subseteq \operatorname{Im} f$ and so $\ker g = \operatorname{Im} f$.

Exactness at C: if $M = C$, g_* surjective $\implies \exists \beta \in \operatorname{Hom}(M, B)$ such that $\operatorname{id}_C = g_* \beta = g\beta \implies g$ surjective. □

Remark. Observe that we have used f_* injective $\implies f$ injective; and g_* surjective $\implies g$ surjective, and these hypothesis are not used anywhere else, thus the result still holds for right exactness or left exactness alone.

Proposition 3.2.4. *Let M, N and L be R-modules. There is an isomorphism* $\operatorname{Hom}(M \otimes_R N, L) \cong \operatorname{Hom}(M, \operatorname{Hom}(N, L))$.

Proof. Let $f : \operatorname{Hom}(M \otimes_R N, L) \longrightarrow \operatorname{Hom}(M, \operatorname{Hom}(N, L))$ be such that, for $\alpha \in \operatorname{Hom}(M \otimes_R N, L)$, $f(\alpha) : M \ni m \mapsto f(\alpha)(m) \in \operatorname{Hom}(N, L)$ and $f(\alpha)(m) : N \ni n \mapsto f(\alpha)(m)(n) = \alpha(m \otimes n) \in L$.

And let $g : \operatorname{Hom}(M, \operatorname{Hom}(N, L)) \longrightarrow \operatorname{Hom}(M \otimes_R N, L)$ be such that, for $\beta \in \operatorname{Hom}(M, \operatorname{Hom}(N, L))$, $g(\beta) : M \otimes_R N \ni m \otimes n \mapsto \beta(m)(n) \in L$.

It is easy to check that they are R-module homomorphisms and, by construction, $gf(\alpha) = \alpha$ and $fg(\beta) = \beta \implies f$ and g are isomorphisms. □

Remark. Observe that the isomorphism is natural with respect to N, in the sense that, if $h : N \longrightarrow N'$, then the following diagram commutes:

$$
\begin{array}{ccc}
\operatorname{Hom}(M \otimes_R N, L) & \xrightarrow{\operatorname{Hom}(1 \otimes h, L)} & \operatorname{Hom}(M \otimes_R N', L) \\
\cong \downarrow & & \downarrow \cong \\
\operatorname{Hom}(M, \operatorname{Hom}(N, L)) & \xrightarrow[\operatorname{Hom}(M, \operatorname{Hom}(h, L))]{} & \operatorname{Hom}(M, \operatorname{Hom}(N', L))
\end{array}
$$

Which can be seen by taking $\alpha \in \operatorname{Hom}(M \otimes_R N, L)$ and for each $m \otimes n \in M \otimes_R N$ we have

$$(f \circ \operatorname{Hom}(1 \otimes h, L))(m)(n) = \alpha(m \otimes h(n)) = (\operatorname{Hom}(M, \operatorname{Hom}(h, L)) \circ f)(m)(n).$$

Since both morphisms are equal evaluated at $m \otimes n$, $\forall m \otimes n \in M \otimes_R N$, by linearity, they are equal in $M \otimes_R N$ and thus the morphisms are equal and the diagram is commutative.

We will prove now Proposition 1.2.14. Recall its statement:

Proposition 1.2.14. *Let*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be a short exact sequence of R-modules, and let N be an R-module. Then

$$M' \otimes_R N \xrightarrow{f \otimes \mathrm{id}} M \otimes_R N \xrightarrow{g \otimes \mathrm{id}} M'' \otimes_R N \longrightarrow 0$$

is exact.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}(N, \mathrm{Hom}(M'', L)) & \longrightarrow & \mathrm{Hom}(N, \mathrm{Hom}(M, L)) & \longrightarrow & \mathrm{Hom}(N, \mathrm{Hom}(M', L)) \\ & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 \longrightarrow & \mathrm{Hom}(N \otimes_R M'', L) & \longrightarrow & \mathrm{Hom}(N \otimes_R M, L) & \longrightarrow & \mathrm{Hom}(N \otimes_R M', L) \end{array}$$

The first row is exact because of Proposition 3.2.1 (applied two times). The isomorphisms and the commutativity of the diagram imply exactness of the second row. This is true for all L, hence by Proposition 3.2.3, we have that

$$M' \otimes_R N \xrightarrow{f \otimes 1} M \otimes_R N \xrightarrow{g \otimes 1} M'' \otimes_R N \longrightarrow 0$$

is exact. □

Proposition 3.2.5. *Let $B = \bigoplus_i B_i$ be an R-module. Then B is flat $\Longleftrightarrow$ each B_i is flat.*

Proof. By Proposition 1.2.14, it is sufficient to show that, given an injection $f : M \longrightarrow N$, then $\mathrm{id}_B \otimes f : B \otimes_R M \longrightarrow B \otimes_R N$ is injective $\Longleftrightarrow$ each $\mathrm{id}_{B_i} \otimes f : B_i \otimes_R M \longrightarrow B_i \otimes_R N$ is injective. This commutative diagram

$$\begin{array}{ccc} (\oplus_i B_i) \otimes_R M & \xrightarrow{\mathrm{id}_B \otimes f} & (\oplus_i B_i) \otimes_R N \\ \cong \downarrow & & \downarrow \cong \\ \oplus_i (B_i \otimes_R M) & \xrightarrow{\oplus_i (\mathrm{id}_{B_i} \otimes f)} & \oplus_i (B_i \otimes_R N) \end{array}$$

which is given by the bilinearity of the tensor product, shows exactly the required condition. □

Corollary 3.2.6. *A free R-module is flat.*

Proof. If M is an R-module, $R \otimes_R M \cong M$ since every element of $R \otimes_R M$ clearly can be written as $1 \otimes m$ for some $m \in M$. Thus, $R \otimes_R -$ is an exact functor and R is flat. Now, by Proposition 3.2.5, any free R-module is flat.

This can also be proved straightaway as in Example 1.2.17. □

Proposition 3.2.7. *Let P be a projective R-module. Then the functor $P \otimes -$ is exact. This is, projective R-modules are flat.*

Proof. P is a direct summand of a free R-module, hence by Corollary 3.2.6 and Proposition 3.2.5 it is flat, and thus $P \otimes -$ is an exact functor. □

3.3 Specific rings

In this section we will study the structure of finitely generated projective R-modules in two specific cases: when R is a principal ideal domain and when R is a local ring. What we will prove is that, in both cases, finitely generated projective R-modules are actually free.

Principal ideal domains

Recall that finitely generated R-modules can be classified when the ring is a principal ideal domain.

Theorem 3.3.1 (Structure Theorem for finitely generated modules over a PID)**.** *Let R be a PID and let M be a finitely generated R-module. Then there are uniquely determined nonzero ideals $(d_1) \supseteq \cdots \supseteq (d_n)$ such that*

$$M \cong R^k \oplus \bigoplus_i R/(d_i).$$

This theorem allows us to write a first proof of the following result.

Theorem 3.3.2. *Let R be a PID. Then every finitely generated projective R-module is free.*

Proof. Let P be a finitely generated projective R-module. By Proposition 3.1.5 and Theorem 3.3.1, it is clear that $P \cong R^k$.

Alternatively, $P \oplus Q \cong R^n$ and applying Theorem 3.3.1 to P, Q and R^n, it follows that $P \cong R^k$ with $k \leq n$. □

Instead of using Theorem 3.3.1, we can prove Theorem 3.3.2 independently with the following result.

Proposition 3.3.3. *Let R be a PID. Then, every submodule M of R^n is free.*

Proof. We will prove by induction that $M \cong R^k$ for some $k \leq n$. For $n = 0$, the statement holds. Assume it holds for $n-1$ and let $\pi : R^n \longrightarrow R$ the projection on the last coordinate. Then $\pi(M)$ is a submodule of R, i.e., an ideal of R. If $\pi(M) = 0$, then M is a submodule of $\ker \pi \cong R^{n-1}$, so it is free by hypothesis. If $\pi(M) \neq 0$, then it is a nonzero ideal of R, so it is generated by one element, and the homomorphism that takes this element to 1 in R is an isomorphism (because R is a PID), this is, $\pi(M) \cong R$. Therefore, the short exact sequence

$$0 \longrightarrow \ker \pi|_M \hookrightarrow M \xrightarrow{\pi|_M} \pi(M) \longrightarrow 0$$

splits, so $M \cong \ker \pi|_M \oplus R$ and $\ker \pi|_M$ is a submodule of R^{n-1}, hence $M \cong R^{k'} \oplus R \cong R^k$ with $k = k' + 1 \leq (n-1) + 1 = n$. □

Local rings

Theorem 3.3.4. *Let R be a local ring. Then every finitely generated projective R-module is free.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. Let P be a finitely generated projective R-module. $P/\mathfrak{m}P$ is a vector space over $R/\mathfrak{m}$. Take a basis $e_1, \ldots, e_n$ of this space and let $x_1, \ldots, x_n$ be elements of P such that its projection to the quotient module are the elements of the base. By Corollary 1.2.10, we know that $x_1, \ldots, x_n$ generate P. We have a short exact sequence

$$0 \longrightarrow \ker \pi \overset{i}{\hookrightarrow} R^n \xrightarrow{\pi} P \longrightarrow 0$$

with R^n the free module over the generators of P. Since P is projective, the sequence splits, so $R^n \cong P \oplus Q$ (with $Q = \ker \pi$). Tensoring by $R/\mathfrak{m}$, we have $(R/\mathfrak{m}) \otimes R^n \cong (R/\mathfrak{m} \otimes P) \oplus (R/\mathfrak{m} \otimes Q) \Longrightarrow (R/\mathfrak{m})^n \cong P/\mathfrak{m}P \oplus Q/\mathfrak{m}Q$. Recall that $P/\mathfrak{m}P$ has dimension n as an $R/\mathfrak{m}$-vector space, hence $Q/\mathfrak{m}Q = 0$, this is, $\mathfrak{m}Q = Q$ and, by Nakayama's Lemma, $Q = 0$. Therefore $P \cong R^n$. □

Actually, Kaplansky showed in [Kapl] the following result:

Theorem 3.3.5. *Let R be a local ring. Then every projective R-module is free.*

Which means that it is not needed for the module to be finitely generated. The proof is obviously more complex and can be found in [Knight], Theorem 3.3.4.

The following results can be used to see how a projective module localizes at prime ideals.

Proposition 3.3.6. *Let R be a ring and $\mathfrak{p}$ a prime ideal. Then if P is a finitely generated projective R-module, $P_{\mathfrak{p}}$ is free.*

Proof. $P \oplus Q \cong R^n$ for some Q, tensoring by $R_{\mathfrak{p}}$, by Proposition 1.3.7, $P_{\mathfrak{p}} \oplus Q_{\mathfrak{p}} \cong (R_{\mathfrak{p}})^n$, so $P_{\mathfrak{p}}$ is a finitely generated projective $R_{\mathfrak{p}}$-module, but $R_{\mathfrak{p}}$ is a local ring, hence $P\mathfrak{p}$ is free by Theorem 3.3.4. □

Proposition 3.3.7. *Let R be a ring and $\mathfrak{p}$ a prime ideal. Let $S_f = \{f^n \mid f \in R,\ n \geq 0\}$. If P is a finitely generated R-module, there is a $t \in R - \mathfrak{p}$ such that $P[\frac{1}{t}] := S_t^{-1}P$ is free.*

Proof. We know $P_{\mathfrak{p}} \cong (R_{\mathfrak{p}})^n$ by Proposition 3.3.6. Let $f_{\mathfrak{p}} : (R_{\mathfrak{p}})^n \longrightarrow P_{\mathfrak{p}}$ be an isomorphism, and put the generators of $(R_{\mathfrak{p}})^n$ and $P_{\mathfrak{p}}$ over a common denominator. By clearing denominators, $f_{\mathfrak{p}}$ lifts to a morphism $f : R^n \longrightarrow P$ (such that it induces the morphism $f_{\mathfrak{p}}$ when localizing at $\mathfrak{p}$). Since $\operatorname{coker} f$ is finitely generated and $(\operatorname{coker} f)_{\mathfrak{p}} = \operatorname{coker}(f_{\mathfrak{p}}) = 0$, we can take s the product of the elements of $R - \mathfrak{p}$ that annihilate the generators of $\operatorname{coker} f$. For this s, $f[\frac{1}{s}] : (R[\frac{1}{s}])^n \longrightarrow P[\frac{1}{s}]$ has also $\operatorname{coker}(f[\frac{1}{s}]) = 0$. Notice that $P[\frac{1}{s}]$ is projective, because tensoring $P \oplus Q \cong R^m$ by $R[\frac{1}{s}]$ we get $P[\frac{1}{s}] \oplus Q[\frac{1}{s}] \cong (R[\frac{1}{s}])^m$. Hence, $f[\frac{1}{s}]$ has a section, so $(R[\frac{1}{s}])^n \cong P[\frac{1}{s}] \oplus M$ (with $M = \ker f[\frac{1}{s}]$), and $M_{\mathfrak{p}} = 0$ since $((R[\frac{1}{s}])^n)_{\mathfrak{p}} \cong (R_p f)^n \otimes [\frac{1}{s}] \cong P_{\mathfrak{p}} \otimes R[\frac{1}{s}] \cong (P[\frac{1}{s}])_{\mathfrak{p}}$. Hence, there is a $s' \in R - \mathfrak{p}$ such that $s'M = 0$ and therefore $f[\frac{1}{t}] : (R[\frac{1}{t}])^n \longrightarrow P[\frac{1}{t}]$ with $t = ss'$ is an isomorphism. □

These propositions, together with the notion of $\operatorname{Spec} R$, describe the property that finitely generated projective R-modules are locally free. Moreover, we have seen that the map

$$\begin{aligned} \operatorname{rk}(P) : \operatorname{Spec} R &\longrightarrow \mathbb{Z} \\ \mathfrak{p} &\longmapsto \operatorname{rk}(P_{\mathfrak{p}}) \end{aligned}$$

that assigns the rank of $P_{\mathfrak{p}}$ to each point $\mathfrak{p}$ is continuous, since the rank of $P_{\mathfrak{p}}$ is locally constant. If $\operatorname{Spec} R$ is connected, then $\operatorname{rk}(P)$ is constant, this is, we can assign a rank to each finitely generated projective R-module.

3.4 Projective resolutions

Projective resolutions are a construction that can be made for R-modules (or for some objects of an abelian category) that allows us to study the R-modules in terms of projective R-modules. In the last chapter we will understand this approach in more detail.

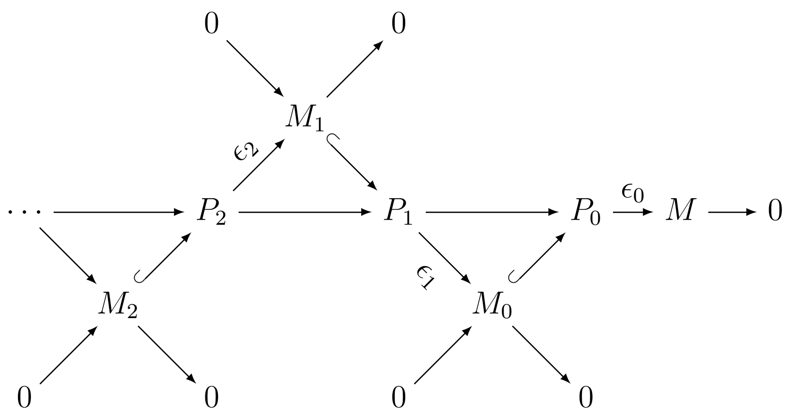
Definition 3.4.1. Let M be an R-module (or, more generally, an object of an abelian category). A *projective resolution* of M is a sequence $P_\bullet$ such that

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is exact, and the P_i are projective R-modules (respectively, projective objects of the abelian category). If $P_i = 0$ for $i > n$ it is said that the projective resolution is of *finite-type* or that it is a *finite projective resolution*, and has length n.

Proposition 3.4.2. *Every R-module M has a projective resolution.*

Proof. Let P_0 be a projective R-module such that we have a homomorphism $\epsilon_0 : P_0 \longrightarrow M$ that is surjective (it always exists, e.g., take P_0 the free R-module over the generators of M and ϵ the projection onto M). Let $M_0 = \ker \epsilon$. We can construct the rest of the sequence inductively: choose a projective P_n and a surjection $\epsilon_n : P_n \longrightarrow M_{n-1}$ and define $M_n := \ker \epsilon_n$.



The resolution consists of the P_i and the maps are the compositions of the ϵ_i with the inclusions of the kernels. The sequence is exact because the image of each morphism is, by construction, the kernel of the next one. □

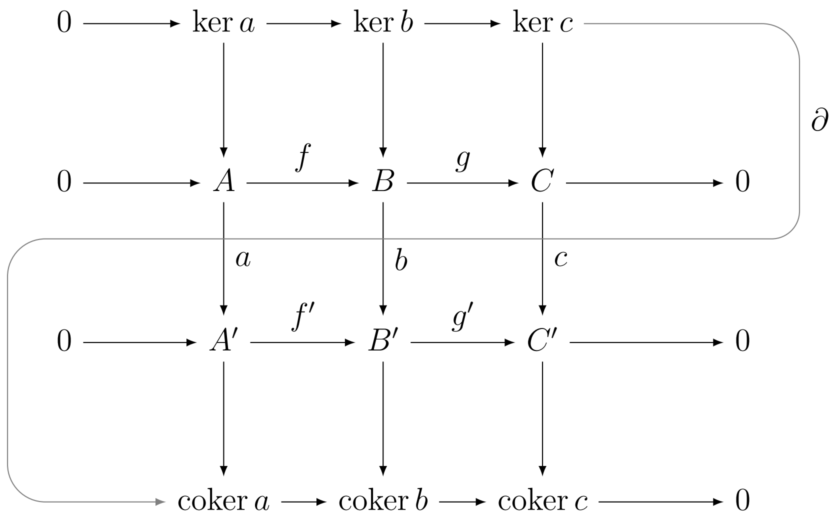
Remark. It can be seen that the existence of projective resolutions is unique up to chain homotopy (see [WeibHA] Theorem 2.2.6).

The following result is a basic lemma in homological algebra. The proof is a simple diagram chase.

Lemma 3.4.3 (Snake Lemma). *Let*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \longrightarrow & 0
\end{array}$$

be a commutative diagram with exact rows. Then, the kernels and cokernels of the morphisms a, b and c make up an exact sequence as follows:



The connecting morphism ∂ *is defined by* $\partial := f'^{-1}bg^{-1}$.

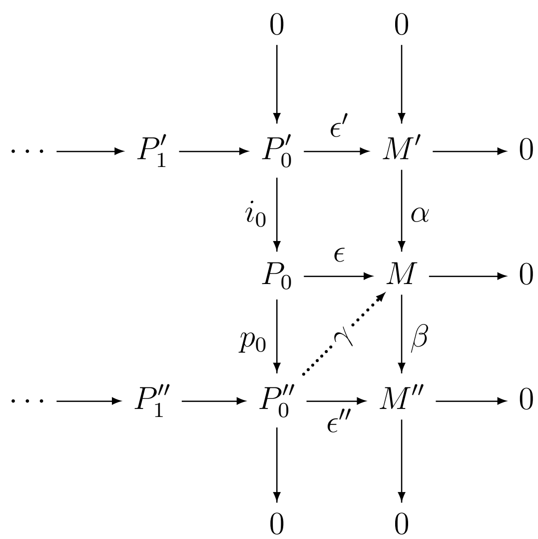
Remark. Notice that the Snake Lemma is true in any abelian category by the remark of the Freyd-Mitchell Embedding Theorem.

Proposition 3.4.4 (Horseshoe Lemma)**.** *Let R be a ring. If*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

is a short exact sequence of R-modules such that M' and M'' have projective resolutions of length n, then M has a projective resolution of length n.

Proof. Let $P'_\bullet \longrightarrow M' \longrightarrow 0$ and $P''_\bullet \longrightarrow M'' \longrightarrow 0$ be the corresponding projective resolutions. Since P''_0 is projective and β is surjective, there is a morphism $\gamma : P''_0 \longrightarrow M$ such that $\beta\gamma = \epsilon''$. Let $P_0 = P'_0 \oplus P''_0$ and $\epsilon = \alpha\epsilon' \oplus \gamma$.



Hence, the squares commute and we have two exact columns. Then, the Snake Lemma applies, and shows that $\operatorname{coker}\epsilon = 0$, i.e, ϵ is surjective, and that

$$0 \longrightarrow \ker\epsilon' \longrightarrow \ker\epsilon \longrightarrow \ker\epsilon'' \longrightarrow 0$$

is exact. If we replace the original short exact sequence by the latter, the argument can be repeated (with projective resolutions of $\ker\epsilon'$ and $\ker\epsilon''$ of length $n-1$; notice that $P_1' \longrightarrow \ker\epsilon'$ and $P_1'' \longrightarrow \ker\epsilon''$ are surjective by the exactness of the projective resolutions), and the statement holds by induction. It is clear that if M' and M'' have projective resolutions of length n, then $P_i = 0 \oplus 0 = 0$, $\forall i > n$, so M has a projective resolution of length n. □

Definition 3.4.5. A ring R is *regular* if it is Noetherian, and every finitely generated R-module has a finite projective resolution.

Remark. The term "regular" comes from algebraic geometry:

Let X be the affine variety defined in $\mathbb{A}^n = \mathbb{C}^n$ by the zeros of a finite set of polynomials

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \quad\vdots \\ f_r(x_1, \dots, x_n) = 0 \end{cases}$$

The ring $\mathbb{C}[X] = \mathbb{C}[x_1, \dots, x_n]/\langle f_1, \dots, f_r\rangle$ is called the ring of algebraic functions on X. Then, the ring $R = \mathbb{C}[X]$ is regular $\iff$ X is nonsingular.

Example 3.4.6. i) Every field is a regular ring, since all finitely generated modules over a field are free, so the identity $M \longrightarrow M$ gives a finite projective resolution.

ii) Moreover, a principal ideal domain R is a regular ring. Indeed, we have short exact sequences

$$0 \longrightarrow R \xrightarrow{\;d_i\;} R \longrightarrow R/(d_i) \longrightarrow 0,$$

and the identity map gives also a projective resolution for R^n, as in the previous example. Hence, by Theorem 3.3.1 and the Horseshoe Lemma, we can build a projective resolution of finite length for any finitely generated R-module (or without the Horseshoe Lemma, $R^{n+k} \longrightarrow M$ is exhaustive with n, k from Theorem 3.3.1, and its kernel is a direct sum of n principal ideals, which is isomorphic to R^n).

Proposition 3.4.7. *Let R be a regular ring and S a multiplicatively closed subset of R. Then $S^{-1}R$ is regular.*

Proof. Let M be a finitely generated $S^{-1}R$-module and let $m_1/s_1, \dots m_r/s_r$ be a set of generators of M. Put all generators to a common denominator and let $n_1, \dots, n_r$ be the numerators. Let N be the R-module generated by the n_i. Then, N is finitely generated and such that $S^{-1}N = M$. Since R is regular, we have a finite projective resolution

$$0 \longrightarrow P_k \longrightarrow \cdots \longrightarrow P_0 \longrightarrow N \longrightarrow 0.$$

Notice that $S^{-1}P_i$ is projective since if $P_i \oplus Q_i$ is free, then $S^{-1}R \otimes_R (P_i \oplus Q_i) \cong S^{-1}P_i \oplus S^{-1}Q_i$ is free. Tensoring by $S^{-1}R$ is exact, so we have a finite projective resolution

$$0 \longrightarrow S^{-1}P_k \longrightarrow \cdots \longrightarrow S^{-1}P_0 \longrightarrow S^{-1}N \cong M \longrightarrow 0,$$

hence $S^{-1}R$ is regular. □

Example 3.4.8. If R is a regular ring and $\mathfrak{p}$ is a prime ideal, then $R_\mathfrak{p}$ is regular.

Remark. There is a theorem, due to Serre, that shows a converse of the last example. This is, R is a regular ring $\Longleftrightarrow$ $R_\mathfrak{p}$ is regular $\forall \mathfrak{p} \in \operatorname{Spec} R$ (see [Mat] Theorem 45).

If R is a regular ring, it can be seen that $R[t]$ and $R[t, t^{-1}]$ are also regular rings. The usual proof of these facts needs a characterization of regular rings in terms of the Ext functors defined in homological algebra (see chapters 2 and 3 from [WeibHA] for definitions and properties). We shall not enter into this detail and assume the following results (see Theorem 3.2.3 and Corollary 3.2.4 from [Ros] for proofs):

Theorem 3.4.9 (Hilbert Syzygy Theorem)**.** *If R is (left) regular, then so is $R[t]$.*

And by Proposition 3.4.7:

Corollary 3.4.10. *If R is (left) regular, then so is $R[t, t^{-1}]$.*

3.5 Vector bundles. Swan's Theorem

In this section we will see how the notion of vector bundle and projective module are related. We will show that categories of vector bundles are actually equivalent to a category of projective modules over a certain ring.

Let X be a compact Hausdorff topological space[2]. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $C^{\mathbb{F}}(X)$ the ring of continuous functions over X with values in $\mathbb{F}$, with pointwise addition and multiplication.

[2] X being compact Hausdorff is needed for some results, especially those involving partitions of unity or the use of finite open covers, but most definitions and general results on vector bundles are true for any topological space X.

Definition 3.5.1. A *vector bundle* over X consists of a topological space E and a surjective continuous map $p : E \longrightarrow X$ such that:

i) $p^{-1}(x)$ has an $\mathbb{F}$-vector space structure $\forall x \in X$. The $p^{-1}(x)$ are called the *fibers* of p.

ii) There is an open cover $\{U_\alpha\}$ of X such that there is a homeomorphism $h_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{F}^n$ for each U_α with restrictions $h_\alpha|_{p^{-1}(x)} : p^{-1}(x) \longrightarrow \{x\} \times \mathbb{F}^n$ vector space isomorphisms for every $x \in U_\alpha$. The h_α are called *local trivializations* of the vector bundle.

E is called the *total space* and X the *base space*. The map p is the *projection map*.

Definition 3.5.2. A *trivial* vector bundle is a vector bundle with $E = X \times \mathbb{F}^n$ and $p : X \times \mathbb{F}^n \longrightarrow X$ the projection on the first component.

Example 3.5.3. i) A common example of vector bundle is the tangent bundle TM of a differentiable manifold M, formed by tangent vectors on each point of the manifold.

For example, the tangent bundle of the unit sphere S^n in $\mathbb{R}^{n+1}$ is $p : TS^n \longrightarrow S^n$ with $TS^n = \{(x, v) \in S^n \times \mathbb{R}^{n+1} \mid v \perp x\}$.

ii) The Möbius bundle $\mathcal{M}$, which is the quotient space $I \times \mathbb{R}/\sim$, where $(0, t) \sim (1, -t)$.

Definition 3.5.4. Let $p : E \longrightarrow X$ and $p' : E' \longrightarrow X$ be two vector bundles over X. A morphism between them is given by a continuous map $f : E \longrightarrow E'$ which is linear on each fiber and such that

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & = & X \end{array}$$

is commutative.

It is an isomorphism if f is a homeomorphism and it is a linear isomorphism on each fiber.

Proposition 3.5.5. *In* Definition 3.5.4, *if f is a linear isomorphism on each fiber, then it is an isomorphism.*

Proof. The commutative diagram and the hypothesis guarantee that f is bijective, so we only have to prove that f^{-1} is continuous. We can restrict to an open set $U \subseteq X$ where both vector bundles are trivial and, by composing with the local trivializations, we have $f(x, v) = (x, g_x(v))$, where $g_x \in GL(n, \mathbb{F})$. Here, g_x depends continuously on x, so g_x^{-1} (the entries of g_x depend continuously on x and the entries of g_x^{-1} depend continuously on the entries of g_x) also depends continuously on x and thus $f^{-1}(x, v) = (x, g_x^{-1}(v))$ is continuous. □

Definition 3.5.6. Let $p : E \longrightarrow X$ and $p' : E' \longrightarrow X$ be two vector bundles over X. Their *direct sum* is defined to be $\bar{p} : E \oplus E' \longrightarrow X$, where $E \oplus E' := \{(v, v') \in E \times E' \mid p(v) = p'(v')\}$ and $\bar{p}(v, v') = p(v) = p(v')$.

Proposition 3.5.7. *The direct sum of two vector bundles is a vector bundle.*

Proof. The fibers are clearly vector spaces. To check that it is locally trivial, notice that $\tilde{p} = p \times p' : E \times E' \longrightarrow X \times X$ is a vector bundle over $X \times X$ since the fibers $p^{-1}(x) \times p'^{-1}(x')$ are vector spaces and we have local trivializations $h_\alpha \times h_\beta$. Now, identify X with the diagonal $\Delta_X = \{(x, x) \in X \times X\} \subseteq X \times X$, and $\bar{p} = \tilde{p}|_{\tilde{p}^{-1}(\Delta_X)} : \tilde{p}^{-1}(\Delta_X) = E \oplus E' \longrightarrow \Delta_X \cong X$, which inherits the vector bundle structure from $\tilde{p} : E \times E' \longrightarrow X \times X$. □

Definition 3.5.8. A *partition of unity subordinated to a finite open cover* $\{U_\alpha\}$ is a collection of continuous functions $\varphi_\alpha : X \longrightarrow [0, 1]$ such that $\varphi_\alpha|_{X \setminus U_\alpha} \equiv 0$ and $\sum_\alpha \varphi_\alpha(x) = 1$, $\forall x \in X$. If X is compact Hausdorff, they always exist (see [Rud] Theorem 2.13).

The following proposition will be used to show that the image of the category of vector bundles over X is in $\operatorname{Proj} C^{\mathbb{F}}(X)$, as we state in the last theorem of this section.

Proposition 3.5.9. *Let $p : E \longrightarrow X$ be a vector bundle over X (which is compact Hausdorff). There is an inner product $\langle \cdot, \cdot \rangle$ for the vector bundle. This is, a function $\langle \cdot, \cdot \rangle : E \times E \longrightarrow \mathbb{F}$ such that it is positive definite, sesquilinear (i.e., $\langle a\, u, v \rangle = a \langle u, v \rangle$ and $\langle u, a\, v \rangle = a^* \langle u, v \rangle$) and $\langle u, v \rangle = \langle v, u \rangle^*$.*

Proof. We can take a finite subcover of the U_α, so we have a partition of unity given by φ_α. We define $\langle v, w \rangle = \sum_\alpha \varphi_\alpha(p(v)) \langle v, w \rangle_\alpha$, where $\langle v, w \rangle_\alpha$ is the inner product on each $p^{-1}(U_\alpha)$ given by the inner product in $\mathbb{F}^n$ (obtained with the local trivializations). This inner product is well-defined and continuous on $v, w \in E$ and $x = p(v) = p(w) \in X$. □

Definition 3.5.10. A *section* of a vector bundle $p : E \longrightarrow X$ is a continuous map $s : X \longrightarrow E$ that is a section of p in the usual sense (i.e., $ps = \mathrm{id}_X$). A section can be understood locally as a continuous function on X since $s(x) = (x, \bar{s}(x))$, where $\bar{s}$ is continuous. The set of sections of a vector bundle is denoted by $\Gamma(X, E)$ and it has naturally a $C^{\mathbb{F}}(X)$-module structure.

A special section is the *zero section*, which is such that $\bar{s} = 0$ in the previous notation. Since any vector bundle isomorphism between two vector bundles maps homeomorphically the images of the zero sections of each vector bundle, we can identify nonisomorphic vector bundles by looking at the complement of the image of the zero section. Another way to compare two vector bundles is by studying a section that is nonzero everywhere.

Example 3.5.11. i) The trivial bundle $S^1 \times \mathbb{R}$ is not isomorphic to the Möbius bundle $\mathcal{M}$. Indeed, the complement of the zero section is connected in the second case, but not in the first one.

ii) The tangent bundle TS^{2n} in $\mathbb{R}^{2n+1}$ is not isomorphic to the trivial bundle $S^{2n} \times \mathbb{R}^{2n}$. If they were, a nonzero section of the trivial bundle (e.g., $\bar{s}$ constant) would be mapped to a nonzero section of the tangent bundle by the isomorphism between both vector bundles, but there is no such section as the Hairy Ball Theorem states.

Definition 3.5.12. A *vector subbundle* of a vector bundle $p : E \longrightarrow X$ is a vector bundle $p' : E' \longrightarrow X$ such that $E' \subseteq E$ (so each fiber of the vector subbundle is a vector subspace of the corresponding fiber of the vector bundle).

Proposition 3.5.13. *Let $p : E \longrightarrow X$ be a vector bundle over X. If $p_1 : E_1 \longrightarrow X$ is a vector subbundle of $p : E \longrightarrow X$, then there is a vector subbundle $p_2 : E_2 \longrightarrow X$ such that $E_1 \oplus E_2 \cong E$.*

Proof. Take $E_2 = E_1^{\perp}$, i.e., the orthogonal complement on each fiber with respect to a chosen inner product of E; and $p_2 : E_2 \longrightarrow X$ the natural projection. We must check that E_2 is locally trivial. To do so, we can assume $E = X \times \mathbb{F}^n$ and $E_1 = X \times \mathbb{F}^m$, since it is a local property and E and E_1 are vector bundles. Then E_1 has m independent sections $s_1, \dots, s_m$ near each $x \in X$, that can be extended to a set of n independent sections in E, for some $s_{m+1}, \dots, s_n$ (the independence is true locally because of the continuity of the determinant function). Now, we can transform continuously these sections to an orthogonal set of sections $s'_1, \dots, s'_n$ (Gram-Schmidt using the chosen inner product). Now, the first m sections define a local trivialization for E_1 and the rest define a local trivialization of E_2, so E_2 is a vector bundle. Now, applying Proposition 3.5.5 to the map $(v, w) \mapsto v + w$, we have $E_1 \oplus E_2 \cong E$. □

Proposition 3.5.14. *Let $p : E \longrightarrow X$ be a vector bundle over X (compact Hausdorff). There exists a vector bundle $p' : E' \longrightarrow X$ such that $E \oplus E'$ is a trivial vector bundle over X.*

Proof. Take open sets $\{U_x\}_{x \in X}$ such that E is trivial over U_x. Urysohn's Lemma (see [Rud] Lemma 2.12) gives continuous maps $\varphi_x : X \longrightarrow [0, 1]$ that are 0 outside U_x and 1 at x. In particular, $\{\varphi_x^{-1}(0, 1]\}_{x \in X}$ is an open cover of X. By compactness of X, we can take a finite amount of open sets for some $x_1, \dots, x_k$. Let $g_i : E \longrightarrow \mathbb{F}^n$ with $g_i(v) = \varphi_{x_i}(v) \cdot \pi_i h_i(v)$ with h_i the local trivialization $p^{-1}(U_{x_i}) \longrightarrow U_{x_i} \times \mathbb{F}^n$ and π_i the projection to $\mathbb{F}^n$. Now $g = (g_1, \dots, g_k) : E \longrightarrow \mathbb{F}^{nk}$ is a linear injection on each fiber, since each g_i is a linear injection on the fibers over $\varphi_{x_i}^{-1}(0, 1]$.

Now, take $f = (p, g) : E \longrightarrow X \times \mathbb{F}^{nk}$, and notice that $f(E)$ is a vector subbundle of $X \times \mathbb{F}^{nk}$ with trivializations given by projecting $\mathbb{F}^{nk}$ onto the corresponding $\mathbb{F}^n$ for each $\varphi_{x_i}^{-1}(0, 1]$. Hence, E is isomorphic to a vector subbundle of $X \times \mathbb{F}^{nk}$ and by Proposition 3.5.13, there is a vector subbundle E' such that $E \oplus E' \cong X \times \mathbb{F}^{nk}$. □

Example 3.5.15. The tangent bundle TS^{2n} and the normal bundle NS^{2n}, which is the orthogonal complement of the tangent bundle, and is actually isomorphic to $S^{2n} \times \mathbb{R}$, are such that $TS^{2n} \oplus NS^{2n} \cong S^{2n} \times \mathbb{R}^{2n+1}$.

Theorem 3.5.16 (Swan's Theorem)**.** *Let $p : E \longrightarrow X$ be a vector bundle over X. Then $\Gamma(X, E)$ is a finitely generated projective $C^{\mathbb{F}}(X)$-module, and every finitely generated projective $C^{\mathbb{F}}(X)$-module comes from this construction, for some vector bundle over X. Moreover, it can be seen that $\Gamma(-, X)$ is an equivalence of categories, from the category of vector bundles over X to the category of finitely generated projective $C^{\mathbb{F}}(X)$-modules.*

Proof. We can take a finite open cover of X such that E is trivial over each open set, multiply the basis of local sections of each open set by the functions of a partition of unity subordinated to this cover and extend by 0 outside the open set. These generate $\Gamma(X, E)$, so it is finitely generated. Now, by Proposition 3.5.14, there is a vector bundle E' over X such that $E \oplus E' \cong X \times \mathbb{F}^n$, hence $\Gamma(X, E) \oplus \Gamma(X, E') \cong \Gamma(X, E \oplus E') \cong \Gamma(X, X \times \mathbb{F}^n) \cong C^{\mathbb{F}}(X)^n$, so $\Gamma(X, E)$ is projective.

Now, let P be a finitely generated projective $C^{\mathbb{F}}(X)$-module, so $P \oplus Q \cong C^{\mathbb{F}}(X)^n \cong C(X, \mathbb{F}^n)$ for some Q. So P is a collection of continuous functions from X to $\mathbb{F}^n$. Let $E = \{(x, v_1, \dots, v_n) \in X \times \mathbb{F}^n \mid \exists s \in P : s(x) = (v_1, \dots, v_n)\}$ and $p : E \longrightarrow X$ such that $p(x, v_1, \dots, v_n) = x$. It is clear that $\Gamma(X, E) = P$. We claim that E is a vector bundle over X. The fibers $p^{-1}(x)$ are clearly vector spaces. To see that it is locally trivial, let $e_1, \dots, e_r \in P$ such that $e_1(x), \dots, e_r(x)$ are a basis of $p^{-1}(x)$ for $x \in X$. These are lineally independent over an open set U_P, because of the continuity of the determinant function of r of the coordinates (those that make the determinant non-zero at x) of the e_i. The same can be done for Q, and we have $f_1, \dots, f_{n-r}$ that are linearly independent over U_Q. At $x' \in U = U_P \cap U_Q$, the $e_i(x')$ generate a rank-r free submodule of P over U, and the f_i generate a rank-$(n-r)$ free submodule of Q over U, hence they exhaust P and Q, and so $p^{-1}(U) \cong U \times \mathbb{F}^r$. □

As seen in the proof, trivial bundles are identified with free modules, since $\Gamma(X, X \times \mathbb{F}^n) \cong C^{\mathbb{F}}(X)^n$. We have seen some vector bundles that are not isomorphic to trivial vector bundles, so they are identified with projective modules that are not free.

Example 3.5.17. i) Applying Swan's Theorem to Example 3.5.15, we obtain an example of a projective R-module that is not free. This is, $\Gamma(S^{2n}, TS^{2n})$ is a projective $C^{\mathbb{R}}(S^{2n})$-module which is not free, and its direct sum with $\Gamma(S^{2n}, NS^{2n}) \cong \Gamma(S^{2n}, S^{2n} \times \mathbb{R}) \cong C^{\mathbb{R}}(S^{2n})$ is the free module $(C^{\mathbb{R}}(S^{2n}))^{2n+1}$.

ii) If we apply Swan's Theorem to Example 3.5.11 i), we obtain a projective $C^{\mathbb{R}}(S^1)$-module given by $\Gamma(S^1, \mathcal{M})$, which is not free.

4

The Grothendieck Group K_0

In this chapter, we introduce the K_0 group of a ring, give some examples and basic properties. We will also introduce the relative K_0 group, and the K_0 of a nonunital ring. Most definitions and results in this chapter are found in [Ros].

4.1 Group completion of a semigroup

Recall that an abelian semigroup is a set with an associative binary operation. The aim of this section is to construct a group from a semigroup by adding, in a way, the inverses of the elements of the semigroup; and give some of its properties.

Definition 4.1.1. Let S be an abelian semigroup. The *group completion* or *Grothendieck group* of S is a group S^+ together with a semigroup homomorphism $\varphi : S \longrightarrow S^+$ such that for every abelian group G and semigroup homomorphism $\psi : S \longrightarrow G$, there is a unique group homomorphism $\theta : S^+ \longrightarrow G$ satisfying $\psi = \theta\varphi$. It is unique up to isomorphism.

Proposition 4.1.2. *Let S be an abelian semigroup. Then, its group completion S^+ exists.*

Proof. Let $S^+ := F(S)/\langle [x+y] - [x] - [y] \rangle$ (with $F(S)$ the free group over S). Let $\varphi : S \longrightarrow S^+$ be defined by $\varphi : x \mapsto [x]$.

Then, for every abelian group G and morphism $\psi : S \longrightarrow G$, the morphism $\theta : S^+ \longrightarrow G$ is defined by $\theta([x]) = \psi(x)$ over the set of generators, and extended by linearity. It is well defined because G is an abelian group (hence it respects the equivalence relation), and it is defined in a way so that $\psi(x) = \theta([x]) = \theta(\varphi(x))$, $\forall x \in S$, so it is unique by construction.

The uniqueness (up to isomorphism) of the group completion follows from the universal property. □

Proposition 4.1.3. *Let S be an abelian semigroup. Then:*

i) Every element of S^+ is of the form $[x] - [y]$ for some $x, y \in S$.

ii) $[x] = [y] \iff \exists z \in S$ such that $x + z = y + z$.

Proof. To prove both statements, we give an alternative construction of the group completion:

Let S^+ be the set of equivalence classes $[(x, y)]$ of pairs $(x, y) \in S \times S$ where $(x, y) \equiv (x', y') \iff x + y' + t = x' + y + t$ for some $t \in S$. Addition is defined by $[(x, y)] + [(u, v)] = [(x + u, y + v)]$, which is well-defined (add the equivalence condition for each element to see that it also holds for their sum). The element $[(s, s)]$ (which is equal to $[(s', s')]$, $\forall s' \in S$) is the zero element 0, because $(x + s, y + s) \equiv (x, y)$; and for every element $[(x, y)]$, there is an element $[(y, x)]$ such that $[(x, y)]+[(y, x)] = [x+y, x+y] = 0$, so S^+ is a group.

Let $\varphi(x) = [x+x, x]$, and for any group G and homomorphism $\psi : S \longrightarrow G$, the morphism θ is defined by $\theta([(x, y)]) = \psi(x) - \psi(y)$.

This construction of S^+ satisfies the definition of the group completion of S, hence is isomorphic to the previously given. Notice that $[(x, y)] = \varphi(x)-\varphi(y)$, so the isomorphism between the two constructions of S^+ takes every $[(x, y)]$ to $[x] - [y]$, hence *i)* holds.

To see *ii)*, first notice that $x + z = y + z \Longrightarrow [x] + [z] = [x + z] = [y + z] = [y] + [z] \Longrightarrow [x] = [y]$. For the opposite implication, $[x] = [y] \Longrightarrow [x] - [y] = 0 \Longrightarrow [(x, y)] = 0 = [(s, s)] \Longrightarrow \exists t \in S$ such that $x+z = y+z$ with $z = s+t$. □

Corollary 4.1.4. *S^+ can also be constructed as the equivalence classes of $(x, y) \in S \times S$ by the equivalence $(x, y) \equiv (x + z, y + z)$.* □

Remark. The group completion is actually a functor, with induced morphisms determined by the universal property:

$$\begin{array}{ccc} S_1 & \xrightarrow{\ f\ } & S_2 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ S_1^+ & \xrightarrow[\ f_*\ ]{} & S_2^+ \end{array}$$

With f_* the unique group homomorphism such that $f_*\varphi_1 = \varphi_2 f$. The functoriality follows from commutativity of the squares and the universal property.

Example 4.1.5. i) The most common example of group completion is the group completion of the semigroup $(\mathbb{N}, +)$, which results in the abelian group $\mathbb{Z}$.

Notice that, since $\mathbb{N}$ is actually a monoid, in $(\mathbb{N})^+$, the relation $[m + n] - [m] - [n] = 0$ with $n = m = 0$ reads as $[0 + 0] - [0] - [0] = 0$, so $[0] = 0$. Hence, the negative numbers in $\mathbb{Z}$ are identified with $-[n] = [0] - [n] = [m] - [n + m]$ in $(\mathbb{N})^+$, and they are the additive inverses of $[n] = [n] - [0] = [n + m] - [m]$.

ii) If G is an abelian group, then $G^+ = G$, because $[x - x] - [x] - [-x] = 0 \implies [-x] = -[x]$ and hence all relations in G^+ are already satisfied in G, and the relations of G are also satisfied in G^+.

4.2 K_0 of a ring. Examples

Definition 4.2.1. Let R be a ring. $K_0(R)$ is the Grothendieck group of the semigroup $(\operatorname{Proj} R, \oplus)$ of isomorphism classes of finitely generated projective R-modules. This is, $K_0(R) = (\operatorname{Proj} R)^+$.

Remark (Eilenberg Swindle). There is an argument, by the method called Eilenberg swindle, that shows why it is important to define K_0 from finitely generated projective R-modules and not from all projective R-modules:

Let P be a projective R-module, so $P \oplus Q \cong F$ is free. Then,

$$\begin{aligned} \tilde{F} = F^{\infty} = F \oplus F \oplus \cdots = (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \cong \\ \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \cong \\ \cong P \oplus (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots = P \oplus \tilde{F}, \end{aligned}$$

so if K_0 was defined using all projective R-modules we would have $[P] = [\tilde{F}] - [\tilde{F}] = 0$ for all P projective.

Definition 4.2.2. Let $f : R \longrightarrow R'$ be a ring homomorphism. We define an induced morphism given by the group completion functor applied to the morphism $[P] \mapsto [P_{R'}] = [R' \otimes_R P]$ defined by extension of scalars from $\operatorname{Proj} R$ to $\operatorname{Proj} R'$.

Notice that if $P \oplus Q \cong R^n$, then $(R' \otimes_R P) \oplus (R' \otimes_R Q) \cong R' \otimes_R (P \oplus Q) \cong R' \otimes_R R^n \cong (R')^n$, so $P_{R'}$ is a finitely generated projective R'-module, as it is desired.

This definition of induced morphism makes K_0 into a functor. The functoriality of the extension of scalars from $\operatorname{Proj} R$ to $\operatorname{Proj} R'$ is immediate and functoriality of K_0 follows from the functoriality of the group completion.

Proposition 4.2.3. $K_0(R) \cong (\operatorname{Idem}(R)_{GL(R)})^+$.

Proof. It is an immediate consequence of Proposition 3.1.7. □

Principal ideal domains

Theorem 4.2.4. *Let R be a PID. Then $K_0(R) \cong \mathbb{Z}$.*

Proof. By Theorem 3.3.2, finitely generated projective R-modules are free, hence isomorphic to R^n for some n. Then the isomorphism classes in $\operatorname{Proj} R$ are defined by the rank n (which is well-defined since it can also be seen as the dimension of the vector space $F \otimes_R M$ for an R-module M, with F the field of fractions), and clearly $[R^n] \oplus [R^m] = [R^n \oplus R^m] = [R^{n+m}]$. Hence, $\operatorname{Proj} R \cong \mathbb{N}$ and therefore $K_0(R) = (\operatorname{Proj} R)^+ \cong \mathbb{Z}$. □

Example 4.2.5. Theorem 4.2.4 applies to the following examples.

i) $K_0(\mathbb{Z}) \cong \mathbb{Z}$.

Let $\mathbb{K}$ be a field. Then:

ii) $K_0(\mathbb{K}) \cong \mathbb{Z}$.

iii) $K_0(\mathbb{K}[t]) \cong \mathbb{Z}$.

Local rings

Theorem 4.2.6. *Let R be a local ring. Then $K_0(R) \cong \mathbb{Z}$.*

Proof. It is clear by Theorem 3.3.4, and proceeding as in Theorem 4.2.4. □

Example 4.2.7. i) Fields are also local rings, so we obtain, independently from the result for PIDs, that the K_0 group of fields is $\mathbb{Z}$.

ii) If R is local, then $K_0(R[[t]]) \cong \mathbb{Z}$.

iii) If $\mathfrak{p}$ is a prime ideal of R, then $K_0(R_\mathfrak{p}) \cong \mathbb{Z}$.

Remark. Observe that in the case of local rings and PIDs, we know that every finitely generated projective module is free, which implies $K_0(R) \cong \mathbb{Z}$. Nevertheless, the fact that a ring R is such that $K_0(R) \cong \mathbb{Z}$ does not imply that finitely generated projective R-modules are free. It can be shown, instead, that $K_0(R) \cong \mathbb{Z} \iff$ every finitely generated projective R-module is stably free, i.e., $P \oplus R^n \cong R^m$ for some m, n.

Topological K^0

Algebraic K-theory was mostly inspired by topological K-theory, which studies the structure of vector bundles over a topological space. We briefly present here the definition of the K^0 group of topological K-theory, and its relation with the K_0 group in algebraic K-theory.

Definition 4.2.8. Let X be a compact Hausdorff topological space. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $\text{Vect}_{\mathbb{F}}(X)$ be the set of isomorphism classes of vector bundles over X. Then $K^0(X) := (\text{Vect}_{\mathbb{F}}(X), \oplus)^+$.

Theorem 4.2.9. *Let X be a compact Hausdorff space, and $C^{\mathbb{F}}(X)$ the ring of continuous $\mathbb{F}$-valued functions over X. Then $K^0(X) \cong K_0(C^{\mathbb{F}}(X))$.*

Proof. It follows from Swan's Theorem. □

Example 4.2.10. Let $X = \{x\}$ be a topological space formed by a single point. It is clear that X is compact Hausdorff. Then, $C^{\mathbb{F}}(X) = \mathbb{F}$, since every function over X is given by a single value of $\mathbb{F}$. Therefore, $K^0(X) \cong K_0(\mathbb{F}) \cong \mathbb{Z}$, because $\mathbb{F}$ is a field.

4.3 Properties of K_0

The K_0 construction has some good properties, which are proved in this section.

Additivity

The additive property of K_0 groups allows us to calculate the K_0 group of a product ring by calculating the K_0 group of each factor.

Proposition 4.3.1. *Let $R = R_1 \times R_2$ be the product ring of two commutative rings R_1 and R_2. Then $K_0(R) \cong K_0(R_1) \oplus K_0(R_2)$.*

Proof. Denote the elements of the product ring by $(r_1, r_2) \in R$, with $r_1 \in R_1$ and $r_2 \in R_2$. Addition and multiplication are defined componentwise: $(r_1, r_2) + (r_1', r_2') = (r_1 + r_1', r_2 + r_2')$ and $(r_1, r_2) \cdot (r_1', r_2') = (r_1 r_1', r_2 r_2')$; with zero element $(0, 0)$ and identity element $(1, 1)$. This construction is analogous in the noncommutative case.

Notice that elements $p \in M(R)$ naturally split as $p = (p_1, p_2)$ with $p_1 \in M(R_1)$ and $p_2 \in M(R_2)$ where the entries of p_1 and p_2 are the first and second components of the entries of p respectively. This is, $M(R) \cong M(R_1) \times$

$M(R_2)$. Moreover, $p^2 = p \iff (p_1^2, p_2^2) = (p_1, p_2) \iff p_1^2 = p_1$ and $p_2^2 = p_2$, so $\mathrm{Idem}(R) \cong \mathrm{Idem}(R_1) \times \mathrm{Idem}(R_2)$.

Also note that $g = (g_1, g_2) \in GL(R) \iff \exists g' = (g_1', g_2')$ such that $gg' = \mathrm{id}_{M(R)} = g'g \iff (g_1, g_2) \cdot (g_1', g_2') = (g_1 g_1', g_2 g_2') = (\mathrm{id}_{M(R_1)}, \mathrm{id}_{M(R_2)}) = (g_1' g_1, g_2' g_2) = (g_1', g_2') \cdot (g_1, g_2) \iff \exists g_1'$ such that $g_1 g_1' = \mathrm{id}_{M(R_1)} = g_1' g_1$ and $\exists g_2'$ such that $g_2 g_2' = \mathrm{id}_{M(R_2)} = g_2' g_2$. Hence, $GL(R) \cong GL(R_1) \times GL(R_2)$.

Then, for $p, q \in \mathrm{Idem}(R)$ with $p = (p_1, p_2)$ and $q = (q_1, q_2)$, we have $p \sim q \iff \exists g = (g_1, g_2) \in GL(R)$ such that $(p_1, p_2) = p = gqg^-1 = (g_1, g_2) \cdot (q_1, q_2) \cdot (g_1^{-1}, g_2^{-1}) = (g_1 q_1 g_1^{-1}, g_2 q_2 g_2^{-1}) \iff \exists g_1 \in GL(R_1)$ such that $p_1 = g_1 q_1 g_1^{-1}$ and $\exists g_2 \in GL(R_2)$ such that $p_2 = g_2 q_2 g_2^{-1} \iff p_1 \sim q_1$ and $p_2 \sim q_2$. Hence $\mathrm{Idem}(R)_{GL(R)} \cong \mathrm{Idem}(R_1)_{GL(R_1)} \times \mathrm{Idem}(R_2)_{GL(R_2)}$ or, equivalently, $\mathrm{Proj}\, R \cong \mathrm{Proj}\, R_1 \times \mathrm{Proj}\, R_2$.

Finally, using Proposition 4.1.3 *ii)*, $[p] = [(p_1, p_2)] = [(q_1, q_2)] = [q] \in K_0(R) \iff \exists r = (r_1, r_2) \in \mathrm{Proj}\, R$ such that $(p_1 \oplus r_1, p_2 \oplus r_2) = (p_1, p_2) \oplus (r_1, r_2) = p \oplus r = q \oplus r = (q_1, q_2) \oplus (r_1, r_2) = (q_1 \oplus r_1, q_2 \oplus r_2) \iff \exists r_1 \in \mathrm{Proj}\, R_1$ such that $p_1 \oplus r_1 = q_1 \oplus r_1$ and $\exists r_2 \in \mathrm{Proj}\, R_2$ such that $p_2 \oplus r_2 = q_2 \oplus r_2$. Therefore, $K_0(R) \cong K_0(R_1) \oplus K_0(R_2)$. □

K_0 of a nonunital ring

One important application of the additivity of K_0 is the definition of a K_0 group for a nonunital ring. The additivity is used to prove that when a nonunital ring does have an identity element, then both definitions agree.

Definition 4.3.2. Let I be a nonunital ring (i.e., a ring that does not necessarily have an identity element). Define I_+ as the ring formed by elements (x, n) with $x \in I$ and $n \in \mathbb{Z}$, with multiplication defined by $(x, n) \cdot (y, m) = (xy + ny + mx, mn)$ and componentwise addition. The identity element in I_+ is $(0, 1)$.

Remark. Note that I is an ideal of I_+, and that homomorphisms of nonunital rings $I \longrightarrow I'$ (same as ring homomorphisms without the condition over the identity element) extend uniquely to ring homomorphisms $I_+ \longrightarrow I'_+$ (apply the original homomorphism to the first component and the identity to the second).

Also, notice that there is an split exact sequence

$$0 \longrightarrow I \hookrightarrow I_+ \xrightarrow{\rho} \mathbb{Z} \longrightarrow 0,$$

with section $\mathbb{Z} \longrightarrow I_+$ given by $n \mapsto (0, n)$.

Definition 4.3.3. For a nonunital ring I, we define

$$K_0(I) := \ker(\rho_* : K_0(I_+) \longrightarrow K_0(\mathbb{Z}) \cong \mathbb{Z}).$$

With this definition, K_0 is also a functor.

Remark. If I already has an identity element 1, then I is also a ring and we have two definitions of $K_0(I)$. Notice that in this case, there is an isomorphism $\alpha : I_+ \longrightarrow I \times \mathbb{Z}$ defined by $\alpha(x, n) = (x + n \cdot 1, n)$ since $\alpha((x, n) \cdot (y, n)) = \alpha(xy+ny+mx, mn) = (xy+ny+mx+mn\cdot 1, mn) = (x+n\cdot 1, n)\cdot(y+m\cdot 1, m) = \alpha(x, n \cdot \alpha(y, m))$ and α is clearly bijective. Therefore, by Proposition 4.3.1, $K_0(I_+) \cong K_0(I \times \mathbb{Z}) \cong K_0(I) \oplus K_0(\mathbb{Z})$ and $\ker \rho_* = K_0(I)$, so the usual definition of $K_0(I)$ agrees with the definition for nonunital rings when I is a ring.

Given a nonunital ring homomorphism $f : I \longrightarrow I'$, we know that it extends uniquely to $\tilde{f} = (f, \mathrm{id}) : I_+ \longrightarrow I'_+$, so the following diagram commutes:

$$\begin{array}{ccc} I_+ & \xrightarrow{\rho} & \mathbb{Z} \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ I'_+ & \xrightarrow[\rho']{} & \mathbb{Z} \end{array}$$

Applying the usual definition of K_0 to his diagram preserves the commutativity by functoriality, and including the kernels of ρ and ρ' we have a unique induced morphism between $K_0(I)$ and $K_0(I')$:

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{k} & K_0(I_+) & \xrightarrow{\rho_*} & K_0(\mathbb{Z}) \cong \mathbb{Z} \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle \tilde{f}_*}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ K_0(I') & \xrightarrow[k']{} & K_0(I'_+) & \xrightarrow[\rho'_*]{} & K_0(\mathbb{Z}) \cong \mathbb{Z} \end{array}$$

This is because of the universal property of the kernel $K_0(I')$, and because $\rho'_* \tilde{f}_* k = \mathrm{id}\, \rho_* k = 0$. The functoriality of this induced morphism follows from the universal property. Notice that if I and I' are actually rings, then $K_0(I_+) \cong K_0(I \times \mathbb{Z}) \cong K_0(I) \oplus K_0(\mathbb{Z})$ and also $K_0(I'_+) \cong K_0(I') \oplus K_0(\mathbb{Z})$, and then $\tilde{f}_* = (f'_*, \mathrm{id})$ and $f'_* : K_0(I) \longrightarrow K_0(I')$ makes the diagram commute, hence $f_* = f'_*$ by the universal property.

Morita invariance

Theorem 4.3.4 (Morita invariance)**.** *Let R be a ring. Then $K_0(R) \cong K_0(M(n,R))$, $\forall n > 0$.*

Remark. Although $M(n,R)$ is a noncommutative ring, the definition of K_0 is the same for a general ring $\tilde{R}$: the definition of projective (left) $\tilde{R}$-module is the same, Proposition 3.1.7 still holds (proceed in the same way but taking into account the noncommutativity by taking left $\tilde{R}$-modules and matrices acting on the right). At this point, $\operatorname{Proj}\tilde{R}$ or $\operatorname{Idem}(\tilde{R})_{GL(\tilde{R})}$ are abelian semigroups and $K_0(\tilde{R})$ can be defined in the usual way.

Proof. Observe that $M(k, M(n,R)) \cong M(nk,R)$ and $GL(k, M(n,R)) \cong GL(nk,R)$, hence $M(M(n,R)) = \bigcup_k M(k, M(n,R)) \cong \bigcup_k M(nk,R)$, and by the inclusions $M(i,R) \hookrightarrow M(i+1,R)$, we have $M(M(n,R)) \cong M(R)$. Similarly, $GL(M(n,R)) \cong GL(R)$.

Therefore, $\operatorname{Idem}(M(n,R)) \cong \operatorname{Idem}(R)$ and by Proposition 4.2.3, we have $K_0(M(n,R)) \cong (\operatorname{Idem}(M(n,R))_{GL(M(n,R))})^+ \cong (\operatorname{Idem}(R)_{GL(R)})^+ \cong K_0(R)$. □

Continuity

The notion of continuity when speaking of functors, in this case the K_0 assignation, refers to the preservation of limits. For the case of the K_0 group, we will show that it preserves direct limits.

Theorem 4.3.5. *Let $\{R_\alpha\}_{\alpha\in I}$ and $\{\theta_{\alpha\beta} : R_\alpha \longrightarrow R_\beta\}_{\alpha\leq\beta}$ a directed system of rings and $R = \varinjlim R_\alpha$ (i.e, R is the direct limit over a filtered category I). Then $K_0(R) = \varinjlim K_0(R_\alpha)$. This is, K_0 is a continuous functor.*

Proof. Applying K_0 we have a directed system of abelian groups $\{K_0(R_\alpha)\}_{\alpha\in I}$ and $\{\theta_{\alpha\beta,*} : K_0(R_\alpha) \longrightarrow K_0(R_\beta)\}_{\alpha\leq\beta}$; and a limit $\varinjlim K_0(R_\alpha)$. By the universal property of the direct limit, there is a unique map $u : \varinjlim K(R_\alpha) \longrightarrow K_0(R)$, which composed with the direct limit maps of $\varinjlim K(R_\alpha)$ are the induced morphisms of the direct limit maps of R. We will prove that u is an isomorphism.

Let $p \in \operatorname{Idem}(R)$. Since p has finitely many nonzero entries, which come from rings $R_{\alpha_1}, \dots, R_{\alpha_k}$ (by Proposition 2.2.9 *i)* and Proposition 2.2.10), take $\gamma \geq \alpha_1, \dots, \alpha_k$, hence p is the image of some element of $\operatorname{Idem}(R_\gamma)$ (by Proposition 2.2.9 *ii)* and Proposition 2.2.10, this is true, maybe replacing γ by $\gamma' \geq \gamma$). Thus, $[p] \in K_0(R)$ is in the image of the induced morphism $K_0(R_\gamma) \longrightarrow K_0(R)$, which factors through $\varinjlim K_0(R_\alpha)$, hence $[p]$ is in the image of u. Since the elements $[p]$ generate $K_0(R)$, u is surjective.

Let $x \in \varinjlim K_0(R_\alpha)$ such that $u(x) = 0$. x comes from some $K_0(R_\alpha)$, and by Proposition 4.1.3 *i)* it is of the form $[p] - [q]$, with $p, q \in \text{Idem}(R_\alpha)$. By abuse of notation, we will understand p, q, r as idempotents of R_γ (for any γ) or R, by taking the image or preimage. Then, $[p] - [u] = 0$ in $K_0(R) \implies \exists r \in \text{Idem}(R)$ such that $[p \oplus r]$=$[q \oplus r]$ (by Proposition 4.1.3 *ii)*) in $K_0(R)$, and they lift from the group completion as two conjugate idempotent matrices by some $s \in GL(R)$. r and s must come from some R_β $r \in \text{Idem}(R_\beta)$ and $s' \in GL(R_\beta)$, which can be taken so that $\beta \geq \alpha$. Then $[p] - [q]$ in $K_0(R_\beta)$ is 0, and since the map $K_0(R_\beta) \longrightarrow K_0(R)$ factors through $\varinjlim K_0(R_\alpha)$, then $x = 0$. □

4.4 Relative K_0 and excision

In algebraic topology, singular homology theory is a very powerful tool to study the structure of topological spaces by looking at the "inner" part of the space. In this sense, topological K-theory studies what is outside the space, but still depends on its structure, that is, vector bundles over the topological space. Algebraic K-theory tries to bring this philosophy to the study of rings.

The comparison between these theories would be simply anecdotal if they did not share any or very few of their properties. What we will show in this section is that the notion of relative homology and the Excision Theorem of singular homology have their equivalences in algebraic K-theory.

We will define a relative K_0 group $K_0(R, I)$, for a ring R and an ideal I, that behaves as relative homology in the sense that we have an analogous of the exact sequence

$$H_0(A) \longrightarrow H_0(X) \longrightarrow H_0(X, A)$$

of singular homology; and we will have an excision result similar to the one of singular homology which says that $H_i(X, A) \cong H_i(X \setminus U, A \setminus U)$ under certain conditions. In the case of K-theory, we will see that we can ignore the R/I part of R to calculate the relative K_0 group, and just study the structure of I as a ring without unit.

Let R be a ring and $I \subseteq R$ an ideal.

Definition 4.4.1. The *double of R along I* is the subring $D(R, I) = \{(x, y) \in R \times R \mid x - y \in I\}$ of $R \times R$.

Remark. There is a short exact sequence

$$0 \longrightarrow I \overset{i_2}{\hookrightarrow} D(R, I) \overset{p_1}{\longrightarrow} R \longrightarrow 0$$

which is split, by the section $s : R \cong \Delta_R \hookrightarrow D(R, I)$. The morphism i_2 is the inclusion in the second factor and p_1 is the projection of the first factor.

Definition 4.4.2. The *relative K_0 group* of R and I is defined by

$$K_0(R, I) := \ker(p_{1,*} : K_0(D(R, I)) \longrightarrow K_0(R)).$$

The following lemma will be useful in the proof of the next theorem.

Lemma 4.4.3. *Let $q : M(k, R) \longrightarrow M(k, R/I)$, for any k, be the quotient map applied to each entry. If $A \in GL(n, R/I)$, then $q(\bar{A}) = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ for some $\bar{A} \in GL(2n, R)$.*

Proof. Let $B, C \in M(n, R)$ such that $q(B) = A$ and $q(C) = A^{-1}$. The matrices $\begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix}$ and $\begin{pmatrix} 1_n & 0 \\ -C & 1_n \end{pmatrix}$ are invertible, and so is $\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$.

Take the invertible matrix

$$\bar{A} = \begin{pmatrix} 2B - BCB & BC - 1_n \\ 1_n - BC & C \end{pmatrix} =$$

$$= \begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ -C & 1_n \end{pmatrix} \begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$$

which satisfies

$$q(\bar{A}) = \begin{pmatrix} 2A - AA^{-1}A & AA^{-1} - 1_n \\ 1_n - AA^{-1} & A^{-1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}.$$

□

Theorem 4.4.4. *Let $\tilde{p}_{2,*} : K_0(R, I) \hookrightarrow K_0(D(R, I)) \xrightarrow{p_{2,*}} K_0(R)$ with $p_{2,*}$ the morphism induced by the projection of the second factor. There is a natural exact sequence*

$$K_0(R, I) \xrightarrow{\tilde{p}_{2,*}} K_0(R) \xrightarrow{q_*} K_0(R/I)$$

with q_* the morphism induced by the quotient map.

Proof. Let $[r] - [s] \in K_0(R, I)$, with $r = (r_1, r_2), s = (s_1, s_2) \in \mathrm{Idem}(D(R, I))$. By Proposition 4.3.1, $K_0(R \times R) \cong K_0(R) \oplus K_0(R)$ and $[r] - [s]$ can be seen as $(r_1 - s_1, r_2 - s_2)$ in $K_0(R \times R)$. Hence,

$$q_*\tilde{p}_{2,*}([r] - [s]) = q_*([r_2] - [s_2]) = [q(r_2)] - [q(s_2)] = [\dot{r}_2] - [\dot{s}_2].$$

Since $[r]-[s]\in K_0(R,I)=\ker p_{1,*}$, we have

$$0=q_*p_{1,*}([r]-[s])=q_*([r_1]-[s_1])=[\dot r_1]-[\dot s_1],$$

but $r,s\in \mathrm{Idem}(D(R,I))\Longrightarrow \dot r_1-\dot r_2=0$ and $\dot s_1-\dot s_2=0$, thus

$$[\dot r_2]-[\dot s_2]=[\dot r_1]-[\dot s_1]=0,$$

so $\mathrm{Im}\,\tilde p_{2,*}\subseteq \ker q_*$.

To prove the other inclusion, suppose that $[r]-[s]\in K_0(R)$, with $r,s\in \mathrm{Idem}(R)$, such that $q_*([r]-[s])=[\dot r]-[\dot s]=0$. Then, by Proposition 4.1.3 *ii)*, $\dot r'=\dot r\oplus\dot t\sim\dot s\oplus\dot t=\dot s'$ for some $t\in\mathrm{Idem}(R)\Longrightarrow \dot r'=\dot g\dot s'\dot g^{-1}$ for some $\dot g\in GL(R/I)$, or $\dot r'\oplus 0=(\dot g\oplus\dot g^{-1})(\dot s'\oplus 0)(\dot g^{-1}\oplus\dot g)$. By Lemma 4.4.3, $\dot g\oplus\dot g^{-1}$ lifts to a matrix $h\in GL(R)$. We can replace r and s by $r'\oplus 0$ and $h(s'\oplus 0)h^{-1}$, since the element in $K_0(R)$ remains the same, and reduce to the case where $\dot r=\dot s$, this is, $(r,s)\in\mathrm{Idem}(D(R,I))$. Take $[(r,r)]-[(r,s)]\in K_0(D(R,I))$. Clearly, $p_{1,*}([(r,r)]-[(r,s)])=0$, so $[(r,r)]-[(r,s)]\in K_0(R,I)$ and $\tilde p_{2,*}([(r,r)]-[(r,s)])=[r]-[s]$, hence $\ker q_*\subseteq\mathrm{Im}\,\tilde p_{2,*}$ and therefore $\ker q_*=\mathrm{Im}\,\tilde p_{2,*}$. □

Theorem 4.4.5 (Excision Theorem)**.** *If I as an ideal of R is also seen as a nonunital ring, then $K_0(R,I)\cong K_0(I)$. Hence, $K_0(R,I)$ does not depend on R, but on the structure of I as a nonunital ring.*

Proof. Let $\gamma: I_+\longrightarrow D(R,I)$ be defined by $\gamma(x,n)=(n\cdot 1,n\cdot 1+x)$, and $\iota:\mathbb{Z}\longrightarrow R$ the unique homomorphism from $\mathbb{Z}$ to R. Then

$$\begin{array}{ccc} I_+ & \xrightarrow{\ \rho\ } & \mathbb{Z} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\iota} \\ D(R,I) & \xrightarrow[\ p_1\ ]{} & R \end{array}$$

commutes, so we have the following diagram:

$$\begin{array}{ccccc} K_0(I) & \overset{k}{\hookrightarrow} & K_0(I_+) & \xrightarrow{\ \rho_*\ } & K_0(\mathbb{Z}) \\ {\scriptstyle\gamma_*}\downarrow & & {\scriptstyle\gamma_*}\downarrow & & \downarrow{\scriptstyle\iota_*} \\ K_0(R,I) & \underset{j}{\hookrightarrow} & K_0(D(R,I)) & \xrightarrow[\ p_{1,*}\ ]{} & K_0(R) \end{array}$$

where $\gamma_* : K_0(I) \longrightarrow K_0(R, I)$ is given by the universal property of the kernel $K_0(R, I)$, since $p_{1,*}\gamma_* k = \iota_* \rho_* k = 0$ and $\gamma_* k = j\gamma_*$. We will show that γ_* takes $K_0(I)$ to $K_0(R, I)$ isomorphically.

To see that it is surjective, take $[r] - [s] \in K_0(R, I)$, with $r = (r_1, r_2), s = (s_1, s_2) \in \mathrm{Idem}(D(R, I))$ and $[r_1] = [s_1]$ in $K_0(R)$. We can replace r and s by $(r_1 \oplus t_1, r_2 \oplus t_2)$ and $(g(s_1 \oplus t_1)g^{-1}, g(s_2 \oplus t_2)g^{-1})$ for suitable $t = (t_1, t_2) \in \mathrm{Idem}(D(R, I))$ and $g \in GL(D(R, I))$ and reduce to the case where $r_1 = s_1$. Moreover, we can replace r and s by $r \oplus (\mathrm{id} - r_1, \mathrm{id} - r_1)$ and $s \oplus (\mathrm{id} - r_1, \mathrm{id} - r_1)$ (with id the identity matrix of the same size as r) and now notice that the image of $r_1 \oplus (\mathrm{id} - r_1) \sim \mathrm{id} \oplus 0$ because r_1 is an idempotent, hence we can take r and s to be $(\mathrm{id} \oplus 0, r_2)$ and $(\mathrm{id} \oplus 0, s_2)$ respectively, without changing the initial value of $[r] - [s]$. Since r and s are matrices over $D(R, I)$, the entries of $r_2 - (\mathrm{id} \oplus 0)$ and $s_2 - (\mathrm{id} \oplus 0)$ are in I. Take the element $[(r_2 - (\mathrm{id} \oplus 0), \mathrm{id} \oplus 0)] - [(s_2 - (\mathrm{id} \oplus 0), \mathrm{id} \oplus 0)]$ in $K_0(I_+)$ (by abuse of notation, meaning that the entries of the matrix in $\mathrm{Idem}(I_+)$ are pairs of entries of the matrices $r_2 - (\mathrm{id} \oplus 0)$ and $\mathrm{id} \oplus 0$; and of $s_2 - (\mathrm{id} \oplus 0)$ and $\mathrm{id} \oplus 0$, respectively), which clearly is in $K_0(I)$ and maps to $[r] - [s]$ by γ_*.

To prove injectivity, let $[r] - [s] \in K_0(I)$, so $r, s \in \mathrm{Idem}(I_+)$ and $[\rho(r)] = [\rho(s)] \iff \mathrm{rank}\, \rho(r) = \mathrm{rank}\, \rho(s)$ (by Theorem 4.2.4, since $\mathbb{Z}$ is a PID). By a similar procedure as before, we can replace r and s by $h(r \oplus t \oplus (\mathrm{id} - s))h^{-1}$ and id, this is, assume $s = \mathrm{id}$ of rank k, and $\mathrm{rank}\, \rho(r) = k$. Since $[\rho(\mathrm{id})] = [\mathrm{id}] = [\rho(r)]$, there is a $g \in GL(\mathbb{Z})$ such that $\mathrm{id} = g\rho(r)g^{-1}$ and g can be brought back to $GL(I_+)$ by the section of ρ. Next, replace r by grg^{-1} and notice that $r = (r - \rho(r), \rho(r))$ and $\mathrm{id} = (0, \mathrm{id})$ in the notation of Definition 4.3.2. Now, if $\gamma_*([r] - [\mathrm{id}]) = 0$, then

$$\gamma_*([(r - \rho(r), \rho(r))]) - \gamma_*([(0, \mathrm{id})]) = [(\mathrm{id}, r)] - [(\mathrm{id}, \mathrm{id})] = 0 \in K_0(D(R, I)).$$

Replace (id, r) and $(\mathrm{id}, \mathrm{id})$ by $(\mathrm{id}, r) \oplus t$ and $(\mathrm{id}, \mathrm{id}) \oplus t$, so there is a matrix $(g_1, g_2) \in GL(D(R, I))$ that conjugates (id, r) to $(\mathrm{id}, \mathrm{id})$. Then, the matrix $(\mathrm{id}, g_1^{-1} g_2) \in GL(D(R, I))$ also conjugates (id, r) to $(\mathrm{id}, \mathrm{id})$. In particular $g_1^{-1} g_2$ conjugates r to id in matrices over R, but $(\mathrm{id}, g_1^{-1} g_2) \in GL(D(R, I)) \implies g_1^{-1} g_2 - \mathrm{id}$ has all entries in I, so $g_1^{-1} g_2 \in GL(I_+)$ (in I_+, we recover the identity matrix that makes $g_1^{-1} g_2$ invertible in R, so that it is invertible in I_+), hence $[r] - [\mathrm{id}] = 0$.

Therefore, $K_0(I) \cong K_0(R, I)$ by γ_*. □

Remark. Notice that $K_0(R, R) \cong K_0(R)$. This is similar to relative homology, where $H_i(X, \varnothing) \cong H_i(X)$.

Excision for topological K^0

We will now see an application of the Excision Theorem in topological K-theory. First of all, we extend the definition of the K^0 group to a more general category of topological spaces.

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition 4.4.6. A topological space X is *locally compact* if every point has a neighbourhood that is contained in a compact set. It can be shown that locally compact spaces allow a one-point compactification (see [Dug] Chapter XI, Theorem 8.3). The natural maps in the category of locally compact spaces are *proper* maps, which are those that extend continuously to the one-point compactification.

Definition 4.4.7. Let X be a locally compact space and $X \cup \{\infty\}$ its compactification. Let $C_0^{\mathbb{F}}(X)$ be the nonunital ring of functions vanishing at infinity on X. We define $K^0(X) = K_0(C_0^{\mathbb{F}}(X))$ using the definition for nonunital rings of K_0.

With this definition, we can now formulate a topological version of the Excision Theorem for K^0.

Proposition 4.4.8. *Let X be a compact Hausdorff space and A a closed subspace of X. Then, there is an exact sequence*

$$K^0(X \setminus A) \longrightarrow K^0(X) \longrightarrow K^0(A)$$

induced by the inclusion $A \hookrightarrow X$.

Proof. Let $R = C^{\mathbb{F}}(X)$ and $I = C_0^{\mathbb{F}}(X \setminus A)$ an ideal of R (it is well-defined because open subsets of a locally compact Hausdorff space are locally compact; also notice that it is isomorphic to the ideal of R of functions vanishing on A). By the Tietze Extension Theorem (an equivalent formulation of Urysohn's Lemma), it can be shown that a continuous function on A is the restriction of a continuous function on X, so $R/I \cong C^{\mathbb{F}}(A)$ and $R \longrightarrow R/I$ corresponds to restriction of functions, hence, by the Excision Theorem we have the desired exact sequence. □

5

K_0 of an Abelian Category

In this chapter we present a generalization of the K_0 group, defining it over a category with exact sequences. We will see that we can recover the definition given in the previous chapter by calculating the K_0 group of $\operatorname{Proj} R$. We will give a proof of the three main abstract theorems of K_0: the Devissage Theorem, the Resolution Theorem and the Localization Theorem.

We will also introduce the G_0 group of a ring R, see its relation with $K_0(R)$ and show a proof of the Fundamental Theorem of G_0.

Most parts of this chapter can be found in [WeibK] or [Ros]. Some results are seen in more detail in [Mag] or [Mat] (the latter only for chain conditions). The proof of the Fundamental Theorem of G_0 can be found in [Swan].

5.1 Definitions

K_0 of a category with exact sequences

Let $\mathcal{A}$ be an abelian category.

Definition 5.1.1. A *category with exact sequences* $\mathcal{P}$ is a full additive subcategory of $\mathcal{A}$ such that:

i) If

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$, then $P', P'' \in \mathcal{P} \implies P \in \mathcal{P}$. This is, $\mathcal{P}$ is *closed under extensions.*

ii) $\mathcal{P}$ has full subcategory $\mathcal{P}_0$ such that it is small and the inclusion functor is an equivalence. $\mathcal{P}_0$ is a *small skeleton* of $\mathcal{P}$.

Example 5.1.2. Let R be a ring.

i) Any abelian category with a small skeleton is itself a category with exact sequences.

ii) The category $\mathbf{Proj}\,R$ of finitely generated projective R-modules is a category with exact sequences with small skeleton the set of direct summands in $\{R^n \mid n \in \mathbb{N}\}$. $\mathbf{Proj}\,R$ is not abelian in general.

iii) The category $R\text{-}\mathbf{mod_{fg}}$ of finitely generated R-modules is a category with exact sequences using the same argument of Example 2.4.2 ii). However, it is not usually abelian, since the kernel of a morphism may not be finitely generated. If R is Noetherian, then $R\text{-}\mathbf{mod_{fg}}$ is abelian.

iv) Let $R\text{-}\mathbf{mod_{fpr}}$ be the category of R-modules with a finite-type projective resolution, this is, R-modules M for which there is an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with $P_i \in \mathbf{Proj}\,R$. It is a category with exact sequences. If R is Noetherian and $R\text{-}\mathbf{mod_{fpr}} = R\text{-}\mathbf{mod_{fg}}$, then R is regular (Definition 3.4.5).

The second example is clearly a category with exact sequences since short exact sequences split if the last module is projective. The last example is a category with exact sequences by the Horseshoe Lemma.

Let $\mathcal{P}$ be a category with exact sequences and $\mathcal{P}_0$ its small skeleton.

Definition 5.1.3. An exact sequence in $\mathcal{P}$ is a sequence in $\mathcal{P}$ that is exact in $\mathcal{A}$.

Definition 5.1.4. $K_0(\mathcal{P})$ is defined to be the free abelian group on $\operatorname{Obj}\mathcal{P}_0$ modulo the relation $[P] = [P'] + [P'']$ if there is a short exact sequence

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

in $\mathcal{P}$.

Remark. This definition implies that:

i) $[0] = 0$ (take $P' = 0$ and $P = P''$).

ii) If $P \cong P'$ (so $P'' = 0$) then $[P] = [P']$.

iii) $[P' \oplus P''] = [P'] + [P'']$ (take $P = P' \oplus P''$).

iv) If $\mathcal{P}'$ is equivalent to $\mathcal{P}$, then $K_0(\mathcal{P}') \cong K_0(\mathcal{P})$ (consequence of ii)).

If $F : \mathcal{P} \longrightarrow \mathcal{P}'$ is an exact functor, then there is an induced morphism $F_* : K_0(\mathcal{P}) \longrightarrow K_0(\mathcal{P}')$ defined by $[P] \mapsto [F(P)]$.

Theorem 5.1.5. *Let R be a ring. Then $K_0(R) \cong K_0(\mathbf{Proj}\, R)$.*

Proof. Actually both definitions are equivalent, since both are abelian groups with a generator $[P]$ for each finitely generated projective R-module, and the relations are equivalent: for both definitions, $[P] = [Q]$ if $P \cong Q$; in $K_0(R)$ we have $[P' \oplus P''] = [P'] + [P'']$, and in $K_0(\mathbf{Proj}\, R)$, $[P] = [P'] + [P'']$ if

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

is exact in $\mathbf{Proj}\, R$, but P'' projective $\Longrightarrow$ the short exact sequence is split, so $P \cong P' \oplus P''$ and therefore $[P' \oplus P''] = [P] = [P'] + [P'']$. □

Definition 5.1.6. An *additive function* between $\mathcal{P}$ and an abelian group G, is a map $f : \mathcal{P} \longrightarrow G$ such that $f(P) = f(P') + f(P'')$ for every short exact sequence

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0.$$

Proposition 5.1.7. *If $f : \mathcal{P} \longrightarrow G$ is an additive function, there is a unique group homomorphism $\tilde{f} : K_0(\mathcal{P}) \longrightarrow G$ such that $\tilde{f}([P]) = f(P)$, $\forall P \in \mathcal{P}$.*

Proof. By definition of K_0, it is the free abelian group over objects of $\mathcal{P}$ modulo the additive relation, hence, any additive function f has at least this relation, and any other relation comes from a unique group homomorphism $\tilde{f} : K_0(\mathcal{P}) \longrightarrow G$, i.e., $\tilde{f}$ is well-defined and uniquely determined by $\tilde{f}([P]) = f(P)$, since the elements $[P]$ generate $K_0(\mathcal{P})$. □

G_0 of a ring

Definition 5.1.8. Let R be a Noetherian ring. Define $G_0(R) := K_0(R\text{-}\mathbf{mod_{fg}})$.

There is a natural morphism $K_0(R) \longrightarrow G_0(R)$ given by $[P] \mapsto [P]$, called the *Cartan homomorphism.*

Proposition 5.1.9. *Let R be a principal ideal domain. Then the Cartan homomorphism is an isomorphism.*

Proof. It is sufficient to see that $G_0(R) \cong \mathbb{Z}$ and that it is generated by $[R]$. Notice that we have exact sequences

$$0 \longrightarrow R \xrightarrow{\ d\ } R \longrightarrow R/dR \longrightarrow 0,$$

so in $G_0(R)$ we have $[R/dR] = [R] - [R] = 0$.

By the Structure Theorem for finitely generated modules over a PID, we have $[M] = [R^n]$, hence $[R]$ is a generator and each R-module can be sent to the rank n, which is well-defined since it can also be seen as the dimension of the vector space $F \otimes_R M$, with F the field of fractions. □

Definition 5.1.10. Let $f : R \longrightarrow S$ be a homomorphism between two Noetherian rings.

i) Any S-module M can be seen as an R-module by the action $r \cdot m = f(r)m$. If S is finitely generated as an R-module, the forgetful functor S-$\mathbf{mod_{fg}} \longrightarrow R$-$\mathbf{mod_{fg}}$ (forget the S-module structure) is exact and induces a contravariant morphism $f_* : G_0(S) \longrightarrow G_0(R)$, called *transfer homomorphism.*

ii) If the functor $S \otimes_R - : R$-$\mathbf{mod_{fg}} \longrightarrow S$-$\mathbf{mod_{fg}}$ is exact, i.e., if S is flat as an R-module, it induces a covariant morphism $f^* : G_0(R) \longrightarrow G_0(S)$, called *base change homomorphism.*

5.2 Main theorems

The Devissage Theorem

Theorem 5.2.1 (Schreier's Theorem)**.** *Let A be an object of an abelian category, and let*

$$0 = A_m \subseteq A_{m-1} \subseteq \cdots \subseteq A_1 \subseteq A_0 = A$$

$$0 = A'_n \subseteq A'_{n-1} \subseteq \cdots \subseteq A'_1 \subseteq A'_0 = A$$

be two filtrations of A. Then these filtrations can be refined so that the quotients between two consecutive objects of one filtration are isomorphic to those of the other, up to a permutation. This is, both filtrations have equivalent refinements.

Proof. Let $A_{ij} = A_{i+1} + (A_i \cap A'_j)$ and $A'_{ij} = A'_{j+1} + (A_i \cap A'_j)$, so the original filtrations can be refined by

$$A_{i+1} = A_{i,n} \subseteq A_{i,n-1} \subseteq \cdots \subseteq A_{i,1} \subseteq A_{i,0} = A_i$$

$$A'_{j+1} = A'_{m,j} \subseteq A'_{m-1,j} \subseteq \cdots \subseteq A'_{1,j} \subseteq A'_{0,j} = A'_j.$$

Notice that

$$A_{i,j+1} + (A_i \cap A'_j) = A_{i+1} + (A_i \cap A'_{j+1}) + (A_i \cap A'_j) = A_{i+1} + (A_i \cap A'_j) = A_{ij}$$

and likewise

$$A'_{i+1,j} + (A_i \cap A'_j) = A'_{ij}.$$

Also, we have

$$A_{i,j+1} \cap (A_i \cap A'_j) = (A_{i+1} + (A_i \cap A'_{j+1})) \cap (A_i \cap A'_j) = (A_{i+1} \cap A'_j) + (A_i \cap A'_{j+1})$$

and likewise

$$A'_{i+1,j} \cap (A_i \cap A'_j) = (A_{i+1} \cap A'_j) + (A_i \cap A'_{j+1}).$$

Hence, by Proposition 1.2.4 *iii)*, we have

$$\frac{A_{ij}}{A_{i,j+1}} = \frac{A_{i,j+1} + (A_i \cap A'_j)}{A_{i,j+1}} \cong \frac{A_i \cap A'_j}{A_{i,j+1} \cap (A_i \cap A'_j)} = \frac{A_i \cap A'_j}{(A_{i+1} \cap A'_j) + (A_i \cap A'_{j+1})}$$

and similarly

$$\frac{A'_{ij}}{A'_{i+1,j}} \cong \frac{A_i \cap A'_j}{(A_{i+1} \cap A'_j) + (A_i \cap A'_{j+1})}$$

so

$$\frac{A_{ij}}{A_{i,j+1}} \cong \frac{A'_{ij}}{A'_{i+1,j}}$$

and we are done. □

Theorem 5.2.2 (Devissage Theorem)**.** *Let $\mathcal{B} \subseteq \mathcal{A}$ be an (exact) abelian subcategory of a small category $\mathcal{A}$, such that $\mathcal{B}$ is closed under subobjects and quotients, and every $A \in \mathcal{A}$ has a $\mathcal{B}$-filtration, i.e., a chain*

$$0 = A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1 \subseteq A_0 = A$$

such that $A_{i-1}/A_i \in \mathcal{B}$.

Then the inclusion functor induces an isomorphism $K_0(\mathcal{B}) \cong K_0(\mathcal{A})$.

Proof. Let $i_* : K_0(\mathcal{B}) \longrightarrow K_0(\mathcal{A})$ be the induced morphism, which is given by $[B] \mapsto [B]$.

Let $A \in \mathcal{A}$, and take a $\mathcal{B}$-filtration

$$0 = A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1 \subseteq A_0 = A.$$

We have short exact sequences

$$0 \longrightarrow A_i \hookrightarrow A_{i-1} \longrightarrow A_{i-1}/A_i \longrightarrow 0,$$

for $i \in \{1, \ldots, n\}$. Hence, $[A_{i-1}] - [A_i] = [A_{i-1}/A_i]$. Now observe that

$$[A] = \sum_{i=1}^{n}([A_{i-1}] - [A_i]) = \sum_{i=1}^{n}[A_{i-1}/A_i],$$

and since $A_{i-1}/A_i \in \mathcal{B}$, the element $\sum_{i=1}^{n}[A_{i-1}/A_i] \in \mathcal{B}$ maps to $[A]$.

Now, choose a $\mathcal{B}$-filtration for each A in $\mathcal{A}$ and let $f : \mathcal{A} \longrightarrow K_0(\mathcal{B})$ be defined by $f(A) = \sum_{i=1}^{n}[A_{i-1}/A_i]$. To see that it is well-defined, observe that if

$$0 = A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1 \subseteq A_0 = A$$

is a $\mathcal{B}$-filtration of A, and we add an object $\bar{A}$ such that $A_i \subseteq \bar{A} \subseteq A_{i-1}$, then there is a short exact sequence

$$0 \longrightarrow \bar{A}/A_i \hookrightarrow A_{i-1}/A_i \longrightarrow A_{i-1}/\bar{A} \longrightarrow 0,$$

in $\mathcal{B}$, so $[A_{i-1}/A_i] = [A_{i-1}/\bar{A}] + [\bar{A}/A_i]$ in $K_0(\mathcal{B})$. By induction, any refinement of the given filtration yields the same element $f(A)$. Schreier's Theorem states that any two filtrations of A have equivalent refinements, hence f is well-defined.

Let

$$0 \longrightarrow A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$, and let

$$0 = A'_r \subseteq A'_{r-1} \subseteq \cdots \subseteq A'_1 \subseteq A'_0 = A'$$

$$0 = A''_s \subseteq A''_{s-1} \subseteq \cdots \subseteq A''_1 \subseteq A''_0 = A''$$

be $\mathcal{B}$-filtrations of A' and A''. Then,

$$0 = \alpha(A'_r) \subseteq \alpha(A'_{r-1}) \subseteq \cdots \subseteq \alpha(A'_1) \subseteq \alpha(A'_0) = \ker \beta =$$

$$= \beta^{-1}(A''_s) \subseteq \beta^{-1}(A''_{s-1}) \subseteq \cdots \subseteq \beta^{-1}(A''_1) \subseteq \beta^{-1}(A''_0) = A$$

is a $\mathcal{B}$-filtration of A. Think of α as the inclusion of the kernel of β; and think of β as the quotient map to the cokernel of α, and use $(M/L)/(N/L) \cong M/N$. Then, it is clear that $f(A) = f(A') + f(A'')$. Hence, by Proposition 5.1.7, there is a group homomorphism $\tilde{f} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B})$ such that $\tilde{f}([A]) = f(A) = \sum_{i=1}^{n}[A_{i-1}/A_i]$.

Finally, we have $i_*(\tilde{f}([A])) = i_*(\sum_{i=1}^{n}[A_{i-1}/A_i]) = \sum_{i=1}^{n}[A_{i-1}/A_i] = [A]$. For each B in $\mathcal{B} \subseteq \mathcal{A}$, a $\mathcal{B}$-filtration of B is $0 = B_1 \subseteq B_0 = B$, so $\tilde{f}(i_*([B])) = \tilde{f}([B]) = [B]$. Therefore, i_* and $\tilde{f}$ are inverse isomorphisms, and so $K_0(\mathcal{B}) \cong K_0(\mathcal{A})$. □

Corollary 5.2.3. *Let R be a Noetherian ring and $I \subseteq R$ a nilpotent ideal (i.e., such that $I^n = 0$ for some n). Then $G_0(R/I) \cong G_0(R)$.*

Proof. Apply the Devissage Theorem to (R/I)-$\mathbf{mod_{fg}} \subseteq R$-$\mathbf{mod_{fg}}$. Notice that any finitely generated R-module M has a filtration

$$0 = MI^n \subseteq MI^{n-1} \subseteq \cdots \subseteq MI^2 \subseteq MI \subseteq M,$$

and all the quotients MI^k/MI^{k+1} are finitely generated R/I-modules. Hence, $G_0(R/I) \cong G_0(R)$. □

Corollary 5.2.4. *Let $s \in R$ and R-$\mathbf{mod}_s := \{M \in R$-$\mathbf{mod_{fg}} \mid \exists n : Ms^n = 0\}$. This is, every finitely generated R-module has a filtration*

$$0 = Ms^n \subseteq Ms^{n-1} \subseteq \cdots \subseteq Ms^2 \subseteq Ms \subseteq M,$$

with $Ms^i/Ms^{i+1} \in (R/sR)$-$\mathbf{mod_{fg}}$. Then, by the Devissage Theorem, *$K_0(R$-$\mathbf{mod}_s) \cong G_0(R/sR)$.* □

The Resolution Theorem

Let $\mathcal{M}, \mathcal{P} \subseteq \mathcal{A}$ be categories with exact sequences and $\mathcal{A}$ an abelian category.

Assume $\mathcal{P}$ is a full subcategory of $\mathcal{M}$ such that every object M of $\mathcal{M}$ has a finite resolution by objects of $\mathcal{P}$, i.e., an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

with $P_i \in \mathcal{P}$.

Proposition 5.2.5. *Let $\alpha : M' \longrightarrow M$ be a morphism in $\mathcal{M}$ and*

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a finite resolution of M by objects of $\mathcal{P}$. If $\mathcal{M}$ and $\mathcal{P}$ both contain the kernels of morphisms that are epis in $\mathcal{A}$, then M' has a finite resolution by objects of $\mathcal{P}$ and there are morphisms α_i making the following diagram commute:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & P'_m & \longrightarrow & \cdots & \longrightarrow & P'_n & \longrightarrow & \cdots & \longrightarrow & P'_0 & \xrightarrow{\epsilon'} & M' & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle \alpha_n} & & & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
 & & & & 0 & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_0 & \xrightarrow[\epsilon]{} & M & \longrightarrow & 0
\end{array}$$

Proof. Let B be the kernel of the morphism $(\epsilon, -\alpha) : P_0 \oplus M' \longrightarrow M$ (which is epi because ϵ is epi, hence is in $\mathcal{M}$). Since B has a resolution by objects of $\mathcal{P}$, in particular we can find an object $P_0' \in \mathcal{P}$ such that we have an epi $P_0' \longrightarrow B$. Take $\epsilon' : P_0' \longrightarrow B \hookrightarrow P_0 \oplus M' \longrightarrow M'$, which is epi because it is a composition of epis (since ϵ is epi, M' is included in $B = \ker(\epsilon, -\alpha)$), and $\alpha_0 : P_0' \longrightarrow B \hookrightarrow P_0 \oplus M' \longrightarrow P_0$.

Now, let $Z_0 = \ker \epsilon$ and $Z_0' = \ker \epsilon'$. There is an induced map from $\alpha_0' : Z_0' \longrightarrow Z_0$ (because the constructed square commutes). Since we have a resolution of Z_0 by objects of $\mathcal{P}$:

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow Z_0 \longrightarrow 0,$$

the same argument applies, and composing the obtained maps with the kernels we can construct inductively a diagram

$$\begin{array}{ccccccccccc}
Z_n' & \longrightarrow & P_n' & \longrightarrow & \cdots & \longrightarrow & P_0' & \xrightarrow{\epsilon'} & M' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_n} & & & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_0 & \xrightarrow[\epsilon]{} & M & \longrightarrow & 0
\end{array}$$

Finally, take a resolution of Z_n' by objects of $\mathcal{P}$ to complete the diagram. □

For the next results, we need to introduce the concept of homology and some of its properties. Let $(C_\bullet, d_\bullet)$ be a sequence in $\mathcal{M}$ such that $d_i d_{i+1} = 0$ (this is, a *chain complex*; it is called *bounded* if it is finite). In R-**mod**, homology is defined to be, for each i, the quotient $H_i(C_\bullet) = \ker d_i / \operatorname{Im} d_{i+1}$. For a general abelian category, the fact that $d_i d_{i+1} = 0$ allows the existence of a monic $\operatorname{Im} d_{i+1} \longrightarrow \ker d_i$. Define the *homology* as $H_i(C_\bullet) = \operatorname{coker}(\operatorname{Im} d_{i+1} \longrightarrow \ker d_i)$. Notice that $H_i(C_\bullet) = 0 \iff$ the sequence is exact at C_i.

Actually, homology is a functor, taking maps $f_\bullet : (C_\bullet, d) \longrightarrow (C_\bullet', d')$ (defined as a map $f_i : C_i \longrightarrow C_i'$ such that $f_{i-1} d_i = d_i' f_i$, $\forall i$) to $f_* : H_*(C_\bullet) \longrightarrow H_*(C_\bullet')$ (given by maps $f_{*,i} : H_i(C_\bullet) \longrightarrow H_i(C_\bullet')$). Moreover, homology preserves additivity, i.e., if $C_\bullet = C_\bullet' \oplus C_\bullet''$, this is, $C_i = C_i' \oplus C_i''$, then $H_i(C_\bullet) = H_i(C_\bullet') \oplus H_i(C''\bullet)$.

Let $f_\bullet : (C_\bullet, d) \longrightarrow (C_\bullet', d')$ be a morphism, then we define the *mapping cone* of f as the complex $\operatorname{Cone}(f) = (C_\bullet'', d'')$, where $C_i'' = C_{i-1} \oplus C_i'$ and $d_i'' = \begin{pmatrix} -d_{i-1} & 0 \\ f & d_i' \end{pmatrix}$. It can be seen that $d_i'' d_{i+1}'' = 0$ and that $\operatorname{Cone}(f)$ is an exact sequence if f induces an isomorphism in homology (this is, f is a *quasi-isomorphism*).

Definition 5.2.6. Let $C_\bullet$ be a bounded chain complex in $\mathcal{M}$. Its *Euler characteristic* is

$$\chi(C_\bullet) = \sum_{i=0}^{n} (-1)^i [C_i] \in K_0(\mathcal{M}).$$

Proposition 5.2.7. *Let $C_\bullet$ be a bounded chain complex in $\mathcal{M}$. If $\mathcal{M}$ is closed under kernels of epis in $\mathcal{A}$, and contains the homology of $C_\bullet$, then*

$$\chi(C_\bullet) = \sum_{i=0}^{n} (-1)^i [H_i(C_\bullet)].$$

Proof. Let $Z_i = \ker d_i$ and $B_i = \operatorname{Im} d_{i+1}$. Consider the following short exact sequences in $\mathcal{A}$:

$$0 \longrightarrow Z_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0,$$

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i(C_\bullet) \longrightarrow 0.$$

Since $B_k = 0$ for a small enough k, by induction, the Z_{i+1} are in $\mathcal{M}$ because of the former short exact sequence and the B_{i+1} are in $\mathcal{M}$ because of the latter. Now, by the definition of $K_0(\mathcal{M})$, the short exact sequences show that:

$$\sum_{i=0}^{n} (-1)^i [H_i(C_\bullet)] = \sum_{i=0}^{n} (-1)^i Z_i - \sum_{i=0}^{n} (-1)^i B_i =$$

$$= \sum_{i=0}^{n} (-1)^i Z_i + \sum_{i=1}^{n+1} (-1)^i B_{i-1} = \sum_{i=0}^{n} (-1)^i [C_i] = \chi(C_\bullet).$$

□

Since exact sequences have zero homology, the following result is an immediate consequence of Proposition 5.2.7.

Corollary 5.2.8. *If $\mathcal{M}$ is closed under kernels of epis in $\mathcal{A}$ and $C_\bullet$ is a finite exact sequence, then $\chi(C_\bullet) = 0$.* □

Corollary 5.2.9. *If $\mathcal{M}$ is closed under kernels of epis in $\mathcal{A}$ and $f : C_\bullet \longrightarrow C'_\bullet$ is a quasi-isomorphism between bounded complexes, then*

$$\chi(C_\bullet) = \chi(C'_\bullet).$$

Proof. Notice that $\chi(C'_\bullet) - \chi(C_\bullet) = \chi(\operatorname{Cone}(f))$, but since f is a quasi-isomorphism, $\operatorname{Cone}(f)$ is exact, and hence $\chi(\operatorname{Cone}(f)) = 0$. Therefore $\chi(C_\bullet) = \chi(C'_\bullet)$. □

Proposition 5.2.10. *Suppose $\mathcal{M}$ and $\mathcal{P}$ are closed under kernels of epis in $\mathcal{A}$. Let $P_\bullet \longrightarrow M$ and $P'_\bullet \longrightarrow M$ be two finite resolutions of $M \in \mathcal{M}$ by objects of $\mathcal{P}$. Then $\chi(P_\bullet) = \chi(P'_\bullet)$ in $K_0(\mathcal{P})$.*

Proof. We have a resolution $P_\bullet \oplus P'_\bullet \longrightarrow M \oplus M$. Consider the diagonal morphism $\Delta = (\mathrm{id}, \mathrm{id}) : M \longrightarrow M \oplus M$ and apply Proposition 5.2.5:

$$\begin{array}{ccccccccccccc}
0 \longrightarrow P''_m \longrightarrow \cdots \longrightarrow & P''_n & \longrightarrow \cdots \longrightarrow & P''_0 & \xrightarrow{\epsilon''} & M & \longrightarrow 0 \\
& \downarrow{\scriptstyle (\alpha_n, \alpha'_n)} & & \downarrow{\scriptstyle (\alpha_0, \alpha'_0)} & & \downarrow{\scriptstyle \Delta} & \\
0 \longrightarrow & P_n \oplus P'_n & \longrightarrow \cdots \longrightarrow & P_0 \oplus P'_0 & \xrightarrow[\epsilon \oplus \epsilon']{} & M \oplus M & \longrightarrow 0
\end{array}$$

The squares commute and since resolutions are exact, the homology of the complexes $P_\bullet$, $P'_\bullet$ and $P''_\bullet$ (leaving out the M) is zero everywhere except at 0 where they have homology M and (α_0, α'_0) precisely induce the morphism Δ, hence α_* and α'_* are quasi-isomorphisms so $\chi(P_\bullet) = \chi(P''_\bullet) = \chi(P'_\bullet)$ by Corollary 5.2.9. □

Theorem 5.2.11 (Resolution Theorem)**.** *With the previous hypothesis over $\mathcal{M}$ and $\mathcal{P}$, this is, they are categories with exact sequences, $\mathcal{P}$ is a full subcategory of $\mathcal{M}$ and every object of $\mathcal{M}$ has a finite resolution by objects of $\mathcal{P}$; and if $\mathcal{M}$ and $\mathcal{P}$ are both closed under kernels of epis in $\mathcal{A}$, then the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{M}$ induces an isomorphism $K_0(\mathcal{P}) \cong K_0(\mathcal{M})$.*

Proof. Since $\mathcal{P}$ is a full subcategory of $\mathcal{M}$ and both are categories with exact sequences, the inclusion is exact, hence it induces a morphism $i_* : K_0(\mathcal{P}) \longrightarrow K_0(\mathcal{M})$.

Let $\varphi : K_0(\mathcal{M}) \longrightarrow K_0(\mathcal{P})$ be the morphism defined by $[M] \mapsto \chi(P_\bullet)$ with $P_\bullet \longrightarrow M$ a finite resolution of M by objects of $\mathcal{P}$. The map is well defined by Proposition 5.2.10. Notice that, since $0 \longrightarrow P_\bullet \longrightarrow M \longrightarrow 0$ is exact, $[M] = \chi(P_\bullet)$ in $K_0(\mathcal{M})$, hence $i_*(\varphi([M])) = [M]$. On the other hand, if $P \in \mathcal{P}$ then $P \longrightarrow P$ is a finite resolution of itself, so $\varphi(i_*([P])) = [P]$. Thus, i_* and φ are inverse isomorphisms and therefore $K_0(\mathcal{P}) \cong K_0(\mathcal{M})$. □

Corollary 5.2.12. *Let R be a regular ring. Then the Cartan morphism is an isomorphism. This is, $K_0(R) \cong G_0(R)$.*

Proof. If R is regular, then $R\text{-}\mathbf{mod}_{\mathbf{fg}} = R\text{-}\mathbf{mod}_{\mathbf{fpr}}$, therefore $G_0(R) = K_0(R\text{-}\mathbf{mod}_{\mathbf{fg}}) = K_0(R\text{-}\mathbf{mod}_{\mathbf{fpr}})$ and notice that $\mathbf{Proj}\, R \subseteq R\text{-}\mathbf{mod}_{\mathbf{fpr}}$ satisfies the hypothesis of the Resolution Theorem, hence the inclusion functor induces an isomorphism, and by Theorem 5.1.5, $K_0(R) \cong K_0(\mathbf{Proj}\, R) \cong K_0(R\text{-}\mathbf{mod}_{\mathbf{fpr}}) = G_0(R)$, which is given precisely by the Cartan morphism. □

The Localization Theorem

Theorem 5.2.13 (Localization Theorem)**.** *Let $\mathcal{A}$ be a small abelian category and $\mathcal{B} \subseteq \mathcal{A}$ a Serre subcategory of $\mathcal{A}$. Then the following sequence is exact:*

$$K_0(\mathcal{B}) \xrightarrow{i_*} K_0(\mathcal{A}) \xrightarrow{T_*} K_0(\mathcal{A}/\mathcal{B}) \longrightarrow 0,$$

where the first morphism is induced by the inclusion and the second one is induced by the quotient morphism of Theorem 2.4.6 *(recall that both morphisms are exact).*

Proof. By definition of the quotient $T : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{B}$, it is clear that $T_* : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A}/\mathcal{B})$ is surjective, because the objects of $\mathcal{A}/\mathcal{B}$ are the objects of $\mathcal{A}$, and the composition is zero because $T(\mathcal{B}) \cong 0$. Therefore, if $\Gamma = \operatorname{coker} i_*$, with $q : K_0(\mathcal{A}) \longrightarrow \Gamma$, there is a unique morphism $u : \Gamma \longrightarrow K_0(\mathcal{A}/\mathcal{B})$ such that $uq = T_*$, and u is surjective because T_* is surjective.

Notice that since q is surjective, then $u : \Gamma \longrightarrow K_0(\mathcal{A}/\mathcal{B})$ is defined by $u : q([A]) \mapsto uq([A]) = T_*([A]) = [T(A)]$. Let $\gamma : \mathcal{A}/\mathcal{B} \longrightarrow \Gamma$ be defined by $\gamma : T(A) \mapsto q([A])$. It is well-defined because T is a bijection on objects. We will prove that γ is additive and thus there will be a morphism $\tilde{\gamma} : K_0(\mathcal{A}/\mathcal{B}) \longrightarrow \Gamma$ such that $\tilde{\gamma} : [T(A)] \mapsto q([A])$, so it will be, by construction, an inverse map to u, which will show that $K_0(\mathcal{A}/\mathcal{B}) \cong \Gamma = \operatorname{coker} i_*$, and therefore the sequence will be exact.

First notice that, if $T(A_1) \cong T(A_2)$, then $\gamma(T([A_1])) = \gamma(T([A_2]))$, or equivalently, $q([A_1]) = q([A_2])$. To see this, let $A_1 \xleftarrow{f} A \xrightarrow{g} A_2$ be morphisms in $\mathcal{A}$, with f a $\mathcal{B}$-iso, such that they represent an isomorphism in $\mathcal{A}/\mathcal{B}$. Since $T(f)$ and $h = T(g)(T(f))^{-1}$ are isomorphisms in $\mathcal{A}/\mathcal{B}$, then $hT(f) = (T(g)(T(f))^{-1})T(f) = T(g)$ is an isomorphism, hence g is a $\mathcal{B}$-iso (its kernel and its cokernel are zero in $\mathcal{A}/\mathcal{B}$). In $\mathcal{A}$ we have the following exact sequences:

$$0 \longrightarrow \ker f \hookrightarrow A \xrightarrow{f} A_1 \longrightarrow \operatorname{coker} f \longrightarrow 0,$$

$$0 \longrightarrow \ker g \hookrightarrow A \xrightarrow{g} A_2 \longrightarrow \operatorname{coker} g \longrightarrow 0.$$

Hence, their Euler characteristic in $K_0(\mathcal{A})$ is zero, so $[A] = [A_1] + [\ker f] - [\operatorname{coker} f] = [A_2] + [\ker g] - [\operatorname{coker} g]$. Since $[\ker f]$, $[\ker g]$, $[\operatorname{coker} f]$ and $[\operatorname{coker} g]$ come from $K_0(\mathcal{B})$ through i_*, in the cokernel of i_*, this is, in Γ, they are zero, therefore $q([A]) = q([A_1]) = q([A_2])$.

Now, let

$$0 \longrightarrow T(A_0) \xrightarrow{\alpha} T(A_1) \xrightarrow{\beta} T(A_2) \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}/\mathcal{B}$, and let $A_1 \xleftarrow{f} A \xrightarrow{g} A_2$ be morphisms in $\mathcal{A}$, with f a $\mathcal{B}$-iso, such that they represent β in $\mathcal{A}/\mathcal{B}$. Since f is a $\mathcal{B}$-iso, the exact sequence

$$0 \longrightarrow \ker f \hookrightarrow A \xrightarrow{f} A_1 \longrightarrow \operatorname{coker} f \longrightarrow 0$$

in $\mathcal{A}$ shows that $q([A]) = q([A_1])$. Consider the following exact sequence in $\mathcal{A}$:

$$0 \longrightarrow \ker g \hookrightarrow A \xrightarrow{g} A_2 \longrightarrow \operatorname{coker} g \longrightarrow 0.$$

Apply the exact functor T, and observe that $T(f)$ is an isomorphism, hence we have a diagram like so:

$$\begin{array}{ccccccccc} 0 \longrightarrow & T(\ker g) & \hookrightarrow & T(A) & \xrightarrow{T(g)} & T(A_2) & \longrightarrow & T(\operatorname{coker} g) & \longrightarrow 0 \\ & & \searrow & \cong \downarrow T(f) & \nearrow_{\beta} & & & & \\ & & & T(A_1) & & & & & \end{array}$$

The fact that $T(f)$ is an isomorphism and that β is an epimorphism, shows that $T(\operatorname{coker} g) = 0$, this is, $\operatorname{coker} g \in \mathcal{B}$, and that $T(\ker g) \cong T(A_0)$, so $q([\ker g]) = q([A_0])$ (we proved this fact before). Therefore, from the original sequence in $\mathcal{A}$ we have $[A] = [A_2] + [\ker g] - [\operatorname{coker} g]$ in $K_0(\mathcal{A})$, hence in Γ we have:

$$q([A_1]) = q([A]) = q([A_2]) + q([\ker g]) = q([A_2]) + q([A_0]).$$

Thus, γ is additive, so the universal property gives a morphism $\tilde{\gamma}$, inverse of u, and therefore the sequence

$$K_0(\mathcal{B}) \xrightarrow{i_*} K_0(\mathcal{A}) \xrightarrow{T_*} K_0(\mathcal{A}/\mathcal{B}) \longrightarrow 0$$

is exact. □

Corollary 5.2.14. *Let R be a Noetherian ring. Then R-$\mathbf{mod_{fg}}/R$-$\mathbf{mod}_s \cong R[\frac{1}{s}]$-$\mathbf{mod_{fg}}$ (*Example 2.4.9*) and by the* Localization Theorem *and using* Corollary 5.2.4*, we have an exact sequence*

$$G_0(R/sR) \longrightarrow G_0(R) \longrightarrow G_0(R[\tfrac{1}{s}]) \longrightarrow 0.$$

□

Corollary 5.2.15. *By* Corollary 5.2.14*, if we take $R[t]$ as the ring, with R a Noetherian ring, and $s = t$, observe that $R[t]/tR[t] \cong R$, so we have an exact sequence*

$$G_0(R) \longrightarrow G_0(R[t]) \longrightarrow G_0(R[t, t^{-1}]) \longrightarrow 0.$$

□

5.3 The Fundamental Theorem of G_0

Let R be a Noetherian ring.

Lemma 5.3.1. *Every finitely generated R-module M has a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

in which every quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$, where every $\mathfrak{p}_i$ is a prime ideal. In particular, each quotient is annihilated by $\mathfrak{p}_i$.

Proof. If $M = 0$, then the filtration is trivial and it satisfies the lemma. Suppose $M \neq 0$. Let Σ be the set of ideals $\mathrm{Ann}(m) = \{r \in R \mid rm = 0\}$ for $0 \neq m \in M$. Σ is not empty since it contains the zero ideal, hence it has a maximal element because R is Noetherian. Let $\mathfrak{p}_1 = \mathrm{Ann}(x_1)$ be this maximal element. We will show that $\mathfrak{p}_1$ is prime. Suppose $ab \in \mathfrak{p}_1$ and $b \notin \mathfrak{p}_1$, so $abx_1 = 0$ and $bx_1 \neq 0$. Notice that $\mathrm{Ann}(x_1) \subseteq \mathrm{Ann}(bx_1)$, and since $\mathrm{Ann}(x_1)$ is maximal in Σ, $\mathrm{Ann}(bx_1) = \mathrm{Ann}(x_1) = \mathfrak{p}_1$. Thus, $a \in \mathrm{Ann}(bx_1) = \mathfrak{p}_1$, so $\mathfrak{p}_1$ is prime. This also implies that the submodule of M generated by x_1 is isomorphic to $R/\mathfrak{p}_1$ with $\mathfrak{p}_1$ a prime ideal.

Let M_1 be this submodule, and $M_0 = 0$ so $M_1/M_0 \cong R/\mathfrak{p}_1$. If $M/M_1 \neq 0$, we can repeat the same procedure to M/M_1 obtain an x_2 and a submodule M_2 of M generated by x_1 and x_2, such that $M_2/M_1 \cong R/\mathfrak{p}_2$; and so on, building an (strictly) ascending chain of submodules of M, and since R is Noetherian it must happen that $M_n = M$ in a finite number of steps. □

Let S be a multiplicative subset of R.

Notice that the inclusions $R/\mathfrak{p}$-$\mathbf{mod_{fg}} \subseteq R$-$\mathbf{mod_{fg}}$ (for every prime ideal $\mathfrak{p}$) and the functor $S^{-1}R \otimes_R -$ are exact, hence we have induced morphisms $G_0(R/\mathfrak{p}) \longrightarrow G_0(R)$ and $G_0(R) \longrightarrow G_0(S^{-1}R)$.

Proposition 5.3.2. *There is an exact sequence:*

$$\bigoplus_{\mathfrak{p} \cap S \neq \varnothing} G_0(R/\mathfrak{p}) \longrightarrow G_0(R) \longrightarrow G_0(S^{-1}R) \longrightarrow 0.$$

Proof. It is clear that every finitely generated $S^{-1}R$-module comes from some R-module, so the last map is surjective. An $R/\mathfrak{p}$-module, seen as an R-module is annihilated by $\mathfrak{p}$, which meets S, so localizing with respect to S results in the zero module.

To see that the sequence is exact at $G_0(R)$, we use Lemma 5.3.1. Let M be a finitely generated R-module and

$$0 = M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$$

a filtration for which every M_i/M_{i+1} is annihilated by a prime ideal $\mathfrak{p}_i$ and let $s \in S$ such that $sM = 0$ (we take s the product of the s_i which annihilate the generators of M), so that the image of $[M]$ is zero. In $G_0(R)$, we have

$$[M] = \sum [M_i/M_{i+1}],$$

and multiplication by s makes the filtration trivial, so s annihilates all the M_i/M_{i+1}, this is, it belongs to each $\mathfrak{p}_i$, i.e., the $\mathfrak{p}_i$ meet S. The M_i/M_{i+1} are $R/\mathfrak{p}_i$-modules, hence $[M]$ comes from an element of $\bigoplus_{\mathfrak{p} \cap S \neq \varnothing} G_0(R/\mathfrak{p})$. □

The following proof of the Fundamental Theorem of G_0 is due to Alexander Grothendieck and it does not work in the noncommutative case.

Theorem 5.3.3 (Fundamental Theorem of G_0)**.** *The inclusions*

$$R \overset{i}{\hookrightarrow} R[t] \overset{j}{\hookrightarrow} R[t, t^{-1}]$$

for a Noetherian ring R, induce isomorphisms

$$G_0(R) \cong G_0(R[t]) \cong G_0(R[t, t^{-1}]).$$

Proof. Let i^* and j^* be the induced morphisms, and let $s = j^* i^*$. Consider the following diagram:

$$\begin{array}{ccc} G_0(R) & \xrightarrow{\quad i^* \quad} & G_0(R[t]) \\ & {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle j^*} & \\ & G_0(R[t, t^{-1}]) & \end{array}$$

The morphism j^* is precisely the one in Corollary 5.2.15, therefore it is surjective.

We define a morphism $f : G_0(R[t, t^{-1}]) \longrightarrow G_0(R)$ as follows. Let M be a finitely generated $R[t, t^{-1}]$-module and consider multiplication by $1 - t$ as a morphism from M to itself. We have an exact sequence

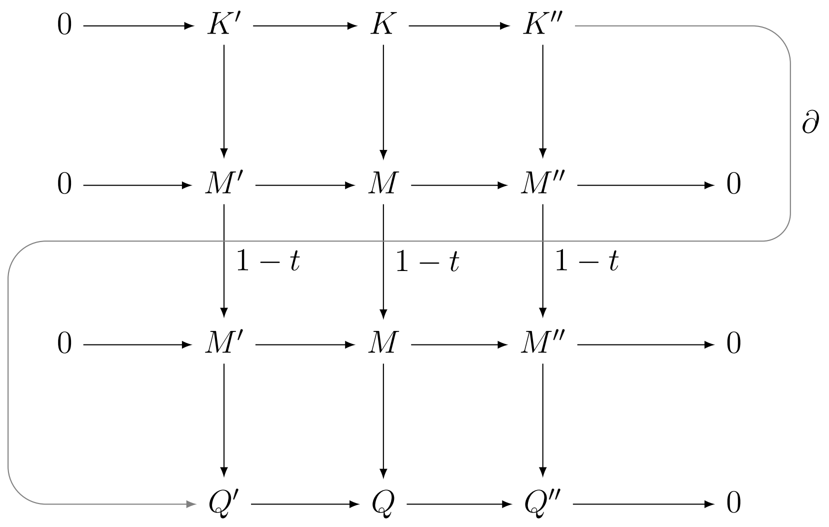
$$0 \longrightarrow K \longrightarrow M \xrightarrow{\;1-t\;} M \longrightarrow Q \longrightarrow 0,$$

where K and Q are the kernel and cokernel respectively of the said morphism. Since $1 - t$ takes the kernel to zero, the action of t is trivial in K, and hence so is the action of t^{-1} (it is the inverse of t in $R[t, t^{-1}]$). Also notice that $Q \cong M/(1 - t)M$, so the same situation occurs in Q. Therefore, K and Q can be seen as finitely generated R-modules. For these modules, define

$f([M]) = [Q] - [K]$. Notice that it is well defined because if there is a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in $R[t, t^{-1}]$, then we can construct a diagram



for which the Snake Lemma applies, hence we have an exact sequence of the kernels and cokernels, connected by ∂, which has Euler characteristic equal to zero. Therefore, the relations in $G_0(R[t, t^{-1}])$ are preserved in $G_0(R)$:

$$[Q] - [K] = ([Q'] - [K']) + ([Q''] - [K'']).$$

Let N be a finitely generated R-module. The image of $[N]$ by s takes it to $[R[t, t^{-1}] \otimes_R N]$. Multiplying $R[t, t^{-1}] \otimes_R N$ by $1 - t$ acts on the $R[t, t^{-1}]$ part, and it cannot annihilate any nonzero element, since that would mean that the degree of the element with respect to t is n and $n + 1$ at the same time. Hence, $R[t, t^{-1}] \otimes_R N \xrightarrow{1-t} R[t, t^{-1}] \otimes_R N$ has kernel zero, and its cokernel is $(R[t, t^{-1}] \otimes_R N)/((1 - t)R[t, t^{-1}] \otimes_R N)$, this is, $t = 1$ in the quotient, so $Q \cong R \otimes_R N \cong N$ and therefore $f(s([N])) = [N]$. This reasoning shows that s is a section of f, so s is injective.

If we show that s is surjective, the commutativity of the first diagram will imply that s, i^* and j^* are all isomorphisms. Since $s = j^* i^*$ and j^* is surjective, it will suffice to show that i^* is surjective. Suppose i^* is not surjective. Then, the set of ideals I for which $G_0(R/I) \longrightarrow G_0(R/I[t])$ is not an isomorphism (notice that i^* is already injective because s is injective) is not empty (at least $I = 0$), so there is a maximal element I_0 satisfying this condition. Replace R

by R/I_0 so that for any nonzero ideal I of R, $G_0(R/I) \longrightarrow G_0(R/I[t])$ is an isomorphism.

We separate the rest of the proof in two cases: either R is a domain, or R is not a domain.

If R is not a domain, take $S = R$, so that $S^{-1}R = 0$. Use Proposition 5.3.2 to construct the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
\bigoplus_{\mathfrak{p}} G_0(R/\mathfrak{p}) & \longrightarrow & G_0(R) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\bigoplus_{\mathfrak{p}} G_0(R/\mathfrak{p}[t]) & \longrightarrow & G_0(R[t]) & \longrightarrow & 0
\end{array}$$

The commutativity of the diagram and the fact that the map on the left is surjective (because 0 is not prime) implies that the one on the right is also surjective, which contradicts the assumption.

If R is a domain, take $S = R - \{0\}$, so $F = S^{-1}R$ is the field of fractions of R. Again, we can build a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\bigoplus_{\mathfrak{p}\cap S\neq\varnothing} G_0(R/\mathfrak{p}) & \longrightarrow & G_0(R) & \longrightarrow & G_0(F) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{\mathfrak{p}\cap S\neq\varnothing} G_0(R/\mathfrak{p}[t]) & \longrightarrow & G_0(R[t]) & \longrightarrow & G_0(F[t]) & \longrightarrow & 0
\end{array}$$

The map on the left is an isomorphism because $0 \cap S = \varnothing$. Since F and $F[t]$ are PIDs, by Proposition 5.1.9, $G_0(F) \cong G_0(F[t]) \cong Z$, so the map on the right is an isomorphism. Now the Five Lemma implies that $G_0(R) \longrightarrow G_0(R[t])$ is surjective, again, a contradiction. □

An interesting case of this theorem is when the ring R is regular. Recall Corollary 5.2.12, i.e., the Cartan homomorphism is an isomorphism for regular rings. By the Fundamental Theorem of G_0, and the Hilbert Syzygy Theorem an its corollary, it immediately follows:

Corollary 5.3.4. *Let R be a regular ring. Then*

$$K_0(R) \cong K_0(R[t]) \cong K_0(R[t, t^{-1}]).$$

□

Some other direct consequences of the Fundamental Theorem of G_0 are stated below.

Corollary 5.3.5. *Let* $\Bbbk$ *be a field. Then*

$$G_0(\Bbbk[t_1, \dots, t_n]) \cong \mathbb{Z}.$$

Proof. By induction $G_0(\Bbbk[t_1, \dots, t_n]) \cong G_0(\Bbbk[t_1, \dots, t_{n-1}]) \cong \cdots \cong G_0(\Bbbk[t_1]) \cong G_0(\Bbbk) \cong \mathbb{Z}$. □

Recall that fields are regular rings, hence from the last two results we have the following statement.

Corollary 5.3.6. *Let* $\Bbbk$ *be a field. Then*

$$K_0(\Bbbk[t_1, \dots, t_n]) \cong \mathbb{Z}.$$

□

From this last statement it follows that, if $R = \Bbbk[t_1, \dots, t_n]$, then finitely generated projective R-modules are stably free. At this point Serre's problem arises naturally: is it true that finitely generated projective R-modules are free? The answer is affirmative, and it was proved independently by Daniel Quillen and Andrei Suslin, and it is now known as the Quillen-Suslin Theorem.

Theorem 5.3.7 (Quillen-Suslin Theorem)**.** *Let* $\Bbbk$ *be a field and* $R = \Bbbk[t_1, \dots, t_n]$*, then every finitely generated projective* R*-module is free.*

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