BACHELOR'S DEGREE THESIS

Degree in Mathematics

Title: Boundary regularity for the fractional heat equation

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Academic year: 2013/2014

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Boundary regularity for the fractional heat equation

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September 2014

Abstract

In this dissertation we present an introduction to nonlocal operators, and in particular, we study the fractional heat equation, which involves the fractional Laplacian, $(-\Delta)^s$. In the first chapters we make a review of known classical results in the topic and present some elementary proofs of widely known propositions. After that, we introduce modern results on the elliptic problem for the fractional Laplacian. In particular, we present the results for the boundary regularity of the solution to the elliptic problem obtained by Ros-Oton and Serra, and we use them to derive the main original result of the dissertation. We show that a solution u of the homogeneous fractional heat equation on a bounded domain Ω fulfils that $u \in C^s(\mathbb{R}^n)$ and that u/δ^s can be extended Hölder continuously up to $\overline{\Omega}$, where $\delta(x) = \text{dist}(x, \partial\Omega)$. Furthermore, we are able to discuss the non-homogeneous case and obtain a similar result when the non-homogeneous term is time independent. Finally, we show an application and an extension of the result obtained. We are able to show that the Pohozaev identity holds for the solution of the fractional heat equation for positive times, and we extend the main result obtained for the fractional Laplacian to other nonlocal stable operators under certain conditions.

Contents

Chapter 1

Introduction

The aim of this work is to present an introduction to the fractional heat equation (analogue to the heat equation using the fractional Laplacian) and related results, and obtain new regularity results for this equation. More precisely, we extend the conclusions achieved by Ros-Oton and Serra in [39] from the elliptic problem to the parabolic problem[1].

Our dissertation concerns with the regularity of solutions to nonlocal equations, one of the hot topics in nonlinear analysis nowadays. Some of the most important contributions of the last years are obtained by Caffarelli and Silvestre in [11, 12, 46, 10], where they studied the regularity of solutions to nonlinear nonlocal equations. Furthermore, Frank and Lenzmann studied the uniqueness and nondegeneracy of solutions to equations in the whole space $\mathbb{R}^n$ [25]. Other examples are, for instance, Cabré and Sire, who established some results regarding the existence of solutions and regularity issues among other properties in semilinear equations with the fractional Laplacian in $\mathbb{R}^n$, [8, 9]; and Chang and González, who studied the relation between the fractional Laplacian operator and a class of conformally covariant operators in conformal geometry [14]. Other important works in the field are, for example, [40, 3, 21, 45].

As we will see later, the fractional Laplacian represents the infinitessimal generator of a Lévy stable diffusion process. This is why problems with this operator have been also widely studied in Probability [16, 15, 6, 5].

The fractional Laplacian and other nonlocal operators can be found in numerous fields, such as mathematical finance or fluid mechanics; and can be used to describe, for example, flame propagation and chemical reactions in liquids, geophysical fluid dynamics or anomalous diffusion in plasmas. For instance, in Finance, the obstacle problem for the fractional Laplacian can be used to model the price of American options [17]; while the pricing for European options can be modeled, after a change of variables, through a heat equation of the form,

$$\partial_t u + Lu = 0. \tag{1.1}$$

Here, L is a nonlocal operator, infinitessimal generator of a Lévy process. When L is the fractional Laplacian, this is the fractional heat equation, studied in this work.

[1] When we talk about the elliptic problem we mean the problem involving the elliptic equation $(-\Delta)^s u = g(x)$. On the other hand, when talking about the parabolic problem we refer to the one with the expression containing a temporal derivative, i.e., $\partial_t u + (-\Delta)^s u = f(x,t)$.

In Physics, it is no longer rare to find examples involving nonlocal operators, and in particular, the fractional Laplacian. In quantum mechanics, the square root of the Laplacian, $(-\Delta)^{1/2}$, is used as the square of the relativistic momentum operator, in contrast with the ordinary Laplacian for the square of the standard momentum. Thus, one could define a *fractional Hamiltonian* H_{2s} as the sum of the potential term and the momentum term; and use it to derive properties, for instance, for the hydrogen atom. This approach has been used to check the stability of relativistic matter with magnetic fields [26]. This is accomplished by checking that the corresponding Hamiltonian operator (which is basically a fractional Laplacian) is bounded from below when applied to the state functions of a system.

Furthermore, one could also use the Lévy processes to model the behaviour of matter at quantum levels, thus obtaining a fractional Schrödinger equation of the form,

$$i\hbar\frac{\partial\psi(r,t)}{\partial t} = H_{2s}\psi(r,t), \tag{1.2}$$

where H_{2s} is the fractional Hamiltonian,

$$H_{2s} = D_{2s}(-\hbar\Delta)^s + V(r,t), \tag{1.3}$$

V is the potential term and D_{2s} is a constant with the appropriate dimensions.

At this point, one could check some properties, such as the hermiticity of the operator (which will be checked in this dissertation in Lemma 3.4). One could also write the time-independent Schrödinger equation, and study the fractional Bohr atom and its energy levels. In like manner, it is possible to study the fractional oscillator, and all the other aspects characteristic of the standard Hamiltonian. For a good introduction to the fractional Schrödinger equation see [32]; and for more information see also [34, 44, 26].

In this dissertation we focus on the fractional heat equation, which has been studied in order to characterize the temporal evolution of functions under the influence of certain processes.

Let us exemplify this by supposing that a particle that moves following a Lévy process $\{X_t, t \geq 0\}$ dies or disappears when exiting a domain Ω. The notion of Lévy process will be introduced and explained in Chapter 2; but now, we can think of it as a random walk with possibly long jumps at random times. Under these circumstances, the probability distribution of the expected position of the particle after a time t is a function $u(x,t)$ that solves the problem

$$\begin{cases} \partial_t u + Lu &= 0 \quad \text{in } \Omega,\ t > 0 \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0. \end{cases} \tag{1.4}$$

where L is a nonlocal operator (explained below) associated to the particular Lévy process. For the fractional Laplacian we will illustrate this fact in section 2.3, through an heuristic derivation.

From a physical point of view, numerous numerical approaches have been made, for example by Zoia et al. in [48], and its applications in quantum mechanics and statistical physics are widely known, for instance, in collisional kinetics equations for appropriate time scales [35, 29].

Let us now present what nonlocal operators are, from a very intuitive point of view. Nonlocal operators are a certain kind of operators such that the value of the image of a function at a certain point depends on other points rather than just a neighbourhood of the selected point. That is, if L is a nonlocal operator, being $u : \mathbb{R}^n \to \mathbb{R}$ a function, and fixing $x_0 \in \mathbb{R}^n$, then the value of $Lu(x_0)$ depends on the value of $u(x)$ in other points outside a neighbourhood of x_0. Nonlocal operators are named like that in contrast with the more typical local operators, where the value of the image of the operator at a certain point depends only on the value of the function *near* the point. It is important to remember that this first approximation is purely intuitive, and by *near* we actually mean that we could take any neighbourhood of the point.

The most relevant example of nonlocal operator is the fractional Laplacian, $(-\Delta)^s$. The role played by the ordinary Laplacian, Δ, as the most important and typical example of operator for second order elliptic PDEs and the diffusion equation, is played by the fractional Laplacian in nonlocal problems of the same kind. Explicitly, the fractional Laplacian has the following expression, clearly nonlocal,

$$\begin{aligned}(-\Delta)^s u =& c_{n,s}\mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy \qquad (1.5)\\ :=& c_{n,s}\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^n\setminus\overline{B_\epsilon(x)}} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy, \qquad (1.6)\end{aligned}$$

where the integral, here and in the whole essay, has to be understood in a principle value sense. From now on, it will be omitted the "PV" notation. Moreover, $s \in (0,1)$ (and actually, when $s \uparrow 1$ we recover the ordinary Laplacian).

Nonlocal problems usually present a similar structure to PDE problems, requiring boundary conditions and specifying the equation in a bounded domain. For example, the Dirichlet problem for the fractional Laplacian is

$$\begin{cases} (-\Delta)^s u = g(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \qquad (1.7)$$

Notice that, in this problem, the "boundary conditions" are prescribed outside the domain, in contrast with typical PDE problems, where the boundary conditions are prescribed only on the boundary of the domain.

Similarly, the fractional heat equation in a bounded domain $\Omega \subset \mathbb{R}^n$ would read like this:

$$\begin{cases} \partial_t u + (-\Delta)^s u = 0 & \text{in } \Omega,\ t>0 \\ u = 0 & \text{in } \mathbb{R}^n\setminus\Omega,\ t\geq 0 \\ u(x,0) = u_0(x) & \text{in } \Omega,\ \text{for } t=0, \end{cases} \qquad (1.8)$$

for some initial condition u_0. Notice that the structure, as it has been already stated, is very similar to the ordinary heat equation, but the prescription $u = 0$ is made in $\mathbb{R}^n \setminus \Omega$ instead of on $\partial\Omega$.

As said before, the main aim of this work is to study the boundary regularity of solutions to the fractional heat equation (1.8), extending the results of Ros-Oton and Serra [39] for the elliptic problem (1.7) to the parabolic problem (1.8).

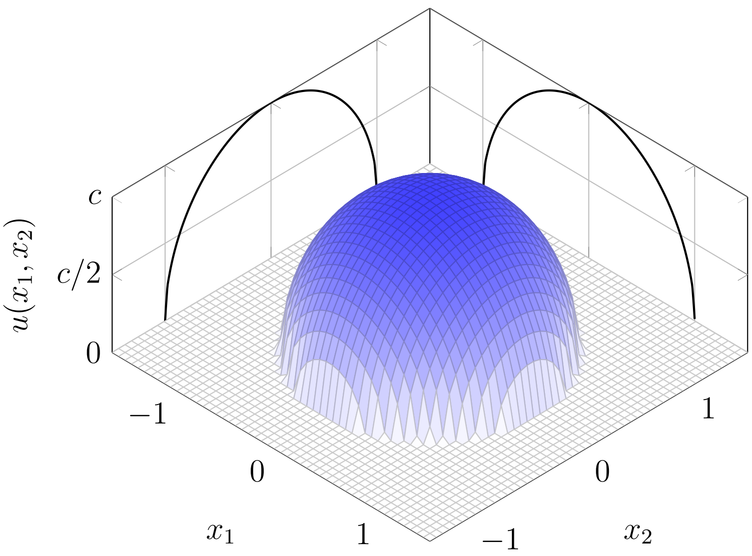

Figure 1.1: Visual representation of the solution to the problem (1.7) for $\Omega = B_1$, $g \equiv 1$, $n = 2$ and $s = 1/2$, $u(x_1, x_2) = c(1 - x_1^2 - x_2^2)^{1/2}$, together with the projections of the convex hull. As it can be seen, inside $\overline{B}_1$ the solution is half of an ellipsoid.

The regularity of solutions to the fractional heat equation has been already studied in the last years. Indeed, Chen-Kim-Song [15] established sharp two-sided estimates for the heat kernel of (1.8) in $C^{1,1}$ domains, and Bogdan-Grzwywny-Ryznar in [6] proved estimates in more general domains, strengthening the results of Chen-Kim-Song. Nevertheless, while these results imply some regularity up to the boundary, none of them establishes a result like the ones obtained by Ros-Oton and Serra in [39] for the elliptic problem, which are explained below.

Although, generally, it is impossible to find an explicit solution for problem (1.7); it has a simple expression when $\Omega = B_1$ and $g(x) \equiv 1$ in Ω. In that situation,

$$u(x) = c(1 - |x|^2)^s \tag{1.9}$$

solves the problem (for some constant $c > 0$). Notice that the solution is not smooth (C^∞) up to the boundary, but it is only $C^s(\overline{\Omega})$ [2]. See Figure 1.1 for a visualization of the solution for this particular case. Moreover, it turns out that this boundary behaviour is the same for all solutions, in the sense that any solution to (1.7) satisfies

$$-C\delta^s \le u \le C\delta^s \text{ in } \Omega, \tag{1.10}$$

where

$$\delta(x) = \mathrm{dist}(x, \partial\Omega). \tag{1.11}$$

The main result of [39] states that, for any $g \in L^\infty(\Omega)$, the ratio u/δ^s is Hölder continuous up to the boundary. In particular, they obtain an estimate of the form

$$\|u/\delta^s\|_{C^\alpha(\overline{\Omega})} \le C\|g\|_{L^\infty(\Omega)}, \tag{1.12}$$

for some $\alpha > 0$ small.

[2]We are using here the Hölder notation (remember, $s \in (0, 1)$).

As we will see, something similar occurs with the fractional heat equation, and the aim of this work is to prove it. Namely, our main result reads as follows:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be any bounded $C^{1,1}$ domain, and $s \in (0,1)$. Let $u_0 \in L^2(\Omega)$, and let u be the solution to the fractional heat equation*

$$\begin{cases} \partial_t u + (-\Delta)^s u &= 0 \quad \text{in } \Omega,\ t > 0 \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &= u_0(x) \quad \text{in } \Omega,\ \text{for } t = 0. \end{cases} \tag{1.13}$$

Then,

1. *For each $t_0 > 0$,*

$$\sup_{t \geq t_0} \|u(\cdot,t)\|_{C^s(\mathbb{R}^n)} \leq C_1(t_0)\|u_0\|_{L^2(\Omega)}. \tag{1.14}$$

2. *For each $t_0 > 0$,*

$$\sup_{t \geq t_0} \left\| \frac{u(\cdot,t)}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})} \leq C_2(t_0)\|u_0\|_{L^2(\Omega)}, \tag{1.15}$$

where $\delta(x) = dist(x, \partial\Omega)$, and $\alpha > 0$ is small.

The constants C_1 and C_2 depend only on t_0, n, s and Ω, and blow up as $t_0 \downarrow 0$.

Let us now explain how the work is divided, and the main results for each section. As explained next, the first chapters correspond to purely theoretical background, necessary to introduce the topic and present some known results. The most important chapters of the present dissertation are Chapter 4 and Chapter 6.

- **Chapter 2** contains the main theoretical background necessary to introduce nonlocal equations. We present an overview of nonlocal operators and nonlocal equations, particularly the fractional Laplacian, and its relation with Lévy processes. Thus, we first introduce Lévy processes and define *stable processes*. After that, we present an heuristic argument to derive the fractional heat equation from a random walk process allowing arbitrary long jumps. Finally, we give a proof of the nonlocal expression of the fractional Laplacian from its definition through Fourier transform.

- **Chapter 3** corresponds to "Preliminaries and known results". In this chapter, we present the most important results that will be used in the dissertation, and some other known results are also presented and proved. In particular, we introduce the fractional Sobolev spaces and we prove that the fractional Laplacian has a sequence of eigenfunctions forming a Hilbert base of $L^2(\Omega)$ for any given domain Ω. Moreover, we also state and prove the maximum principle and the comparison principle for the elliptic and parabolic problem. The facts that the eigenfunctions form a basis of L^2 and the maximum principle for this operators are well known, but are usually proved for more general operators, using sometimes stronger techniques and theorems [22]. In this case, however, we present simple proofs using tools specific for the fractional Laplacian.

After that, we treat classical interior regularity and expose the main known results, presenting some ideas that are widely used in the field. At this point, we will also explain the main result by Ros-Oton and Serra, which is the cornerstone and motivation of the whole dissertation.

Finally, at the end of Chapter 3 we also present the known results regarding the homogeneous fractional heat equation, obtained in [15].

- **Chapter 4** is, probably, the most important chapter in the work, together with Chapter 6. There, we present the main results of the dissertation and the corresponding proofs. Firstly, we prove regularity up to the boundary for the eigenfunctions of the fractional Laplacian, establishing the first original result of the dissertation. Considering a bounded and $C^{1,1}$ domain Ω, we are able to bound the L^∞ norm of the eigenfunctions in $\mathbb{R}^n$ by its L^2 norm and a multiplicative constant, giving explicitly its dependence on the eigenvalue, λ. After that, we proceed finding the expression of the solution of the fractional heat equation in terms of the eigenfunctions, and we also prove the uniqueness of the solution. This allows us to prove the main original result of the dissertation, Theorem 1.1.

  In addition, we are able to establish some results regarding the regularity in the temporal domain, and a proposition regarding the interior regularity for the solutions of the fractional heat equation problem for positive times.

- **Chapter 5** is the natural continuation of Chapter 4. We there expose an introduction to the non-homogeneous problem, that is, problem (1.8) with a right-hand side $f(x,t)$. In particular, we study the uniqueness and existence of solution in this case, under the assumption of a bounded non-homogeneous term. To do so, we introduce the Duhamel's principle for the fractional Laplacian, and check its validity. While we do not reach any conclusion in the general $f(x,t)$, we here present an original result regarding the case when f is time independent and bounded, $f = f(x)$. The statement of the proposition is very similar to the one presented in Theorem 1.1, but instead of considering $\|u_0\|_{L^2(\Omega)}$ we consider $\|u_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(\Omega)}$. We then give some ideas for the general case, and some approaches done by the author.

  Finally, we discuss some boundary inequalities for the solution of the fractional heat equation, $u(x,t)$, obtaining that for positive times $t > 0$, there are constants c and C depending only on n, s, t and Ω such that

$$0 < c(t)\delta^s(x) \le u(x,t) \le C(t)\delta^s(x) \tag{1.16}$$

  as long as there is a nonnegative and not constant zero initial condition and a nonnegative non-homogeneous term.

- **Chapter 6** is the last and one of the two most important chapters of the dissertation. In it, we proceed to show an application and an extension of the main result of the work, Theorem 1.1.

We first prove the Pohozaev identity for the solution of the homogeneous fractional heat equation for positive times (see Theorem 6.1). That is, if $u = u(\cdot, t)$ is the solution, we prove that, for any $t > 0$

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u dx = \frac{2s-n}{2} \int_\Omega u(-\Delta)^s u dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma, \tag{1.17}$$

where ν is the unit outward normal. As we can see, it is essential the fact that in the main theorem of the dissertation, Theorem 1.1, we had proven that $\frac{u}{\delta^s}$ can be extended continuously up to the boundary of the domain Ω.

We finally extend the results obtained in Chapter 4 to other nonlocal operators. We deal with the more general parabolic problem

$$\begin{cases} \partial_t u + Lu &= 0 & \text{in } \Omega,\ t > 0 \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &= u_0(x) & \text{in } \Omega,\ \text{for } t = 0, \end{cases} \tag{1.18}$$

where L is a nonlocal operator of the form,

$$Lu(x) = \int_{\mathbb{R}^n} (u(x) - u(x+y)) \frac{a(y/|y|)}{|y|^{n+2s}} dy. \tag{1.19}$$

As we will see, these operators are infinitesimal generators of stable Levy processes. In this chapter we prove Theorem 1.1 for this new problem, under the additional assumption

$$0 < \lambda \leq a(\theta) \leq \Lambda,\ \forall \theta \in S^{n-1}, \tag{1.20}$$

for some $0 < \lambda \leq \Lambda$ (see Theorem 6.4).

In all, the main original results of the dissertation are Theorem 1.1, Corollary 4.8, Theorem 6.3 and Theorem 6.4.

Chapter 2

Nonlocal operators and the fractional Laplacian

In this chapter we present the first concepts regarding the fractional Laplacian and nonlocal operators in general. Firstly, we introduce the Lévy processes and their relation with nonlocal operators, as well as the definition of *stable process*. After that, we present an heuristic argument to obtain the fractional heat equation, which follows from a discretization of the plane. Finally, we define the fractional Laplacian in two different ways, checking that its definition as a nonlocal operator is equivalent to its definition by means of the Fourier transform.

2.1 Nonlocal equations and Lévy processes

Nonlocal equations arise naturally from the properties of Lévy processes. As an intuitive approach, Lévy processes are stochastic processes with independent and stationary increments that represent the motion of a point particle as an extension of Brownian motion, with independent displacements and statistically identical over time intervals with the same length. Examples of Lévy processes include the Brownian motion itself, the Poisson process, stable processes and subordinators. Again, intuitively, they consist of paths that can present jump discontinuities of random size at random moments.

Lévy processes can be used to model many physical systems, as well as in engineering, economics and ecology.

More formally, the definition of a Lévy process is the following, defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$:

Definition 2.1 (Lévy process)**.** A Lévy process $X = (X_t, t \geq 0)$ is a real-valued (or $\mathbb{R}^n$ valued) stochastic process that fulfils the following requirements:

1. $P(X_0 = 0) = 1$ almost surely
2. The random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent $\forall n \geq 1$, for $0 \leq t_0 < t_1 < t_2 < \cdots < t_n$ (Independent increments)
3. The random variable $X_{t+s} - X_t$ has the same distribution as X_s, for $s, t \geq 0$ (Stationary increments)

4. $\forall \epsilon > 0$ and $\forall s \geq 0$, $\lim_{t\to s} P(|X_t - X_s| > \epsilon) = 0$ (Stochastic continuity)

It is possible also to add another condition including almost sure right continuity and left limits for the process, but this depends on the author. Also, the first condition is redundant once one defines the third one, considering $s = 0$.

The second property (Independent increments) indicates that there is no memory in the stochastic process, and the third property (Stationary increments) refers to the fact that the distribution of $X_{t+s} - X_t$ depends only on the time interval length s, implying that the process is autonomous. Finally, the Stochastic continuity is what makes it a process that *rarely* jumps, i.e., there is a null probability to obtain a significant jump of length $\epsilon > 0$.

Before proceeding to the relation between Lévy processes and nonlocal equations, it is necessary to introduce the Lévy-Khintchine formula. To do so, recall that the characteristic function of a random variable, X, is defined by $\phi_X : \mathbb{R}^n \to \mathbb{C}$

$$\phi_X(z) := \mathbb{E}[e^{iz\cdot X}].$$

For a Lévy proces, it is defined by

$$\phi_t(z) := \phi_{X_t}(z) = \mathbb{E}[e^{iz\cdot X_t}].$$

Then, the Lévy-Khintchine formula states that:

Theorem 2.1 (Lévy-Khintchine formula)**.** *If $X = (X_t, t \geq 0)$ is a Lévy process, then*

$$\phi_t(z) = e^{t\eta(z)}, \quad z \in \mathbb{R}^n, \tag{2.1}$$

where $\eta(z)$ is a function given by

$$\eta(z) = ib\cdot z - \frac{1}{2}z\cdot Az + \int_{\mathbb{R}^n} (e^{iz\cdot y} - 1 - iz\cdot y\chi_{B_1}(y))\nu(dy), \tag{2.2}$$

where B_1 is the unit ball, b is a vector, A is a nonnegative definite matrix and ν is a Lévy measure.

The proof of the theorem uses the fact that the Lévy processes are infinitely divisible, i.e., it is possible to express $X_t = Y_1 + \ldots Y_m$ where Y_j are independent identically distributed random variables, for any $m \in \mathbb{N}$. To see this, simply consider $h = t/m$ and take $Y_j = X_{jh} - X_{(j-1)h}$. Using the first three properties of the definition of a Lévy process, it follows the infinite divisibility.

Moreover, ν is a $\mathbb{R}^n$ measure known as the *Lévy measure*, defined as follows:

Definition 2.2 (Lévy measure)**.** Let $X = (X_t, t \geq 0)$ a Lévy process on $\mathbb{R}^n$. Then, the measure ν on $\mathbb{R}^n$ defined as follows is the Lévy measure associated to X:

$$\nu(C) = \mathbb{E}[\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in C\}], \quad C \in \mathcal{B}(\mathbb{R}^n), \tag{2.3}$$

where

$$\Delta X_t = X_t - \lim_{s\uparrow t} X_s. \tag{2.4}$$

That is, the measure of a set C is the expected number of jumps with size in C per unit time.

Any Lévy measure, moreover, fulfils:

$$\int_{\mathbb{R}^n} (1 \wedge |y|^2)\nu(dy) < \infty, \tag{2.5}$$

where $\min\{a, b\} = a \wedge b$. A Lévy process almost surely defines a semigroup $\{U_t, t \geq 0\}$ acting on functions $u : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$(U_t u)(x) = \mathbb{E}[u(x + X_t)], \tag{2.6}$$

which is a semigroup thanks to the properties of a Lévy process, since U_0 is the identity and $U_{t+s} = U_t \circ U_s$ because $X_t - X_s$ is distributed as X_{t-s}. Since it is a semigroup, it has an infinitessimal generator given by L and defined by:

$$Lu = \lim_{t \downarrow 0} \frac{\mathbb{E}[u(x + X_t)] - u(x)}{t}. \tag{2.7}$$

This linear operator describes the semigroup since the Feller property is satisfied ($\lim_{t\downarrow 0} U_t u(x) = u(x),\ \forall x$). Moreover, it is possible to express an evolution of the function u over the time by means of this operator, defining $u(x, t) := (U_t u)(x)$, and assuming $u(x, t)$ is regular enough, it follows that

$$\partial_t u = Lu \quad \forall (x, t) \in \mathbb{R}^n \times [0, \infty). \tag{2.8}$$

And using the Lévy-Khintchine formula, it is possible to express the infinitessimal generator as

$$Lu = b \cdot \nabla u(x) + tr(A \cdot D^2 u) + \int_{\mathbb{R}^n} \{u(x + y) - u(x) - y \cdot \nabla u(x) \chi_{B_1}(y)\}\, \nu(dy), \tag{2.9}$$

where the integral is taken in principle value. The first term corresponds to the drift, the second term to the diffusion and the third term to the jump part. As it can be seen, from the Lévy processes it has naturally arisen a nonlocal equation.

The process $\{X_t, t \geq 0\}$ is said to be a pure jump process when $A = 0$ and $b = 0$. Typical assumptions when studying these processes are

$$\nu(dy) = K(y)dy \tag{2.10}$$

and

$$K(y) = K(-y), \tag{2.11}$$

with $K \geq 0$ and $\int_{\mathbb{R}^n} (1 \wedge |y|^2) K(y) dy < \infty$.

Under these assumptions, the infinitessimal generator can be written as:

$$Lu = \int_{\mathbb{R}^n} (u(x + y) - u(x)) K(y) dy. \tag{2.12}$$

Note that the term in $\nabla u(x)$ has vanished, since integrating the scalar product $y \cdot \nabla u(x)$ with respect to y in the unit ball multiplied by an even function yields to zero. In particular, when

$$K(y) = c|y|^{-n-2s}, \tag{2.13}$$

then L is the fractional Laplacian, as it will be seen in section 2.4.

Let us see now how Levy processes lead to nonlocal problems like 1.7 and 1.8.

For example, suppose that we are dealing with a Lévy process $\{X_t, t \geq 0\}$ with infinitessimal generator L, and a particle following it. We first want to know the expected time at which the particle, starting at x, will escape the domain. We name this time $u(x)$. Then, $u(x)$ solves the following problem

$$\begin{cases} Lu = 1 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.14}$$

Now, we want to know the expected running payoff of the particle when escaping the domain if it starts at x, assuming a payoff function on the domain $g(x)$. If we name again this expected running payoff as $u(x)$, then u solves the problem

$$\begin{cases} Lu = g(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.15}$$

Finally, let us recall the example presented in the introduction. Suppose now that we want the probability distribution of the expected position of the particle after a time t, and we name this probability distribution $u(x,t)$. Then, u is a function that solves the following problem

$$\begin{cases} \partial_t u + Lu = 0 & \text{in } \Omega,\ t > 0 \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0, \end{cases} \tag{2.16}$$

which will be illustrated for the fractional Laplacian in section 2.3 through an heuristic derivation.

2.2 Stable processes

Once the Lévy processes have been defined, let us introduce a class of specially relevant kind of Lévy process, the *α-stable* Lévy processes. These processes are the ones satisfying certain scaling properties.

When talking about stability we refer to the fact that the sum of two independent stable random variables, X and Y, identically distributed, is also stable, and $X+Y$ has the same distribution as X and Y, when renormalized by $2^{-1/\alpha}$. Here, α is called the stability exponent, and $\alpha \in (0,2]$. When $\alpha = 2$ we have that the ordinary normal distributions fulfil the property.

For $\alpha \in (0,2)$, stable processes are the equivalent to Gaussian random processes when dealing with infinite variance random variables. Indeed, the Generalized Central Limit Theorem states that the distribution of the sum of infinite variance random variables converges in density to a stable process under certain assumptions (see [43], [33] or [2] for a precise statement).

Stable distributions model random aggregations of small and independent perturbations. They are used, for example, in financial mathematics, internet traffic statistics, in models of random scalar fields, signal processing among others; see [43, 36] and references therein.

Let us introduce the precise definition of stable Lévy process.

Definition 2.3. (α-stable Lévy process) Let $X = (X_t, t \geq 0)$ be a Lévy process on $\mathbb{R}^n$. It is called *α-stable* if

$$X_1 \stackrel{d}{=} \frac{1}{t^{1/\alpha}} X_t, \;\; \forall t > 0, \tag{2.17}$$

that is, there exist a property of self-similarity saying that X_b is distributed like $\frac{1}{t^{1/\alpha}} X_{bt}$ for $b, t > 0$.

The infinitessimal generator of α-stable Lévy processes are uniquely determined by a finite measure on the unit sphere S^{n-1}; often referred as the *spectral measure* of the process. When this measure is absolutely continuous with respect to the classical measure of the sphere and is symmetric, stable processes have nonlocal operators associated as infinitessimal generators (when $\alpha < 2$), of the form

$$Lu(x) = \int_{\mathbb{R}^n} (u(x) - u(x+y)) \frac{a(y/|y|)}{|y|^{n+2s}} dy, \tag{2.18}$$

where $s \in (0,1)$, the stability exponent is $\alpha = 2s$. Here, a is any nonnegative function $a \in L^1(S^{n-1})$ satisfying $a(y) = a(-y)$ (a is symmetric).

The most simple and important case of stable process, as it has been already introduced, is when it is radially symmetric (isotropic). In this case we have that the infinitessimal generator is a multiple of the fractional Laplacian,

$$(-\Delta)^s u = c_{n,s} \int_{\mathbb{R}^n} \frac{(u(x) - u(x+y))}{|y|^{n+2s}} dy. \tag{2.19}$$

2.3 Heuristic argument to obtain the heat equation in bounded domains

There exist numerous arguments regarding the heat equation with the Laplacian, most of them involving a discretization of the space and basic probability, that eventually yields to the well known expression. In the case of the fractional Laplacian, there are also some heuristic argumentations that yield the expected result, assuming a certain kind of kernel and through the discretization of the space. We summarize here this heuristic argument, explained and discussed in [47] to obtain the equation for the fractional Laplacian for a random walk with possibly long jumps, in the homogeneous case with $\mathbb{R}^n$ domain. A similar proof can be found in [23].

As in the Laplacian case, the first thing to do is to consider a lattice in $\mathbb{R}^n$, and consider the points on $h\mathbb{Z}^n$ for $h > 0$ (eventually we will want $h \to 0$). As in the random walk case, suppose a point particle walks from one point of the lattice to another with a time step τ. In the typical discrete random walk, the point particle is only allowed to walk towards adjacent points, but now it will be able to jump to any other point, with a certain probability.

In order to describe the probability to make arbitrary long jumps, it is defined a function, $\mathcal{K} : \mathbb{R}^n \to [0, \infty)$, that will weight the probability of each jump. We want this function to be symmetric with respect to the origin (even), so $\mathcal{K}(z) = \mathcal{K}(-z), \;\; \forall z \in \mathbb{R}^n$. In particular, this function describes the probability for a particle to perform a jump of "size" $z \in \mathbb{R}^n$, and assuming symmetry in the probabilities, the even condition follows naturally. Given $m, \overline{m} \in \mathbb{Z}^n$, the probability to jump from hm to $h\overline{m}$ (or vice-versa) is given by $\mathcal{K}(m-\overline{m}) = \mathcal{K}(\overline{m}-m)$. Also, since $\mathcal{K}$ represents a probability, it must satisfy:

$$\sum_{m\in\mathbb{Z}^n} \mathcal{K}(m) = 1. \tag{2.20}$$

Now, suppose $u(x,t)$ is the probability that the particle is found at $x \in \mathbb{R}^n$ at the instant $t \in [0,\infty)$ (later on, this will be the probability density that there is a particle with these coordinates). In the discrete time, this is simply a probability evaluated on the lattice points. Then, after a time τ, we have:

$$u(x,t+\tau) = \sum_{m\in\mathbb{Z}^n} \mathcal{K}(m)u(x+hm,t), \tag{2.21}$$

since the probability to be at x at a time $t+\tau$ is the sum of probabilities to be at the points $x+hm$ at time t weighted according to the "size" of the jump necessary to reach x from each position.

And now, just subtract from both sides $u(x,t)$, and divide by τ

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \sum_{m\in\mathbb{Z}^n} \frac{\mathcal{K}(m)}{\tau}(u(x+hm,t)-u(x,t)). \tag{2.22}$$

At this point, it is necessary to make $h \to 0$, and $\tau \to 0$. Assume there exists a relation between them such that $\tau = h^{2s}$ $(s > 0)$, so that they both tend to 0 simultaneously (but possibly at different velocities). It is wanted the right hand side of the previous expression to converge, and its resemblance with a Riemann sum is obvious. It would be necessary a factor h^n on the right-hand side, as well as a dependence on $\mathcal{K}(hm)$ instead of $\mathcal{K}(m)$, so we want to assume

$$\frac{\mathcal{K}(m)}{\tau} = h^n\mathcal{K}(mh), \tag{2.23}$$

maybe with a constant multiplicative factor. An example of function $\mathcal{K}$ following the previous expression is

$$\mathcal{K}(z) = \frac{1}{|z|^{n+2s}},$$

imposing $\mathcal{K}(0) = 0$ and a multiplicative constant so that (2.20) is fulfilled. Now we have the following Riemann sum (up to multiplicative constant):

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = h^n \sum_{m\in\mathbb{Z}^n} \mathcal{K}(hm)(u(x+hm,t)-u(x,t)). \tag{2.24}$$

Taking a continuous limit and assuming a good convergence, $\tau \downarrow 0$, it follows (up to a multiplicative factor, again)

$$\partial_t u(x,t) = \int_{\mathbb{R}^n} \frac{u(x+y,t)-u(x,y)}{|y|^{n+2s}}dy = -(-\Delta)^s u(x,t), \text{ in } \mathbb{R}^n,\ t>0, \tag{2.25}$$

where we impose $s \in (0,1)$ so that the integral exists in principle value. As it can be seen, we have obtained a heat equation with a fractional Laplacian (acting on x variables):

$$\partial_t u + (-\Delta)^s u = 0, \text{ in } \mathbb{R}^n,\ t>0, \tag{2.26}$$

where it is needed to fix an initial condition (for $t = 0$).

Assume now that we are dealing with a bounded domain situation, $\Omega \subset \mathbb{R}^n$. That is, if the particle escapes the domain, it disappears or dies. Now, how does the previous expression of probability density evolve with time?

Basically, bounding the particle in the domain implies that the probability density function outside of it is exactly 0. So we impose a boundary condition, outside the domain, and fix it at 0. Notice that, differently from the random walk, here it is necessary to impose the boundary condition in $\mathbb{R}^n \setminus \Omega$ and not only in $\partial\Omega$. This is due to the fact that now the particle can jump, and it could appear at any point outside the boundary; which means that the particle somehow disappears or dies. It is necessary, then, to establish a "boundary condition" in the whole $\mathbb{R}^n \setminus \Omega$.

$$\left\{ \begin{array}{rcll} \partial_t u + (-\Delta)^s u &=& 0 & \text{in } \Omega,\ t > 0 \\ u &=& 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &=& u_0(x) & \text{in } \Omega,\ \text{for } t = 0, \end{array} \right. \tag{2.27}$$

where u_0 is the initial condition, fixed at time $t = 0$.

We could even consider a non-homogeneous term. Namely, suppose that the system somehow has the ability to introduce particles at each position at a certain rate, $f(x,t)$, which depends obviously on the position and on the time. $f(x,t)$ would represent the probability per unit time that a particle is introduced at the system in the position x at time t. Then, the solution u would describe the evolution of the probability density to have a particle at a given point of the domain at a certain time. It is even possible to assume that f depends on u, i.e., the probability of adding particles depends directly on the probability to have particles with certain coordinates.

In order to introduce this change, simply rewrite expression (2.21) as follows:

$$u(x, t+\tau) = \left(\sum_{m \in \mathbb{Z}^n} \mathcal{K}(m) u(x + hm, t) \right) + \tau f(x,t). \tag{2.28}$$

Once we divide by τ, the term $f(x,t)$ (or $f(x,t,u)$) no longer depends on the parameters τ, h, which are the ones that will tend to 0. Therefore, we could recover the expression (2.27) and write it as

$$\left\{ \begin{array}{rcll} \partial_t u + (-\Delta)^s u &=& f(x,t) & \text{in } \Omega,\ t > 0 \\ u &=& 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &=& u_0(x) & \text{in } \Omega,\ \text{for } t = 0, \end{array} \right. \tag{2.29}$$

where we could even impose conditions in $\mathbb{R}^n \setminus \Omega$ to fix the probability density outside the domain.

2.4 The fractional Laplacian and the Fourier transform

The fractional Laplacian, $(-\Delta)^s$, is a pseudo-differential operator that can be derived naturally from the standard Laplacian and the use of the Fourier transform. First, recall that given $u : \mathbb{R}^n \to \mathbb{R}$, the Fourier transform is defined by

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{-2\pi i x\cdot\xi} dx, \tag{2.30}$$

$$\mathcal{F}^{-1}(\hat{u})(x) = u(x) = \int_{\mathbb{R}^n} \hat{u}(\xi) e^{2\pi i \xi\cdot x} d\xi. \tag{2.31}$$

From which it is easy to derive

$$\mathcal{F}(\partial_j u) = \widehat{\partial_j u} = i\xi_j(\mathcal{F}(u)),$$

and therefore

$$\mathcal{F}(-\Delta u) = |\xi|^2(\mathcal{F}u).$$

At this point, the definition of the fractional Laplacian follows naturally as

$$(-\Delta)^s u := \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)) \tag{2.32}$$

for $s \in (0, 1)$ (the standard Laplacian is recovered for $s = 1$).

Proposition 2.2. *The fractional Laplacian, $(-\Delta)^s$, is a nonlocal operator that can be written as:*

$$(-\Delta)^s u = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy, \tag{2.33}$$

where $c_{n,s}$ is a constant depending only on n and s.

Note that the previous proposition is usually taken as a definition, and one then has to check the equivalence with the Fourier transform expression. Before proceeding to the proof, the following lemma is introduced:

Lemma 2.3. *Given $u : \mathbb{R}^n \to \mathbb{R}$, and $K : \mathbb{R}^n \to [0, +\infty)$; then*

$$\mathcal{L}u(x) := \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy = \mathcal{F}^{-1}(\mathcal{S}(\mathcal{F}u)) \tag{2.34}$$

for some function $\mathcal{S} : \mathbb{R}^n \to \mathbb{R}$, given by

$$\mathcal{S}(\xi) = 2\int_{\mathbb{R}^n} (\cos(\xi\cdot y) - 1)K(y)dy. \tag{2.35}$$

Proof. By definition:

$$\mathcal{S}(\xi)(\mathcal{F}u)(\xi) = \mathcal{F}(\mathcal{L}u)(\xi) = \mathcal{F}\left(\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy\right).$$

And now, operating on the right-hand side and using properties of the Fourier transform:

$$\begin{aligned}
\mathcal{F}(\mathcal{L}u)(\xi) &= \int_{\mathbb{R}^n} \left(\mathcal{F}(u(x+y) + u(x-y) - 2u(x))\right) K(y)dy \\
&= \int_{\mathbb{R}^n} (e^{i\xi\cdot y} + e^{-i\xi\cdot y} - 2)(\mathcal{F}u)(\xi)K(y)dy \\
&= (\mathcal{F}u)(\xi)\int_{\mathbb{R}^n} (e^{i\xi\cdot y} + e^{-i\xi\cdot y} - 2)K(y)dy \\
&= 2(\mathcal{F}u)(\xi)\int_{\mathbb{R}^n} (\cos(\xi\cdot y) - 1)K(y)dy.
\end{aligned}$$

□

At this point it is possible to proceed with the proof of Proposition 2.2.

Proof of Proposition 2.2. First of all, notice:

$$\begin{aligned}(-\Delta)^s u(x) =& c_{n,s}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}dy = c_{n,s}\int_{\mathbb{R}^n}\frac{u(x)-u(x+y)}{|y|^{n+2s}}dy\\ =& -\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\frac{u(x+y)-u(x)}{|y|^{n+2s}}dy - \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\frac{u(x-y)-u(x)}{|y|^{n+2s}}dy\\ =& -\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy,\end{aligned}$$

where it has been used that

$$\int_{\mathbb{R}^n}\frac{u(x+y)-u(x)}{|y|^{n+2s}}dy = \int_{\mathbb{R}^n}\frac{u(x-y)-u(x)}{|y|^{n+2s}}dy,$$

which follows simply performing a change of variables, y by $-y$. The expression obtained is the same as in (2.34), with a previous constant and $K(y) = -\frac{c_{n,s}}{2}|y|^{-(n+2s)}$. Therefore, the proposition will follow if

$$|\xi|^{2s} = \bar{c}\int_{\mathbb{R}^n}\frac{1-\cos(\xi\cdot y)}{|y|^{n+2s}}dy, \tag{2.36}$$

for some constant $\bar{c}$. To do so, it is important to see that the right-hand side of the expression is rotationally invariant: changing ξ by $R\xi$ for some rotation matrix R does not change the expression, since the scalar product becomes $(R\xi)\cdot y = \xi\cdot(R^T y)$, and it is possible to perform a change of variable and put $R^T y$ instead of y. Since $|y|^{n+2s}$ is indeed rotationally invariant, this change does not modify the expression, and therefore one can put, without lose of generality, $\xi = |\xi|e_1$.

Now write $\eta = (\eta_1,\ldots,\eta_n)$, $\eta = |\xi|y$,

$$\int_{\mathbb{R}^n}\frac{1-\cos(|\xi|y_1)}{|y|^{n+2s}}dy = \int_{\mathbb{R}^n}\frac{1-\cos(\eta_1)}{|\eta/|\xi||^{n+2s}}\frac{d\eta}{|\xi|^n} = |\xi|^{2s}\int_{\mathbb{R}^n}\frac{1-\cos(\eta_1)}{|\eta|^{n+2s}}d\eta.$$

So it is enough to prove that $\int_{\mathbb{R}^n}\frac{1-\cos(\eta_1)}{|\eta|^{n+2s}}d\eta$ is finite. First, notice that

$$\int_{\mathbb{R}^n\setminus B_\epsilon}\frac{1-\cos(\eta_1)}{|\eta|^{n+2s}}d\eta < +\infty, \qquad \forall\epsilon>0,$$

since $s>0$. Moreover, for ϵ small enough, it follows that

$$\frac{1-\cos(\eta_1)}{|\eta|^{n+2s}} \le \frac{|\eta_1|^2}{|\eta|^{n+2s}} \le \frac{1}{|\eta|^{n-2(1-s)}},$$

which is integrable in B_ϵ since $s<1$. □

Moreover, it has been possible to determine the value of the constant, $c_{n,s}$, which happens to be

$$c_{n,s} = \left(\int_{\mathbb{R}^n}\frac{1-\cos(\eta_1)}{|\eta|^{n+2s}}d\eta\right)^{-1}. \tag{2.37}$$

And this integral can be computed explicitly:

$$c_{n,s} = \frac{s 2^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2}\Gamma(1-s)}, \tag{2.38}$$

where Γ represents the well known gamma function. There, now it is possible to write the fractional Laplacian in the following way:

$$(-\Delta)^s u(x) = -\frac{1}{2} c_{n,s} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \forall x \in \mathbb{R}^n, \tag{2.39}$$

where now the integral, for smooth functions u, is not necessary in a principle value sense, because the singularity has been removed. The numerator in the fraction inside the integral can be bounded by $|y|^2 \|D^2 u\|_{L^\infty}$ near the origin, so the fraction is integrable near 0.

Also note that the fractional Laplacian as a nonlocal operator corresponds to a certain kind of Lévy process, though the infinitessimal generator. Namely, using the notation in (2.2), for $A = 0$ (no Brownian component), $b = 0$ (no drift) and using

$$\nu(dy) = \frac{dy}{|y|^{n+2s}} \tag{2.40}$$

for $s \in (0, 1)$, then the infinitessimal generator of the Lévy process is given by $(-\Delta)^s$.

Moreover, it is also possible to define the inverse of the fractional Laplacian of order $s \in (0, 1)$, noted as $(-\Delta)^{-s}$. If $0 < 2s < n$, then one can write the following explicit formula in integral form, as long as u is integrable enough:

$$(-\Delta)^{-s} u(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{u(x-y)}{|y|^{n-2s}} dy. \tag{2.41}$$

Notice that this expression corresponds to the Riesz potential $I_{2s}u$ of the locally integrable function u; where $C_{n,s}$ is some constant depending only on n and s.

Chapter 3

Preliminaries and known results

In this chapter we first introduce the fractional Sobolev spaces, which are the natural spaces when dealing with problems with the fractional Laplacian. After that, we proceed to see that the eigenfunctions of the elliptic problem with the fractional Laplacian form a Hilbert basis of L^2, and we prove the maximum principle for both the elliptic and the parabolic problem. We then state the main results currently known for the elliptic problem with the fractional Laplacian. Concretely, we introduce the classical results for interior regularity, and more modern results regarding the regularity up to the boundary. Finally, the known results for the homogeneous fractional heat equation are also presented.

3.1 Introduction to fractional Sobolev spaces

First of all, let us define the fractional Sobolev spaces. Intuitively, one could say that we expect a Banach space somehow found "between" $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

Definition 3.1 (Fractional Sobolev space)**.** The fractional Sobolev space $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{n/p+s}} \in L^p(\Omega \times \Omega) \right\}. \tag{3.1}$$

The norm considered in this space is the following:

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)} + [u]_{W^{s,p}(\Omega)}, \tag{3.2}$$

where $[u]_{W^{s,p}(\Omega)}$ is the *Gagliardo* seminorm of u,

$$[u]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}. \tag{3.3}$$

An equivalent norm would be:

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}. \tag{3.4}$$

Multiple results regarding these spaces can be found in the literature. Some of them are presented and proved in the survey [19]. Just like in the case with $s = 1$, fixing $p = 2$ yields to a special case, where the fractional Sobolev space happens to be also a Hilbert space. Using the same notation as in Sobolev spaces, we will write $H^s(\Omega)$ instead of $W^{s,2}(\Omega)$. It is rather easy to check the parallelogram identity, using that

$$|a+b|^2 + |a-b|^2 = 2|a|^2 + 2|b|^2,$$

for $a = u(x) - u(y)$, $b = v(x) - v(y)$, in order to see that

$$[u+v]^2_{H^s(\Omega)} + [u-v]^2_{H^s(\Omega)} = 2[u]^2_{H^s(\Omega)} + 2[v]^2_{H^s(\Omega)}. \tag{3.5}$$

An alternative definition of the fractional Sobolev spaces for the case $p = 2$ can be now introduced, using Fourier transforms, and in all $\mathbb{R}^n$

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < \infty \right\}. \tag{3.6}$$

Proposition 3.1. *The definitions of $H^s(\mathbb{R}^n)$ stated in* (3.1) *for $p = 2$ and $\Omega = \mathbb{R}^n$, and in* (3.6) *are equivalent.*

Proof. The proof follows by simple calculation from Plancherel's theorem, which states that the inner product in $L^2(\mathbb{R}^n)$ is invariant under Fourier transforms.

Therefore, it is only necessary to check whether

$$[u]_{H^s(\mathbb{R}^n)} = C \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi, \tag{3.7}$$

for some constant C that may depend on the dimension n and s. This follows by simple computation:

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x+y)-u(y)|^2}{|x|^{n+2s}} dxdy \\
&= \int_{\mathbb{R}^n} \frac{1}{|x|^{n+2s}} \left(\int_{\mathbb{R}^n} |u(x+y)-u(y)|^2 dy \right) dx \\
&= \int_{\mathbb{R}^n} \frac{1}{|x|^{n+2s}} \|u(x+\cdot) - u(\cdot)\|_{L^2(\mathbb{R}^n)} dx.
\end{aligned}$$

Now use the Plancherel's theorem again, and the results from (2.37), where $c_{n,s}$ was a finite constant, it follows

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{1}{|x|^{n+2s}} \|u(x+\cdot) - u(\cdot)\|_{L^2(\mathbb{R}^n)} dx &= \int_{\mathbb{R}^n} \frac{1}{|x|^{n+2s}} \|\mathcal{F}(u(x+\cdot) - u(\cdot))\|_{L^2(\mathbb{R}^n)} dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot x} - 1|^2 |\mathcal{F}u(\xi)|^2}{|x|^{n+2s}} d\xi dx \\
&= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|1-\cos(\xi\cdot x)|^2 |\mathcal{F}u(\xi)|^2}{|x|^{n+2s}} d\xi dx \\
&= \frac{2}{c_{n,s}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi dx.
\end{aligned}$$

□

At this point, it is possible to interpret the fact that fractional Sobolev spaces $H^s(\mathbb{R}^n)$ are somehow found between $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, as stated at the beginning of the section. To assert that, we refer to the following proposition.

Proposition 3.2. *Let $u \in H^s(\mathbb{R}^n)$ for some $s \in (0,1)$. Then*

$$\|u\|^2_{H^s(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \frac{2}{c_{n,s}}\|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^n)}. \tag{3.8}$$

Proof. It is enough to check whether $[u]^2_{H^s(\mathbb{R}^n)} = \frac{2}{c_{n,s}}\|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^n)}$, by one of the equivalent forms of the H^s-norm. We use the previous proposition and the Plancherel's theorem again, together with the definition of fractional Laplacian:

$$\|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^n)} = \|\mathcal{F}(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^n)} = \||\xi|^s\mathcal{F}u\|^2_{L^2(\mathbb{R}^n)}. \tag{3.9}$$

And now, by the previous proposition, $[u]^2_{H^s(\mathbb{R}^n)} = \frac{2}{c_{n,s}}\|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^n)}$. □

In order to finish the introduction to fractional Sobolev spaces, we here present an analogue of the Sobolev inequality for this particular kind of spaces; the fractional Sobolev inequality.

Proposition 3.3 (Fractional Sobolev inequality). *Let $u \in H^s(\mathbb{R}^n)$, and let $p \in [1, \frac{n}{s})$. Then*

$$\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{pn}{n-ps}}dx\right)^{\frac{n-ps}{pn}} \leq C\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dydx\right)^{\frac{1}{p}}, \tag{3.10}$$

where C is a constant that depends only on n, p, and $s \in (0,1)$.

3.2 Eigenfunctions of the fractional Laplacian

In this section and the following ones (the maximum principle for regular solutions and for weak solutions) we deal with widely known results that have been proven many times for more general operators and using stronger techniques. In this case we present simple proofs using basic tools specific of the fractional Laplacian.

In this first section we show that the fractional Laplacian has eigenfunctions forming a Hilbert basis of $L^2(\Omega)$. The problem is to find the solutions

$$\begin{cases} (-\Delta)^s\phi &= \lambda\phi \quad \text{in } \Omega \\ \phi &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.11}$$

and see if they are a basis of L^2. To do so, first consider the following problem:

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.12}$$

where $f \in L^2(\Omega)$. Let us see that this problem has a unique weak solution, $u \in H^s$. To do so, firstly it is needed to define the concept of weak solution for the fractional Laplacian case. In the ordinary Laplacian, the weak solution is obtained multiplying both sides by an arbitrary function $v \in H^1_0$ and integrating. Then it is used that in a weak formulation $\langle -\Delta u, v\rangle = \langle \nabla u, \nabla v\rangle$ (all scalar products considered are and will be in L^2). It is possible to perform a similar reasoning for the fractional Laplacian:

Lemma 3.4. *Given u and v regular enough functions with compact support, then*

$$\langle(-\Delta)^s u, v\rangle = \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}dxdy = \langle u, (-\Delta)^s v\rangle. \quad (3.13)$$

Proof. Simply write:

$$\langle(-\Delta)^s u, v\rangle = c_{n,s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} v(x)\frac{(u(x)-u(y))}{|x-y|^{n+2s}}dydx,$$

$$\langle(-\Delta)^s u, v\rangle = c_{n,s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} v(y)\frac{(u(y)-u(x))}{|y-x|^{n+2s}}dxdy,$$

and add both expressions. □

In particular, the fractional Laplacian operator is self-adjoint. Using the previous lemma, it is possible to define the scalar product $\langle(-\Delta)^s u, v\rangle$ for functions $u, v \in H^s(\mathbb{R}^n)$ with null boundary or exterior conditions outside Ω. In order to simplify notation, define $H^s_*(\Omega)$ or simply H^s_* as the space of $H^s(\mathbb{R}^n)$ functions with null boundary or exterior conditions in $\mathbb{R}^n \setminus \Omega$, i.e., $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Then, the weak formulation of problem (3.12) is obtained multiplying by v and integrating over $\mathbb{R}^n$.

Definition 3.2. (weak solution) We say that $u \in H^s_*(\Omega)$ is a weak solution of the problem (3.12) if

$$\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} = \int_{\Omega} fv, \qquad \forall v \in H^s_*(\Omega). \quad (3.14)$$

We define $a(u, v)$ as the clearly bilinear form of the left-hand side of the equation, and at this point we want to apply Lax-Milgram theorem. To do so, it is first necessary to check that a is continuous and coercive. We previously introduce the following well-known statement, the fractional Poincaré inequality, which is an extension of the corresponding inequality for ordinary Sobolev spaces; preceded by a simple lemma.

Lemma 3.5. *Let f be a measurable function. Let $0 < p < q$, then*

$$\|f\|_{L^p(\Omega)} \leq \|f\|_{L^q(\Omega)}|\Omega|^r, \quad (3.15)$$

for r such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

Proof. Recall Hölder's inequality, for $p', q' \in [1, +\infty]$ and $1 = \frac{1}{p'} + \frac{1}{q'}$

$$\|\tilde{f}\tilde{g}\|_{L^1(\Omega)} \leq \|\tilde{f}\|_{L^{p'}(\Omega)}\|\tilde{g}\|_{L^{q'}(\Omega)}, \quad (3.16)$$

and use $\tilde{g} \equiv 1$, and $\tilde{f} = |f|^p$.

□

And the fractional Poincaré inequality would read as

Proposition 3.6 (Fractional Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain, and let $u \in H^s_*(\Omega)$. Then*

$$\int_\Omega |u(x)|^2 dx \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dydx, \tag{3.17}$$

where C is a constant that depends only on $|\Omega|$, n and $s \in (0,1)$.

Proof. We will only prove the case $n > 2s$. To do so, fix $p = 2$ and use fractional Sobolev inequality. Then use the previous lemma with $p = 2$, $q = \frac{2n}{n-2s}$. □

We can now prove the following.

Proposition 3.7. *For each $f \in L^2(\Omega)$, the problem*

$$\begin{cases} (-\Delta)^s u = f & in\ \Omega \\ u = 0 & in\ \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.18}$$

admits a unique weak solution $u \in H^s_*$.

Proof. First of all, let us check that the bilinear form $a : H^s_* \times H^s_* \to \mathbb{R}$ defined by

$$a(u,v) := \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \tag{3.19}$$

is continuous and coercive.

The continuity follows trivially if one realises that it is possible to apply Cauchy-Schwarz to $a(u,v)$, since it is a bilinear, symmetric and positive semidefinite form

$$a(u,v)^2 \leq a(u,u)a(v,v) = \frac{1}{4}[u]^2_{H^s}[v]^2_{H^s} \leq \|u\|^2_{H^s}\|v\|^2_{H^s}.$$

To check coercivity, the fractional Poincaré inequality is used.

$$\begin{aligned} a(u,u) &= \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} \\ &\geq \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} + \frac{1}{4} \int_\Omega \int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} \\ &\geq \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} + \frac{1}{4C}\|u\|^2_{L^2} \geq \overline{C}\|u\|^2_{H^s}, \end{aligned}$$

for some constant $\overline{C}$.

Therefore, by Lax-Milgram theorem, the problem (3.18) has a unique solution $u \in H^s_*$ for every $f \in L^2(\Omega)$, and it satisfies

$$\|u\|_{H^s} \leq C\|f\|_{L^2(\Omega)}. \tag{3.20}$$

for some constant C.

□

And finally, we are able to state the main proposition of the section, which states that the eigenfunctions form a Hilbert basis of $L^2(\Omega)$.

Proposition 3.8. *The fractional Laplacian with null exterior conditions in a bounded and Lipschitz domain Ω has a set of eigenfunctions forming a Hilbert basis of $L^2(\Omega)$.*

Proof. Thanks to the previous proposition, we now see that it is possible to define the operator inverse of the problem (3.14) or (3.18) in a weak sense, which would be the inverse of the fractional Laplacian. We name this operator $T : L^2(\Omega) \to H^s_*$, and is such that $T(f) = u$ (in the previous notation). It is clearly linear, and by (3.20) it is also continuous.

We want to apply the spectral theorem to the operator T. To do so, first extend $T : L^2(\Omega) \to L^2(\Omega)$. This is possible since $H^s_* \subset L^2(\Omega)$, and this inclusion is compact (see [19, Theorem 7.1]),

$$L^2(\Omega) \xrightarrow{T} H^s_* \overset{i}{\hookrightarrow} L^2(\Omega). \tag{3.21}$$

Since T is continuous and i is compact, $i \circ T$ is compact, and is the extension of T to $L^2(\Omega)$ (which will be also named T). It has been already seen that $(-\Delta)^s$ is self-adjoint, so T is also self-adjoint. By the spectral theorem, there exists a sequence of eigenfunctions of T with eigenvalues going to 0 that form a Hilbert basis of $L^2(\Omega)$. I.e., in the weak sense, we have

$$\begin{cases} (-\Delta)^s \phi_k &= \lambda_k \phi_k \quad \text{in } \Omega \\ \phi_k &= 0 \quad\quad\; \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.22}$$

with ϕ_k basis of $L^2(\Omega)$, and λ_k eigenvalues, $\lambda_k \to +\infty$. The eigenvalues are all positive since $\langle (-\Delta)^s u, u \rangle \geq 0 \ \ \forall u \in H^s$.

□

3.3 The maximum principle for the fractional Laplacian

The maximum principle for the diffusion equation is a well known result with multiple applications that explains the behaviour of a solution as time goes by. It is consistent with what we intuitively perceive when understanding the function as a probability, temperature or concentration. For example, the maximum principle allows us to bound a particular solution or to find estimates through the comparison principle.

For the fractional heat equation one could also expect a maximum principle, since the intuition still allows us to see the function as a probability evolution. Actually, any operator derived from a Lévy process has a maximum principle, and so does the fractional Laplacian. The proof of the theorem is easier than in the ordinary Laplacian problem, since the fact that the operator is nonlocal simplifies some computations.

We begin with the following key lemma:

Lemma 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $u \in C^2(\Omega) \cap C^0(\mathbb{R}^n)$ and $u \leq 0$ in $\mathbb{R}^n \setminus \Omega$ (resp. $u \geq 0$ in $\mathbb{R}^n \setminus \Omega$). Suppose $x_0 \in \Omega$ such that $\max_{x \in \Omega} u(x) = u(x_0)$ (resp. $\min_{x \in \Omega} u(x) = u(x_0)$). Then, $(-\Delta)^s u(x_0) \geq 0$ (resp. $(-\Delta)^s u(x_0) \leq 0$). Moreover, if $(-\Delta)^s u(x_0) = 0$, then $u \equiv u(x_0)$ constant in $\mathbb{R}^n$.*

Proof. We just do the maximum case. The minimum case follows exchanging u by $-u$.

Consider the definition of fractional Laplacian,

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy, \tag{3.23}$$

and notice that if x_0 is a maximum for u, then the function being integrated is always nonnegative, and therefore $(-\Delta)^s u(x_0) \geq 0$.

If $(-\Delta)^s u(x_0) = 0$, and since $u(x_0) \geq u(y)$ and u is continuous, we have $u(x_0) = u(y)$ for all $y \in \mathbb{R}^n$ (the function is constant). □

Now we can proceed with the proof of the maximum principle. There will be stated two different maximum principles, one for the Dirichlet problem for the fractional Laplacian and another for the fractional heat equation, i.e., the elliptic and the parabolic case.

For the fractional Laplacian the maximum principle basically states that, if the fractional Laplacian of a function with Dirichlet exterior conditions is nonpositive, then the maximum of the function is attained in the boundary of the domain (or exterior).

Before proceeding to state the maximum principle, we first present a useful proposition, which leads to a *comparison principle* for the Dirichlet problem for the fractional Laplacian. The proposition can be proved using the maximum principle trivially, but it is presented separately because it is the most common way in which the statement of the maximum principle will be used.

Proposition 3.10. *Let $\Omega \subset \mathbb{R}^n$ bounded domain, and $u \in (C^2(\Omega) \cap C^0(\overline{\Omega}))$ a solution of the following problem,*

$$\begin{cases} (-\Delta)^s u &= g \quad \text{in } \Omega \\ \qquad u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.24}$$

where $g \geq 0$ in Ω. Then $u \geq 0$ in $\mathbb{R}^n$.

Proof. Let us argue by contradiction. Assume there are points where $u < 0$. In particular, since $\overline{\Omega}$ is compact and u is continuous, there exists a global minimum $x_0 \in \overline{\Omega}$ where $u(x_0) < 0$. By the previous lemma, at this point $(-\Delta)^s u(x_0) \leq 0$. If $(-\Delta)^s u(x_0) = 0$, then $u \equiv 0$ (since $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$), and then $g \equiv 0$. If $(-\Delta)^s u(x_0) < 0$, but $g \geq 0$, in particular $g(x_0) \geq 0$; a contradiction again. Therefore, $u \geq 0$ in $\mathbb{R}^n$. □

Notice that the regularity conditions of the hypothesis are fulfilled if g and Ω are regular enough, and they are sufficient to assure that the fractional Laplacian can be applied at any point. Now, it is possible to state the maximum principle, that is derived immediately from the previous proposition:

Theorem 3.11 (Maximum principle for the fractional Laplacian)**.** *Let $\Omega \subset \mathbb{R}^n$ bounded domain, and $u \in (C^2(\Omega) \cap C^0(\mathbb{R}^n))$. Then,*

1. *If* $(-\Delta)^s u \leq 0$ *in* Ω *and* $u \leq 0$ *in* $\mathbb{R}^n \setminus \Omega$*,*

$$\sup_{\mathbb{R}^n} u = \sup_{\mathbb{R}^n \setminus \Omega} u. \tag{3.25}$$

2. *If* $(-\Delta)^s u \geq 0$ *in* Ω *and* $u \geq 0$ *in* $\mathbb{R}^n \setminus \Omega$*:*

$$\inf_{\mathbb{R}^n} u = \inf_{\mathbb{R}^n \setminus \Omega} u. \tag{3.26}$$

Proof. Let us see 1., and to see 2. just exchange u by $-u$.

Suppose that the statement is not true. Then we would have

$$\sup_{\mathbb{R}^n} u = \sup_{\overline{\Omega}} u = \max_{\overline{\Omega}} u. \tag{3.27}$$

Since $\overline{\Omega}$ is compact, and u is continuous, the maximum is attained at some point $x_0 \in \overline{\Omega}$. Since we arguing by contradiction, $x_0 \in \Omega$, because if it was on $\partial\Omega$ the statement would be true. Now, we can apply the previous lemma, so we get $(-\Delta)^s u(x_0) \geq 0$. If $(-\Delta)^s u(x_0) = 0$, then u is constant, and the statement would be true. Therefore, $(-\Delta)^s u(x_0) > 0$, which contradicts the hypothesis. Hence,

$$\sup_{\mathbb{R}^n} u = \sup_{\mathbb{R}^n \setminus \Omega} u. \tag{3.28}$$

□

We can highlight the main differences with the maximum principle for local operators, such as the ordinary Laplacian. First of all, it is important to see that instead of seeing that the maximum in $\overline{\Omega}$ is in $\partial\Omega$, we have seen that the maximum in all $\mathbb{R}^n$ is outside the domain, in $\mathbb{R}^n \setminus \Omega$. Since we are not considering compact sets (like $\overline{\Omega}$ or $\partial\Omega$), the maximum principle has to be stated using the supremum instead of the maximum. We keep the nomenclature because in most cases the exterior condition will have its maximum in the boundary, $\partial\Omega$, for example when it is constant.

Probably, the main use of the maximum principle is through the comparison principle:

Proposition 3.12 (Comparison principle for the elliptic problem)**.** *Let* $\Omega \subset \mathbb{R}^n$ *bounded domain, and let* $u_1, u_2 \in (C^2(\Omega) \cap C^0(\mathbb{R}^n))$ *such that*

$$\begin{cases} (-\Delta)^s u_1 &= g_1 \quad \text{in } \Omega \\ u_1 &= f_1 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.29}$$

$$\begin{cases} (-\Delta)^s u_2 &= g_2 \quad \text{in } \Omega \\ u_2 &= f_2 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.30}$$

where $g_1 \geq g_2$ *and* $f_1 \geq f_2$ *everywhere in* Ω *and* $\mathbb{R}^n \setminus \Omega$ *respectively. Then*

$$u_1 \geq u_2 \text{ in } \mathbb{R}^n. \tag{3.31}$$

Proof. Simply consider $u_1 - u_2$ and check that by linearity it fulfils the hypothesis of the maximum principle (case 2). I.e.:

$$\begin{cases} (-\Delta)^s(u_1 - u_2) & \geq & 0 & \text{in } \Omega \\ u_1 - u_2 & \geq & 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.32}$$

The infimum in $\mathbb{R}^n$ is found in $\mathbb{R}^n \setminus \Omega$, where $u_1 - u_2 \geq 0$, so this is fulfilled in $\mathbb{R}^n$, reaching the conclusion we wanted. □

Notice that usually the comparison principle is used in Dirichlet problems, so that the exterior conditions are fixed to zero (actually, when used in this dissertation, it will be treated this way).

Now we can proceed to introduce the maximum principle for the fractional heat equation. Again, the statement is very intuitive: if the non-homogeneous term of the fractional heat equation is nonpositive, then the maximum of the solution is attained in the initial condition, or in the boundary of the domain. The process to prove it is similar to the previous one. Before starting, we introduce a bit of notation to simplify things:

Suppose $\Omega \in \mathbb{R}^n$ is a bounded domain, and $T > 0$.

Definition 3.3. We define the following concepts regarding the domain of the solution:

i. $Q_T := \Omega \times (0, T) \subset \mathbb{R}^{n+1}$.

ii. Lateral boundary of Q_T: $\partial_L Q := \partial\Omega \times [0, T]$.

iii. Parabolic boundary of Q_T: $\partial_p Q_T := (\Omega \times \{0\}) \cup \partial_L Q_T$.

Notice that, differently from what we did in the elliptic case, the value of the function in the lateral boundary has always been fixed to 0. The value of the function in $\Omega \times \{0\}$ corresponds to the initial condition. We first state the following proposition:

Proposition 3.13. *Let $\Omega \subset \mathbb{R}^n$ bounded domain, and let $u(x,t)$ be a function that is C^2 in x and C^1 in t for $(x,t) \in \Omega \times (0,T)$, and C^0 in both x and t for $(x,t) \in \overline{\Omega} \times [0,T]$; and u is a solution of*

$$\begin{cases} \partial_t u + (-\Delta)^s u & = & f(x,t) & \text{in } \Omega,\ t > 0 \\ u & = & 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) & = & u_0(x) & \text{in } \Omega,\ \text{for } t = 0, \end{cases} \tag{3.33}$$

with $f \geq 0$ in $\overline{Q_T}$, and $u_0 \geq 0$ in Ω. Then $u \geq 0$ in $\overline{Q_T}$.

Proof. Consider $0 < T' < T$, and $\overline{Q_{T'}}$, and let us argue by contradiction. Assume $u < 0$ somewhere in $\overline{Q_{T'}}$. Since $u \in C^0(\overline{Q_{T'}})$, and $\overline{Q_{T'}}$ compact, there exist $(x_0, t_0) \in \overline{Q_{T'}}$ such that $u(x_0, t_0) = \min_{\overline{Q_{T'}}} u < 0$. Since $u \geq 0$ in $\partial_P \overline{Q_{T'}} \subset \partial_P \overline{Q_T}$, we have $(x_0, t_0) \notin \partial_P \overline{Q_{T'}}$.

1. If $(x_0, t_0) \in Q_{T'}$ is a minimum, $u_t(x_0, t_0) = 0$ and by Lemma 3.9, since $u(\cdot, t_0) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and is zero outside the domain, we have $(-\Delta)^s u(x_0, t_0) \leq 0$. If $(-\Delta)^s u(x_0, t_0) = 0$, then $u(\cdot, t_0) \equiv 0$, which is a contradiction with $u(x_0, t_0) < 0$, therefore $(-\Delta)^s u(x_0, t_0) < 0$. But $0 \leq f(x_0, t_0) = u_t(x_0, t_0) + (-\Delta)^s u(x_0, t_0) < 0$. Contradiction.

2. If $(x_0, t_0) \in \Omega \times \{T'\}$ is the minimum, then $u_t(x_0, t_0) \leq 0$ (the function has to be approaching decreasing), and by Lemma 3.9 again $(-\Delta)^s u(x_0, t_0) < 0$; contradiction.

Therefore, $u \geq 0$ in $Q_{T'}$. Now consider Q_T: $u \geq 0$ in $\overline{Q_{T'}}$ $\forall T' < T$. By continuity, $u \geq 0$ in $\overline{Q_T}$.

□

It must be pointed out that the previous statement is also true if we fix an always nonnegative exterior condition, and the proof would be the same, exchanging $\partial_P Q_T$ by $(\mathbb{R}^n \times \{0\}) \cup (\mathbb{R}^n \setminus \Omega) \times [0, T]$. Now we can state the maximum principle for the fractional heat equation (which again could be stated fixing the appropriate sign for the exterior condition):

Theorem 3.14 (Maximum principle for the fractional heat equation)**.** *Let $\Omega \subset \mathbb{R}^n$ bounded domain, $T > 0$ and let u be a function with the same regularity as in the previous proposition and Dirichlet (zero) exterior conditions. Then:*

1. If $u_t + (-\Delta)^s u \leq 0$ in Ω, $t \in [0, T]$:

$$\max_{\overline{Q_T}} u = \max_{\partial_P Q_T} u. \tag{3.34}$$

2. If $u_t + (-\Delta)^s u \geq 0$ in Ω, $t \in [0, T]$:

$$\min_{\overline{Q_T}} u = \min_{\partial_P Q_T} u. \tag{3.35}$$

Proof. Let us see 2., and 1. follows using $v := -u$ in 2. If $u(x, 0) \geq 0$ in 2., then we use the previous proposition to see $u \geq 0$ in $\overline{Q_T}$, and since $\partial_P Q_T \subset \overline{Q_T}$ and $u|_{\partial_P Q_T} \equiv 0$, $\min_{\overline{Q_T}} u = \min_{\partial_P Q_T} u = 0$.

Otherwise, proceed with the same argument used in the proof of the previous proposition. If it is not true that $u \geq 0$ everywhere in Q_T, so there exists $(x_0, t_0) \in \overline{Q_T}$ such that $\min_{\overline{Q_T}} u = u(x_0, t_0) < 0$. By the proof of the previous proposition, it is not possible that there exists a negative minimum in $Q_T \cup (\Omega \times \{T\})$, therefore, the minimum in $\overline{Q_T}$ must be in $\partial_P Q_T$. □

Again, like in the elliptic case, one of the main uses of the maximum principle for the fractional heat equation is through the comparison principle, which can be stated in the following way.

Proposition 3.15 (Comparison principle for the parabolic problem)**.** *Let $\Omega \subset \mathbb{R}^n$ bounded domain, and let $u_1(x, t)$ and $u_2(x, t)$ be functions that are C^2 in x and C^1*

in t for $(x,t) \in \Omega \times (0,T)$, and C^0 in both x and t for $(x,t) \in \overline{\Omega} \times [0,T]$; being u_1 and u_2 such that

$$\begin{cases} \partial_t u_1 + (-\Delta)^s u_1 = f_1(x,t) & \text{in } \Omega,\ t > 0 \\ u_1 = 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u_1(x,0) = u_{0,1}(x) & \text{in } \Omega,\ \text{for } t = 0, \end{cases} \tag{3.36}$$

$$\begin{cases} \partial_t u_2 + (-\Delta)^s u_2 = f_2(x,t) & \text{in } \Omega,\ t > 0 \\ u_2 = 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u_2(x,0) = u_{0,2}(x) & \text{in } \Omega,\ \text{for } t = 0, \end{cases} \tag{3.37}$$

where $f_1 \geq f_2$ and $u_{0,1} \geq u_{0,2}$ everywhere in $\Omega \times \mathbb{R}^+$ and Ω respectively. Then

$$u_1 \geq u_2,\ \text{in } \mathbb{R}^n,\ \text{for } t \geq 0. \tag{3.38}$$

Proof. Just like in the elliptic case, we define $v := u_1 - u_2$. Clearly, v solves the following problem, where $f_v(x,t) := f_1(x,t) - f_2(x,t)$ and $v_0(x) := u_{0,1}(x) - u_{0,2}(x)$.

$$\begin{cases} \partial_t v + (-\Delta)^s v = f_v(x,t) & \text{in } \Omega,\ t > 0 \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ v(x,0) = v_0(x) & \text{in } \Omega,\ \text{for } t = 0, \end{cases} \tag{3.39}$$

Under these circumstances, $f_v \geq 0$ and $v_0 \geq 0$ in $\Omega \times \mathbb{R}^*$ and Ω. We now apply the Proposition 3.13, whose hypothesis are clearly fulfilled here. Hence, $v = u_1 - u_2 \geq 0$, as we wanted to see. □

3.4 The maximum principle for weak solutions

While in the previous section we have proved the maximum principle both for the elliptic and parabolic problem, we required a strong hypothesis involving the regularity of the solution. However, as we will see next, the maximum principle and the comparison principle also hold for a more general class of solutions: weak solutions.

We here present an introduction on how to see this in the elliptic and parabolic case. We have only defined weak solution for the elliptic problem, so it will be necessary to introduce the concept for the parabolic problem too.

Let us start with the simple case though. For simplicity, we will fix exterior conditions to zero. Under this assumption, the maximum principle is equivalent to the following proposition,

Proposition 3.16. *Let $\Omega \subset \mathbb{R}^n$ bounded domain, and u a weak solution of the following problem,*

$$\begin{cases} (-\Delta)^s u = g & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.40}$$

where $g \geq 0$ a.e. in Ω. Then $u \geq 0$ a.e. in $\mathbb{R}^n$.

Proof. We say that $u \in H^s_*(\Omega)$ is a weak solution of the previous problem if

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} = \int_{\Omega} gv, \qquad \forall v \in H^s_*(\Omega). \tag{3.41}$$

Split $u = u^+ - u^-$, where u^+ and u^- are the positive and negative parts of u, i.e., $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := \max\{-u(x), 0\}$. Now let us argue by contradiction, and suppose that the set of points where u^- is non zero has positive measure, by choosing $v = u^-$.

The right-hand side of the previous equation is clearly nonnegative, since $g \geq 0$ a.e. and $v \geq 0$, so that $\int_\Omega gv \geq 0$. Now, we will see that the left-hand side is negative, so that the contradiction is reached.

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} < \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u^+(x)-u^+(y))(u^-(x)-u^-(y))}{|x-y|^{n+2s}},$$

where it has been used that $(u^-(x) - u^-(y))(v(x) - v(y)) \geq 0$, and since it is non zero for a set with positive measure in $\mathbb{R}^n \times \mathbb{R}^n$, the integral is positive.

Now, see that $(u^+(x)-u^+(y))(u^-(x)-u^-(y)) \leq 0$, since in the sets $\{u^+(x), u^-(x)\}$ and $\{u^+(y), u^-(y)\}$ there is at least one element equal to zero, and the other is nonnegative. Therefore,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u^+(x)-u^+(y))(u^-(x)-u^-(y))}{|x-y|^{n+2s}} \leq 0, \tag{3.42}$$

and we have reached a contradiction. □

Similarly, one could state the comparison principle for weak solutions,

Proposition 3.17 (Comparison principle for weak solutions to the elliptic problem)**.** *Let $\Omega \subset \mathbb{R}^n$ bounded domain, and let u_1, u_2 such that are weak solutions of the following problems,*

$$\begin{cases} (-\Delta)^s u_1 &= g_1 \quad \text{in } \Omega \\ u_1 &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.43}$$

$$\begin{cases} (-\Delta)^s u_2 &= g_2 \quad \text{in } \Omega \\ u_2 &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.44}$$

where $g_1 \geq g_2$ a.e. in Ω. Then

$$u_1 \geq u_2 \text{ a.e. in } \mathbb{R}^n.with \tag{3.45}$$

Proof. Simply consider the problem

$$\begin{cases} (-\Delta)^s (u_1 - u_2) &= g_1 - g_2 \quad \text{in } \Omega \\ u_1 - u_2 &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.46}$$

and apply the previous proposition to the function $u_1 - u_2$. □

And now, let us proceed with the parabolic case. First of all, let us introduce the notion of weak solution for the parabolic problem. We here denote $H_*^{s,1}$ as the set of functions in variables x and t such that are H_*^s with respect to x and H^1 with respect to t.

Definition 3.4 (weak solution for the parabolic problem)**.** We say that $u \in H_*^{s,1}(\Omega \times [0,T])$ is a weak solution of the problem

$$\begin{cases} \partial_t u + (-\Delta)^s u &= f(x,t) \quad \text{in } \Omega,\ t > 0 \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &= u_0(x) \quad \text{in } \Omega,\ \text{for } t = 0, \end{cases} \tag{3.47}$$

if

$$\int_\Omega \int_0^T u_t v + \frac{c_{n,s}}{2} \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} = \int_0^T \int_\Omega fv, \tag{3.48}$$

for all $v \in H_*^{s,1}(\Omega \times [0,T])$.

We can now proceed to prove the maximum principle for the parabolic problem. Like in the elliptic case, it is equivalent to the following statement.

Proposition 3.18. *Let $\Omega \subset \mathbb{R}^n$ bounded domain, $T > 0$, and u a weak solution of the following problem,*

$$\begin{cases} \partial_t u + (-\Delta)^s u &= f(x,t) \quad \textit{in } \Omega,\ t > 0 \\ u &= 0 \quad \textit{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &= u_0(x) \quad \textit{in } \Omega,\ \textit{for } t = 0, \end{cases} \tag{3.49}$$

where $u_0 \geq 0$ a.e. in Ω and $f \geq 0$ a.e. in $\Omega \times [0,T]$. Then $u \geq 0$ a.e. in $\mathbb{R}^n \times [0,T]$.

Proof. The idea of the proof is the same as in the elliptic problem, arguing by contradiction. Take $v = u^-$, the negative part (nonpositive) of $u = u^+ - u^-$. Suppose u^- is nonzero in a set of positive measure. We know that

$$\int_\Omega \int_0^T u_t v + \frac{c_{n,s}}{2} \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} = \int_0^T \int_\Omega fv, \tag{3.50}$$

and in the elliptic problem we have already seen that $\int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} < 0$, since it is negative for a set of times with positive measure; and $\int_0^T \int_\Omega fv \geq 0$. If we are able to see that $\int_\Omega \int_0^T u_t v \leq 0$ we will have finished, because we will have reached a contradiction.

Let us write it as

$$\int_\Omega \int_0^T u_t v = \int_\Omega \int_0^T (u^+)_t u^- - \int_\Omega \int_0^T (u^-)_t u^- = -\frac{1}{2} \int_\Omega \int_0^T \frac{\partial}{\partial t}((u^-)^2), \tag{3.51}$$

where $\int_\Omega \int_0^T (u^+)_t u^-$ is zero because either one factor inside the integral or the other is zero for each time almost everywhere. Now we have,

$$\int_\Omega \int_0^T u_t v = -\frac{1}{2} \int_\Omega \left((u^-)^2(\cdot,T) - (u_0^-)^2\right) = -\frac{1}{2} \int_\Omega (u^-)^2(\cdot,T) \leq 0, \tag{3.52}$$

where we have used that $u_0^- \equiv 0$, because the initial condition is nonnegative. We have reached a contradiction, and therefore, $u \geq 0$ a.e. in $\mathbb{R}^n \times [0,T]$. □

And the comparison principle for weak solutions.

Proposition 3.19 (Comparison principle for weak solutions to the parabolic problem)**.** *Let $\Omega \subset \mathbb{R}^n$ bounded domain, and let $u_1(x,t)$ and $u_2(x,t)$ be weak solutions of the following problems*

$$\begin{cases} \partial_t u_1 + (-\Delta)^s u_1 &= f_1(x,t) & \text{in } \Omega,\ t>0 \\ u_1 &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u_1(x,0) &= u_{0,1}(x) & \text{in } \Omega,\ \text{for } t=0, \end{cases} \tag{3.53}$$

$$\begin{cases} \partial_t u_2 + (-\Delta)^s u_2 &= f_2(x,t) & \text{in } \Omega,\ t>0 \\ u_2 &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u_2(x,0) &= u_{0,2}(x) & \text{in } \Omega,\ \text{for } t=0, \end{cases} \tag{3.54}$$

where $f_1 \geq f_2$ and $u_{0,1} \geq u_{0,2}$ a.e. in $\Omega \times \mathbb{R}^+$ and Ω respectively. Then

$$u_1 \geq u_2,\ \text{a.e. in } \mathbb{R}^n,\ \text{for } t \geq 0. \tag{3.55}$$

Proof. Proceed exactly as in the elliptic case or the parabolic case for solutions regular enough. □

Therefore, in all, we have been able to state and prove the maximum principle and the comparison principle for solutions when they are just weak, rather than requiring a certain regularity as an hypothesis.

3.5 Classical interior regularity for the elliptic problem

For nonlocal operators there are some classical results regarding the regularity, presented in this section, and modern results (some of them less than five years old) presented in the following section. Classical results usually deal with interior regularity, while modern results will be treating regularity up to the boundary.

To introduce the results we also introduce the Hölder notation. First, recall the Hölder norm, defined as:

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + \max_{|\beta|=k} |D^\beta f|_{C^{0,\alpha}}, \tag{3.56}$$

where $|f|_{C^{0,\alpha}}$ is the Hölder coefficient, $|f|_{C^{0,\alpha}} = \sup_{x\neq y \in \Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$, and k is an integer. The second term corresponds to the Hölder seminorm, and it will be denoted by $[f]_{C^{k+\alpha}}$.

In this dissertation, we will use the Hölder notation. If $k+\alpha$ is not an integer, we will denote $C^{k+\alpha} := C^{k,\alpha}$. If it is an integer, it must be corrected appropriately for every situation.

The classical estimates for the fractional Laplacian refer to a problem where the fractional Laplacian of a function is known in the unitary ball, i.e.,

$$(-\Delta)^s u = g \text{ in } B_1 \subset \mathbb{R}^n. \tag{3.57}$$

Now, the main known result regarding interior regularity is the following:

$$\|u\|_{C^{\alpha+2s}(B_{1/2})} \leq C\left(\|g\|_{C^{\alpha}(B_1)} + \|u\|_{L^{\infty}(\mathbb{R}^n)}\right), \tag{3.58}$$

where C is a constant that depends on s and n. This estimate can be seen, for example, in [31] (see also [46]), and it is true whenever $\alpha + 2s$ is not an integer. By means of this inequality it is possible to obtain the following classical result for interior regularity:

Theorem 3.20. *Let Ω be a bounded domain of $\mathbb{R}^n$, and consider the problem*

$$\begin{cases} (-\Delta)^s u = g & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.59}$$

If $\delta(x) = dist(x, \partial\Omega)$, for each $\rho > 0$ define $\Omega_\rho := \{x \in \Omega : \delta(x) > \rho\}$. Then, if $\alpha + 2s$ is not an integer,

$$\|u\|_{C^{\alpha+2s}(\Omega_\rho)} \leq C\|g\|_{C^{\alpha}(\Omega)}, \tag{3.60}$$

where C is a constant that depends only on n, s, Ω, α and ρ.

We do not give a compete proof of the previous theorem, but we introduce the idea on how to do it. In fact, this idea is widely used when dealing with estimates, and it is interesting to present it here.

A way to extend a result like the one in (3.58) to general theorems like Theorem 3.20 is to consider, firstly, equation (3.58) in a general ball $B_r \subset \Omega$ with radius small enough. Precisely, using the notation from Theorem 3.20, if r is smaller than ρ, then every point in Ω_ρ has a neighbourhood in the form of a ball of radius $r/2$ contained in Ω where the solution is $C^{\alpha+2s}$ and fulfils a bound similar to (3.58). Now, the constant, however, depends also on r (which, at the same time, depends on ρ). See Figure 3.1 for a visualization of the described configuration.

It only remains to see how inequality (3.58) changes in small balls, for the fractional Laplacian with Dirichlet conditions. It is well known that the term $\|u\|_{L^\infty(\mathbb{R}^n)}$ can be bound by $\|g\|_{L^\infty(\Omega)}$ (see Lemma 3.21 below), which can be trivially bound by $\|g\|_{C^\alpha(\Omega)}$.

Consider the equation

$$(-\Delta)^s u = g \text{ in } B_r \subset \mathbb{R}^n. \tag{3.61}$$

We would like to rewrite it in B_1 and use the estimates we already know, to express them in terms of $\|u\|_{C^{\alpha+2s}(B_{r/2})}$ and $\|g\|_{C^\alpha(B_r)}$. To do so, define $\tilde{u}(x) = u(rx)$ and $\tilde{g}(x) = g(rx)$. Expressing equation (3.61) in terms of $\tilde{u}$ and $\tilde{g}$ we get

$$(-\Delta)^s \tilde{u} = r^{2s}\tilde{g} \text{ in } B_1 \subset \mathbb{R}^n, \tag{3.62}$$

which can be easily seen using the definition with the integral of the fractional Laplacian, and changing variables. Notice that the term r^{2s} is consistent with the fact that the fractional Laplacian behaves like a fractional derivative (for the ordinary Laplacian, we would obtain the term r^2). Now we got the following estimate:

$$\|\tilde{u}\|_{C^{\alpha+2s}(B_{1/2})} \leq C\left(r^{2s}\|\tilde{g}\|_{C^{\alpha}(B_1)} + \|u\|_{L^{\infty}(\mathbb{R}^n)}\right). \tag{3.63}$$

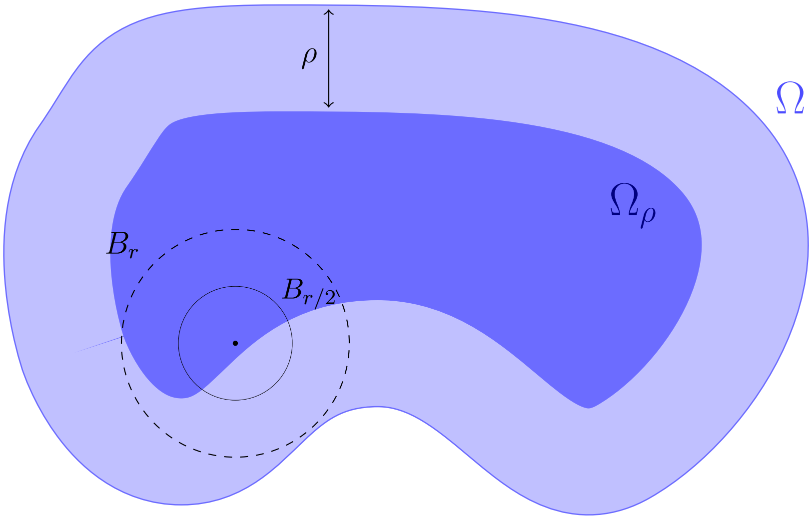

Figure 3.1: Sketch of a general bounded domain Ω, Ω_ρ and a ball B_r contained in Ω with radius $r < \rho$.

Suppose $\alpha = k + \beta$ and $\alpha + 2s = k' + \beta'$ for $k, k' \in \mathbb{N} \cup \{0\}$ and $\beta, \beta' \in (0,1)$; so $0 \le k' - k \le 2$. Then, using that r is small enough ($r < 1$),

$$\begin{aligned}\|\tilde{u}\|_{C^{\alpha+2s}(B_{1/2})} =& \|\tilde{u}\|_{C^{k'}(B_{1/2})} + [\tilde{u}]_{C^{\alpha+2s}(B_{1/2})} \\ \ge& r^{k'}\|u\|_{C^{k'}(B_{r/2})} + r^{k'+\alpha+2s}[u]_{C^{\alpha+2s}(B_{r/2})} \\ \ge& r^{k'+\alpha+2s}\|u\|_{C^{\alpha+2s}(B_{r/2})},\end{aligned}$$

and

$$\begin{aligned}\|\tilde{g}\|_{C^{\alpha}(B_1)} =& \|\tilde{g}\|_{C^{k}(B_1)} + [\tilde{g}]_{C^{\alpha}(B_1)} \\ \le& \|g\|_{C^{k}(B_r)} + r^{k+\alpha}[g]_{C^{\alpha}(B_r)} \\ \le& \|g\|_{C^{\alpha}(B_r)}.\end{aligned}$$

Putting all together now:

$$\|u\|_{C^{\alpha+2s}(B_{r/2})} \le \frac{C}{r^{k'+\alpha}}\left(\|g\|_{C^{\alpha}(B_r)} + \frac{1}{r^{2s}}\|u\|_{L^\infty(\mathbb{R}^n)}\right). \tag{3.64}$$

From which, imposing exterior conditions, it is possible to obtain an estimate like the one in Theorem 3.20, with a constant depending on r, and therefore, ρ.

We can now show that, indeed, in problem (3.59), $\|u\|_{L^\infty(\mathbb{R}^n)}$ can be bound by $\|g\|_{L^\infty(\Omega)}$.

Lemma 3.21. *Let Ω be a bounded domain of $\mathbb{R}^n$, and consider the problem*

$$\left\{\begin{array}{rcll}(-\Delta)^s u &=& g & \text{in } \Omega \\ u &=& 0 & \text{in } \mathbb{R}^n \setminus \Omega.\end{array}\right. \tag{3.65}$$

Then, there exist a constant C such that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le C\|g\|_{L^\infty(\Omega)}. \tag{3.66}$$

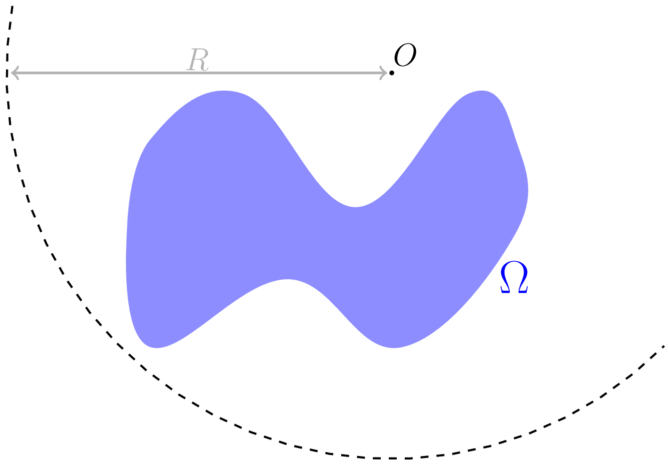

Figure 3.2: Sketch of a given domain Ω and an appropriate R as exposed in the proof of Lemma 3.21.

Proof. In order to prove this, we will use the maximum principle for the elliptic problem, previously introduced and proved. We want to find an appropriate function w such that

$$\begin{cases} (-\Delta)^s w & \geq & \|g\|_{L^\infty(\Omega)} & \text{in } \Omega \\ w & \geq & 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.67}$$

so that by the comparison principle we obtain $w \geq u$, and in particular, $\|w\|_{L^\infty(\Omega)} \geq \|u\|_{L^\infty(\Omega)}$. Consider the problem with explicit known solution:

$$\begin{cases} (-\Delta)^s w & = & \|g\|_{L^\infty(\Omega)} & \text{in } B_R \\ w & = & 0 & \text{in } \mathbb{R}^n \setminus B_R, \end{cases} \tag{3.68}$$

whose solution (which is nonnegative by the maximum principle, again) is, for some constant C,

$$w(x) = C\|g\|_{L^\infty(\Omega)}(R^2 - |x|^2)^s. \tag{3.69}$$

Now, take R such that $\Omega \subset B_R$ (exists because Ω is bounded. See Figure 3.2 as an example), in order to make w a solution of problem (3.67). We now have that $\|w\|_{L^\infty(\Omega)} = C\|g\|_{L^\infty(\Omega)}$ for some C (depending on n, s and Ω), as we wanted to see. □

3.6 Modern regularity up to the boundary for the elliptic equation

The results in the previous section deal with the *interior* regularity of the solution. Barely, how regular it is when the points considered are far enough from the boundary, so that it does not interfere. But, what about the regularity *up to the boundary*? Results regarding this regularity are quite new, and in this case they are extracted from [39]. Actually, the results here presented obtained by Ros-Oton and Serra are the main motivation of the whole dissertation, and are the ones that will be extended to the fractional heat equation.

In general, it is interesting to wonder how does the solution go to zero as it approaches the boundary (since we will be imposing Dirichlet conditions). To illustrate how an ordinary solution of the fractional Laplacian with Dirichlet conditions looks, we recover here the example presented in Chapter 1. Consider the standard fractional Laplacian problem, where the non-homogeneous term is constant fixed to 1 and the domain is the unitary ball,

$$\begin{cases} (-\Delta)^s u &= 1 \quad \text{in } B_1 \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus B_1. \end{cases} \tag{3.70}$$

This is one of the very few problems that involves the fractional Laplacian and has an explicit and simple solution. In this case, it is known that $u(x) = c(1-|x|^2)^s$ (remember $s \in (0,1)$) is a solution of the problem. As we could expect from the interior regularity results from the previous section, this solution is $C^\infty(B_1)$, smooth in the interior. Moreover, it is continuous up to the boundary, but not Lipschitz. In fact, it is $C^s(\overline{\Omega})$, and therefore $C^s(\mathbb{R}^n)$, but not better (not $C^{s'}(\overline{\Omega})$ for any $s' > s$).

Studying the regularity up to the boundary is interesting and essential in other aspects, apart from itself. For example, it is necessary to deduce the Pohozaev identity for the fractional Laplacian, [40], which is an extension of the *Integration by parts* formula for other operators; it is also useful in overdetermined problems with the fractional Laplacian, [18, 20], and in free boundary and obstacle problems, [13, 46].

Ros-Oton and Serra in [39] obtained the following two results, which will be essential in the development of the dissertation. Both refer to the following problem,

$$\begin{cases} (-\Delta)^s u &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.71}$$

The first proposition refers to the regularity of the function alone up to the boundary:

Proposition 3.22 ([39]). *Let $\Omega \subset \mathbb{R}^n$ be any bounded $C^{1,1}$ domain, $s \in (0,1)$, and u be a solution of* (3.71)*. If $g \in L^\infty(\Omega)$; then $u \in C^s(\mathbb{R}^n)$. Moreover,*

$$\|u\|_{C^s(\mathbb{R}^n)} \leq C\|g\|_{L^\infty(\Omega)}, \tag{3.72}$$

where the constant C depends only on Ω and s.

It is easy to see that the previous result implies the following inequalities,

$$-C\delta^s \leq u \leq C\delta^s, \text{ in } \Omega, \tag{3.73}$$

for some positive constant C (at the end of Chapter 5 we see an equivalent result for the parabolic problem).

The second and most important result by Ros-Oton and Serra refers to the regularity of u/δ^s up to the boundary:

Theorem 3.23 ([39]). *Let $\Omega \subset \mathbb{R}^n$ be any bounded $C^{1,1}$ domain, $s \in (0,1)$, and u be the solution of* (3.71)*. If $g \in L^\infty(\Omega)$, then $u/\delta^s \in C^\alpha(\overline{\Omega})$ for some small $\alpha \in (0,1)$, where $\delta(x) = dist(x, \partial\Omega)$, being $\partial\Omega$ the boundary of the domain. In particular, $u/\delta^s|_\Omega$ can be extended continuously to $\overline{\Omega}$. Moreover,*

$$\|u/\delta^s\|_{C^\alpha(\overline{\Omega})} \leq C\|g\|_{L^\infty(\Omega)}, \tag{3.74}$$

where the constants α and C depend only on Ω and s.

Going back to the previous example, (3.70), it is interesting to see what happens with the ratio u/δ^s (recall $\delta(x) = \text{dist}(x, \partial\Omega)$). In this case, $\delta(x) = 1 - |x|$ and therefore $u(x)/\delta^s(x) = (1 + |x|)^s$. Notice that this ratio is Hölder continuous in $\overline{\Omega}$ (not only continuous, also smooth), just like the previous theorem states.

3.7 Results for the fractional heat equation

In this section we present some known results regarding the fractional heat equation, and in particular, the heat kernel. Although they will not be used in this dissertation, it is interesting to expose them, since there is little information about boundary regularity for the fractional heat equation in the literature.

Definition 3.5. The fractional heat kernel in $\mathbb{R}^n$, $p(x,t)$, is the solution of the following problem,

$$\begin{cases} \partial_t p + (-\Delta)^s p &= 0 \quad \text{in } \mathbb{R}^n, \ t > 0 \\ p(x,0) &= \delta_0 \quad \text{in } \mathbb{R}^n, \ \text{for } t = 0, \end{cases} \tag{3.75}$$

where δ_0 is the Dirac's delta centred at the origin.

The fractional heat kernel, therefore, is the fundamental solution of the fractional heat equation. Notice that, equivalently, the fractional heat kernel $p(x,t)$ is such that its Fourier transform can be written as $\hat{p}(\xi, t) = e^{-t|\xi|^{2s}}$. This can be seen simply expressing the previous problem in the Fourier side. It is also known that $p(x,t)$ is C^∞ in x, and C^∞ in t for positive times. Furthermore, the following rescaling identity holds:

$$p(x,t) = t^{-\frac{n}{2s}} p(t^{-\frac{1}{2s}} x, 1), \tag{3.76}$$

which is easy to see expressing p as the inverse Fourier transform of $e^{-t|\xi|^{2s}}$, and changing variables.

There is not a simple expression for $p(x,t)$, however some bounds exist. In this case, we present a known proposition (see, for example, [5], or the introduction of [15]).

Proposition 3.24. *There exists a constant C such that the following inequalities hold:*

$$C^{-1}\left(t^{\frac{n}{2s}} \wedge \frac{t}{|x|^{n+2s}}\right) \le p(x,t) \le C\left(t^{\frac{n}{2s}} \wedge \frac{t}{|x|^{n+2s}}\right). \tag{3.77}$$

We also denote $p(x,y,t) = p(y-x,t)$. Now, $p(x,y,t)$ is the transition density of the diffusion process whose infinitessimal generator is the fractional Laplacian $(-\Delta)^s$.

Again, we would be interested in treating bounded domains, and the regularity there. Classical estimates like the ones found before can be also written for the fractional heat equation (see again [31], and also the more recent work [30]). I.e., consider the equation

$$\partial_t u + (-\Delta)^s u = f(x,t) \text{ in } B_1 \times (-1, 0]. \tag{3.78}$$

Then, the following estimate holds:

$$\sup_{t\in[-\frac{1}{2},0]} \|u(\cdot,t)\|_{C^{\alpha+2s}(B_{1/2})} \leq C(\|f\|_{C^\alpha(B_1\times(-1,0])} + \|u\|_{L^\infty(\mathbb{R}^n\times(-1,0])}), \tag{3.79}$$

where C is a constant that depends on s and n.

Using similar arguments to the ones used for the estimates for the fractional Laplacian, it is possible to state the following theorem:

Theorem 3.25. *Let Ω be a bounded domain of $\mathbb{R}^n$, and consider the problem*

$$\begin{cases} \partial_t u + (-\Delta)^s u &= f(x,t) & \text{in } \Omega, t>0 \\ u &= 0 & \text{in } \mathbb{R}^n\setminus\Omega, t\geq 0 \\ u(x,0) &= u_0 & \text{in } \Omega,\ \text{for } t=0. \end{cases} \tag{3.80}$$

If $\delta(x) = dist(x,\partial\Omega)$, for each $\rho>0$ define $\Omega_\rho := \{x\in\Omega : \delta(x)>\rho\}$, $T>0$ and $t_0\in(0,T)$, then

$$\sup_{t\in[t_0,T]} \|u(\cdot,t)\|_{C^{\alpha+2s}(\Omega_\rho)} \leq C(\|f\|_{C^\alpha(\Omega\times(0,T))} + \|u_0\|_{L^\infty(\Omega)}), \tag{3.81}$$

where C is a constant that depends on n, s, Ω, ρ and T.

Hence, the solution of the fractional heat equation is smooth inside the domain, for positive times. This will be proved later for the homogeneous fractional heat equation using only regularity of the elliptic case.

Regarding the regularity up to the boundary there are no results analogue to Theorem 3.23. The only boundary regularity results for the fractional heat equation refer to the heat kernel, and are stated next.

It is possible to define the fractional heat kernel for the fractional heat equation in a domain Ω, $p_\Omega(x,y,t)$, with zero exterior conditions. For this situation, there are bounds apparently similar to the ones found for the heat kernel in $\mathbb{R}^n$ (but much more difficult to obtain). Chen, Kim and Song established in 2010 the following result, divided into two different parts. The first statement describes the behaviour of the heat kernel for small times t, while the second one is for large times.

Theorem 3.26 ([15])**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded $C^{1,1}$ open set, and $\delta(x) = dist(x,\partial\Omega)$. Then:*

1. *For all $T>0$, on $\Omega\times\Omega\times(0,T]$, there exist positive constants C_1 and C_2 depending on s, n and Ω such that*

$$C_1 B_\Omega(x,y,t) \leq p_\Omega(x,y,t) \leq C_2 B_\Omega(x,y,t), \tag{3.82}$$

for B_Ω defined by

$$B_\Omega(x,y,t) := \left(1\wedge\frac{\delta^s(x)}{\sqrt{t}}\right)\left(1\wedge\frac{\delta^s(y)}{\sqrt{t}}\right)\left(t^{-\frac{n}{2s}}\wedge\frac{t}{|x-y|^{d+2s}}\right). \tag{3.83}$$

2. *For all* $T > 0$ *on* $\Omega \times \Omega \times [T, \infty)$ *there are positive constants* C_1 *and* C_2 *depending on* s*,* n *and* Ω *such that*

$$C_1 e^{-\lambda_1 t} \delta^s(x) \delta^s(y) \le p_\Omega(x, y, t) \le C_2 e^{-\lambda_1 t} \delta^s(x) \delta^s(y), \qquad (3.84)$$

where λ_1 *is the first eigenvalue of the fractional Laplacian in* Ω *with zero boundary or exterior conditions.*

From the previous expressions it is also possible to obtain bounds for the Green function in the specified domains. Moreover, the last theorem implies inequalities of the form

$$c\delta^s \le u \le C\delta^s, \text{ in } \Omega, \qquad (3.85)$$

for any fixed positive time and positive initial condition (as we will see in Chapter 5).

The previous theorem corresponds to one of the main results known currently regarding the fractional heat equation in bounded domains. This result was extended to more general domains in [6] in 2010.

In all we have seen that there exist results for the regularity of the solution, but none of them refers to the ratio u/δ^s, for example. Later, it will be shown that a very similar result to the one for the fractional Laplacian also holds for the homogeneous fractional heat equation.

Chapter 4

A first original result: boundary regularity for the fractional heat equation

In this chapter we proceed to prove Theorem 1.1, and we present the proof of this result. To do so, we first study the regularity for the eigenfunctions of the fractional Laplacian in rather general domains, using the results previously introduced and new ideas that will be explained.

After that, we write the general solution for the fractional heat equation in terms of its eigenfunctions, so that the regularity of this solution can be reduced to study the regularity of the eigenfunctions. Thus, from the expression of the solution and knowing the appropriate bounds for the eigenfunctions we obtain bounds up to the boundary for the solution of the fractional heat equation.

4.1 Regularity of the eigenfunctions

At this point, let us try to improve the regularity conditions up to the boundary on the eigenfunctions. Recall that, by construction, the eigenfunctions belong to $L^2(\Omega)$. Here, we will prove that they are, in fact, in $L^\infty(\Omega)$, and we obtain a bound for its L^∞ norm. This bound should be obtained in terms of the eigenvalue λ of the eigenfunction, since this expression will be needed to prove Theorem 1.1.

The main results of this section is stated in the following proposition and the posterior corollary, where we find that the eigenfunctions in $C^{1,1}$ bounded domains have C^s regularity in $\mathbb{R}^n$, and the ratio of the function and the term δ^s is Hölder continuous up to the boundary. Recall that the eigenfunctions refer to the following problem,

$$\left\{\begin{array}{rcll}(-\Delta)^s\phi & = & \lambda\phi & \text{in } \Omega\\ \phi & = & 0 & \text{in } \mathbb{R}^n\setminus\Omega.\end{array}\right. \tag{4.1}$$

Proposition 4.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded $C^{1,1}$ domain, $s\in(0,1)$, and ϕ any eigenfunction of problem* (4.1)*, with eigenvalue λ. Then, $\phi\in L^\infty(\Omega)$. Moreover*

$$\|\phi\|_{L^\infty(\Omega)}\leq\ C\lambda^{w-1}\|\phi\|_{L^2(\Omega)}, \tag{4.2}$$

for some constant C depending only on n, s and Ω, and some $w \in \mathbb{N}$, depending only on n and s.

And the corollary, which will be proved using Proposition 3.22 and Theorem 3.23.

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain, $s \in (0,1)$, and ϕ any eigenfunction of problem (4.1), with eigenvalue λ. Then, $\phi \in C^s(\mathbb{R}^n)$ and $\phi/\delta^s \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$. Moreover*

$$\|\phi\|_{C^s(\mathbb{R}^n)} \leq C\lambda^w \|\phi\|_{L^2(\Omega)}, \tag{4.3}$$

$$\|\phi/\delta^s\|_{C^\alpha(\overline{\Omega})} \leq C\lambda^w \|\phi\|_{L^2(\Omega)}, \tag{4.4}$$

for some constant C depending only on n, s and Ω, and some $w \in \mathbb{N}$, depending only on n and s.

In order to prove Proposition 4.1, we will use the following result, which follows immediately from [38, Proposition 1.4]:

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain, $s \in (0,1)$, $g \in L^2(\Omega)$, and u be the weak solution of*

$$\begin{cases} (-\Delta)^s u &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{4.5}$$

Then,

1. *Let $2 \leq p < \frac{n}{2s}$, and assume $g \in L^p(\Omega)$. Then there exists a constant C, depending only on n, s and p, such that*

$$\|u\|_{L^q(\Omega)} \leq C\|g\|_{L^p(\Omega)}, \quad q = \frac{np}{n-2ps}. \tag{4.6}$$

2. *Let $\frac{n}{2s} < p < \infty$. Then, there exists a constant C, depending only on n, s, and p, such that:*

$$\|u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^p(\Omega)}. \tag{4.7}$$

3. *If $p = \frac{n}{2s}$, then there exists a constant C, depending only on n, s and q, such that:*

$$\|u\|_{L^q(\Omega)} \leq C\|g\|_{L^p(\Omega)}, \quad \forall q < \infty. \tag{4.8}$$

And now, we can proceed with the proof of Proposition 4.1.

Proof of Proposition 4.1. In order to prove this result, consider the problem (4.5), with $g = \lambda\phi$, being ϕ an eigenfunction and λ its eigenvalue. It has been already proved that $\phi \in L^2(\Omega)$. If $\frac{n}{2s} < 2$, apply Proposition 4.3 second result with $p = 2$ to get

$$\|\phi\|_{L^\infty(\Omega)} \leq C\lambda\|\phi\|_{L^2(\Omega)}. \tag{4.9}$$

If $\frac{n}{2s} = 2$, first use the third result from Proposition 4.3 and consider $q > \frac{n}{2s}$, which reduces to the previous case. Therefore, in this situation $w = 3$.

Suppose $\frac{n}{2s} > 2$. Apply Proposition 4.3, first result.

$$\|\phi\|_{L^q(\Omega)} \leq C\lambda\|\phi\|_{L^p(\Omega)}, \quad q = \frac{np}{n-2ps}. \tag{4.10}$$

Now the constant C does not depend on λ. If $\phi \in L^p(\Omega)$, then $\phi \in L^q(\Omega)$ (for $2 \leq p < \frac{n}{2s}$). Take an initial $p_0 = 2$.

Define $p_{k+1} = \frac{np_k}{n-2p_k s}$. This sequence is obviously increasing, has no fixed points while $n > 2p_k s$ and implies the following chain of inequalities,

$$C^{(0)}\|\phi\|_{L^{p_0}(\Omega)} \geq C^{(1)}\lambda^{-1}\|\phi\|_{L^{p_1}(\Omega)} \geq \cdots \geq C^{(k+1)}\lambda^{-k-1}\|\phi\|_{L^{p_{k+1}}(\Omega)}. \tag{4.11}$$

As long as $n > 2p_k s$. Define N as the index of the first time $n \leq 2p_N s$. We know that $\phi \in L^{p_N}(\Omega)$, with $p_N \geq \frac{n}{2s}$. If $p_N = \frac{n}{2s}$, consider $p'_N = p_N - \epsilon$, then $q = \frac{np'_N}{n-2p'_N s}$ is, for some $\epsilon > 0$, larger than $\frac{n}{2s}$. It is possible to conclude that $\phi \in L^Q(\Omega)$, for some $Q > \frac{n}{2s}$.

Now use the second result from Proposition 4.3, to see $\phi \in L^\infty(\Omega)$.

$$\|\phi\|_{L^\infty(\Omega)} \leq C\lambda\|\phi\|_{L^{p_N}(\Omega)}, \tag{4.12}$$

which by the previous chain of inequalities implies

$$\|\phi\|_{L^\infty(\Omega)} \leq C\lambda^{w-1}\|\phi\|_{L^2(\Omega)}, \tag{4.13}$$

where $w = N+2$ if $p_N > \frac{n}{2s}$ and $w = N+3$ if $p_N = \frac{n}{2s}$; as we wanted to see. □

And now, let us prove Corollary 4.2.

Proof of Corollary 4.2. Use the result from Proposition 3.22 to see

$$\|\phi\|_{C^s(\mathbb{R}^n)} \leq C\lambda\|\phi\|_{L^\infty(\Omega)} \leq C\lambda^w\|\phi\|_{L^2(\Omega)}, \tag{4.14}$$

so $\phi \in C^s(\mathbb{R}^n)$. By the same reason, using Theorem 3.23 we have $\phi/\delta^s \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$, with w as stated.

$$\|\phi/\delta^s\|_{C^\alpha(\overline{\Omega})} \leq C\lambda\|\phi\|_{L^\infty(\Omega)} \leq C\lambda^w\|\phi\|_{L^2(\Omega)}. \tag{4.15}$$

□

From the previous proof, the possible values for w arise, depending only on n and s. They can be summarized in the following remark:

Remark 4.1. In the previous Proposition 4.1, the values for w depending only on n and s are:

1. If $\frac{n}{2s} < 2$, $w = 2$.
2. If $\frac{n}{2s} = 2$, $w = 3$.
3. If $\frac{n}{2s} > 2$, let us define $\{p_k\}_{k\geq 0}$ a sequence, $p_{k+1} = \frac{np_k}{n-2p_k s}$ and $p_0 = 2$. Also define $N := \min\{k \in \mathbb{N} : n \leq 2p_k s\}$.

(a) If $p_N > \frac{n}{2s}$, $w = N + 2$.

(b) If $p_N = \frac{n}{2s}$, $w = N + 3$.

Finally, we want to show even more regularity for the eigenfunctions. Actually, we want to see that they are C^∞ in the interior of the domain.

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain, $s \in (0,1)$, and ϕ any eigenfunction of the elliptic problem* (4.5),

$$\begin{cases} (-\Delta)^s\phi &= \lambda\phi \quad \text{in } \Omega \\ \phi &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{4.16}$$

where λ is the eigenvalue. Then, $\phi \in C^\infty(\Omega) \cap C^s(\overline{\Omega})$.

Proof. We already know that $\phi \in C^s(\mathbb{R}^n)$ by Corollary 4.2. It remains to see $\phi \in C^\infty(\Omega)$.

To prove it, we refer to Theorem 3.20. Actually, it can be easily seen from the idea of the proof of this theorem (and the fact that comes from a local result) that we could restate it so that the bound is

$$\|u\|_{C^{\alpha+2s}(\Omega_\rho)} \leq C\|g\|_{C^\alpha(\Omega_{\rho'})}, \text{ for } \rho' < \rho, \tag{4.17}$$

with the notation of the theorem. With eigenfunctions we would have

$$\|\phi\|_{C^{\alpha+2s}(\Omega_\rho)} \leq C\lambda\|\phi\|_{C^\alpha(\Omega_{\rho'})}, \text{ for } \rho' < \rho, \tag{4.18}$$

for any $\alpha > 0$. We can now write the following chain of inequalities

$$\|\phi\|_{C^s(\Omega_{\rho_0})} \geq C^{(1)}\lambda^{-1}\|\phi\|_{C^{3s}(\Omega_{\rho_1})} \geq \cdots \geq C^{(k)}\lambda^{-k}\|\phi\|_{C^{(2k+1)s}(\Omega_{\rho_k})} \tag{4.19}$$

for $\rho_0 < \rho_1 < \cdots < \rho_k$, and where the constants $C^{(i)}$ depend on n, s, Ω and $\{\rho_j\}_{j=1,\dots,k}$. Suppose $\rho_i = \left(2 - \frac{1}{2^i}\right)\rho_0$, we have now

$$\|\phi\|_{C^{(2k+1)s}(\Omega_{2\rho_0})} \leq \|\phi\|_{C^{(2k+1)s}(\Omega_{\rho_k})} \leq C^{(k)}\lambda^k\|\phi\|_{C^s(\Omega_{\rho_0})}. \tag{4.20}$$

Using the result from Corollary 4.2 we have

$$\|\phi\|_{C^{(2k+1)s}(\Omega_{2\rho_0})} \leq C^{(k)}\lambda^{k+w}\|\phi\|_{L^2(\Omega)}. \tag{4.21}$$

So we have concluded that for any k, $\phi \in C^{(2k+1)s}(\Omega_{2\rho_0})$. Since $2\rho_0$ can be as small as wanted, this implies that $\phi \in C^\infty(\Omega)$. □

4.2 Solutions of the homogeneous fractional heat equation

We have studied the regularity of the eigenfunctions of the fractional Laplacian, and now it is time for us to use them to find an expression for the solutions of the fractional heat equation in terms of these eigenfunctions. This is achieved through a procedure very similar to the one usually performed for the ordinary heat equation: separation of variables. First, recall the problem we are dealing with,

$$\begin{cases} \partial_t u + (-\Delta)^s u &= 0 & \text{in } \Omega,\ t > 0 \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &= u_0(x) & \text{in } \Omega,\ \text{for } t = 0. \end{cases} \tag{4.22}$$

And let us proceed to prove the following proposition, referring to the expression of the solution, its uniqueness and a first step towards its regularity.

Proposition 4.5. *Consider the fractional heat equation* (4.22)*, for an initial condition at* $t = 0$*,* $u(x,0) = u_0(x) \in L^2(\Omega)$*, being* u_k *its coefficients in the orthonormal basis* $\{\phi_k\}_k$*, and for a bounded domain* Ω*. Then,*

1. *The solution of the problem* (4.22) *is of the form*

$$u(x,t) = \sum_{k>0} u_k \phi_k(x) e^{-\lambda_k t}. \tag{4.23}$$

2. $u(\cdot,t) \to u_0(x)$ *as* $t \downarrow 0$ *in* $L^2(\Omega)$ *norm.*

3. $u(\cdot,t)$ *is the only solution (in* $L^2(\Omega)$*) fulfilling the previous condition.*

Proof. We begin with 1. Proceed assuming $u(x,t) = A(x)B(t)$, with $A(x) = 0,\ x \in \mathbb{R}^n \setminus \Omega$. The first equation becomes,

$$A(x)B'(t) + B(t)(-\Delta)^s A(x) = 0. \tag{4.24}$$

This implies

$$\begin{cases} -(-\Delta)^s A(x) = -\lambda A(x) \\ B'(t) = -\lambda B(t), \end{cases} \tag{4.25}$$

for some constant $\lambda \in \mathbb{R}$. The first expression corresponds to the known eigenvalues-eigenfunctions problem for the fractional Laplacian. The only solutions are for $A(x) = \phi_k(x)$ and $\lambda = \lambda_k > 0$ for some k, being ϕ_k the eigenfunctions of the fractional Laplacian, and λ_k the corresponding eigenvalues. Here the "$-\lambda$" notation is justified. Plus, $B(t) = B(0)e^{-\lambda_k t}$. A general solution will be written as linear combinations of all the possibilities,

$$u(x,t) = \sum_{k>0} c_k \phi_k(x) e^{-\lambda_k t}. \tag{4.26}$$

The coefficients are found thanks to the initial value condition. For $t = 0$, $u(x,0) = u_0(x)$ is simply the linear combination with c_k coefficients and the corresponding eigenfunctions, which will be normalized to unitary norm in $L^2(\Omega)$. Thanks to the spectral theorem we already know the eigenfunctions are a basis of $L^2(\Omega)$, so if $u_0(x) \in L^2(\Omega)$, we can express it as linear combinations of the eigenfunctions,

$$u_0(x) = \sum_{k>0} u_k \phi_k(x), \quad \|\phi\|_{L^2(\Omega)} = 1. \tag{4.27}$$

In all, the solution is

$$u(x,t) = \sum_{k>0} u_k \phi_k(x) e^{-\lambda_k t}. \tag{4.28}$$

As we wanted to see. Notice that this expression converges (in L^2) for all $t \geq 0$ since for $t = 0$ does,

$$\|u(x,t)\|_{L^2(\Omega)} = \sum_{k>0} u_k^2 e^{-2\lambda_k t}. \tag{4.29}$$

Now let us see the uniqueness for the solution:
$u(x,t)$ is a solution by construction. To prove 2. we use Parseval,

$$\begin{aligned} \|u(\cdot,t) - u_0\|^2_{L^2(\Omega)} &= \left\| \sum_{k>0} u_k \phi_k (e^{-\lambda_k t} - 1) \right\|^2_{L^2(\Omega)} \\ &= \sum_{k>0} u_k^2 (e^{-\lambda_k t} - 1)^2. \end{aligned}$$

Each term of the last series goes to 0 as $t \downarrow 0$, and they can be also bounded uniformly by the terms of a convergent series $u_k^2(e^{-\lambda_k t} - 1)^2 < u_k^2$, since $\sum u_k^2 < \infty$. Therefore $\|u(\cdot,t) - u_0\|^2_{L^2(\Omega)} \to 0$ as $t \downarrow 0$.

To see 3. consider two functions u_1, u_2, both solutions of the problem. Define $v := u_1 - u_2$; v is solution of the fractional heat equation with zero initial condition, so $\|v(\cdot,t)\|_{L^2(\Omega)} \to 0$ as $t \downarrow 0$. Proceed with the integral method,

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|v\|^2_{L^2(\Omega)} &= \int_{\mathbb{R}^n} v v_t dx \\ &= \langle v, v_t \rangle \\ &= \langle v, -(-\Delta)^s v \rangle \\ &= -\frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x-y|^{n+2s}} dx dy \leq 0. \end{aligned}$$

So v has decreasing L^2 norm, and its initial value is 0, therefore v is 0 always in L^2 sense.

□

4.3 Proof of a first original result: Theorem 1.1

Finally, in this section, we can proceed with the proof of Theorem 1.1.

First of all, however, we introduce the following result regarding the eigenvalues of the fractional Laplacian. The asymptotic behaviour of these eigenvalues is well-known, and numerous works deal with the study of the behaviour of the eigenvalues for general domains. One example is the paper by Frank and Geisinger, [24], which treats the asymptotics of the sum of eigenvalues under very weak assumptions regarding the regularity of the domain. The result that will be used in this dissertation is the following.

Proposition 4.6 ([4], see also [24]). *Let λ_k be the k-th eigenvalue of the s-fractional Laplacian on a bounded $C^{1,1}$ domain, Ω. Then:*

$$\lambda_k = C_{n,s}|\Omega|^{-\frac{2s}{n}} k^{\frac{2s}{n}}(1+o(1)) \tag{4.30}$$

where $C_{n,s}$ is a constant depending only on the subindexs (we will denote $C_{n,s,\Omega} := C_{n,s}|\Omega|^{-\frac{2s}{n}}$).

We here recall the statement of our main result, already stated in the introduction.

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be any bounded $C^{1,1}$ domain, and $s \in (0,1)$. Let $u_0 \in L^2(\Omega)$, and let u be the solution to the fractional heat equation*

$$\begin{cases} \partial_t u + (-\Delta)^s u &= 0 & \text{in } \Omega,\ t>0 \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &= u_0(x) & \text{in } \Omega,\ \text{for } t=0. \end{cases} \tag{4.31}$$

Then,

1. *For each $t_0 > 0$,*

$$\sup_{t \geq t_0} \|u(\cdot,t)\|_{C^s(\mathbb{R}^n)} \leq C_1(t_0)\|u_0\|_{L^2(\Omega)}. \tag{4.32}$$

2. *For each $t_0 > 0$,*

$$\sup_{t \geq t_0} \left\| \frac{u(\cdot,t)}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})} \leq C_2(t_0)\|u_0\|_{L^2(\Omega)}, \tag{4.33}$$

where $\delta(x) = dist(x, \partial\Omega)$, and $\alpha > 0$ is small.

The constants C_1 and C_2 depend only on t_0, n, s and Ω, and blow up as $t_0 \downarrow 0$.

Proof of Theorem 1.1. Let us prove 1. First, by Proposition 4.5 the solution u can be expressed as

$$u(x,t) = \sum_{k>0} u_k \phi_k e^{-\lambda_k t}. \tag{4.34}$$

Notice that we already know that the solution will always be in $L^2(\Omega)$, since the L^2 norm is decreasing with time (as seen in the expression (4.29)).

We then try to bound $\|u(\cdot,t)\|_{C^s(\mathbb{R}^n)}$, through the expression found in (4.3), and noticing that the sequence $|u_k|$ has a maximum (since it converges to 0) and it satisfies $\max_{k>0}|u_k| \leq \|u_0\|_{L^2(\Omega)}$,

$$\begin{aligned} \|u(\cdot,t)\|_{C^s(\mathbb{R}^n)} &= \left\| \sum_{k>0} u_k \phi_k e^{-\lambda_k t} \right\|_{C^s(\mathbb{R}^n)} \\ &\leq \sum_{k>0} |u_k| \|\phi_k\|_{C^s(\mathbb{R}^n)} e^{-\lambda_k t} \\ &\leq \|u_0\|_{L^2(\Omega)} \sum_{k>0} C_{\lambda_k} \|\phi_k\|_{L^2(\Omega)} e^{-\lambda_k t} \\ &= \|u_0\|_{L^2(\Omega)} C \sum_{k>0} \lambda_k^w e^{-\lambda_k t}. \end{aligned}$$

We have bounded $\|u(\cdot,t)\|_{C^s(\mathbb{R}^n)}$ by an expression decreasing with time (since $\lambda_k > 0$ always). Therefore, it is only needed to consider $\|u(\cdot,t_0)\|_{C^s(\mathbb{R}^n)}$ and bound it using the previous expression.

So it is enough to prove the convergence of the series $\sum_{k>0}\lambda_k^w e^{-\lambda_k t_0}$ considering the asymptotics previously introduced. Let us study the convergence of the tail.

There exists k_0 such that

$$\begin{aligned}\sum_{k\geq k_0}\lambda_k^w e^{-\lambda_k t_0} &< \sum_{k\geq k_0}\left(\frac{3}{2}C_{n,s,\Omega}k^{\frac{2s}{n}}\right)^w e^{-\frac{1}{2}C_{n,s,\Omega}k^{\frac{2s}{n}}t_0}\\ &= \left(\frac{3}{2}\right)^w C_{n,s,\Omega}^w \sum_{k\geq k_0} k^{\gamma w}e^{-\frac{1}{2}C_{n,s,\Omega}k^{\gamma}t_0},\end{aligned}$$

where it has been used the notation introduced in Proposition 4.6. The notation has been simplified, $\gamma = \frac{2s}{n}$.

As well as that, the qualitative convergence with respect to t_0, as t_0 goes to 0, is the same in the last and in the first expression. If $\gamma \geq 1$ the series obviously converges. It remains to see the case $0 < \gamma < 1$. To do it, use the integral criterion (it also works for $\gamma \geq 1$), changing variables $y = x^\gamma$, and $z = \frac{1}{2}C_{n,s,\Omega}yt_0$; and defining $\beta := w + \frac{1-\gamma}{\gamma} = w + \frac{n}{2s} - 1$,

$$\begin{aligned}\int_{k_0}^{\infty} x^{\gamma w}e^{-\frac{1}{2}C_{n,s,\Omega}x^\gamma t_0}dx =& \frac{1}{\gamma}\int_{k_0^\gamma}^{\infty} y^{w+\frac{1-\gamma}{\gamma}}e^{-\frac{1}{2}C_{n,s,\Omega}yt_0}dy\\ =&\frac{2^{\beta+1}}{\gamma C_{n,s,\Omega}^{\beta+1}t_0^{\beta+1}}\int_{\frac{1}{2}C_{n,s,\Omega}t_0k_0^\gamma}^{\infty} z^\beta e^{-z}dz < +\infty.\end{aligned}$$

Notice that it has been possible to bound the expression since $t_0 > 0$. For $t_0 = 0$ the previous procedure is not appropriate, which is consistent with the fact that the initial condition is not necessary in $C^s(\mathbb{R}^n)$. That is, fixed $t_0 > 0$,

$$\|u(\cdot,t_0)\|_{C^s(\mathbb{R}^n)} \leq C(t_0)\|u_0\|_{L^2(\Omega)}, \tag{4.35}$$

where $C(t_0)$ depends on n, s, Ω and t_0. The dependence on t_0 has been found before, as t_0 approaches 0: $C(t_0) = O(t_0^{-w-\frac{n}{2s}})$, for $t_0 \downarrow 0$.

Finally, to prove 2., we use the same argument as used in 1., but now consider

$$\left\|\frac{u(\cdot,t)}{\delta^s}\right\|_{C^\alpha(\overline{\Omega})} = \left\|\sum_{k>0}u_k\frac{\phi_k}{\delta^s}e^{-\lambda_k t}\right\|_{C^\alpha(\overline{\Omega})},$$

and follow the same way, to see that $\left\|\frac{u(\cdot,t)}{\delta^s}\right\|_{C^\alpha(\overline{\Omega})}$ is bounded by an expression equivalent to the one found for 1.

□

From the previous theorem and its proof, it is possible to extend the result to the temporal domain.

Corollary 4.7. *The solution to the fractional heat equation* (4.22)*, u, for an initial condition at $t = 0$, $u(x, 0) = u_0(x) \in L^2(\Omega)$, and for a $C^{1,1}$ bounded domain Ω, satisfies that, for any fixed $x_0 \in \mathbb{R}^n$, $u(x_0, \cdot) \in C^\infty(\mathbb{R}^+)$.*

Proof. The expression $u(x_0, t) = \sum_{k>0} u_k\phi_k(x_0)e^{-\lambda_k t}$ clearly converges uniformly over any compact subset $K \subset \mathbb{R}^+$, as seen in the proof of previous theorem for $w = 0$ (since $\phi \in C^s(\mathbb{R}^n)$. Therefore, since every term is continuous, so it is $u(x_0, t)$. Consider the derivative obtained deriving term by term (due to uniform convergence). $u_t(x_0, t) = -\sum_{k>0} u_k\phi_k(x_0)\lambda_k e^{-\lambda_k t}$; which again converges uniformly as seen before, for $w = 1$. Repeat the iterative process, to see that $u(x_0, \cdot) \in C^\infty(\mathbb{R}^+)$.

Moreover, if Ω is only bounded (and not $C^{1,1}$), then $\phi \in L^\infty(\Omega)$ (as seen in the proof of Proposition 4.1, using Proposition 4.3). Therefore, the same argument holds a.e.; so $u(x_0, \cdot) \in C^\infty(\mathbb{R}^+)$ a.e. $x_0 \in \mathbb{R}^n$. □

And also, we could explicitly write the bound on the norm of the spatial derivatives:

Corollary 4.8. *The solution to the fractional heat equation* (4.22)*, u, for an initial condition at $t = 0$, $u(x, 0) = u_0(x) \in L^2(\Omega)$, and for a $C^{1,1}$ bounded domain Ω, satisfies that*

1. *For each $t_0 > 0$,*

$$\sup_{t\geq t_0} \left\| \frac{\partial^j u}{\partial t^j}(\cdot, t) \right\|_{C^s(\mathbb{R}^n)} \leq C_1^{(j)} \|u_0\|_{L^2(\Omega)}, \ \forall j \in \mathbb{N} \tag{4.36}$$

2. *For each $t_0 > 0$,*

$$\sup_{t\geq t_0} \left\| \frac{1}{\delta^s} \frac{\partial^j u}{\partial t^j}(\cdot, t) \right\|_{C^\alpha(\overline{\Omega})} \leq C_2^{(j)} \|u_0\|_{L^2(\Omega)}, \ \forall j \in \mathbb{N} \tag{4.37}$$

   where $\delta(x) = dist(x, \partial\Omega)$, and $\alpha > 0$ is small.

The constants $C_1^{(j)}$ and $C_2^{(j)}$ depend only on t_0, n, s, j and Ω, and blow up as $t_0 \downarrow 0$.

Proof. To prove this corollary we can simply use the same argument used in the proof of Theorem 1.1, but now considering the expression of $\frac{\partial^j u}{\partial t^j}(\cdot, t)$ obtained deriving term by term (which can be done because the series expansion is uniformly convergent, as seen in the previous corollary).

$$\frac{\partial^j u}{\partial t^j}(\cdot, t) = (-1)^j \sum_{k>0} \lambda_k^j u_k\phi_k(x)e^{-\lambda_k t}, \tag{4.38}$$

and proceed with the proof of Theorem 1.1 using $w + j$ instead of w. □

Notice that from the previous corollary would trivially arise Corollary 4.7, since we see that all the temporal derivatives are continuous.

Finally, using the ideas from the proof of the main theorem and the results from Proposition 4.4, we can also prove the following result, regarding interior regularity for the solution of the fractional heat equation.

Proposition 4.9. *Let $\Omega \subset \mathbb{R}^n$ be any bounded $C^{1,1}$ domain, and $s \in (0,1)$. Let $u_0 \in L^2(\Omega)$, and let u be the solution to the fractional heat equation*

$$\left\{\begin{array}{rcll} \partial_t u + (-\Delta)^s u &=& 0 & \text{in } \Omega,\ t>0 \\ u &=& 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &=& u_0(x) & \text{in } \Omega,\ \text{for } t=0. \end{array}\right. \tag{4.39}$$

Then, $u(\cdot,t) \in C^\infty(\Omega)$ for any $t>0$.

Proof. Recall from the proof of Proposition 4.4 that

$$\|\phi\|_{C^{(2j+1)s}(\Omega_\rho)} \leq C^{(j)} \lambda^{j+w} \|\phi\|_{L^2(\Omega)}, \tag{4.40}$$

where $\Omega_\rho = \{x \in \Omega : \delta(x) > \rho\}$, and for some constant $C^{(j)}$ depending only on j, n, s, Ω and ρ. Now, proceeding as in the proof of Theorem 1.1 we have (for unitary eigenfunctions)

$$\begin{aligned} \|u(\cdot,t)\|_{C^{(2j+1)s}(\Omega_\rho)} &= \left\| \sum_{k>0} u_k \phi_k e^{-\lambda_k t} \right\|_{C^{(2j+1)s}(\Omega_\rho)} \\ &\leq \sum_{k>0} |u_k| \|\phi_k\|_{C^{(2j+1)s}(\Omega_\rho)} e^{-\lambda_k t} \\ &\leq \|u_0\|_{L^2(\Omega)} \sum_{k>0} C^{(j)} \lambda_k^{j+w} \|\phi_k\|_{L^2(\Omega)} e^{-\lambda_k t} \\ &= \|u_0\|_{L^2(\Omega)} C^{(j)} \sum_{k>0} \lambda_k^{j+w} e^{-\lambda_k t}. \end{aligned}$$

From which we obtain

$$\|u(\cdot,t)\|_{C^{(2j+1)s}(\Omega_\rho)} \leq C^{(j)}(t) \|u_0\|_{L^2(\Omega)}, \tag{4.41}$$

for some constant $C^{(j)}(t)$ depending only on j, t, n, s, Ω and ρ, which blows up as $t \downarrow 0$. This can be done for any j, and for any $\rho > 0$, therefore $u(\cdot,t) \in C^\infty(\Omega)$ for $t > 0$.

□

Chapter 5

Non-homogeneous fractional heat equation

This chapter is dedicated to study and discuss a similar result for the non-homogeneous fractional heat equation. To do so, we begin introducing the Duhamel's principle for this case, and use it to derive existence of the solution. After that, we are able to state a similar result to Theorem 1.1 but considering the non-homogeneous time independent fractional heat equation. We then discuss some approaches made by the author towards reaching a similar conclusion for the general case. Finally, we state some boundary inequalities regarding the solution of the non-homogeneous fractional heat equation, that allow us to see that the exponent in the term $\delta(x)$ used in the whole dissertation is actually optimal.

The non-homogeneous fractional heat equation corresponds to the situation where the expression of the parabolic equation involving the fractional Laplacian is no longer homogeneous. The problem now becomes

$$\begin{cases} \partial_t u + (-\Delta)^s u &= f(x,t) & \text{in } \Omega,\ t>0 \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &= u_0(x) & \text{in } \Omega,\ \text{for } t=0, \end{cases} \tag{5.1}$$

for a $C^{1,1}$ bounded domain Ω, $u_0 \in L^2(\Omega)$ the initial condition, and $f(x,t)$ the non-homogeneous term whose regularity will be treated later. Intuitively, and from a probabilistic point of view, recall from the heuristic derivation that the non-homogeneous term is the rate of probability for a particle to appear at a certain location, at a certain moment.

Proposition 5.1. *If $u(x,t)$ is a solution for the non-homogeneous fractional heat equation problem* (5.1)*, such that $u(\cdot,t) \to u_0(x)$ as $t \downarrow 0$ in $L^2(\Omega)$ norm, then $u(\cdot,t)$ is the only solution (in $L^2(\Omega)$), for each $t \in \mathbb{R}$.*

Proof. It follows exactly the same way as the uniqueness for the homogeneous case, studied in Proposition 4.5, applying the integral or energy method. □

5.1 Duhamel's principle

In order to find a solution to the previous problem, and study its regularity, we will apply the same approach used for differential operators: Duhamel's principle.

Firstly, split the problem (5.1) into two different problems.

$$\begin{cases} \partial_t u_1 + (-\Delta)^s u_1 &= 0 & \text{in } \Omega,\ t>0 \\ u_1 &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u_1(x,0) &= u_0(x) & \text{in } \Omega,\ \text{for } t=0, \end{cases} \tag{5.2}$$

$$\begin{cases} \partial_t u_2 + (-\Delta)^s u_2 &= f(x,t) & \text{in } \Omega,\ t>0 \\ u_2 &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u_2(x,0) &= 0 & \text{in } \Omega,\ \text{for } t=0. \end{cases} \tag{5.3}$$

The solution to the problem (5.1) will be $u = u_1 + u_2$. The regularity, existence and uniqueness of u_1 has been studied in the previous section. It remains to see the existence and regularity of u_2.

Theorem 5.2. *A solution of the problem*

$$\begin{cases} \partial_t u_2 + (-\Delta)^s u_2 &= f(x,t) & \text{in } \Omega,\ t>0 \\ u_2 &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u_2(x,0) &= 0 & \text{in } \Omega,\ \text{for } t=0, \end{cases} \tag{5.4}$$

can be expressed as

$$u_2(x,t) = \int_0^t v_\zeta(x,t)d\zeta, \tag{5.5}$$

where $v_\zeta(x,t)$ is the solution of the problem

$$\begin{cases} \partial_t v_\zeta + (-\Delta)^s v_\zeta &= 0 & \text{in } \Omega,\ t>\zeta \\ v_\zeta &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq \zeta \\ v_\zeta(x,\zeta) &= f(x,\zeta) & \text{in } \Omega,\ \text{for } t=\zeta. \end{cases} \tag{5.6}$$

Proof. To proceed with the proof, check the three expressions that should fulfil the solution. The last two are the easiest.

$$u_2(x,0) = \int_0^0 v_\zeta(x,t)d\zeta = 0, \tag{5.7}$$

and, for $x \in \mathbb{R}^n \setminus \Omega$, $v_\zeta(x,t)$ is 0 by definition, for any $t \geq \zeta$,

$$u_2(x,t) = \int_0^t v_\zeta(x,t)d\zeta = 0, \quad \forall x \in \mathbb{R}^n \setminus \Omega. \tag{5.8}$$

Then notice

$$\partial_t u_2(x,t) = v_t(x,t) + \int_0^t \partial_t v_\zeta(x,t)d\zeta$$

and

$$\begin{aligned} (-\Delta)^s u_2(x,t) &= (-\Delta)^s \int_0^t v_\zeta(x,t)d\zeta \\ &= c_{n,s} \int_{\mathbb{R}^n} \frac{\int_0^t (v_\zeta(x,t)(x) - v_\zeta(x,t)(y))d\zeta}{|x-y|^{n+2s}} dy \\ &= c_{n,s} \int_{\mathbb{R}^n} \int_0^t \frac{(v_\zeta(x,t)(x) - v_\zeta(x,t)(y))}{|x-y|^{n+2s}} d\zeta dy \\ &= \int_0^t (-\Delta)^s v_\zeta(x,t)d\zeta. \end{aligned}$$

Adding the previous two expression one gets

$$\begin{aligned}\partial_t u_2(x,t) + (-\Delta)^s u_2(x,t) &= v_t(x,t) + \int_0^t (\partial_t v_\zeta(x,t) + (-\Delta)^s v_\zeta(x,t))d\zeta \\ &= v_t(x,t) = f(x,t).\end{aligned}$$

Therefore, $u_2(x,t) = \int_0^t v_\zeta(x,t)d\zeta$ is indeed a solution of the problem (5.3)

□

Proposition 5.3. *If $f \in L^\infty(\Omega \times (0,T))$ for any $T > 0$, then the problem* (5.1) *admits a unique solution.*

Proof. The solution will be $u = u_1 + u_2$. We already know $u_1(\cdot,t)$ is in $L^2(\Omega)$ for each $t \geq 0$, and $u_1(\cdot,t) \to u_0(x)$ as $t \downarrow 0$ in $L^2(\Omega)$. Let us check a similar expression for u_2 as defined in the previous theorem and then applying Proposition 5.1 it will follow the uniqueness.

$$\begin{aligned}\|u_2(\cdot,t)\|_{L^2(\Omega)} = \left\| \int_0^t v_\zeta(x,t)d\zeta \right\|_{L^2(\Omega)} &\leq \int_0^t \|v_\zeta(x,t)\|_{L^2(\Omega)}\, d\zeta \\ &\leq \int_0^t \|v_\zeta(x,\zeta)\|_{L^2(\Omega)}\, d\zeta \\ &= \int_0^t \|f(x,\zeta)\|_{L^2(\Omega)}\, d\zeta \\ &\leq t|\Omega|^{1/2}\|f\|_{L^\infty(\Omega\times[0,t])} < \infty,\end{aligned}$$

where it has been used that $\|v_\zeta(x,t)\|_{L^2(\Omega)} \leq \|v_\zeta(x,\zeta)\|_{L^2(\Omega)}$ for $t \geq \zeta$, since in problem (5.6), the $L^2(\Omega)$ norm of the solution decreases as time goes by, as seen in (4.29).

So we have that $u_2(\cdot,t)$ is in $L^2(\Omega)$ for each $t \geq 0$, and $\|u_2(\cdot,t)\|_{L^2(\Omega)} \downarrow 0$ (so $u_2(\cdot,t) \to 0$) as $t \downarrow 0$. Therefore, $u(\cdot,t) \in L^2(\Omega)$ and $u(\cdot,t) \to u_0(x)$ as $t \downarrow 0$ in $L^2(\Omega)$. □

5.2 The time independent case

While in this dissertation we do not present a theorem regarding the regularity up to the boundary for u and u/δ^s in the general non-homogeneous case, we here present a result for the situation where the non-homogeneous term is time independent, i.e., $f = f(x)$. The result obtained is very similar to what we would expect for the general f, except that in this situation we will only consider the L^∞ norm in the spatial domain. The main proposition of this section, and the corresponding proof, is the following.

Proposition 5.4. *Let $\Omega \subset \mathbb{R}^n$ be any bounded $C^{1,1}$ domain, and $s \in (0,1)$. Let $u_0 \in L^2(\Omega)$, $f \in L^\infty(\Omega)$, and let u be the solution to the fractional heat equation*

$$\begin{cases} \partial_t u + (-\Delta)^s u &= f(x) \quad \text{in } \Omega,\ t > 0 \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &= u_0(x) \quad \text{in } \Omega,\ \text{for } t = 0. \end{cases} \tag{5.9}$$

Then,

1. *For each* $t_0 > 0$,

$$\sup_{t\geq t_0} \|u(\cdot,t)\|_{C^s(\mathbb{R}^n)} \leq C_1(t_0)\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(\Omega)}\right). \tag{5.10}$$

2. *For each* $t_0 > 0$,

$$\sup_{t\geq t_0} \left\| \frac{u(\cdot,t)}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})} \leq C_2(t_0)\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(\Omega)}\right), \tag{5.11}$$

where $\delta(x) = dist(x, \partial\Omega)$, *and* $\alpha > 0$ *is small.*

The constants C_1 *and* C_2 *depend only on* t_0, n, s *and* Ω, *and blow up as* $t_0 \downarrow 0$.

Proof. The key step in this proof is to realize that the solution to the fractional heat equation presented can be expressed as a sum of two functions, by simple linearity,

$$u(x,t) = v(x,t) + w(x) \tag{5.12}$$

where w is the solution of the elliptic problem

$$\begin{cases} (-\Delta)^s w &= f(x) \quad \text{in } \Omega \\ w &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{5.13}$$

and v is the solution of the homogeneous problem

$$\begin{cases} \partial_t v + (-\Delta)^s v &= 0 \quad \text{in } \Omega,\ t > 0 \\ v &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ v(x,0) &= u_0(x) - w(x) \quad \text{in } \Omega,\ \text{for } t = 0. \end{cases} \tag{5.14}$$

By the results from Ros-Oton and Serra stated in Proposition 3.22 and Theorem 3.23 (obtained from [39]), we can write the following bounds,

$$\|w\|_{C^s(\mathbb{R}^n)} \leq C\|f\|_{L^\infty(\Omega)} \tag{5.15}$$

and

$$\left\| \frac{w}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})} \leq C\|f\|_{L^\infty(\Omega)} \tag{5.16}$$

From now on we will only do the first part of the statement, 1. By Theorem 1.1, we have for each $t_0 \geq 0$

$$\sup_{t\geq t_0} \|v(\cdot,t)\|_{C^s(\mathbb{R}^n)} \leq C(t_0)\|u_0 - w\|_{L^2(\Omega)} \leq C(t_0)(\|u_0\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}). \tag{5.17}$$

We already know $\|w\|_{L^2(\Omega)} \leq |\Omega|^{1/2}\|w\|_{L^\infty}$, and as seen in Lemma 3.21, $\|w\|_{L^\infty} \leq C\|f\|_{L^\infty(\Omega)}$. In all, we have obtained

$$\begin{aligned} \sup_{t\geq t_0} \|u(\cdot,t)\|_{C^s(\mathbb{R}^n)} \leq& \sup_{t\geq t_0} \|v(\cdot,t)\|_{C^s(\mathbb{R}^n)} + \|w\|_{C^s(\mathbb{R}^n)} \\ \leq& C'(t_0)\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(\Omega)}\right) + C\|f\|_{L^\infty(\Omega)} \\ \leq& C(t_0)\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(\Omega)}\right), \end{aligned}$$

for some constant $C(t_0)$ depending only on n, s, Ω and t_0, that blows up as $t_0 \downarrow 0$.

To prove the second statement, 2., it is possible to follow exactly the same path. □

5.3 Some comments on the time dependent case

We have not been able to obtain nor prove a result like the one presented for the time independent case in the general non-homogeneous problem. The original reason why the Duhamel's principle has been introduced in this chapter (apart from the fact that it helps us find a solution) was to obtain the result for the general case from the homogeneous problem. While we have not been able to obtain such result, we here present the path we tried to follow.

The aim was to obtain a bound (of the appropriate norms) for the solution of

$$\begin{cases} \partial_t u_2 + (-\Delta)^s u_2 &= f(x,t) & \text{in } \Omega,\ t>0 \\ u_2 &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u_2(x,0) &= 0 & \text{in } \Omega,\ \text{for } t=0, \end{cases} \tag{5.18}$$

involving a constant depending on a time t_0 and also involving $\|f\|_{L^\infty(\Omega\times(0,T))}$, for some $T>0$. From Duhamel's principle, we already know

$$u_2(x,t) = \int_0^t v_\zeta(x,t)d\zeta, \tag{5.19}$$

where $v_\zeta(x,t)$ is the solution of the problem

$$\begin{cases} \partial_t v_\zeta + (-\Delta)^s v_\zeta &= 0 & \text{in } \Omega,\ t>\zeta \\ v_\zeta &= 0 & \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq \zeta \\ v_\zeta(x,\zeta) &= f(x,\zeta) & \text{in } \Omega,\ \text{for } t=\zeta. \end{cases} \tag{5.20}$$

This last problem already fulfils a condition on its solution, which is the one that we want to use to find the result for the non-homogeneous case. We already know that, for each $t_0>0$, and $t\in(0,T)$,

$$\sup_{t\geq t_0+\zeta} \|v_\zeta(\cdot,t)\|_{C^s(\mathbb{R}^n)} \leq C_1(t_0)\|f(\cdot,\zeta)\|_{L^2(\Omega)}, \tag{5.21}$$

and

$$\sup_{t\geq t_0+\zeta} \left\| \frac{v_\zeta(\cdot,t)}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})} \leq C_2(t_0)\|f(\cdot,\zeta)\|_{L^2(\Omega)}. \tag{5.22}$$

Now writing it in terms of the L^∞ norm, and particularizing for the first time possible we have

$$\|v_\zeta(\cdot,t_0+\zeta)\|_{C^s(\mathbb{R}^n)} \leq C_1(t_0)|\Omega|^{1/2}\|f\|_{L^2(\Omega\times(0,T))}, \tag{5.23}$$

and

$$\left\| \frac{v_\zeta(\cdot,t_0+\zeta)}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})} \leq C_2(t_0)|\Omega|^{1/2}\|f\|_{L^2(\Omega\times(0,T))}. \tag{5.24}$$

At this point, we know that from the norm of the expression of the solution we need to find a bound involving the norm of the interior of the integral. Simply use the triangular inequality for integrals, to get

$$\|u_2(\cdot,t)\|_{C^s(\mathbb{R}^n)} = \left\| \int_0^t v_\zeta(x,t)d\zeta \right\|_{C^s(\mathbb{R}^n)} \leq \int_0^t \|v_\zeta(\cdot,t)\|_{C^s(\mathbb{R}^n)}\, d\zeta, \tag{5.25}$$

and

$$\left\| \frac{u_2(\cdot,t)}{\delta^s} \right\|_{C^s(\mathbb{R}^n)} = \left\| \int_0^t \frac{v_\zeta(x,t)}{\delta^s(x)} d\zeta \right\|_{C^s(\mathbb{R}^n)} \leq \int_0^t \left\| \frac{v_\zeta(\cdot,t)}{\delta^s} \right\|_{C^s(\mathbb{R}^n)} d\zeta. \tag{5.26}$$

Now, we use the bounds found before, (5.23) and (5.24), to get

$$\|u_2(\cdot,t)\|_{C^s(\mathbb{R}^n)} \leq |\Omega|^{1/2} \|f\|_{L^2(\Omega\times(0,T))} \int_0^t C_1(t-\zeta) d\zeta, \tag{5.27}$$

and

$$\left\| \frac{u_2(\cdot,t)}{\delta^s} \right\|_{C^s(\mathbb{R}^n)} \leq |\Omega|^{1/2} \|f\|_{L^2(\Omega\times(0,T))} \int_0^t C_2(t-\zeta) d\zeta. \tag{5.28}$$

And here is where the prove fails. The dependence of the constant with time is of the form $t^{-\frac{n}{2s}-w}$, for $w \in \mathbb{N}$. The exponent is always lower than -1, so the integral does not converge; thus, we have not reached any bound.

One could think that the problem relies on the expression found for the bound of the constant, from which follows an exponent too negative ($-\frac{n}{2s}-w$). However, using the methods here presented, it is not possible to improve the exponent any further than $-\frac{n}{2s}$; since this term comes from the expression of the eigenvalues, associated with the domain. Another option could be using a refinement on the coefficients u_k under the basis of eigenfunctions of the initial condition.

This could be seen using another approach also tried by the author, which consists in considering the expression in the form of the eigenfunctions, following exactly the same way as in the homogeneous case. That is, we know that for any $t \geq \zeta$,

$$v_\zeta(x,t) = \sum_{k>0} f_k(\zeta)\phi_k(x) e^{-\lambda_k(t-\zeta)} \tag{5.29}$$

where $f_k(\zeta) = \int_\Omega f(y,\zeta)\phi_k(y)dy$. We already know that the series is uniformly convergent over compacts with respect to t, so we can proceed

$$u_2(x,t) = \int_0^t v_\zeta(x,t)d\zeta = \sum_{k>0} \left\{ \int_\Omega \left(\int_0^t f(y,\zeta) e^{\lambda_k \zeta} d\zeta \right) \phi_k(y)dy \right\} \phi_k(x) e^{-\lambda_k t}. \tag{5.30}$$

And now it would be interesting to find the appropriate bound for

$$\int_\Omega \left(\int_0^t f(y,\zeta) e^{\lambda_k \zeta} d\zeta \right) \phi_k(y)dy, \tag{5.31}$$

so that we could use the same approach as in the homogeneous case. All the bounds tested by the author lead to divergent series.

While it has not been possible to obtain the general result for the non-homogeneous case we strongly believe that there exists a bound of the form

$$\sup_{T\geq t\geq t_0} \|u(\cdot,t)\|_{C^s(\mathbb{R}^n)} \leq C_1(t_0) \left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(\Omega\times(0,T))} \right), \tag{5.32}$$

$$\sup_{T\geq t\geq t_0} \left\| \frac{u(\cdot,t)}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})} \leq C_2(t_0) \left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(\Omega\times(0,T))} \right), \tag{5.33}$$

for each $t_0 > 0$ and for $T > 0$. These bounds could be obtained through the comments we have already stated in this section, or rewriting the original prove by Ros-Oton and Serra in [39] for the elliptic case adapted for the parabolic case. This last option might be much more complicated, but would lead almost certainly to a conclusion.

5.4 An application of Theorem 1.1: boundary bounds for solutions

The first immediate application of Theorem 1.1 allows us to check whether s is the optimum exponent for $\delta(x)$ when approaching the boundary. I.e., we want to check if it is true that there exist constants $c(t_0, T)$ and $C(t_0)$ such that $c(t_0, T)\delta^s(x) \leq u(x, t_0) \leq C(t_0)\delta^s(x)$ for all $T \geq t \geq t_0$, being u a solution of the fractional heat equation under the appropriate conditions (for example, positivity of the solution).

One of the inequalities will follow thanks to Theorem 1.1, while the other will follow immediately using Theorem 3.26 by Chen, Kim and Song [15].

Suppose we are dealing with the problem,

$$\begin{cases} \partial_t u + (-\Delta)^s u &= f(x,t) \quad \text{in } \Omega,\ t > 0 \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ u(x,0) &= u_0(x) \quad \text{in } \Omega,\ \text{for } t = 0, \end{cases} \tag{5.34}$$

and assume $f \in L^\infty(\Omega \times \mathbb{R}^+)$ for some bounded domain Ω, at least $C^{1,1}$. The first result can be stated in the following way.

Proposition 5.5. *Suppose u is a weak solution of problem (5.34), Ω is a bounded and $C^{1,1}$ domain and $t_0 > 0$. Then,*

$$|u(x,t)| \leq C(t_0)(\|f\|_{L^\infty(\Omega\times\mathbb{R}^+)} + \|u_0\|_{L^2(\Omega)})\delta^s(x),\ \forall t \geq t_0 > 0,\ \forall x \in \Omega, \tag{5.35}$$

for some constant $C(t_0)$ depending only on n, s, Ω and t_0.

Proof. Suppose v is the solution of

$$\begin{cases} \partial_t v + (-\Delta)^s v &= C \quad \text{in } \Omega,\ t > 0 \\ v &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\ v(x,0) &= v_0 \quad \text{in } \Omega,\ \text{for } t = 0, \end{cases} \tag{5.36}$$

for $v_0(x) := |u_0(x)|$ and $C = \|f\|_{L^\infty(\Omega\times\mathbb{R}^+)}$. We are now interested in using the comparison principle.

We have $u_0 \leq v_0$ and $f(x,t) \leq C$, so by the comparison principle $u(x,t) \leq v(x,t)$. Moreover, $-u$ solves the fractional heat equation with $-f$ and $-u_0$. Since $-f \leq C$ and $-u_0 \leq v_0$, we also have $-u(x,t) \leq v(x,t)$ and therefore $|u(x,t)| \leq v(x,t)$. On the other hand, we know that by Proposition 5.4, we have

$$\sup_{t\geq t_0} \left\| \frac{v(\cdot,t)}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})} \leq C(t_0)\left(\|u_0\|_{L^2(\Omega)} + C\right), \tag{5.37}$$

where it has been used that $\|u_0\|_{L^2(\Omega)} = \|v_0\|_{L^2(\Omega)}$.

Now using that,

$$\frac{v(x,t)}{\delta^s(x)} \leq \left\| \frac{v(\cdot,t)}{\delta^s} \right\|_{L^\infty(\overline{\Omega})} \leq \sup_{t\geq t_0} \left\| \frac{v(\cdot,t)}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})}, \ \forall t \geq t_0,\ \forall x \in \Omega, \tag{5.38}$$

it follows the desired result. □

Remark 5.1. Notice that we could have also proved the result using the C^s continuity. That is, Proposition 5.4 also tells us that,

$$\sup_{t\geq t_0} [v(\cdot,t)]_{C^s(\mathbb{R}^n)} \leq \sup_{t\geq t_0} \|v(\cdot,t)\|_{C^s(\mathbb{R}^n)} \leq C(t_0)\left(\|u_0\|_{L^2(\Omega)} + C\right). \tag{5.39}$$

Moreover, consider the definition of $[v(\cdot,t)]_{C^s(\mathbb{R}^n)}$,

$$[v(\cdot,t)]_{C^s(\mathbb{R}^n)} = \sup_{x\neq y\in\mathbb{R}^n} \frac{|v(x,t)-v(y,t)|}{|x-y|^s}. \tag{5.40}$$

Now, for any fixed $x \in \Omega$, define $y_0(x)$ such that $\text{dist}(x,\partial\Omega) = \text{dist}(x, y_0(x))$. Since $\partial\Omega$ is closed, $y_0(x) \in \partial\Omega$, and therefore $v_0(y_0(x),t) \equiv 0$ for all t. We then have,

$$\begin{aligned}
\frac{|v(x,t)|}{\delta(x)^s} =& \frac{|v(x,t)-v(y_0(x),t)|}{|x-y_0(x)|^s} \\
\leq& \sup_{x\neq y\in\mathbb{R}^n} \frac{|v(x,t)-v(y,t)|}{|x-y|^s} \\
=& [v(\cdot,t)]_{C^s(\mathbb{R}^n)} \\
\leq& C(t_0)\left(\|u_0\|_{L^2(\Omega)} + C\right), \ \forall t \geq t_0,\ \forall x \in \Omega,
\end{aligned}$$

as we wanted to see.

Let us now proceed to prove the second inequality. To do so, we refer to the result obtained by Chen, Kim and Song in [15], already stated in Theorem 3.26. Let us rewrite here the statement we need.

Theorem (Theorem 3.26)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ open set, and $\delta(x) = dist(x,\partial\Omega)$. Then, for each $t_0 > 0$ there are positive constants $C_1(t_0)$ and $C_2(t_0)$ depending on s, n, t_0 and Ω such that*

$$C_1(t_0)e^{-\lambda_1 t}\delta^s(x)\delta^s(y) \leq p_\Omega(x,y,t) \leq C_2(t_0)e^{-\lambda_1 t}\delta^s(x)\delta^s(y), \forall t \geq t_0, \forall x,y \in \Omega, \tag{5.41}$$

where λ_1 is the first eigenvalue of the fractional Laplacian in Ω with zero boundary or exterior conditions.

Recall that p_Ω is the fractional heat kernel in the domain Ω. So we can state the inequality:

Proposition 5.6. *Let Ω be a bounded and $C^{1,1}$ domain and $T > t_0 > 0$. Suppose u is a weak solution of problem*

$$\begin{cases}
\partial_t u + (-\Delta)^s u &= f(x,t) \quad \text{in } \Omega,\ t > 0 \\
u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,\ t \geq 0 \\
u(x,0) &= u_0(x) \quad \text{in } \Omega,\ \text{for } t = 0,
\end{cases} \tag{5.42}$$

and also suppose $f \geq 0$ *and* $u_0 \geq 0$ *in their domains, and* $\|u_0\|_{L^2(\Omega)} > 0$ *(so that* u_0 *it is not 0 in* L^2 *sense). Then,*

$$u(x,t) \geq c(t_0,T)\delta^s(x) > 0, \ \forall T \geq t \geq t_0 > 0, \ \forall x \in \Omega, \tag{5.43}$$

for some constant $c(t_0,T)$ *depending only on* n*,* s*,* Ω*,* t_0 *and* T*.*

Proof. Again, we need to use the comparison principle. It will be enough to see the result for $f \equiv 0$, since $f \geq 0$ and the solution for $f \equiv 0$ will be smaller than for $f \geq 0$ by the comparison principle.

The solution, using the fractional heat kernel, is

$$u(x,t) = \int_\Omega p_\Omega(x,y,t)u_0(y)dy. \tag{5.44}$$

Now, using one of the inequalities of (5.41), we have

$$u(x,t) \geq \int_\Omega c'(t_0)\delta^s(x)\delta^s(y)e^{-\lambda_1 t}u_0(y)dy, \forall t \geq t_0, \forall x \in \Omega. \tag{5.45}$$

If $u_0(y)$ is not identically 0 almost everywhere, then

$$\int_\Omega \delta^s(y)u_0(y)dy > 0, \tag{5.46}$$

and, therefore, we obtain the desired result,

$$u(x,t) \geq c'(t_0)\delta^s(x)e^{-\lambda_1 T}\int_\Omega \delta^s(y)u_0(y)dy, \forall T \geq t \geq t_0, \forall x \in \Omega. \tag{5.47}$$

□

Chapter 6

Further original results: more general nonlocal operators and Pohozaev identity

In this chapter we proceed to introduce an application and an extension of the main theorem of the dissertation, Theorem 1.1. In particular, as an application we show the Pohozaev identity for the solution of the homogeneous fractional heat equation for positive times. After that, as an extension, we prove that the present results can be extended to other nonlocal operators under certain conditions of stability and ellipticity. We argue and prove that, in fact, the method presented in this work allow us to extend results from the elliptic to the parabolic problem, so that they can be applied for other operators rather than just $(-\Delta)^s$.

6.1 The Pohozaev identity for the fractional heat equation

The classical Pohozaev identity is an expression originally obtained and used by Pohozaev in [37] for the solutions of semilinear equations, $-\Delta u = \lambda f(u)$. It reads like this:

$$\int_\Omega \left\{ nF(u) - \frac{n-2}{2} u f(u) \right\} = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) d\sigma, \tag{6.1}$$

where $\frac{\partial u}{\partial \nu}$ is the exterior normal derivative, ν is the unit outward normal and F is the primitive of f, $F(\mu) = \int_0^\mu f$.

Although it was originally derived to prove nonexistence of solutions for supercritical f, numerous other applications have arisen since the work [37], and currently it is widely used.

It can be obtained through the divergence theorem, and in particular, it follows from the identity

$$\int_\Omega (x \cdot \nabla u)\Delta u = \frac{2-n}{2} \int_\Omega u\Delta u + \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) d\sigma, \tag{6.2}$$

which holds for u with $u \equiv 0$ in $\partial\Omega$.

For the fractional Laplacian, even if there is no divergence theorem for such operator, the following Pohozaev identity was found by Ros-Oton and Serra, [40]:

If u is a solution of the elliptic problem, then

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu)d\sigma, \quad (6.3)$$

where now, instead of using the boundary term $\frac{\partial u}{\partial \nu}$, it has been replaced by $\frac{u}{\delta^s}$.

Here, it is possible to notice that the regularity up to the boundary is essential. The fact that $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$ allows us to evaluate the integral on $\partial\Omega$. In addition, we see that when fixing $s = 1$ in the equation (6.3) we recover (6.2), since $u/\delta|_{\partial\Omega} = \partial u/\partial\nu$ and $\Gamma(2) = 1$.

This identity is the only known integration by parts type formula for a nonlocal operator involving a local boundary term.

For the ordinary Laplacian, the Pohozaev identity follows easily from integration by parts or the divergence theorem. However, in this nonlocal framework these tools are not available, and the proof of Ros-Serra of (6.3) is much more difficult.

In this section we show that the same identity (6.3) holds for solutions $u(x,t)$ of the homogeneous fractional heat equation. To do so, we will use the results from Ros-Oton and Serra from [41]. The main result we will need, corresponding to [41, Proposition 1.6], is the following. Recall that we defined $\Omega_\rho = \{x \in \Omega : \delta(x) \geq \rho\}$.

Theorem 6.1 ([41]). *Let Ω be a bounded and $C^{1,1}$ domain. Assume that u is a $H^s(\mathbb{R}^n)$ function which vanishes in $\mathbb{R}^n \setminus \Omega$, and satisfies*

1. *$u \in C^s(\mathbb{R}^n)$ and, for every $\beta \in [s, 1+2s)$, u is of class $C^\beta(\Omega)$ and*
$$[u]_{C^\beta(\Omega_\rho)} \leq C\rho^{s-\beta}, \ \forall\rho \in (0,1). \quad (6.4)$$
2. *The function $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, there exists $\alpha \in (0,1)$ such that $u/\delta^s \in C^\alpha(\overline{\Omega})$. In addition, for all $\beta \in [\alpha, s+\alpha]$, it holds the estimate*
$$[u/\delta^s]_{C^\beta(\Omega_\rho)} \leq C\rho^{\alpha-\beta}, \ \forall\rho \in (0,1). \quad (6.5)$$
3. *$(-\Delta)^s u$ is uniformly bounded in Ω.*

Then, the following identity holds

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu)d\sigma, \quad (6.6)$$

where ν is the unit outward normal to $\partial\Omega$ at x, and Γ is the Gamma function.

It might seem like the hypothesis are quite specific. Concretely, they were found to be consistent with the elliptic problem for the fractional Laplacian. The following proposition, from [39, Corollary 1.6] is the connection between the previous theorem and the elliptic problem. We here denote $f \in C^{0,1}_{loc}(\overline{\Omega}\times\mathbb{R})$, implying that f is Lipschitz in any compact subset of the domain.

Proposition 6.2 ([39]). *Let Ω be a bounded and $C^{1,1}$ domain, $f \in C^{0,1}_{loc}(\overline{\Omega}\times\mathbb{R})$, and u be a solution of*

$$\begin{cases} (-\Delta)^s u &= f(x,u) & \text{in } \Omega \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{6.7}$$

where $\delta(x) = dist(x, \partial\Omega)$. Then,

1. *$u \in C^s(\mathbb{R}^n)$, and for every $\beta \in [s, 1+2s)$, u is of class $C^\beta(\Omega)$ and*

$$[u]_{C^\beta(\Omega_\rho)} \leq C\rho^{s-\beta}, \ \forall \rho \in (0,1). \tag{6.8}$$

2. *The function $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, there exists $\alpha \in (0,1)$ such that $u/\delta^s \in C^\alpha(\overline{\Omega})$. In addition, for al $\beta \in [\alpha, s+\alpha]$, it holds the estimate*

$$[u/\delta^s]_{C^\beta(\Omega_\rho)} \leq C\rho^{\alpha-\beta}, \ \forall \rho \in (0,1). \tag{6.9}$$

The constants α and C depend only on Ω, s, f, $\|u\|_{L^\infty(\mathbb{R}^n)}$, and β.

Remark 6.1. When $f(x,u) = \lambda u$ it can be seen from the proof of this proposition that then $C = \bar{C}\lambda\|u\|_{L^\infty(\Omega)}$, where $\bar{C}$ depends only Ω, s, n and β.

As it can be seen, the conclusions of Proposition 6.2 are the hypothesis of Theorem 6.1. In this section we need to adapt the previous proposition to the heat equation, so that we can use Theorem 6.1 in order to obtain the Pohozaev identity. In all, the result of this section is the following theorem.

Theorem 6.3. *Let Ω be any bounded $C^{1,1}$ domain, and let $u(x,t)$ be a solution of*

$$\begin{cases} \partial_t u + (-\Delta)^s u &= 0 & \text{in } \Omega, \ t > 0 \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega, \ t \geq 0 \\ u(x,0) &= u_0(x) & \text{in } \Omega, \ \text{for } t = 0, \end{cases} \tag{6.10}$$

where $u_0 \in L^2(\Omega)$. Then, for any fixed $t > 0$, the following identity holds,

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu) d\sigma, \tag{6.11}$$

where we have used the notation $u = u(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$.

Proof. In Theorem 1.1 we have checked that indeed u/δ^s can be extended continuously up to the boundary for positive times, so that the boundary integral makes sense.

For the fixed $t > 0$, we will use Theorem 6.1 to prove the identity. It is enough to check that the hypothesis are fulfilled. Firstly, u for positive times is always in $H^s(\mathbb{R}^n)$, since it is a solution of the fractional heat equation. In addition, to check the third hypothesis we could see that $\partial_t u = -(-\Delta)^s u$ and in Corollary 4.8 we had shown that all temporal derivatives are $C^s(\mathbb{R}^n)$ for positive times, so in particular, are uniformly bounded in Ω.

In order to check hypothesis 1. and 2., we will use Proposition 6.2 for the elliptic case. We assume $f(x,u) = \lambda_k u$, so that the solution is the corresponding eigenfunction. Therefore, we have the following bounds (using Remark 6.1),

$$[\phi_k]_{C^\beta(\Omega_\rho)} \leq C\lambda_k \|\phi_k\|_{L^\infty(\Omega)}\rho^{s-\beta}, \ \forall \rho \in (0,1), \ \beta \in [s, 1+2s), \tag{6.12}$$

and

$$[\phi_k/\delta^s]_{C^\beta(\Omega_\rho)} \leq C\lambda_k \|\phi_k\|_{L^\infty(\Omega)}\rho^{\alpha-\beta}, \ \forall \rho \in (0,1), \ \beta \in [\alpha, s+\alpha], \tag{6.13}$$

where ϕ_k is the k-th eigenfunction of the fractional Laplacian problem in Ω, and C depends on n, s, Ω and β. Proceed like in the proof of Theorem 1.1, expressing $u(x,t) = \sum_{k>0} u_k\phi_k e^{-\lambda_k t}$, being u_k the coefficients of u_0 in the basis $\{\phi_k\}_{k>0}$,

$$[u(\cdot,t)]_{C^\beta(\Omega_\rho)} \leq \sum_{k>0} u_k[\phi_k]_{C^\beta(\Omega_\rho)}e^{-\lambda_k t} \leq C\rho^{s-\beta}\sum_{k>0} u_k\lambda_k\|\phi_k\|_{L^\infty(\Omega)}e^{-\lambda_k t}, \tag{6.14}$$

similarly,

$$[u(\cdot,t)/\delta^s]_{C^\beta(\Omega_\rho)} \leq C\rho^{\alpha-\beta}\sum_{k>0} u_k\lambda_k\|\phi_k\|_{L^\infty(\Omega)}e^{-\lambda_k t}. \tag{6.15}$$

Now, we recall the result from Proposition 4.1, which stated

$$\|\phi_k\|_{L^\infty(\Omega)} \leq C\lambda_k^{w-1}\|\phi_k\|_{L^2(\Omega)} \tag{6.16}$$

for some $w \in \mathbb{N}$, and C depending only on n, s and Ω. Hence, assuming unitary eigenfunctions,

$$[u(\cdot,t)]_{C^\beta(\Omega_\rho)} \leq C\rho^{s-\beta}\sum_{k>0} u_k\lambda_k^w e^{-\lambda_k t} \tag{6.17}$$

and

$$[u(\cdot,t)/\delta^s]_{C^\beta(\Omega_\rho)} \leq C\rho^{\alpha-\beta}\sum_{k>0} u_k\lambda_k^w e^{-\lambda_k t}. \tag{6.18}$$

Which are expressions almost identical to the ones found in the proof of Theorem 1.1. Proceeding the same way, we reach

$$[u(\cdot,t)]_{C^\beta(\Omega_\rho)} \leq C(t)\|u_0\|_{L^2(\Omega)}\rho^{s-\beta}, \tag{6.19}$$

and

$$[u(\cdot,t)/\delta^s]_{C^\beta(\Omega_\rho)} \leq C(t)\|u_0\|_{L^2(\Omega)}\rho^{\alpha-\beta}, \tag{6.20}$$

for some constant $C(t)$ depending only on n, s, Ω, β and t, that blows up when $t \downarrow 0$. So, for any fixed $t > 0$, we have seen that hypothesis of Theorem 6.1 are fulfilled, and therefore,

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu)d\sigma, \tag{6.21}$$

where we have now again used that $u = u(\cdot,t)$ for a fixed $t > 0$. □

6.2 Extension to other nonlocal operators

In this dissertation we have showed that from a regularity result for the elliptic problem up to the boundary, it is possible to obtain a regularity result for the parabolic problem, following the method introduced.

Most of the results presented in this work can be generalized to other nonlocal operators. For example, the maximum principle is still true for general nonlocal operators, and it will be used as a known result (the proof would be very similar to the one presented, using the nonlocal expression of the operator).

In this section we will extend the result presented in Theorem 1.1 to a more general class of operators: stable operators, already introduced in section 2.2 from Chapter 2.

Before that, let us recall what the infinitessimal generator of a stable process is, and let us also state the ellipticity conditions. A $2s$-stable Lévy process has as an infinitessimal generator the operator L, defined by

$$Lu(x) = \int_{\mathbb{R}^n} (u(x) - u(x+y)) \frac{a(y/|y|)}{|y|^{n+2s}} dy, \tag{6.22}$$

where $s \in (0,1)$ and $a \in L^1(S^{n-1})$, is nonnegative and symmetric. Here, $a(\theta)d\theta = d\mu(\theta)$, $\theta \in S^{n-1}$ denotes the spectral measure. We will require that a fulfils an additional condition, known as the uniform ellipticity condition,

$$0 < \lambda \leq a(\theta) \leq \Lambda, \ \forall \theta \in S^{n-1}, \tag{6.23}$$

for some λ, Λ.

The nonlocal operators that we will use will have to fulfil the two previous conditions, (6.22) and (6.23). The main result of this section is the following theorem, an extension of Theorem 1.1.

Theorem 6.4. *Let $\Omega \subset \mathbb{R}^n$ be any bounded $C^{1,1}$ domain, and $s \in (0,1)$. Let $u_0 \in L^2(\Omega)$, and let u be the solution to the fractional heat equation with a general operator*

$$\begin{cases} \partial_t u + Lu &= 0 & \text{in } \Omega, \ t > 0 \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega, \ t \geq 0 \\ u(x,0) &= u_0(x) & \text{in } \Omega, \ \text{for } t = 0, \end{cases} \tag{6.24}$$

where L is a nonlocal operator of the form (6.22) *and* (6.23)*. Then,*

1. *For each $t_0 > 0$,*

$$\sup_{t \geq t_0} \|u(\cdot,t)\|_{C^s(\mathbb{R}^n)} \leq C_1(t_0)\|u_0\|_{L^2(\Omega)}. \tag{6.25}$$

2. *For each $t_0 > 0$,*

$$\sup_{t \geq t_0} \left\| \frac{u(\cdot,t)}{\delta^s} \right\|_{C^{s-\epsilon}(\overline{\Omega})} \leq C_2(t_0)\|u_0\|_{L^2(\Omega)}, \tag{6.26}$$

where $\delta(x) = dist(x, \partial\Omega)$, and for any $\epsilon > 0$.

The constants C_1 and C_2 depend only on t_0, L, n, s and Ω, and blow up as $t_0 \downarrow 0$.

The proof of this theorem follows using the same ideas as in the fractional Laplacian problem, but instead, we will have to extend some of the results presented in this work. In particular, we will need to generalize Proposition 4.3 and state it for nonlocal operators of the previous form, as can be seen in Proposition 6.10 below.

Results for the regularity up to the boundary for the elliptic problem can also be generalized. The main results used in this dissertation are Proposition 3.22 and Theorem 3.23, which have been extended to nonlocal operators L of the previous form by the same authors, Ros-Oton and Serra, in [41].

It is important to notice that, in this more recent paper [41], the authors prove that $u/\delta^s \in C^{s-\epsilon}(\overline{\Omega})$ for all $\epsilon > 0$, for u a solution of the elliptic problem with the appropriate conditions (already stated in this work), where now the Hölder exponent $s-\epsilon$ is optimal or almost optimal for $g \in L^\infty(\Omega)$. So, using [41], not only we can extend the results of this dissertation to other nonlocal operators, but we also improve the results of the present work. Hence, in the second part of the statement of Theorem 1.1, we could exchange α by $s-\epsilon$, gaining, this way, regularity.

The theorem by Ros-Oton and Serra is the following, analogous to Theorem 3.23 for more general operators.

Theorem 6.5 ([41]). *Let $\Omega \subset \mathbb{R}^n$ be any bounded $C^{1,1}$ domain, $s \in (0,1)$, L an operator fulfilling (6.22) and (6.23) and u be the solution of the elliptic problem*

$$\left\{ \begin{array}{rcll} Lu & = & g & in\ \Omega \\ u & = & 0 & in\ \mathbb{R}^n \setminus \Omega. \end{array} \right. \tag{6.27}$$

If $g \in L^\infty(\Omega)$, then $u/\delta^s \in C^{s-\epsilon}(\overline{\Omega})$ for all $\epsilon > 0$, where $\delta(x) = dist(x, \partial\Omega)$, being $\partial\Omega$ the boundary of the domain. In particular, $u/\delta^s|_\Omega$ can be extended continuously to $\overline{\Omega}$. Moreover,

$$\|u/\delta^s\|_{C^{s-\epsilon}(\overline{\Omega})} \leq C\|g\|_{L^\infty(\Omega)}, \tag{6.28}$$

where the constant C depends only on Ω and s.

In the next subsections we proceed to prove Theorem 6.4.

6.2.1 Existence of eigenfunctions and asymptotic behaviour of eigenvalues

The aim of this subsection is to prove the following.

Proposition 6.6. *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain, and L an operator of the form (6.22), with $a \in L^1(S^{n-1})$. Then,*

1. *There exist a sequence of eigenfunctions forming a Hilbert basis of L^2.*

2. *If $\{\lambda_k\}_{k\in\mathbb{N}}$ is the sequence of eigenvalues associated to the eigenfunctions of L in increasing order, then*

$$\lim_{k\to\infty} \lambda_k k^{-\frac{2s}{n}} = C, \tag{6.29}$$

for some constant C depending on n, s, L and Ω.

One of the things we will need to check refers to the asymptotic behaviour of the eigenvalues. For the fractional Laplacian we had that

$$\lim_{k\to\infty} \lambda_k k^{-\frac{2s}{n}} = C_{n,s,\Omega}, \tag{6.30}$$

for some constant $C_{n,s,\Omega}$ depending on the subindexs. We here want to see that, if L is the infinitessimal generator of a stable Lévy process, then the asymptotic behaviour of its eigenvalues is the same (with the constant depending, also, on the operator). To do so, we refer to a result by Geisinger [27] regarding the Weyl's law for fractional differential operators.

In this subsection, given an operator L, $A(\xi)$ will denote its Fourier symbol (i.e., $\widehat{Lu}(\xi) = A(\xi)\hat{u}(\xi)$). For the fractional Laplacian, we recall that the Fourier symbol is $A_{(-\Delta)^s}(\xi) = |\xi|^{2s}$. Let us state the result by Geisinger, first introducing two conditions on A (extracted directly from [27]).

1. There is a function $A_0 : \mathbb{R}^n \to \mathbb{R}$ with the following three properties. A_0 is homogeneous of degree $\alpha > 0$: $A_0(\nu\xi) = \nu^\alpha A_0(\xi)$ for $\xi \in \mathbb{R}^n$ and $\nu > 0$. The set of $\xi \in \mathbb{R}^n$ with $A_0(\xi) < 1$ has finite Lebesgue-measure, and the function A_0 fulfils
$$\lim_{\nu\to\infty} \nu^{-\alpha} A(\nu\xi) = A_0(\xi). \tag{6.31}$$
If the previous convergence is not uniform, we will require an additional property.

2. There are constants $C_0 > 0$ and $M \in \mathbb{N}$ such that for all $\eta \in \mathbb{R}^n$,
$$\sup_{\xi\in\mathbb{R}^n} \left(\frac{1}{2}(A(\xi+\eta) + A(\xi-\eta)) - A(\xi)\right) \le C_0(1+|\eta|)^M. \tag{6.32}$$

Theorem 6.7 ([27])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set of finite volume and assume that A is the symbol of a differential operator L that satisfies the previous two conditions. Then,*
$$\lim_{k\to\infty} \lambda_k k^{-\frac{2s}{n}} = C_{n,s,\Omega}, \tag{6.33}$$
for some constant $C_{n,s,\Omega}$ depending only on the subindexs and L, and $\{\lambda_k\}_{k\in\mathbb{N}}$ the eigenvalues of L in increasing order.

At this point, we want to see that the infinitessimal generator of a stable process fulfils the previous conditions in order to use the theorem by Geisinger. To do so, we use that the Fourier symbol $A(\xi)$ of L can be explicitly written in terms of s and the spectral measure μ, as

$$A(\xi) = \int_{S^{n-1}} |\xi \cdot \theta|^{2s} d\mu(\theta), \tag{6.34}$$

which can be seen, for example, in [43]. We have used here the notation with the spectral measure μ, but as in the previous notation, $d\mu(\theta) = a(\theta)d\theta$. Moreover, the following two bounds hold,

$$0 < \mu_1|\xi|^{2s} \le A(\xi) \le \mu_2|\xi|^{2s} \tag{6.35}$$

for

$$\mu_2 = \int_{S^{n-1}} d\mu(\theta), \qquad \mu_1 = \inf_{\nu\in S^{n-1}} \int_{S^{n-1}} |\nu\cdot\theta|^{2s} d\mu(\theta), \tag{6.36}$$

where these constants are strictly positive for any stable operator with non-degenerate Lévy measure (as we have here).

Lemma 6.8. *Let $a \in L^1(S^{n-1})$. Then, the Gagliardo seminorm (see* (3.3)*) associated to the operator* (6.22)*, L,*

$$[u]_{H^s_L} = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(x+y))^2}{|y|^{n+2s}} a\left(\frac{y}{|y|}\right) dxdy \tag{6.37}$$

is equivalent to the Gagliardo seminorm in $H^s(\mathbb{R}^n)$,

$$[u]_{H^s} = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(x+y))^2}{|y|^{n+2s}} dxdy. \tag{6.38}$$

In particular, norms associated to the operator L are equivalent to norms associated to the fractional Laplacian.

Proof. To prove the lemma, we notice that

$$[u]_{H^s_L} = \langle Lu, u\rangle, \tag{6.39}$$

for the standard inner product in $L^2(\mathbb{R}^n)$. This can be seen writing explicitly the scalar product and Lu in its nonlocal form.

Now, we use Plancherel's theorem, thus obtaining

$$\langle Lu, u\rangle = \langle \widehat{Lu}, \hat{u}\rangle = \int_{\mathbb{R}^n} A(\xi)\hat{u}^2(\xi)d\xi. \tag{6.40}$$

Using the bounds on $A(\xi)$ and $A_{(-\Delta)^s}(\xi) = |\xi|^{2s}$ we have

$$\mu_1\langle \widehat{(-\Delta)^s u}, \hat{u}\rangle \le \langle \widehat{Lu}, \hat{u}\rangle \le \mu_2\langle \widehat{(-\Delta)^s u}, \hat{u}\rangle, \tag{6.41}$$

and using Plancherel's theorem again we get the desired result. □

In particular, the previous Lemma implies the Sobolev inequality for $p = 2$ when dealing with general non-degenerate stable operators.

We can now proceed to prove Proposition 6.6.

Proof of Proposition 6.6. To see the first result we can repeat the proof in Chapter 3 for the fractional Laplacian and it would yield directly, since by the previous lemma we know that the norm associated is equivalent to the one for the fractional Laplacian.

For the second result we will use Theorem 6.7.

Assume that A is the Fourier symbol of the stable process L. It is enough to check the two conditions of the theorem by Geisinger. The first condition trivially holds taking $A_0 = A$, since it is homogeneous, and we have the previous bounds.

We need to check the second condition. To do so, let us state the following inequality, for any $a \geq b \geq 0$ and $s \in (0,1)$,

$$2a^{2s} + 2b^{2s} \geq (a+b)^{2s} + (a-b)^{2s}, \tag{6.42}$$

which follows by concavity, since $s \in (0,1)$,

$$\begin{aligned}(a+b)^{2s} + (a-b)^{2s} =& (a^2+b^2+2ab)^s + (a^2+b^2-2ab)^s \\ \leq& 2(a^2+b^2)^s \\ \leq& 2(a^{2s}+b^{2s}).\end{aligned}$$

From the previous inequality, we obtain

$$|\xi\cdot\theta + \eta\cdot\theta|^{2s} + |\xi\cdot\theta - \eta\cdot\theta|^{2s} \leq 2|\xi\cdot\theta|^{2s} + 2|\eta\cdot\theta|^{2s}, \tag{6.43}$$

and therefore,

$$\begin{aligned}A(\xi+\eta) + A(\xi-\eta) - 2A(\xi) =& \int_{S^{n-1}} \left\{|\xi\cdot\theta + \eta\cdot\theta|^{2s} + |\xi\cdot\theta - \eta\cdot\theta|^{2s} - 2|\xi\cdot\theta|^{2s}\right\} d\mu(\theta) \\ \leq& 2\int_{S^{n-1}} |\eta\cdot\theta|^{2s} d\mu(\theta) \\ \leq& 2|\eta|^{2s}\mu_2 \\ \leq& 2\mu_2(1+|\eta|)^{2s},\end{aligned}$$

as we wanted to see.

□

Therefore, we have proved that the asymptotic behaviour of the eigenvalues is the same as in the fractional Laplacian case, for any operator being the infinitessimal generator of a non-degenerate stable Lévy process. It is important to highlight that the results presented in this subsection have not been proved just for the operators fulfilling the ellipticity condition, but for any operator coming from a non-degenerate stable Lévy process.

6.2.2 Regularity of eigenfunctions

In this section we want to see the following result, analogous to Proposition 4.1, referring to the problem,

$$\left\{\begin{array}{rcll} L\phi &=& \lambda\phi & \text{in } \Omega \\ \phi &=& 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{array}\right. \tag{6.44}$$

Proposition 6.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $s \in (0,1)$, and ϕ any eigenfunction of problem* (6.44)*, with eigenvalue λ. Then, $\phi \in L^\infty(\Omega)$. Moreover*

$$\|\phi\|_{L^\infty(\Omega)} \leq C\lambda^{w-1}\|\phi\|_{L^2(\Omega)}, \tag{6.45}$$

for some constant C depending only on n, s, L and Ω, and some $w \in \mathbb{N}$, depending only on n and s.

In order to see that, we first need to state and prove an extension of Proposition 4.3.

Proposition 6.10. *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain, $s \in (0,1)$, $g \in L^2(\Omega)$, and u be the weak solution of*

$$\begin{cases} Lu &= g \quad in\ \Omega \\ u &= 0 \quad in\ \mathbb{R}^n \setminus \Omega. \end{cases} \tag{6.46}$$

Then,

1. *Let $2 \leq p < \frac{n}{2s}$, and assume $g \in L^p(\Omega)$. Then there exists a constant C, depending only on n, s, L and p, such that*
$$\|u\|_{L^q(\Omega)} \leq C\|g\|_{L^p(\Omega)}, \quad q = \frac{np}{n-2ps}. \tag{6.47}$$
2. *Let $\frac{n}{2s} < p < \infty$. Then, there exists a constant C, depending only on n, s, L and p, such that:*
$$\|u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^p(\Omega)}. \tag{6.48}$$
3. *If $p = \frac{n}{2s}$, then there exists a constant C, depending only on n, s, L and q, such that:*
$$\|u\|_{L^q(\Omega)} \leq C\|g\|_{L^p(\Omega)}, \quad \forall q < \infty. \tag{6.49}$$

To prove the previous statement, let us first introduce a bit elliptic problems with general operators L.

The following known lemma will be essential.

Lemma 6.11 (see [28])**.** *Let $s \in (0,1)$ and $n > 2s$. Suppose L is an operator of the form (6.22) fulfilling the ellipticity condition (6.23), where $2s$ is the stability exponent of the stable process. Then, there exists a function V, fundamental solution of L in the sense that for all g,*

$$u(x) = \int g(y)V(x-y)dy \tag{6.50}$$

satisfies

$$Lu = g \ in\ \mathbb{R}^n. \tag{6.51}$$

In addition, the ellipticity condition implies,

$$\frac{c_1}{|y|^{n-2s}} \leq V(y) \leq \frac{c_2}{|y|^{n-2s}} \tag{6.52}$$

for some c_1 and c_2 positive constants depending only on n and s, and λ and Λ respectively.

Under these assumptions we will define L^{-1} as the operator

$$L^{-1}g(x) = \int g(y)V(x-y)dy \tag{6.53}$$

for the elliptic problem in $\mathbb{R}^n$, where V is given by the previous lemma.

Let us also recall the well known Hardy-Littlewood-Sobolev inequality. It basically states that, if $f \in L^p(\mathbb{R}^n)$, then

$$\|I_{2s}f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \; q = \frac{np}{n-2ps}, \tag{6.54}$$

for some constant C depending only on p, n and s. Here, $I_{2s}f$ stands for the Riesz potential of a locally integrable function, defined by

$$(I_{2s}f)(x) = C_{n,s}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2s}}dy. \tag{6.55}$$

We are now able to provide the following result, previous to the proof of Proposition 6.10.

Lemma 6.12. *Let $s \in (0,1)$, $n > 2s$ and g and u be such that*

$$u = L^{-1}g \text{ in } \mathbb{R}^n, \tag{6.56}$$

for some operator L coming from a $2s$-stable Lévy process, and L^{-1} defined through a fundamental solution V as seen in Lemma 6.11. Suppose $u, g \in L^p(\mathbb{R}^n)$, with $1 \leq p < \frac{n}{2s}$. Then, there exists a constant C depending only on n, s, L and p such that

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}, \qquad q = \frac{np}{n-2ps}. \tag{6.57}$$

Proof. We try to bound $|u|$ by some expression equivalent to a Riesz potential of g, in order to use the Hardy-Littlewood-Sobolev inequality. We use that

$$u(x) = L^{-1}g(x) = \int_{\mathbb{R}^n} g(y)V(x-y)dy, \tag{6.58}$$

Now, since $0 \leq V(x-y) \leq \frac{c_2}{|x-y|^{n-2s}}$, we have

$$|u(x)| \leq \int_{\mathbb{R}^n} |g(y)|V(x-y)dy \leq \int_{\mathbb{R}^n} |g(y)|\frac{c_2}{|x-y|^{n-2s}}dy = C(I_{2s}|g|)(x), \tag{6.59}$$

from which we can derive

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C'\|I_{2s}|g|\|_{L^q(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}, \; q = \frac{np}{n-2ps}, \tag{6.60}$$

as we wanted to see.

□

Remark 6.2. We have not provided any result for the case $n \leq 2s$. This only occurs when $n = 1$ and $s \in [1/2, 1)$, but for $n = 1$ any stable process is the fractional Laplacian, which has already been studied in this work.

We can now prove Proposition 6.10 for a general operator L of the form (6.22) and (6.23).

Proof of Proposition 6.10. Statement 1. can be obtained as follows. Consider the problem

$$Lv = |g| \text{ in } \mathbb{R}^n, \tag{6.61}$$

where g has been extended from Ω to $\mathbb{R}^n$ by zero. We know that there is a $v \geq 0$ solving the problem, since we can define

$$v(x) = \int_\Omega |g(y)|V(x-y)dy \geq 0. \tag{6.62}$$

Now, using the comparison principle we have $-v \leq u \leq v$, since $|g| \geq g \geq -|g|$. This means $\|v\|_{L^q(\mathbb{R}^n)} \geq \|u\|_{L^q(\Omega)}$, and by Lemma 6.12 first result we have that

$$\|v\|_{L^q(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}, \qquad q = \frac{np}{n-2ps}, \tag{6.63}$$

just like we wanted to see.

To see the second statement let us proceed similarly. Define v as before, so that using Hölder's inequality

$$0 \leq |u(x)| \leq v(x) = \int_\Omega |g(y)|V(x-y)dy \leq \|g\|_{L^p(\Omega)} \left(\int_\Omega V(x-y)^{p'}dy\right)^{1/p'}, \tag{6.64}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. We now want to bound $\int_\Omega V(x-y)^{p'}dy$. Consider $B_R(0)$ a ball centred at the origin such that $\Omega \subset B_R$ (exists because Ω is bounded). Then we want to see whether the following integral converges,

$$\int_{B_R(0)} V(y)^{p'}dy, \tag{6.65}$$

which is equivalent to seeing that

$$\int_{B_R(0)} \frac{1}{|y|^{(n-2s)p'}}dy < \infty, \tag{6.66}$$

and this last integral converges because $p > \frac{n}{2s}$.

Therefore, we reach

$$\|u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^p(\Omega)}, \tag{6.67}$$

for some constant C depending only on n, s, L, p and Ω.

Statement 3. follows using the first result, 1., and making $p \uparrow \frac{n}{2s}$ using that

$$\|g\|_{L^p(\Omega)} \leq C\|g\|_{L^r(\Omega)}, \quad r = \frac{n}{2s}, \tag{6.68}$$

by Hölder's inequality on bounded domains. □

And from here, we prove Proposition 6.9 directly,

Proof of Proposition 6.9. The proof is exactly the same as the proof of Proposition 4.1, but now using the result of Proposition 6.10 instead of Proposition 4.3. □

6.2.3 Proof of regularity for general operators: Theorem 6.4

Finally, through Propositions 6.6 and 6.9 and the ideas already introduced, we can proceed to prove Theorem 6.4.

Proof of Theorem 6.4. We already know that the elliptic problem with L has discrete L^2 eigenfunctions and eigenvalues.

In order to see the L^∞ bound for the eigenfunctions, we can use Proposition 6.9.

The $C^s(\mathbb{R}^n)$ bound for ϕ_k and the $C^{s-\epsilon}(\overline{\Omega})$ bound for ϕ_k/δ^s (being ϕ_k the eigenfunctions) follow using the results from [41], stated in Theorem 6.5.

Finally, we need to know the asymptotic behaviour of the eigenvalues. For a general stable operator, we have seen in Proposition 6.6 that their asymptotic behaviour is the same as in the fractional Laplacian. Now, using the reasoning done in the proof of Theorem 1.1 we reach the desired result.

□

Remark 6.3. The only point at which we needed both ellipticity conditions on a, the spectral measure, was when referring to the paper by Ros-Oton and Serra [41]. In a future work, [42], the same authors will show an analogous result for operators coming from general non-degenerate stable processes. Thus, using this forthcoming paper, we could restate the whole section only imposing an upper bound on the measure a, which would be even more general than the current result.

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