Master in Photonics

MASTER THESIS WORK

MODELLING OF GENERAL RELATIVITY PHENOMENA WITH PHOTONICS

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Presented on date 18th July 2014

Registered at

Escola Tècnica Superior d'Enginyeria de Telecomunicació de Barcelona
Modelling of general relativity phenomena with photonics

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Abstract. Metamaterials are man-made artificial materials whose electromagnetic parameters $\varepsilon$ and $\mu$ are determined by its internal structure rather than their substance. An analogy between the electrodynamics of these new materials and general relativity is discussed. Based on this analogy, the possibility to design optical black holes in the laboratory is studied.

Keywords: general relativity, metamaterials, optical black holes

1. Introduction

Recent developments in modern technology have allowed to build artificial materials whose permittivity and permeability can be engineered by incorporating structural elements of subwavelength size. As a result, one can create materials (called metamaterials) with the desired electromagnetic response which offers new opportunities for realizing such exotic phenomena as negative refraction, cloaking, super-lenses for subwavelength imaging, microantennas, etc. [1]. This has become an expanding branch of photonics.

One of the many interesting applications of metamaterials is that they can be used to mimic general relativity phenomena that are difficult to observe directly using the existing astronomical tools [2, 3]. It has been proven that the propagation of electromagnetic waves in curved space-time is formally equivalent to propagation in flat spacetime in a certain nonhomogeneous anisotropic or bianisotropic medium [4]. Based on this equivalence, a metamaterial can be engineered to study a particular phenomenon such as: optical analogues of black holes [3, 5, 6, 7], Schwarzschild spacetime [8], the ‘Big Bang’ and cosmological inflation [9, 10]. Some of these models have been carried out in a laboratory experiment [3, 5, 7]. However, metamaterials are not the only branch of photonics used to model general relativity. Plasmons have also been used to mimic a rotating black hole [11].

In this work, we are going to focus in black holes, which are one of the most fascinating predictions of the general theory of relativity. They are massive celestial objects which bend light towards their center of gravity resulting in either the deflection of light or its trapping. As we have pointed out, this phenomenon can also be interpreted as the bending of light in an equivalent anisotropic metamaterial. Such
systems are of great interest not only from the fundamental point of view but also for the possible applications in optical devices that trap light (solar cells, for example). It is thus important to investigate the associated phenomena under controlled laboratory conditions [3].

Black hole physics has already been mimicked in [3, 5, 6, 7] using a centrally symmetric metamaterial, however, these examples are not solutions of Einstein field equations. On the other hand, in [8] a real black hole is modelled: the Schwarzschild black hole. This is the simplest type of black hole because it is spherically symmetric and it has neither angular momentum nor charge. In this work, we focus in a rotating black hole, the Kerr black hole, which is a solution of Einstein field equations as well. In contrast to previous studies where static space-time metrics have been considered, we investigate a stationary metric.

By making use of the equivalence between curved space-time and the electromagnetic response of the metamaterial, we determine the permittivity and permeability tensors of the related media. Then, we simulate ray trajectories outside the Kerr black hole, and finally we will analyse the deflection angle as a function of the impact parameter comparing the results with the known analytical asymptotes. This bending angle can be measured in optical experiments.

2. Metamaterials

Materials are composed of atoms whose structure defines the different properties of the materials. Electromagnetic properties are essentially determined by two parameters: the relative permittivity \( \varepsilon \) and relative permeability \( \mu \). They measure the response a material has when exposed to an external electromagnetic field. As shown in figure 1, physical behaviour of materials depends drastically on the sign of \( \varepsilon \) and \( \mu \), which is crucial in the definition of the refractive index \( n \):

\[
 n = \pm \sqrt{\varepsilon \mu} \tag{1}
\]

Taking the positive sign in (1), we obtain the classically known expression for the index of refraction. But when the values of the permittivity and permeability are simultaneously negative, the negative sign of the square root must be taken. This is due to the fact that the electrodynamics for negative \( \varepsilon \) and \( \mu \) is not the same as in the case of positive \( \varepsilon \) and \( \mu \).

![Figure 1. Representation of the real part of the relative permittivity \( \varepsilon \) and the permeability \( \mu \) [1].](image-url)
As Veselago pointed out [12], from Maxwell’s curl equations one can see that the electric field $E$, magnetic field $H$ and wave vector $k$ form a right-handed triplet for positive $\varepsilon$ and $\mu$ and a left-handed one for negative $\varepsilon$ and $\mu$. But something else has to be taken into consideration, the Poynting vector

$$S = E \times H$$

always forms a right-handed triplet with $E$ and $H$. Since $S$ and the group velocity $v_g$ have the same direction while the direction of the phase velocity $v_p$ coincides with the direction of $k$, one can conclude that $v_g$ and $v_p$ are antiparallel when $\varepsilon$ and $\mu$ are simultaneously negative. Given that the phase velocity is directly related to the refractive index, the negative sign of the square root in (1) must be taken. Having a negative index of refraction implies some unexpected consequences. For instance, the Snell law is modified, and the Doppler effect and Cherenkov effect are reversed in negative index materials (NIMs) [12, 13].

However, we do not find NIMs in nature. Even if metals have negative $\varepsilon$ and resonant ferromagnetic systems have negative $\mu$, this phenomenon only occurs near a resonance in a small bandwidth at different frequencies $\omega$. Typically, negative $\varepsilon$ occurs at very high frequencies (in metals, for instance, it happens at visible frequencies) but negative $\mu$ is found at much lower frequencies [14]. Nevertheless, modern technology has allowed to build such materials.

The same way materials consist of atoms, metamaterials are composed of structural units that can be called meta-atoms. These meta-atoms and the distance between neighbouring meta-atoms have to be substantially smaller than the wavelength considered so that the inhomogeneities are subwavelength in scale and the metamaterial is macroscopically homogeneous [1, 14]. It has to be possible to distinguish between refraction and diffraction. This is not the case of photonic crystals, whose scattering elements are on the order of the wavelength. It would be inappropriate to describe a photonic crystal as a medium because it is not possible to define $\varepsilon$ and $\mu$ [15].

The precise control of the building blocks of the metamaterials on the submicron scale was what made it technically unachievable. It was not until 2000 when the first negative index metamaterial was created. It was made of straight metallic wires to give the negative $\varepsilon$ and metallic split ring resonators which led to a negative $\mu$ [14, 16]. This metamaterial worked at the microwave scale so the meta-atoms could be several millimetres in size. The challenge nowadays is to reduce the scale to nanometres in order to have a working metamaterial at visible wavelength. The first steps have already been made using high-resolution electron-beam lithography [13].

Although the primary motivation was to build metamaterials with a negative refractive index, nowadays this term has obtained a much broader usage and it refers to artificially designed materials with a subwavelength periodic or quasi-periodic structure [1].

Several practical uses for metamaterials have already been found which include superlenses, invisibility cloaks or ultrasmall optical devices [12, 16, 17].

3. Maxwell equations in curved space-time

In the previous section we have exposed that metamaterials can be designed to have certain particular properties. Using differential geometry and general relativity, an
expression for the permittivity $\varepsilon$ and the permeability $\mu$ can be found. This will allow to build a specific metamaterial for each application.

Consider the free-space Maxwell equations [18):

$$\partial_i D^i = \rho$$  \hspace{1cm} (3)

$$\varepsilon^{ijk} \partial_j H_k = \partial_i D^i + J^i$$  \hspace{1cm} (4)

$$\partial_i B^i = 0$$  \hspace{1cm} (5)

$$\varepsilon^{ijk} \partial_j E_k = -\partial_i B^i$$  \hspace{1cm} (6)

where $\varepsilon^{ijk}$ is the Levi-Civita symbol. These equations can also be written using the Faraday tensor $F_{\mu\nu}$, which contains the electric field $E$ and the magnetic field $B$ [2, 17, 18],

$$\partial_\lambda F_{\mu\nu} = 0$$  \hspace{1cm} (7)

$$\varepsilon_0 \partial_\nu F^{\mu\nu} = J^\mu$$  \hspace{1cm} (8)

where [...] indicates the antisymmetrization. Taking into consideration the metric $g_{\alpha\beta}$, with determinant $g$, we can define an equivalent contravariant tensor $H^{\mu\nu}$ that will contain the $D$ and $H$ fields [2, 17] by

$$F_{\mu\nu} = \frac{1}{\varepsilon_0 \sqrt{-g}} g_{\alpha\beta} g^{\mu\nu} H_{\alpha\beta}$$  \hspace{1cm} (9)

And therefore (8) can be re-written as follows defining $J^\mu = \sqrt{-g} J^\mu$,

$$\partial_\nu H^{\mu\nu} = J^\mu$$  \hspace{1cm} (10)

Considering the $F_{0i}$ component of (9), the constitutive equation for $D$ can be found:

$$D = \varepsilon_0 \varepsilon E + \Gamma \times H$$  \hspace{1cm} (11)

Using the dual tensors $^*F^{\mu\nu}$ and $^*H^{\mu\nu}$ [2, 17, 18], the constitutive relation for $B$ is similarly derived:

$$B = \mu_0 \mu H - \Gamma \times E$$  \hspace{1cm} (12)

The symmetric matrices $\varepsilon$ and $\mu$ and the vector $\Gamma$ are given by [2, 4, 17]

$$\varepsilon^{ij} = \mu^{ij} = -\sqrt{-g} g^{ij}$$  \hspace{1cm} (13)

$$\Gamma_i = \frac{g_{0i}}{g_{00}}$$  \hspace{1cm} (14)

This $\Gamma$ vector describes a magneto-electric coupling between the electric and magnetic fields. The simplest example of a magneto-electric material is a moving medium with velocity proportional to $\Gamma$ because, even if the dielectric responds to the electromagnetic field in its local frame, it is moving and Lorentz transformations mix electric and magnetic fields [2, 17, 18]. Therefore, empty space can be regarded as an anisotropic moving medium.

The opposite case can also be considered: a medium can appear as empty space if the constitutive equations (11), (12) fit (13), (14) of empty flat space in curved coordinates. These media are called transformation media. They map the electromagnetism of this virtual empty space (called electromagnetic space-time) to physical space, which contains the medium [2] (see figure 2).

We need to distinguish two sets of coordinates and three metric tensors: the coordinates $x^{\alpha'}$ and the metric tensor $g_{\alpha'\beta'}$ of electromagnetic space-time which in
physical space appears as $g_{\alpha\beta}$ with coordinates $x^i$. Physical coordinates $x^i$ are determined by a physical metric $\gamma_{ij}$ which can differ from the spatial part of $g_{\alpha\beta}$ since $g_{\alpha\beta}$ is the metric generated by the medium [2].

In physical space, Maxwell’s divergence equations (3), (5) have to be re-written because $\gamma$ (the physical metric determinant) is different from $g$. To do that, $D$ and $B$ have to be multiplied by $\sqrt{\gamma}$ and $\rho$ and $j$, re-escaled [17]. However, when writing Maxwell’s curl equations (4), (6) something else must be taken into consideration. This coordinate transformation that we are performing might turn a right-handed coordinate system into a locally left-handed one. This is because the coordinate directions change handedness when the Jacobian changes from positive to negative. Moreover, $\epsilon^{ijk}$ changes sign under handedness transformations [2] so the permittivity and permeability matrices and the $\Gamma$ have to allow this change [17]

\[
\varepsilon^{ij} = \mu^{ij} = \mp \sqrt{-g} g^{ij} \\
\Gamma_i = \frac{g_{0i}}{\sqrt{\gamma_{00}}} 
\]  

(15)  

(16)

The sign $\mp$ indicates the handedness: the minus sign for right-handed transformations and plus for the locally left-handed ones.

4. Optical Kerr black hole

4.1. Kerr metric in metamaterials

Einstein’s theory of general relativity predicted the existence of massive objects which prevents anything from escaping: neither light nor matter. They are called black holes because they absorb all the light and matter beyond the horizon, reflecting nothing. Not only are they important from the point of view of cosmology, but they may also have possible applications on optical devices that control, slow or trap light [5].

There are many black holes models, the simplest one of them being the Schwarzschild black hole because it is spherically symmetric and static [19]. The metamaterial equivalent to this black hole was studied in 2010 by Chen, Miao and Li [8]. They found the corresponding $\varepsilon$ and $\mu$ and performed a simulation.
We study another kind of black hole, one that rotates: the Kerr black hole. This black hole is described by the following metric in Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) [20]

\[
ds^2 = -dt^2 + \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + \left( r^2 + a^2 \right) \sin^2 \theta d\phi^2 + \frac{2Mr}{\Sigma} (a \sin^2 \theta d\phi - dt)^2
\]

where \(\Delta \equiv r^2 - 2Mr + a^2\) and \(\Sigma \equiv r^2 + a^2 \cos^2 \theta\). The Kerr metric depends on two parameters: \(M\), the mass of the black hole and \(a\), the angular momentum per mass unit measured from infinity. It should be pointed out that when \(a = 0\), we obtain the Schwarzschild metric. The Kerr metric is stationary, since it does not depend explicitly on time. On the other hand, it is not static, since it is not invariant under time reversal due to the coupling term \(dt d\phi\). It is also axisymmetric i.e. it does not explicitly depend on \(\phi\) [19].

The Kerr metric is singular at \(r_{\pm} = M \pm \sqrt{M^2 - a^2}\), when \(\Delta = 0\). The radius \(r_{+}\) turns out to be the event horizon of the Kerr black hole. That means that all matter or light which go beyond \(r_{+}\) will be captured by the black hole and will not be able to escape. We should note that not all values of \(a\) and \(M\) correspond to a Kerr black hole. The horizon \(r_{+}\) only exists for \(a \leq M\), therefore the angular momentum of the black hole cannot be larger than \(M^2\) [19].

In a Kerr black hole there is another interesting surface, called the stationary limit surface where the metric component \(g_{00} = 0\). This happens at \(r_e(\theta) = M + \sqrt{M^2 - a^2 \cos^2 \theta}\) [19]. Inside this surface, no particle can remain fixed in a certain position, it is forced to rotate around the black hole in the sense of its rotation [21]. The volume comprised between the horizon \(r_{+}\) and the stationary limit surface \(r_e(\theta)\) is called the ergosphere and it is a characteristic phenomenon of the Kerr black hole.

For simplicity we will restrict our analysis to the equatorial plane of the metric where \(\theta = \pi/2\), which contains all the essential physics. And for convenience, we work in Cartesian coordinates, therefore \(\gamma = 1\). We can calculate \(\epsilon\) and \(\mu\) from (15):

\[
\begin{align*}
\epsilon_{xx} &= \mu_{xx} = \frac{1}{\Delta} \left( x^2 + y^2 + \frac{2Mx^2}{(x^2 + y^2)^2} \alpha \right) \\
\epsilon_{yy} &= \mu_{yy} = \frac{1}{\Delta} \left( x^2 + y^2 + \frac{2My^2}{(x^2 + y^2)^2} \alpha \right) \\
\epsilon_{xy} &= \mu_{xy} = \frac{2Mxy}{\Delta(x^2 + y^2)^2} \alpha
\end{align*}
\]

where we have defined \(\alpha \equiv a^4/(r - 2M) - r^3\), and the vector \(\Gamma\) is obtained from (16) in the form

\[
\begin{align*}
\Gamma_x &= \frac{2May}{(x^2 + y^2)(r - 2M)} \\
\Gamma_y &= \frac{2May}{(x^2 + y^2)(r - 2M)}
\end{align*}
\]

Thus, the curved spacetime associated with a spinning black hole is represented by the bianisotropic medium characterized by (18) and (19). Alternatively, (18) and (19) may be interpreted as the medium parameters necessary to give rise to a curved spacetime geometry associated with the spinning Kerr black hole. To realise it experimentally, it would be wise to create a core for \(r \leq r_{+}\) with slightly different values, giving \(\epsilon\) and \(\mu\) an imaginary part, to avoid the singularity \(\Delta = 0\). With these
parameters we could proceed to simulate the electrodynamics of the material and study the behaviour of incident light rays.

4.2. Photon orbits

From the metric (17) in the equatorial plane, the equations of motion can be found. In this case, they are [19]

\[
\frac{1}{L^2} \left( \frac{dr}{d\lambda} \right)^2 = \frac{1}{b^2} - W(r, b) \tag{20}
\]

\[
\frac{d\phi}{d\lambda} = \frac{L}{\Delta} \left( 1 - \frac{2M}{r} + \frac{2Ma}{rb} \right) \tag{21}
\]

where \( L \) is the angular momentum of the photon and \( b \) corresponds to the impact parameter. The effective potential \( W(r, b) \) is defined as [19]

\[
W(r, b) = \frac{1}{r^2} \left[ 1 - \left( \frac{a}{b} \right)^2 - \frac{2M}{r} \left( 1 - \frac{a}{b} \right)^2 \right] \tag{22}
\]

This is a central potential in which three kinds of orbits are possible: capture, deflection and an unstable circular orbit around the black hole (which will eventually fall or escape). From (20), we can see that the effective total energy that will make this differentiation possible is \( 1/b^2 \).

Due to the rotation, direct (\( b > 0 \), co-rotating) and retrograde orbits (\( b < 0 \), counter-rotating) behave differently. One can see that the effective potential (22) will not be the same in these two cases. Therefore, the critical impact parameter \( b_c \), the one corresponding to the maximum of the potential which separates capture trajectories from the deflection ones, will be different in each case and this produces an anisotropy. In figure 3 we present our simulation of the three kinds of trajectories we have mentioned along with their corresponding effective potential.

[Figure 3: Trajectories and potential for \( b = 5M \) (red), \( b = 2.8445M \) (green) and \( b = 2.5M \) (blue) in a Kerr black hole of \( a = 0.9M \).]

In figure 4 we show an example of set of trajectories obtained numerically. In this example, we have chosen a range of impact parameters which are smaller than
the critical one for the retrograde case and larger, for the direct one. This shows the
mentioned anisotropy: for $b > 0$, light is deflected by the black hole and for $b < 0$, it
is captured. We can also see that light can go inside the ergosphere and then escapes
from the black hole. Moreover, in the case $b < 0$ we can see how light changes its
rotation sense when it enters the ergosphere.

\[
\Delta \phi = \frac{4M}{b} \quad (23)
\]

From a photonics point of view, figure 3(a) and figure 4 should be regarded as
the capture or deflection of light by an optical black hole in a metamaterial which has
a permittivity and permeability like (18) and a magneto-electrical coupling like (19).
This device can capture light from any direction, only depending on the value of its
impact parameter $b$ and its sign. Furthermore, the capture condition does not depend
on the wavelength, so we have a nearly perfect absorption as long as the structural
units of the metamaterial are subwavelength [6]. It should be noted that although the
medium parameters are static, they model a spinning gravitational black hole.

4.3. Deflection angle

The main application this kind of metamaterials have is to study general relativity
phenomena in a laboratory. One of the possible experiments which could be carried
out is the study of the deflection angle. The advantage of performing this experiment
rather than only looking at the trajectories is that the deflection angle does not depend
on the coordinates, it is a physical measurement.

We have calculated the deflection angle solving numerically the equations of
movement (20) and (21) and compared it with the known analytical asymptotes.
Firstly, we have studied the limit of the Schwarzschild black hole ($a \to 0$). We have
checked that our results agree with the cases of both the strong deflection limit ($b \to b_c$)
and the weak deflection limit ($b \gg b_c$).

In the weak deflection limit (WDL), the angle is found with Einstein’s term [19]
$\Delta \phi = \frac{4M}{b}$

For the strong deflection limit (SDL), Darwin [22] made the first approach of the
bending angle. However, since then some better and more general approximations
have been made \[23, 24\]

\[
\Delta \phi = \ln \left[216(7 - 4\sqrt{3}) - \ln \left(\frac{b}{b_c} - 1\right) - \pi\right] \tag{24}
\]

In figure 5(a), we present the numerical simulation with the two asymptotic limits. We can see that the simulation is in good agreement with the limit cases.

\[
\begin{align*}
\Delta \phi/\pi &= 0.0 \quad \text{WDL, } a=0 \quad \text{SDL, } a=0 \\
0 &\leq \frac{b}{b_c} \leq 1.0
\end{align*}
\]

(a) Deflection angle for a Schwarzschild black hole (solid line) compared with the asymptotes from (23) and (24).

\[
\begin{align*}
\Delta \phi/\pi &= 0.0 \quad \text{WDL, } b<0 \\
0 &\leq \frac{b}{b_c} \leq 1.0
\end{align*}
\]

(b) Deflection angle for a Kerr hole with \(a=0.5\) compared with the Schwarzschild simulation and the WDL asymptote from (25).

\[
\begin{align*}
\Delta \phi/\pi &= 0.0 \quad \text{WDL, } b>0 \\
0 &\leq \frac{b}{b_c} \leq 1.0
\end{align*}
\]

Then we have studied the Kerr black hole when \(a \neq 0\) for both \(b > 0\) and \(b < 0\). We have compared it as well with the WDL which is, with rotation, \[25, 26\]

\[
\Delta \phi = \frac{4M}{|b|} + \left(\frac{15\pi}{4} - \frac{bu}{|b|}\right) \frac{M^2}{b^2} \tag{25}
\]

The first-order term is Einstein’s term (23), however, the second-order term depends on the angular momentum of the black hole \(a\) and the sign of \(b\). This shows the asymmetric behaviour of this kind of black holes. In figure 5(b), the numerical simulation is shown along with this limit and we can see that there is a good agreement as well. A comparison between the two Kerr cases and the Schwarzschild one is also shown. We observe that the three scenarios produce different bendings. When \(b > 0\), the approaching particle co-rotates with the black hole, so it helps the particle deflect more than it would without rotation. On the other hand, when \(b < 0\), the particle counter-rotates with the black hole and it is less deflected.

5. Conclusions

To summarize, we have introduced the concept of metamaterial and used that they can be designed to have almost any \(\varepsilon\) and \(\mu\) we wish to find a recipe to calculate them in terms of the metric \(g_{\alpha\beta}\). With these parameters, we have seen that the electrodynamics of metamaterials is analogous to gravitation. To illustrate this, we have studied and simulated the case of a rotating black hole with Kerr geometry. We
have found that with the adequate $\varepsilon$, $\mu$ and $\Gamma$ we could build a metamaterial acting as an effective broad-band omnidirectional absorber, an optical black hole. Moreover, this black hole has a different capture condition for co-rotating and counter-rotating light. Finally, we have numerically calculated the deflection angle and compared it with the known asymptotes.

The modelling of general relativity with metamaterials may become a new method to study cosmology phenomena since it would be possible to simulate the universe in a laboratory. One may even discover new physical phenomena on metamaterials which have not been found in cosmology yet. Furthermore, it opens new perspectives in topics such as Hawking radiation, time delay and gravitational lensing. In addition, interesting devices could be developed using these phenomena, for instance, the optical black hole. It might prove to be useful in photovoltaic cells since it can absorb light from any direction. These devices can also be extended to other wavelengths of the electromagnetic spectra or to different kinds of waves (such as acoustic waves).

Acknowledgments

I would like to thank my advisor, Dr. Oleg Bulashenko, for his help and guidance without which this project would have not been possible. I also thank my co-supervisor, Prof. Crina Cojocaru for her supervision. I thank Manel Bosch for his helpful comments. Finally, I am indebted to my family for their support.

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