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Computation of Periodic Orbits and Their Invariant Manifolds in the RTBP

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Introduction

This project was elaborated as the final thesis for the ‘MASTER IN ADVANCED MATHEMATICS AND MATHEMATICAL ENGINEERING’ at the Polytechnic University of Catalonia. The aim of the project is to study periodic orbits and invariant manifolds of the RTBP (restricted three-body problem) from the point of view of the theory of dynamical systems. The thesis covers analytical deductions and also numerical computations, and it is focused mainly on Hill’s Problem, a variation of the RTBP devised originally to study the orbits of the Moon.

The method of study is based on the paper by Simó & Stucchi [1]. To understand and apply numerically the contents of this paper it has been needed to refer to several writings that cover related topics. These are listed in the references section and the most relevant are mentioned in this introduction.

The contents are organized in the following way. In the first chapter it is covered the basic knowledge about the RTBP that will be used later. It starts with the deduction of the equations from simple physical principles, and covers the changes of variables needed for later computations. Changes of variables are a very essential topic within celestial mechanics. This is done from the point of view of the theory of Hamiltonian Systems. Many of the ideas in this chapter have been studied from the classic book on celestial mechanics by Victor Szebehely [3]. Other more simple ideas, related to the two body problem or to general basic notions on celestial mechanics, come from the book by the same author [4].

The second chapter describes Hill’s problem and the Levi-Civita regularization (as described in the work of Alessandra Celleti [7]), that allows to work arbitrarily close to a singularity in a system. For this chapter several code routines have been implemented also, to deduce and study things that would be impossible to study analytically. In particular, it has been implemented a routine that finds the angle of intersection of the invariant manifolds over a Poincaré section, allowing to check numerically the existence of chaotic behavior. The way in which these routines have been implemented is explained in detail in chapter four, the last chapter of the thesis which includes most of the programming part of it. Also, for
chapter two, it has been devised proofs of some results that are just mentioned in [1], like Proposition 2.7.1 about the existence and location of the families of Hill’s periodic orbits. This chapter constitutes a big part of all the work done during the project.

In chapter three, it is described an attempt to use averaging theory to work with Hill’s problem. Originally, the aim of this section was to check if the averaged system approximates the dynamics of Hill’s problem. The attempt did not work as expected, but the chapter is still included because averaging theory has interest by its own, and because it just gives us an excuse to study the dynamics of the averaged system, in spite it is seen that it does not represent the original system. There is also a final appendix that includes some of the general results of Hamiltonian Systems and Dynamical Systems theory that are used during the previous sections. Many other minor results are just mentioned whenever needed to apply, and are not covered in the appendix.

Weekly meetings with both of the thesis directors, Tere Martínez-Seara and Pau Martín, have happened during the project duration. I’m very thankful for the considerable amount of time they have invested and their help. Also I want to thank Mercè Ollé for the help she provided for the numerical part of the project, and to Carles Simò for replying a couple of my emails regarding his paper [1], despite he did not know me and did not know about this project either.
Chapter 1

Changes of Coordinates in the RTBP

1.1 Introduction to the RTBP

1.1.1 Restricted three-body problem in sidereal coordinates

The following is a description of the three-body problem and in particular of the restricted case. The three-body problem describes the motion of three bodies moving in the space only under the influence of their mutual gravitation. We will refer to the positions of these bodies as $\vec{q}_1$, $\vec{q}_2$, and $\vec{q}_3$, and $q_{ij}$ will be the distance from the body $i$ to the body $j$. In the restricted case, one the three bodies will be considered of negligible mass. Although the problem of course can be applied to any bodies, usually in the restricted problem it is tradition to refer to the biggest body as the Sun and to the second biggest as Jupiter. This is just an analogy of our solar system. Also one may call the two main bodies ‘the primaries’ and even sometimes, ‘the primary’ and ‘the secondary’. Another classical model is to take the primary to be the Earth, the secondary the Moon, and the massless body (we will see what this means in a moment) an artificial satellite.

One more convention is that in celestial mechanics normally, instead of the gravitational constant $G$ it is used the Gaussian gravitation constant $K$. This is defined as $K = \sqrt{M_\odot G}$ where $M_\odot$ is the mass of the Sun. But in some particular units $M_\odot = 1$, so for us $K = \sqrt{G}$. In these units, the gravitational potential can be written as

$$V = K^2 \left( \frac{m_1 m_2}{q_{12}} + \frac{m_2 m_3}{q_{23}} + \frac{m_1 m_3}{q_{31}} \right)$$
and sometimes it is called the self-potential not to confuse it with another potential that will appear later in the Hamiltonian. And we have the equations of motion

\[
\begin{align*}
\dddot{q}_1 &= K^2 \frac{m_2}{q_{12}^3} (\vec{q}_2 - \vec{q}_1) + K^2 \frac{m_3}{q_{13}^3} (\vec{q}_3 - \vec{q}_1) \\
\dddot{q}_2 &= K^2 \frac{m_1}{q_{12}^3} (\vec{q}_1 - \vec{q}_2) + K^2 \frac{m_3}{q_{23}^3} (\vec{q}_3 - \vec{q}_2) \\
\dddot{q}_3 &= K^2 \frac{m_1}{q_{13}^3} (\vec{q}_1 - \vec{q}_3) + K^2 \frac{m_2}{q_{23}^3} (\vec{q}_2 - \vec{q}_3).
\end{align*}
\] (1.1)

Note that for deducing these equations we just need to apply Newton’s law of universal gravitation and Newton’s second law.

These are three times three equations of second order so it is a system of 18th order. It is known that there are ten first integrals only (refer to [9]), so the problem is not integrable. For the restricted problem, what we do is to consider that one of the masses, say $m_3$, is infinitesimal, and call it the massless or infinitesimal particle. Under this assumption, the two first equations become,

\[
\begin{align*}
\dddot{q}_1 &= K^2 \frac{m_2}{||\vec{q}_2 - \vec{q}_1||^3} (\vec{q}_2 - \vec{q}_1) \\
\dddot{q}_2 &= K^2 \frac{m_1}{||\vec{q}_2 - \vec{q}_1||^3} (\vec{q}_1 - \vec{q}_2)
\end{align*}
\] (1.2)

while the third equation remains as it is. That is, the two first equations correspond to the two body problem for the primaries. Therefore these two bodies describe, as it is known, Keplerian orbits. We will consider that they describe a circular orbit.

The important thing to note is that the first two equations have been uncoupled. But some comment must be done on the third equation. We are assuming $m_3$ to be sufficiently small to uncouple the equations (it has no effect on $m_1$ and $m_2$) but still it is not null. This means that the third equation actually is exact while the other two equations are approximated.

We are neglecting the effect of the third body over the Sun and Jupiter or the primaries. Also note that the degree of approximation is given by the smallness of

$$\frac{m_3}{q_{13}^3} (\vec{q}_3 - \vec{q}_1) \quad \text{compared to} \quad \frac{m_2}{q_{12}^3} (\vec{q}_2 - \vec{q}_1)$$

and

$$\frac{m_3}{q_{23}^3} (\vec{q}_3 - \vec{q}_2) \quad \text{compared to} \quad \frac{m_1}{q_{12}^3} (\vec{q}_1 - \vec{q}_2).$$

So what makes ‘good’ or ‘bad’ the approximation is not only the smallness of $m_3$, but also if it is close enough to $m_1$ and $m_2$. With this simplification we may solve the first two equations, and plug the solution into the third equation. The resultant equation constitutes the restricted three-body problem. But this problem is not integrable either (but it
can be studied numerically.

The third equation describes a system of two and a half or three and a half degrees of freedom (considering also time), depending on the dimension of the space where the motion happens (two or three). Assuming that \( m_1 \) and \( m_2 \) perform a circular orbit and that the motion of \( m_3 \) occurs in the plane of rotation of the other two bodies, then the problem is known as ‘the restricted planar and circular three-body problem’ (RTBP). This is maybe the most studied approximation. When \( m_1 \) and \( m_2 \) move in elliptic motion then we have the elliptic three-body problem which is also studied. The elliptic problem is also called ‘pseudo-restricted’ problem, and also sometimes is considered the three dimensional case (non-planar).

So the equation,

\[
\ddot{q}_3 = K^2 m_1 \frac{q_1}{q_{13}^2} (\bar{q}_1 - \bar{q}_3) + K^2 m_2 \frac{q_2}{q_{23}^2} (\bar{q}_2 - \bar{q}_3) \tag{1.3}
\]

and the assumption that the two primaries perform circular orbits around each other constitutes the RTBP in sidereal coordinates (inertial rectangular coordinates).

For convenience we separate the variables, \( \bar{q}_3 = (X, Y) \)

\[
\begin{align*}
\ddot{X} &= -K^2 m_1 \frac{X-X_1}{R_1^2} - K^2 m_2 \frac{X-X_2}{R_2^2} \\
\ddot{Y} &= -K^2 m_1 \frac{Y-Y_1}{R_1^2} - K^2 m_2 \frac{Y-Y_2}{R_2^2}
\end{align*}
\tag{1.4}
\]

where

\[
\begin{align*}
R_1 &= \left[ (X - X_1)^2 + (Y - Y_1)^2 \right]^{1/2} \\
R_2 &= \left[ (X - X_2)^2 + (Y - Y_2)^2 \right]^{1/2}
\end{align*}
\tag{1.5}
\]

We want also to express these equations in dimensionless coordinates.

Considering \( M = m_1 + m_2 \), \( l \) the distance between the primaries, and \( n \) the angular velocity of the two primaries revolving around each other (what is known in celestial mechanics as the mean motion), the first thing we do is to equate the gravitational and centripetal forces over the primaries to obtain,

\[
K^2 \frac{m_1 m_2}{l^2} = m_1 \rho_1 n^2 = m_2 \rho_2 n^2 \tag{1.6}
\]

where \( \rho_{1,2} \) are the radii of rotation of each mass and \( l = \rho_1 + \rho_2 \). This can be done because, as we said, for the two primaries we have the two body problem. From (1.6) we obtain the
expressions
\[
\begin{align*}
K^2m_1 &= \rho_2n^2l^2 \\
K^2m_2 &= \rho_1n^2l^2
\end{align*}
\] (1.7)
and by adding them both,
\[
K^2(m_1 + m_2) = n^2l^3.
\] (1.8)
This is a form of Kepler’s third law. And from these expressions we obtain also,
\[
\rho_1 = \frac{m_2}{m_1}l, \quad \rho_2 = \frac{m_1}{m_1}l.
\]
Applying the following changes of variables
\[
\xi = \frac{X}{l}, \quad \eta = \frac{Y}{l}, \quad \tau = nt, \quad \mu_1 = \frac{m_1}{M}, \quad \mu_2 = \frac{m_2}{M},
\]
the positions of the primaries are
\[
\begin{align*}
(\xi_1, \eta_1) &= (\mu_2 \cos nt, \mu_2 \sin nt) \\
(\xi_2, \eta_2) &= (-\mu_1 \cos nt, -\mu_1 \sin nt)
\end{align*}
\]
and the equations become,
\[
\begin{align*}
\ddot{\xi} &= -\mu_1 \frac{\xi - \mu_2 \cos t}{d_1^2} - \mu_2 \frac{\xi + \mu_1 \cos t}{d_2^2} \\
\ddot{\eta} &= -\mu_1 \frac{\eta - \mu_2 \sin t}{d_1^2} - \mu_2 \frac{\eta + \mu_1 \sin t}{d_2^2}
\end{align*}
\] (1.9)
where
\[
\begin{align*}
d_1^2 &= (\xi - \mu_2 \cos t)^2 + (\eta - \mu_2 \sin t)^2 \\
d_2^2 &= (\xi + \mu_1 \cos t)^2 + (\eta + \mu_1 \sin t)^2.
\end{align*}
\] (1.10)
In the three-body problem of course the total energy is preserved (kinetic energy + potential energy). But in the restricted case, as we are neglecting the influence of the infinitesimal body over the primaries, it is not. In the next sections we will find how to write the problem in such a way that we can find a first integral. This could be considered
as defining a new energy that is not equivalent to the sum of potential and kinetic energy. And that will define at the end a Hamiltonian system.

### 1.1.2 Restricted three-body problem in synodical coordinates

Now, from equations (1.4), the system will be transformed to a rotating reference frame, what is known as synodical coordinates. We want to pass from the coordinates $X$ and $Y$ of (1.4) to new coordinates $x$ and $y$ that rotate along with $m_1$ and $m_2$ (primaries). Observe that we have to replace by

$$
\begin{align*}
X_1 &= \rho_1 \cos nt \\
Y_1 &= \rho_1 \sin nt
\end{align*}
$$

and

$$
\begin{align*}
X_2 &= -\rho_2 \cos nt \\
Y_2 &= -\rho_2 \sin nt
\end{align*}
$$

so

$$
\begin{align*}
\dot{X} &= -K^2 m_1 \frac{X - \rho_1 \cos nt}{R_1} - K^2 m_2 \frac{X + \rho_2 \cos nt}{R_2} \\
\dot{Y} &= -K^2 m_1 \frac{Y - \rho_1 \sin nt}{R_1} - K^2 m_2 \frac{Y + \rho_2 \sin nt}{R_2}.
\end{align*}
$$

Considering complex numbers the computations are easier. For the old variables we write $Z = X + iY$ and for the new $z = x + iy$. And $Z = z e^{nti}$, $Z_1 = \rho_1 e^{nti}$, $Z_2 = -\rho_2 e^{nti}$. We can compute,

$$
R_1 = [(X - X_1)^2 + (Y - Y_1)^2]^{1/2} = |ze^{nti} - \rho_1 e^{nti}| = |z - \rho_1| = [(x - \rho_1)^2 + y^2]^{1/2}
$$

$$
R_2 = [(X - X_2)^2 + (Y - Y_2)^2]^{1/2} = |z e^{nti} + \rho_2 e^{nti}| = |z + \rho_2| = [(x + \rho_2)^2 + y^2]^{1/2}.
$$

And also we can compute the left hand side of the equation in the new variables,

$$
\dot{Z} = \dot{z} e^{nti} + i n z e^{nti} \Rightarrow \ddot{Z} = \ddot{z} e^{nti} + i n \dot{z} e^{nti} + i n z \ddot{e}^{nti} + (in)^2 ze^{nti} = 
$$

$$
= (\ddot{z} + 2in \dot{z} - n^2 z)e^{nti},
$$

and the right hand side just by substitution

$$
-K^2 \left[ m_1 \frac{z - \rho_1}{|z - \rho_1|} + m_2 \frac{z + \rho_2}{|z + \rho_2|} \right].
$$

Putting all together it has been obtained

$$
\ddot{z} + 2in \dot{z} - n^2 z = -K^2 \left[ m_1 \frac{z - \rho_1}{|z - \rho_1|} + m_2 \frac{z + \rho_2}{|z + \rho_2|} \right].
$$

(1.13)
and just it is left to split this expression into its real and imaginary parts.

\[
\begin{cases}
\ddot{x} - 2n\dot{y} - n^2 x = -K^2 \left[m_1 \frac{x - \rho_1}{r_1^3} + m_2 \frac{x + \rho_2}{r_2^3} \right] \\
\ddot{y} + 2n\dot{x} - n^2 y = -K^2 \left[m_1 \frac{y}{r_1^3} + m_2 \frac{y}{r_2^3} \right]
\end{cases}
\] (1.14)

or

\[
\begin{cases}
\ddot{x} - 2n\dot{y} - n^2 x = -K^2 \left[m_1 \frac{x - \rho_1}{r_1^3} + m_2 \frac{x + \rho_2}{r_2^3} \right] \\
\ddot{y} + 2n\dot{x} - n^2 y = -K^2 \left[m_1 \frac{y}{r_1^3} + m_2 \frac{y}{r_2^3} \right]
\end{cases}
\] (1.15)

Note that a derivative of order one that was not before has appeared, so this may look like a bad change of coordinates. This is because the new coordinates are not inertial (coriolis forces appear). But in a moment we will see that we have got a first integral.

Defining

\[ F = \frac{n^2}{2} (x^2 + y^2) + K^2 \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right) \]

and differentiating

\[
\begin{align*}
\frac{\partial F}{\partial x} &= n^2 x - K^2 \left[m_1 \frac{x - \rho_1}{r_1^3} + m_2 \frac{x + \rho_2}{r_2^3} \right] \\
\frac{\partial F}{\partial y} &= n^2 y - K^2 \left[m_1 \frac{y}{r_1^3} + m_2 \frac{y}{r_2^3} \right]
\end{align*}
\]

we may write the system in a simplified way as

\[
\begin{cases}
\ddot{x} - 2n\dot{y} = \frac{\partial F}{\partial x} \\
\ddot{y} + 2n\dot{x} = \frac{\partial F}{\partial y}
\end{cases}
\] (1.16)

This is the system in synodical coordinates. The good thing is that now the system does not depend on time, and it is easy to find a first integral.

From (1.16), multiply the first equation times \( \dot{x} \) and the second times \( \dot{y} \), add both to get

\[
\dot{x} \ddot{x} + \dot{y} \ddot{y} = \frac{\partial F}{\partial x} \dot{x} + \frac{\partial F}{\partial y} \dot{y} = \frac{dF}{dt},
\]

integrating

\[
\frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 = F - C^* \]

where \( C^* \) is an arbitrary constant. This is known as the Jacobi integral. These equations
are also usually converted to their dimensionless counterpart. The changes to proceed are the same as before,

\[
\begin{align*}
\xi &= \frac{x}{l}, \quad \eta = \frac{y}{l}, \quad \tau = nt \\
r_1 &= \frac{R_1}{l}, \quad r_2 = \frac{R_2}{l}, \quad \mu_1 = \frac{m_1}{M}, \quad \mu_2 = \frac{m_2}{M},
\end{align*}
\]

to obtain

\[
\begin{cases}
\ddot{\xi} - 2\dot{\eta} = \bar{\Omega}\xi \\
\ddot{\eta} + 2\dot{\xi} = \bar{\Omega}\eta.
\end{cases}
\]

(1.17)

and where we have also applied Kepler’s third law (1.8), and defined

\[
\bar{\Omega} = \frac{1}{2}(\xi^2 + \eta^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}
\]

that has been obtained with \( \bar{\Omega} = \frac{F}{r_{\text{m}}^2} \), also

\[
\begin{cases}
r_1^2 = (\xi - \mu_2)^2 + \eta^2 \\
r_2^2 = (\xi + \mu_1)^2 + \eta^2.
\end{cases}
\]

(1.18)

Observe that if \( \tilde{q} = (\xi, \eta) \), now the Jacobi integral is

\[
\frac{1}{2}\|\dot{\tilde{q}}\|^2 - \frac{1}{2}\|\tilde{q}\|^2 - \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} = -\bar{C}.
\]

Another convention is to add to \( \bar{\Omega} \) a constant to make the expression more symmetric,

\[
\Omega = \bar{\Omega} + \frac{1}{2}\mu_1\mu_2
\]

so

\[
\Omega = \frac{1}{2}(\xi^2 + \eta^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} + \frac{1}{2}\mu_1\mu_2 = \frac{1}{2}\left[\mu_1 r_1^2 + \mu_2 r_2^2 + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right].
\]

Differentiating,

\[
\Omega_\xi = \xi - \frac{\mu_1(\xi - \mu_2)}{r_1^3} - \frac{\mu_2(\xi + \mu_2)}{r_2^3}
\]

\[
\Omega_\eta = \eta \left[1 - \frac{\mu_1}{r_1^3} - \frac{\mu_2}{r_2^3}\right].
\]
The system is now

\[
\begin{align*}
\ddot{\xi} - 2\dot{\eta} &= \Omega \xi \\
\ddot{\eta} + 2\dot{\xi} &= \Omega \eta
\end{align*}
\] (1.19)

or

\[
\begin{align*}
\ddot{\xi} - 2\dot{\eta} &= \xi - \frac{\mu_1(\xi - \mu_2)}{r_1^2} - \frac{\mu_2(\xi + \mu_2)}{r_2^2} \\
\ddot{\eta} + 2\dot{\xi} &= \eta \left[ 1 - \frac{\mu_1}{r_1^2} - \frac{\mu_2}{r_2^2} \right].
\end{align*}
\] (1.20)

With all these changes the new Jacobi\(^1\) constant is

\[
C = \frac{C^*}{\mu_1 \mu_2} + \frac{1}{2} \mu_1 \mu_2.\]

The equations still depend on two parameters. But as \(\mu_1 + \mu_2 = 1\), we can take \(\mu_1 = 1 - \mu\) and \(\mu_2 = \mu\) with \(\mu < \frac{1}{2}\). By doing so the mass on the right of the center of gravity (\(\mu_1\)) is the biggest. It is just another convention more, so we may refer to this mass as the primary or the Sun and to the other as the secondary or Jupiter. And using the expressions for the radii (1.18), it turns out that the position for \(\mu_1\) is \((\mu, 0)\) and for \(\mu_2\) is \((\mu - 1, 0)\).

Equations (1.20) can easily be converted to a system of first order. By taking \(\gamma = \dot{\xi}\) and \(\delta = \dot{\eta}\)

\[
\begin{align*}
\dot{\xi} &= \gamma \\
\dot{\eta} &= \delta. \\
\dot{\gamma} &= 2\delta + \Omega \xi \\
\dot{\delta} &= -2\gamma + \Omega \eta
\end{align*}
\] (1.21)

Dropping all the changes of name of the variables and rewriting the equations with \(x\) and \(y\),

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= \Omega x \\
\ddot{y} + 2\dot{x} &= \Omega y
\end{align*}
\]

\[
\begin{align*}
\ddot{x}_1 &= x_3 \\
\ddot{x}_2 &= x_4 \\
\ddot{x}_3 &= 2x_4 + \Omega x_1 \\
\ddot{x}_4 &= -2x_3 + \Omega x_2 \\
\text{with } x_1 &= x, x_2 = y, x_3 = \dot{x}, x_4 = \dot{y}
\end{align*}
\] (1.22)

But these are not the equations of motion of a Hamiltonian system yet.

\(^1C = -\dot{\xi}^2 - \dot{\eta}^2 + 2\Omega(\xi, \eta)\)
1.2 The Restricted Problem as a Hamiltonian System

The restricted three body problem in synodical coordinates is not a Hamiltonian system. However, it can be done a change of coordinates in (1.22) to transform it into a Hamiltonian system. The change is

\[
\begin{align*}
    x_1 &= x_1 \\
    x_2 &= x_2 \\
    y_1 &= x_3 - x_2 \\
    y_2 &= x_4 + x_1
\end{align*}
\]  

(1.23)

and the equations become

\[
\begin{align*}
    \dot{x}_1 &= y_1 + x_2 \\
    \dot{x}_2 &= y_2 - x_1 \\
    \dot{y}_1 &= y_2 - x_1 + \Omega_{x_1} \\
    \dot{y}_2 &= -x_2 - y_1 + \Omega_{x_2}
\end{align*}
\]  

(1.24)

with

\[
\Omega(x_1, x_2) = \frac{1}{2} \left( x_1^2 + x_2^2 \right) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2} \mu(1 - \mu) 
\]  

(1.25)

and for which can be found the Hamiltonian,

\[
H(x_1, x_2, y_1, y_2) = \frac{1}{2} \left( y_1^2 + y_2^2 \right) + x_2y_1 - x_1y_2 - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} - \frac{1}{2} \mu(1 - \mu).
\]  

(1.26)

The Hamiltonian is a first integral of the system and it can be seen that it is easily related to the Jacobi integral. Rewriting the Hamiltonian in the non-canonical variables,

\[
H(x_1, x_2, x_3, x_4) = \frac{1}{2} \left( x_1^2 + x_2^2 \right) + \frac{1}{2} \left( -2x_2x_3 + x_2^2 + 2x_1x_4 + x_1^2 \right) +
\]

\[
+ x_2x_2 - x_3^2 - x_1x_4 - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} - \frac{1}{2} \mu(1 - \mu) =
\]

\[
= \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} (x_1^2 + x_2^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} - \frac{1}{2} \mu(1 - \mu) =
\]

\[
= \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) - \Omega(x_1, x_2) = \frac{-C}{2}.
\]

So \( C = -2H \).
1.3 Canonical Changes of Variables

1.3.1 Generating Functions

In many situations, it is more relevant the phase space also called 'the space of states of energy' than the configuration space. The n-dimensional space of configurations corresponds to the position of points in n coordinates in the dynamical system. We will have to add n more coordinates that correspond to the momenta of inertia. Points in the phase space do not correspond to configurations but to states of the system. The equations of motion are of second order, therefore specifying only the configurations is not enough as an initial condition set. A configuration only determines the state of the system at a given time. But to know its future states it is also needed the momentum, since for this purpose 2n initial conditions will be needed. Note that at a given point in the configuration space, many orbits originate and this does not violate the uniqueness of solutions even in the autonomous case.

If \((q_i, p_i)\) for \(i = 1, \ldots, n\) are conjugated variables for a Hamiltonian \(H\), we say that they are canonical coordinates. We are interested in transformations that will take us from canonical coordinates to canonical coordinates. In general we will have transformations like

\[
Q_1 = Q_1(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \\
\vdots \\
Q_n = Q_n(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \\
P_1 = Q_n(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \\
\vdots \\
P_n = Q_n(q_1, \ldots, q_n, p_1, \ldots, p_n, t)
\]

and we want that after the transformation,

\[
\dot{Q}_i = \frac{\partial \tilde{H}}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q_i}
\]

for a new Hamiltonian \(\tilde{H}\) that is also to be found.

Hamilton’s principle of least action states that the development in time for a mechanical system is such that, the integral of the difference between the kinetic and the potential
energy (the Lagrangian), is stationary. Mathematically

$$\delta \int_{t_1}^{t_2} \mathcal{L} \, dt = 0$$

where $\mathcal{L}$ is the Lagrangian of the system. And the Lagrangian is related to the Hamiltonian with the expression $H = \dot{q}_i p_i - \mathcal{L}$, using Einstein’s summation convention.

By Hamilton’s principle,

$$\delta \int_{t_1}^{t_2} (\dot{q}_i p_i - H) \, dt = 0 \quad \text{and we want} \quad \delta \int_{t_1}^{t_2} (\dot{Q}_i P_i - \tilde{H}) \, dt = 0.$$ 

Doing the difference of both expressions

$$\delta \int_{t_1}^{t_2} (\dot{q}_i p_i - \dot{Q}_i P_i - H + \tilde{H}) \, dt = 0$$

and introducing a function $W$ such that

$$\frac{dW}{dt} = \dot{q}_i p_i - \dot{Q}_i P_i - H + \tilde{H} \quad \text{(1.27)}$$

we have

$$\delta \int_{t_1}^{t_2} \frac{dW}{dt} \, dt = \delta [W(t_2) - W(t_1)] = 0.$$ 

Equation (1.27) normally is written in form of differentials

$$p_i dq_i - P_i dQ_i = dW + (H - \tilde{H}) \, dt \quad \text{(1.28)}$$

and $W$ is the so called ‘generating function’ and depends on $2n$ variables plus time. This means that it can be defined in four ways,

$$W_1 = W_1(q, Q, t)$$

$$W_2 = W_2(q, P, t)$$

$$W_3 = W_3(p, Q, t)$$

$$W_4 = W_4(p, P, t)$$

each one being a different type of generating function.
Canonical transformation of type II: $W_2(q, P, t)$:

$$dW_2 = \frac{\partial W_2}{\partial q_i} dq_i + \frac{\partial W_2}{\partial P_i} dP_i + \frac{\partial W_2}{\partial t} dt$$

and plugging this expression into (1.28)

$$p_i dq_i - P_i dQ_i = \frac{\partial W_2}{\partial q_i} dq_i + \frac{\partial W_2}{\partial P_i} dP_i + \left( \frac{\partial W_2}{\partial t} + H - \tilde{H} \right) dt$$

choosing

$$\frac{\partial W_2}{\partial q_i} = p_i \text{ and } \frac{\partial W_2}{\partial P_i} = Q_i$$

we get

$$0 = d(P_i Q_i) + \left( \frac{\partial W_2}{\partial t} + H - \tilde{H} \right) dt$$

or

$$\tilde{H} = H + \frac{\partial W_2}{\partial t} - \frac{d(Q_i P_i)}{dt} = H + \frac{\partial W_2}{\partial t}.$$

Here we find the problem that we don’t know how to get rid of the term $\frac{d(Q_i P_i)}{dt}$. But observe that if instead $W_2$ we use $\tilde{W}_2 = W_2 - Q_i P_i$, the integral deduced from Hamilton’s principle is equally satisfied,

$$\delta \int_{t_1}^{t_2} dW_2 / dt dt = \delta \int_{t_1}^{t_2} d\tilde{W}_2 / dt dt + \delta (Q_i P_i) \big|_{t_1}^{t_2} = \delta \int_{t_1}^{t_2} d\tilde{W}_2 / dt dt.$$

This means that if the integral above is zero for $\tilde{W}_2$, then it is also zero for $W_2$. $\tilde{W}_2$ is known as the generator of the generating function. In fact we can reach to the same conclusion but noticing that

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H) dt = 0 \iff \delta \int_{t_1}^{t_2} (-\dot{p}_i q_i - H) dt = 0$$

and using the second integral, that now does not correspond to Hamilton principle of least action because we are not integrating the Lagrangian, to find the generating function.

We have

$$\begin{cases} \frac{\partial W_2}{\partial q_i} = p_i \\ \frac{\partial W_2}{\partial P_i} = Q_i \\ \tilde{H} = H + \frac{\partial W_2}{\partial t} \end{cases}$$

(1.29)

In Section 1.3.2 it will be shown how to use the generating functions of type 3 with a
1.3 Canonical Changes of Variables

practical example. The deductions for the other types are all similar and lead to the following table (see [3]). The easiest case is type I, because the generator and the generating function coincide.

<table>
<thead>
<tr>
<th>G. Function</th>
<th>( q_i )</th>
<th>( p_1 )</th>
<th>( Q_i )</th>
<th>( P_i )</th>
<th>( H - \dot{H} )</th>
<th>Generator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_1(q_i, Q_i, t) )</td>
<td>( \frac{\partial W_1}{\partial q_i} )</td>
<td>( \frac{\partial W_1}{\partial Q_i} )</td>
<td>( -\frac{\partial W_1}{\partial t} )</td>
<td>( \frac{\partial W_1}{\partial t} )</td>
<td>( W_1 = W_1 )</td>
<td></td>
</tr>
<tr>
<td>( W_2(q_i, P_i, t) )</td>
<td>( \frac{\partial W_2}{\partial q_i} )</td>
<td>( \frac{\partial W_2}{\partial P_i} )</td>
<td>( -\frac{\partial W_2}{\partial t} )</td>
<td>( \frac{\partial W_2}{\partial t} )</td>
<td>( W_2 = W_2 - Q_i P_i )</td>
<td></td>
</tr>
<tr>
<td>( W_3(p_i, Q_i, t) )</td>
<td>( -\frac{\partial W_3}{\partial p_i} )</td>
<td>( \frac{\partial W_3}{\partial Q_i} )</td>
<td>( -\frac{\partial W_3}{\partial t} )</td>
<td>( \frac{\partial W_3}{\partial t} )</td>
<td>( W_3 = W_3 + q_i p_i )</td>
<td></td>
</tr>
<tr>
<td>( W_4(p_i, P_i, t) )</td>
<td>( -\frac{\partial W_4}{\partial p_i} )</td>
<td>( \frac{\partial W_4}{\partial P_i} )</td>
<td>( -\frac{\partial W_4}{\partial t} )</td>
<td>( \frac{\partial W_4}{\partial t} )</td>
<td>( W_4 = W_4 - Q_i P_i + q_i p_i )</td>
<td></td>
</tr>
</tbody>
</table>

Table 1.1 Generating functions.

1.3.2 Polar Canonical Coordinates

If we want to transform a Hamiltonian \( H(q_1, q_2, p_1, p_2) \) system to polar coordinates

\[
q_1 = r \cos \theta, \quad q_2 = r \sin \theta
\]

with a canonical change of variables, we need to establish the conjugated variables of \( r, \theta \) (say \( R, \Theta \)). The following generating function, which is of type 3, can be used for that purpose.

\[
W_3(p_1, p_2, r, \theta) = -p_1 r \cos \theta - p_2 r \sin \theta \tag{1.30}
\]

And we have that, in order that Hamilton’s principle is satisfied, it is needed that

\[
p_i \dot{q}_i - P_i \dot{Q}_i - H + \dot{H} = \frac{dW_3}{dt} \tag{1.31}
\]

As

\[
\frac{dW_3}{dt} = \frac{\partial W_3}{\partial p_1} \frac{dp_1}{dt} + \frac{\partial W_3}{\partial p_2} \frac{dp_2}{dt} + \frac{\partial W_3}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial W_3}{\partial r} \frac{dr}{dt} + \frac{\partial W_3}{\partial t} \tag{1.32}
\]

taking

\[
-\frac{\partial W_3}{\partial p_1} = q_1, \quad -\frac{\partial W_3}{\partial p_2} = q_2
\]

and

\[
-\frac{\partial W_3}{\partial r} = R, \quad -\frac{\partial W_3}{\partial \theta} = \Theta
\]
from (1.31) and (1.32) it follows
\[(p_1q_1)' + (p_2q_2)' - H + \dot{H} = 0.\]

It looks that we don’t know how to kill the first two terms to establish a relationship between the old and the new Hamiltonian. But note that if we take as generator
\[\tilde{W}_3 = W_3 + p_1q_1 + p_2q_2\]
the integral derived from the principle of Hamilton is equally satisfied
\[
\delta \int_{t_1}^{t_2} \frac{d\tilde{W}_3}{dt} dt = \delta \int_{t_1}^{t_2} \left\{ \frac{dW_3}{dt} + (p_iq_i) \right\} dt = \delta \int_{t_1}^{t_2} \frac{dW_3}{dt} dt + \delta (p_iq_i)|_{t_1}^{t_2} = \delta \int_{t_1}^{t_2} \frac{dW_3}{dt} dt =
\]
\[= \delta [W_3(t_2) - W_3(t_1)] = 0.\]

Because \(\frac{\partial W_3}{\partial t} = 0\), we see that to change from the old Hamiltonian to the new one, what is needed is just to replace the variables \((H = \tilde{H})\). And we can obtain the full change to the new variables by differentiating \(W_3\):

\[
\begin{align*}
q_1 &= r \cos \theta \\
q_2 &= r \sin \theta \\
R &= p_1 \cos \theta + p_2 \sin \theta = \frac{q_1 p_1 + q_2 p_2}{r} \\
\Theta &= -p_1 r \sin \theta + p_2 r \cos \theta = q_1 p_2 - q_2 p_1.
\end{align*}
\]

These new variables \(R\) and \(\Theta\), introduced by the canonical change of coordinates, have a physical meaning when in the original variables the conjugated variables are related like \(p_1 = \dot{q}_1\) and \(p_2 = \dot{q}_2\). To see this, note that in this case,

\[
R = \frac{q_1 \dot{q}_1 + q_2 \dot{q}_2}{r} = \frac{d(q_1^2 + q_2^2)}{dt} \frac{1}{2r} = \frac{dr^2}{dt} \frac{1}{2r} = \frac{2r \dot{r}}{2r} = \dot{r}.
\]

So \(R\) is proportional to the linear momentum in the \(r\) direction. Also

\[
\Theta = q_1 \dot{q}_2 - q_2 \dot{q}_1 = r \cos \theta \left( \dot{r} \sin \theta + r \cos \theta \dot{\theta} \right) - r \sin \theta \left( \dot{r} \cos \theta - r \sin \theta \dot{\theta} \right) =
\]
\[= r^2 \left( \cos^2 \theta \dot{\theta} + \sin^2 \theta \dot{\theta} \right) = r^2 \dot{\theta}.
\]

So \(\Theta\) is proportional to the angular momentum.
1.4 Equilibria of the RTBP

Consider the equations of the RTBP in synodical coordinates,

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= \Omega_x \\
\dot{y} + 2\dot{x} &= \Omega_y 
\end{align*}
\]

(1.34)

where

\[
\Omega(x, y) = \frac{x^2 + y^2}{2} + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2} \mu(1 - \mu),
\]

and \( r_1 = \sqrt{(x - \mu)^2 + y^2} \), \( r_2 = \sqrt{(x - \mu + 1)^2 + y^2} \).

If we write them as a first order system we can find expressions for the equilibria of the system. We have

\[
\begin{align*}
\dot{y}_1 &= y_3 \\
\dot{y}_2 &= y_4 \\
\dot{y}_3 &= 2y_4 + \Omega_x \\
\dot{y}_4 &= -2y_3 + \Omega_y
\end{align*}
\]

(1.35)

where \( y_1 = x, y_2 = y, y_3 = \dot{x} \) and \( y_4 = \dot{y} \).

There are two types of equilibria, the collinear equilibria and the triangular equilateral equilibria. For the equilateral equilibria, closed expressions are found easily. These points are found by assuming that \( y_2 \neq 0 \). From equations (1.35), we see that any equilibria must satisfy \( \Omega_x = 0 \) and \( \Omega_y = 0 \). First we compute the derivatives of \( \Omega \):

\[
\begin{align*}
\Omega_x &= x - \frac{(1-\mu)(x-\mu)}{r_1} - \frac{\mu(x+1-\mu)}{r_2} = x(1 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}) - \frac{\mu(1-\mu)}{r_1} - \frac{\mu(1-\mu)}{r_2} \\
\Omega_y &= y(1 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2})
\end{align*}
\]

(1.36)

And from \( \Omega_y = 0 \) and \( y_2 \neq 0 \), it follows that \( 1 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} = 0 \) must be satisfied. Replacing into \( \Omega_x = 0 \) we obtain that \( r_1 = r_2 \) must be satisfied also in order to have zeroes. Now solving

\[
\begin{align*}
1 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} &= 0 \\
r_1 &= r_2
\end{align*}
\]

gives \( r_1 = r_2 = 1 \). For this reason they are called equilateral equilibria, because they form an equilateral triangle with the two primaries. And its computation is trivial now. There
are two (called $L_4$ and $L_5$) with positions

$$\begin{align*}
L_4 &= \left(\mu - \frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0\right) \\
L_5 &= \left(\mu - \frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, 0\right).
\end{align*}$$

(1.37)

On the other side, for the collinear equilibria ($y_2 = 0$) there are not closed expressions because its computation leads to a quintic equation (Euler’s quintic). Here we will reproduce the way in which $L_1$ (the collinear in between the primaries) is found numerically. The other collinear points, $L_2$ and $L_3$, can be found analogously.

Writing $y_1 = \mu - 1 + \xi$ we will search for $\xi$. Now $y_2 = 0$ so $\Omega_y = 0$ is trivially satisfied. And from $\Omega_x = 0$ the following quintic is obtained,

$$p(\xi) = \xi^5 - (3 - \mu)\xi^4 + (3 - 2\mu)\xi^3 - \mu\xi^2 + 2\mu\xi - \mu = 0.$$  

(1.38)

Obviously, it has at least one real zero because the order is odd. But indeed it can be verified that there are not more real zeros. We do it with the following argument. First check that if we replace into the quintic $\xi$ by $-\xi$, Descarte’s rule of signs shows that there are not negative roots. So all roots must be positive. Now check that $p(0) = -\mu$. This means that if we prove that for $\xi > 0$ and $\mu \in (0, \frac{1}{2})$ gives $p'(\xi) > 0$, then there exists an unique root. So we need

$$5\xi^4 - 4(3 - \mu)\xi^3 + 3(3 - 2\mu)\xi^2 - 2\mu\xi + 2\mu > 0.$$  

And this can be done in the following way. Note that the AM-GM inequality implies $\frac{1}{2}\mu\xi^2 + 2\mu \geq 2\mu\xi$. So it suffices to prove that

$$5\xi^4 - 4(3 - \mu)\xi^3 + 3(3 - 2\mu)\xi^2 - \frac{1}{2}2\mu\xi^2 > 0$$

or

$$5\xi^4 + \frac{1}{2}(18 - 13\mu)\xi^2 > 4(3 - \mu)\xi^3.$$  

Using again the AM-GM inequality, observe that

$$5\xi^4 + \frac{1}{2}(18 - 13\mu)\xi^2 \geq \sqrt{10(18 - 13\mu)}\xi^3 = 2\sqrt{\frac{5(18 - 13\mu)}{2}}\xi^3.$$  

So if $2\sqrt{\frac{5(18 - 13\mu)}{2}}\xi^3 > 4(3 - \mu)\xi^3$ is satisfied, then $p'(\xi) > 0$ for $\xi > 0$. And this is true;
1.4 Equilibria of the RTBP

\[ 2\sqrt{\frac{5(18 - 13\mu)}{2}}\xi^3 > 4\mu(3 - \mu)\xi^3 \iff 5(18 - 13\mu) > 8\mu(3 - \mu)^2 \]
\[ \iff 8\mu^2 + 17\mu - 18 < 0. \]

The last expression can be verified easily by considering \( \mu \in (0, \frac{1}{2}) \).

And now, to find the zero according to the parameter \( \mu \) compute

\[ \xi^5 - (3 - \mu)\xi^4 + (3 - 2\mu)\xi^3 = \mu - 2\mu\xi + \mu\xi^2 \]
\[ \xi^3(\xi^2 - (3 - \mu)\xi + (3 - 2\mu)) = \mu - 2\mu\xi + \mu\xi^2 \]
\[ \xi^3 = \frac{\mu - 2\mu\xi + \mu\xi^2}{\xi^2 - (3 - \mu)\xi + (3 - 2\mu)} = \frac{\mu(1 - \xi)^2}{3 - 2\mu - \xi(3 - \mu - \xi)}. \]

So the root can be found with the iterative procedure

\[ \xi_{k+1} = \left[ \frac{\mu(1 - \xi_k)^2}{3 - 2\mu - \xi_k(3 - \mu - \xi_k)} \right]^{\frac{1}{3}} \quad (1.39) \]

and with the initial value \( \xi_0 = \left[ \frac{\mu}{3(1 - \mu)} \right]^{\frac{1}{3}} \). This value corresponds to \( \xi_{k+1} \) taking \( \xi_k = 0 \).

For this value it is known that converges.

For \( L_1 \) and \( L_2 \), the proof of the existence of an unique root is easier, as it can be used Descarte’s rule directly. The fixed point formulas for these three points are:

<table>
<thead>
<tr>
<th>point</th>
<th>formula</th>
<th>initial value</th>
<th>( x_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_1 )</td>
<td>( \xi_{k+1} = \left[ \frac{\mu(1 - \xi)^2}{3 - 2\mu - \xi(3 - \mu - \xi)} \right]^{\frac{1}{3}} )</td>
<td>( \xi_0 = \left[ \frac{\mu}{3(1 - \mu)} \right] )</td>
<td>( \mu - 1 + \xi )</td>
</tr>
<tr>
<td>( L_2 )</td>
<td>( \xi_{k+1} = \left[ \frac{\mu(1 + \xi)^2}{3 - 2\mu + \xi(3 - \mu + \xi)} \right]^{\frac{1}{3}} )</td>
<td>( \xi_0 = \left[ \frac{\mu}{3(1 - \mu)} \right] )</td>
<td>( \mu - 1 - \xi )</td>
</tr>
<tr>
<td>( L_3 )</td>
<td>( \xi_{k+1} = \left[ \frac{(1 - \mu)(1 + \xi)^2}{1 + 2\mu + \xi(2 + \mu + \xi)} \right]^{\frac{1}{3}} )</td>
<td>( \xi_0 = 1 - \frac{7}{12} )</td>
<td>( \mu + \xi )</td>
</tr>
</tbody>
</table>

Table 1.2 Formulas for the collinear equilibria.

To know the stability of the equilibria, we compute the differential matrix of the system:

\[ Df = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_{xx} & \Omega_{xy} & 0 & 2 \\ \Omega_{xy} & \Omega_{yy} & -2 & 0 \end{pmatrix} \quad (1.40) \]
where

\[
\begin{aligned}
\Omega_{xx} &= -(1-\mu)\frac{y^2-2(x-\mu)^2}{r_1^2} - \mu\frac{y^2-2(x+1-\mu)^2}{r_2^2} + 1 \\
\Omega_{yy} &= -(1-\mu)\frac{(x-\mu)^2-2y^2}{r_1^2} - \mu\frac{(x+1-\mu)^2-2y^2}{r_2^2} + 1 \\
\Omega_{xy} &= \frac{3}{2}(1-\mu)\frac{2(x-\mu)y}{r_1^2} - \frac{3}{2}\mu\frac{2(x+1-\mu)y}{r_2^2}
\end{aligned}
\]  

(1.41)

And the characteristic polynomial of the differential is

\[
p_c(L_i) = l^4 + (4 - \Omega_{xx}(L_i) - \Omega_{yy}(L_i))l^2 + \Omega_{xx}(L_i)\Omega_{yy}(L_i) - [\Omega_{xy}]^2 = 0.
\]

For the collinear points \(y = 0\) and therefore \(\Omega_{xy} = 0\). And also note that \(\Omega_{xx} = 0\) in the equilibria \(\frac{1-\mu}{r_1^2} - \frac{\mu}{r_2^2} + x = 0\) so

\[
\mu = x - \frac{1-\mu}{r_1^2} = \mu - 1 + r_2 + \frac{1-\mu}{r_1^2}
\]

where it has been used that \(x = \mu - 1 + r_2\). Replacing this expression into \(\Omega_{yy} \big|_{y=0}\)

\[
\Omega_{yy} \big|_{L_2} = -(1-\mu)\frac{1}{r_1^2} - \frac{1}{r_2} \left(\mu - 1 + r_2 + \frac{1-\mu}{r_1^2}\right) + 1 =
\]

\[
= -(1-\mu)\frac{1}{r_1^2} + (1-\mu)\frac{1}{r_2} - 1 + \frac{1-\mu}{r_1^2}r_2 + 1 = -(1-\mu)\left(\frac{1}{r_1^2} - \frac{1}{r_2} + \frac{1}{r_1^2r_2}\right) =
\]

\[
= -(1-\mu)\left(\frac{1}{r_1^2} - \frac{1}{r_2} + \frac{1}{r_1^2r_2}\right) = (1-\mu)\left(\frac{1}{r_2} + \frac{1}{r_1}\right) (r_1 - \frac{1}{r_2}).
\]

So as \(r_1 < 1\) then \(\Omega_{yy} \big|_{y=0} < 0\).

Defining now

\[
\begin{aligned}
\beta_1 &= 2 - \frac{\Omega_{xx} + \Omega_{yy}}{2} \\
\beta_2 &= -\Omega_{xx}\Omega_{yy}
\end{aligned}
\]
the characteristic polynomial can be expressed as \( l^4 + 2\beta_1 \lambda^2 - \beta_2^2 = 0 \) and

\[
l^2 = -\beta_1 \pm \sqrt{\beta_1^2 + \beta_2^2} = \pm A.
\]

So we will have as roots \( l = \pm \lambda \) and \( l = \pm \nu i \) with \( \lambda, \nu \in \mathbb{R} \). Which means that each collinear point is a saddle center and is unstable.

On the other side, for the equilateral points it can be verified by evaluating the second derivatives of \( \Omega \) that

\[
\begin{align*}
\Omega_{xx}(L_{4,5}) &= \frac{3}{4} \\
\Omega_{xy}(L_4) &= \frac{3\sqrt{3}}{2} (\mu - \frac{1}{2}) = -\Omega_{xy}(L_5) \\
\Omega_{yy}(L_{4,5}) &= \frac{9}{4}
\end{align*}
\]

and the characteristic polynomial can be written as \( l^4 + l^2 + \frac{27}{4} \mu (1 - \mu) = 0 \), and

\[
l^2 = \frac{1}{2} \left\{-1 \pm \sqrt{1 - 27\mu(1 - \mu)}\right\}.
\]

The expression inside the square root \( f(\mu) = 1 - 27\mu(1 - \mu) \) is zero when for the value of \( \mu \),

\[
\mu_{\text{routh}} = \frac{1}{2} - \sqrt{\frac{69}{9}} = 0.038528.
\]

At this value the stability changes. Therefore there are three possibilities:

1. \( 0 < \mu < \mu_{\text{routh}} \) \( \Rightarrow \) \( l^2 = \left\{ \begin{array}{l}
\Lambda_1 < 0 \\
\Lambda_2 < 0
\end{array} \right\} \Rightarrow l = \left\{ \begin{array}{l}
\pm \sqrt{|\Lambda_1|} i \\
\pm \sqrt{|\Lambda_2|} i
\end{array} \right\} \Rightarrow \text{center-center}.
\)

2. \( \mu = \mu_{\text{routh}} \) \( \Rightarrow \) \( l^2 = -\frac{1}{2} \Rightarrow l = \pm \sqrt{-\frac{1}{2}} i \Rightarrow \text{degenerate center}.
\)

3. \( \mu_{\text{routh}} < \mu \leq \frac{1}{2} \) \( \Rightarrow \) \( l^2 = \left\{ \begin{array}{l}
a + bi \\
a - bi
\end{array} \right\} \Rightarrow \text{complex saddle}.
\)
1.5 Symmetry of the Problem

The RTBP satisfies the following symmetry:

\[(x, y, \dot{x}, \dot{y}, t) \iff (x, -y, -\dot{x}, \dot{y}, -t)\]  

(1.42)

which means that a solution has always a symmetric solution with respect to the \( x \) axis, that has opposite \( x \) component of velocity. This is easy to prove. From the equations of
the problem

\[
\begin{align*}
\dot{y}_1 &= y_3 \\
\dot{y}_2 &= y_4 \\
\dot{y}_3 &= 2y_4 + y_1\left(1 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}\right) - \frac{\mu(1-\mu) - r_1}{r_2} \\
\dot{y}_4 &= -2y_3 + y_2\left(1 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}\right)
\end{align*}
\]

(1.43)

where \( y_1 = x, \ y_2 = y, \ y_3 = \dot{x} \) and \( y_4 = \dot{y} \). And \( r_1 = \sqrt{(y_1 - \mu)^2 + y_2^2}, \ r_2 = \sqrt{(y_1 + 1)^2 + y_2^2} \),

it is just needed to assume that \((x, y, \dot{x}, \dot{y}, t)\) is a solution and check that also \((x, -y, -\dot{x}, \dot{y}, -t)\)

is a solution. By simple inspection we see that the signs in the rhs with respect to the lhs

of the equations stay invariant after the change of variables.
Chapter 2

Hill’s Lunar Problem

2.1 Construction of the Problem

Using the RTBP to approximate the orbits of a planet, for example the Earth, in the Sun-Jupiter system, is reasonable. The Sun and Jupiter are very massive compared to the rest of the bodies of the solar system. Indeed it does not work so bad to study such a system as a two-body problem, considering a system only with the Sun and any other planet which we want to know its motion, as long as the influence of the other planets on the one studied is not very big.

But for the Moon this does not work so well. To get realistic approximations of the Moon’s orbit, at least we need to consider the Earth and the Sun. So the simplest approximation involves considering three bodies. A story tells that Newton, after working on the problem of Moon’s orbit, said ‘It causeth my head to ache’.

But a trick will allow us to convert the problem of the orbits of the Moon, into a problem with only two bodies in such a way that it will be still a good approximation. From the equations of motion for the RTBP in synodical coordinates (1.34), Hill’s problem is constructed. The first thing we do is to translate the secondary (in this case we may refer to it as the Earth) to the origin with \( \xi = x + 1 - \mu \) and \( \eta = y \). The equations of motion become,

\[
\begin{cases}
\ddot{\xi} - 2\dot{\eta} = \xi + \mu - 1 - \frac{(1-\mu)(\xi-1)}{r_1^2} - \frac{\mu \xi}{r_2^2} \\
\ddot{\eta} + 2\dot{\xi} = \eta \left[ 1 - \frac{1-\mu}{r_1^2} - \frac{\mu}{r_2^2} \right]
\end{cases}
\]

with

\[
\begin{cases}
r_1 = \left[ (\xi - 1)^2 + \eta^2 \right]^{1/2} \\
r_2 = (\xi^2 + \eta^2)^{1/2}
\end{cases}
\]
As it is done usually in dynamical systems, we want to introduce a small parameter. In this case, as the Earth has such a small mass compared to the Sun. It makes sense to use as small parameter $\mu$. As we want to study orbits near the Earth, that is, for small $\xi$ and $\eta$, we will proceed by making the change of variables $X = \xi/\mu^\alpha$ and $Y = \eta/\mu^\alpha$. After it will be found which $\alpha$ is the appropriate in our case. The equations of motion become

$$
\begin{align*}
\dot{X} - 2\dot{Y} &= X + (\mu - 1)\mu^{-\alpha} + \frac{(\mu - 1)\mu^{-\alpha}(\mu^\alpha X - 1)}{r_1^3} - \frac{\mu X}{r_2^3} \\
\dot{Y} + 2\dot{X} &= Y \left[ 1 - \frac{1 - \mu}{r_1^3} - \frac{\mu}{r_2^3} \right]
\end{align*}
$$

(2.3)

with

$$
\begin{align*}
r_1 &= \left[ (X\mu^\alpha - 1)^2 + \mu^{2\alpha}Y^2 \right]^{1/2} = \left[ (X^2 + Y^2)\mu^{2\alpha} - 2\mu^\alpha X + 1 \right]^{1/2} \\
r_2 &= \mu^\alpha (X^2 + Y^2)^{1/2}
\end{align*}
$$

(2.4)

or

$$
\begin{align*}
\dot{X} - 2\dot{Y} &= X + (\mu - 1)\mu^{-\alpha} + \frac{(\mu - 1)\mu^{-\alpha}(\mu^\alpha X - 1)}{[(X^2 + Y^2)\mu^{2\alpha} - 2\mu^\alpha X + 1]^{1/2}} - \frac{\mu X}{\mu^{3\alpha}(X^2 + Y^2)^{1/2}} \\
\dot{Y} + 2\dot{X} &= Y \left[ 1 - \frac{1 - \mu}{[(X^2 + Y^2)\mu^{2\alpha} - 2\mu^\alpha X + 1]^{1/2}} - \frac{\mu}{\mu^{3\alpha}(X^2 + Y^2)^{1/2}} \right]
\end{align*}
$$

(2.5)

And from (2.5) we see that the most simple choice is $\alpha = \frac{1}{3}$. So by making this choice,

$$
\begin{align*}
\dot{X} - 2\dot{Y} &= X + (\mu - 1)\mu^{-1/3} + \frac{(\mu - 1)\mu^{-1/3}(\mu^{1/3} X - 1)}{[(X^2 + Y^2)\mu^{2/3} - 2\mu^{1/3} X + 1]^{1/2}} - \frac{X}{(X^2 + Y^2)^{1/2}} \\
\dot{Y} + 2\dot{X} &= Y \left[ 1 - \frac{1 - \mu}{[(X^2 + Y^2)\mu^{2/3} - 2\mu^{1/3} X + 1]^{1/2}} - \frac{1}{(X^2 + Y^2)^{1/2}} \right]
\end{align*}
$$

(2.6)

Taylor expanding $r_1^3 = 1 - 3\mu^{1/3} X + O(\mu^{2/3})$ now,

$$
\begin{align*}
(\mu - 1)\mu^{-1/3} + \frac{(\mu - 1)\mu^{-1/3}(\mu^{1/3} X - 1)}{r_1^3} &= \frac{\mu - 1}{\mu^{1/3}} \left( \frac{r_1^3 - 1 + \mu^{1/3} X}{r_1^3} \right) = \\
&= (\mu - 1) \frac{X}{r_1^3} + \frac{\mu - 1}{r_1^3} \frac{r_1^3 - 1}{\mu^{1/3}} = \\
&= (\mu - 1) \frac{X}{1 - 3\mu^{1/3} X + O(\mu^{2/3})} + \frac{\mu - 1}{1 - 3\mu^{1/3} X + O(\mu^{2/3})} \frac{1 - 3\mu^{1/3} X + O(\mu^{2/3}) - 1}{\mu^{1/3}}
\end{align*}
$$

Written in this form, it is easy to take the limit of the expression when $\mu \to 0$ that is $2X$. The other limits in the equations of motion are immediate. So the equations of motion of
Hill’s problem are,
\[
\begin{aligned}
\dot{X} - 2\dot{Y} &= 3X - \frac{X}{(X^2 + Y^2)^{3/2}} \\
\dot{Y} - 2\dot{X} &= -\frac{Y}{(X^2 + Y^2)^{3/2}}
\end{aligned}
\tag{2.7}
\]
or
\[
\begin{aligned}
\dot{X} - 2\dot{Y} &= \frac{\partial \Omega^H}{\partial X} \\
\dot{Y} - 2\dot{X} &= \frac{\partial \Omega^H}{\partial Y}
\end{aligned}
\tag{2.8}
\]
with \( \Omega^H = \frac{1}{2} \left( 3X^2 + \frac{2}{(X^2 + Y^2)^{3/2}} \right) \).

And doing the same trick of Section 1.1.2 we find a Jacobi integral also for Hill’s problem:
\[
C_H = -\dot{X}^2 - \dot{Y}^2 + 2\Omega^H(X, Y). \tag{2.9}
\]
But if we want to relate the value of \( C_H \) with that of \( C \) for the RTBP, more work is needed.

First, we do the translation and the change of variables of Hill’s problem to the ‘old’ Jacobi constant
\[
C = \mu^{2/3} \left( -\dot{X}^2 - \dot{Y}^2 + X^2 + 2\mu^{2/3}X - 2\mu^{-1/3}X + \mu^{4/3} - 2\mu^{1/3} + Y^2 - \frac{2}{[(X^2 + Y^2)\mu^{2/3} - 2\mu^{1/3}X + 1]^{1/2}} + \frac{1}{(X^2 + Y^2)} \right)
\]
then we Taylor expand the denominators resulting in the following expression
\[
\mu^{-2/3}C = -\dot{X}^2 - \dot{Y}^2 + \mu^{2/3} \left( 2X^4 - 6X^2Y^2 + \frac{3}{4}Y^4 \right) + \mu^{1/3} \left( 2X^3 - 3XY^2 \right) + 3X^2 + \frac{2}{(X^2 + Y^2)^{1/2}} + 3 \left( \mu^{-2/3} - \mu^{1/3} \right) + \mathcal{O}(\mu)
\]
and now doing \( C' = \mu^{-2/3}C - 3 \left( \mu^{-2/3} - \mu^{1/3} \right) \)
\[
C' = -\dot{X}^2 - \dot{Y}^2 + \mu^{2/3} \left( 2X^4 - 6X^2Y^2 + \frac{3}{4}Y^4 \right) + \mu^{1/3} \left( 2X^3 - 3XY^2 \right) + 3X^2 + \frac{2}{(X^2 + Y^2)^{1/2}} + \mathcal{O}(\mu)
\]
so \( \tilde{C}_H = \mu^{-2/3} \left( C - 3(1 - \mu) \right) \) for \( \mu \) small before taking the limit. This gives an idea of the relation for \( \mu \) small. And \( \tilde{C}_H \to C_H \) as \( \mu \to 0 \).

Note that the change of variables and taking the limit, is equivalent to making the Sun of infinite size (has mass 1 and the Earth mass 0), and sending it to infinite distance of the Earth (\( r_1 = 1 \) and \( r_2 \) is very small).
2.2 Hill’s Problem as a Hamiltonian System

Doing the same change of variables to the one we did in the RTBP (1.23):

\[
\begin{align*}
q_1 &= x \\
p_1 &= \dot{x} - y \\
q_2 &= y \\
p_2 &= \dot{y} + x
\end{align*}
\]  

(2.10)

the problem is transformed to a Hamiltonian system with Hamiltonian

\[
H_H(q_1, q_2, p_1, p_2) = \frac{1}{2} (p_1^2 + p_2^2) - \frac{1}{(q_1^2 + q_2^2)^{1/2}} + \underbrace{q_2 p_1 - q_1 p_2}_{\text{coriolis forces}} - \underbrace{\frac{q_1^2}{2} + \frac{q_2^2}{2}}_{\text{influence of Sun}}
\]  

(2.11)

and the equations of motion in these variables are

\[
\begin{align*}
\dot{q}_1 &= p_1 + q_2 \\
\dot{q}_2 &= p_2 - q_1 \\
\dot{p}_1 &= p_2 + 2q_1 - \frac{q_1}{(q_1^2 + q_2^2)^{1/2}} \\
\dot{p}_2 &= -p_1 - q_2 - \frac{q_2}{(q_1^2 + q_2^2)^{1/2}}.
\end{align*}
\]  

(2.12)

The coriolis forces roughly correspond to the angular momentum as it was seen before in Section 1.3.2. The last term of the Hamiltonian has to be the influence of the Sun. Observe that without this term \(H_H\) would be the Hamiltonian of the two-body problem.

Also note that the values of energy of (2.11) are related to the Jacobi constant of the problem: \(C_H = \frac{H_H}{2}\).

2.3 Equilibria of the Problem

Unlike the RTBP, in which we have to resort to numerical methods to compute the exact position of the collinear equilibria, in Hill’s problem the collinear equilibria can be computed analytically.

First observe that the triangular equilibria are gone, because we have sent one of the primaries to infinity. Also, as \(L_3\) was on the other side of the Sun, this equilibrium is also gone in the new problem. We are left with \(L_1\) and \(L_2\).

To find these equilibria, from (2.12) impose \(\dot{q}_1 = \dot{q}_2 = \dot{p}_1 = \dot{p}_2 = 0\, \text{so} \, q_1 = p_2, \, q_2 = -p_1,\)
2.3 Equilibria of the Problem

\[
\begin{cases}
-\frac{q_2}{(q_1^2 + q_2^2)^{3/2}} = 0 & \Rightarrow q_2 = 0, p_1 = 0 \\
-\frac{1}{|q_1|^2} + 3q_1 = 0 & \Rightarrow \pm 1 = 3q_1^2.
\end{cases}
\]

Therefore, the positions of the equilibria are:

\[
L_1 = \left(3^{-1/3}, 0, 0, 3^{-1/3}\right)
\]

\[
L_2 = \left(-3^{-1/3}, 0, 0, -3^{-1/3}\right)
\]

To deduce the stability of \(L_1, L_2\), differentiate the right hand side of (2.12):

\[
Df = \begin{bmatrix}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
\frac{2q_1^2 - q_2^2}{(q_1^2 + q_2^2)^{3/2}} + 2 & \frac{3q_1q_2}{(q_1^2 + q_2^2)^{3/2}} & 0 & 1 \\
\frac{3q_1q_2}{(q_1^2 + q_2^2)^{3/2}} & \frac{2q_2^2 - q_1^2}{(q_1^2 + q_2^2)^{3/2}} + 1 & -1 & 0
\end{bmatrix}
\]

evaluate at the equilibria,

\[
Df(L_i) = \begin{bmatrix}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
8 & 0 & 0 & 1 \\
0 & -4 & -1 & 0
\end{bmatrix}
\]

to find the characteristic polynomial \(z^4 - 2z^2 - 27 = 0\) that has roots

\[
\begin{cases}
z_1 = i\sqrt{2\sqrt{7} - 1} \\
z_2 = -i\sqrt{2\sqrt{7} - 1} \\
z_3 = \sqrt{2\sqrt{7} + 1} \\
z_4 = -\sqrt{2\sqrt{7} + 1}.
\end{cases}
\]

So both equilibria are saddle centers.
2.4 Symmetry of the Hill problem

Hill’s problem also satisfies the symmetry seen in Section 1.5, and that can be verified in the same way. In synodical coordinates, the symmetry reads

\[(x, y, \dot{x}, \dot{y}, t) \iff (x, -y, -\dot{x}, \dot{y}, -t)\]  (2.13)

and in canonical variables

\[(q_1, q_2, p_1, p_2, t) \iff (q_1, -q_2, -p_1, p_2, -t)\]  (2.14)

This fact implies that if an orbit intersects the \(q_2 = 0\) axis orthogonally, then it must have a symmetric orbit, and both join into a periodic orbit.

2.5 Levi-Civita Regularization

Now The Levi-Civita regularization will be done to the problem. This regularization will transform the singularity of the problem (the Earth) into an equilibrium. After the regularization, the problem can be studied as close to the Earth as needed. The Levi-Civita regularization consists in three steps:

1. A canonical change of variables, to eliminate the square roots in the denominators.

2. Introduction of the extended phase space. A new variable is introduced, conjugated to the time. This allows to do another canonical change that vanishes the denominators. The new variable corresponds to a fixed energy level of the Hamiltonian.
This will mean that we work on a 'extended' Hamiltonian with two more variables in the energy level zero.

3. Finally, a symplectic transformation is performed to eliminate the new artificial variable introduced.

**step 1: canonical change of coordinates**

Consider the generating function of type III given by

\[ W(p_1, p_2, q_1, q_2) = -p_1 \left( \dot{q}_1^2 - \dot{q}_2^2 \right) - p_2 \left( 2\dot{q}_1\dot{q}_2 \right). \]  

(2.15)

Using the results summarized in table 1.1, we can relate the old and the new variables of the associated change,

\[
\begin{align*}
\frac{\partial W}{\partial p_1} &= - (\dot{q}_1^2 - \dot{q}_2^2) = -q_1, & \frac{\partial W}{\partial q_1} &= -2p_1\dot{q}_1 - 2p_2\dot{q}_2 = -\dot{p}_1 \\
\frac{\partial W}{\partial p_2} &= -2\dot{q}_1\dot{q}_2 = -q_2, & \frac{\partial W}{\partial q_2} &= 2p_1\dot{q}_2 - 2p_2\dot{q}_1 = -\dot{p}_2
\end{align*}
\]

Isolating the old variables from these equations we get the needed changes,

\[
\begin{align*}
q_1 &= \dot{q}_1^2 - \dot{q}_2^2 \\
q_2 &= 2\dot{q}_1\dot{q}_2 \\
p_1 &= \frac{\dot{q}_1\dot{p}_1 - \dot{q}_2\dot{p}_2}{2\dot{p}_2^2} \\
p_2 &= \frac{\dot{q}_1\dot{p}_2 + \dot{q}_2\dot{p}_1}{2\dot{p}_2^2}
\end{align*}
\]

(2.16)

that will be applied to the Hamiltonian \( H_H \) of the problem (2.11). Of course, as the change comes from a generating function, it is canonical. And as the old Hamiltonian \( H_H \) is time independent, the new Hamiltonian will be obtained just by replacing the variables. The generating function has been chosen in such a way, as to produce a change that kills the square roots in the denominators.

\[
H_H(q, p) = \tilde{H}_H(\dot{q}, \dot{p}) =
\]

\[
= \frac{1}{2} \left( \frac{1}{2\dot{p}_2^2} \right)^2 \left( \dot{q}_1^2 \dot{p}_1^2 - 2\dot{q}_1\dot{q}_2\dot{p}_2 + \dot{q}_2^2 \dot{p}_2^2 + \dot{q}_1^2 \dot{q}_2^2 + 2\dot{q}_1\dot{p}_1\dot{q}_2\dot{p}_2 + \dot{q}_2^2 \dot{p}_1^2 \right) - \frac{1}{\dot{p}_2^2} + \\
+ \left( \frac{1}{2\dot{p}_2^2} \right)^2 \left( 2\dot{q}_1^2 \dot{q}_2\dot{p}_1 - 2\dot{q}_1\dot{q}_2^2 \dot{p}_2 - \dot{q}_1^2 \dot{p}_2^2 - \dot{q}_1^2 \dot{q}_2 \dot{p}_1 + \dot{q}_1\dot{q}_2^2 \dot{p}_2 + \dot{q}_2^2 \dot{p}_1^2 \right) - \dot{q}_1^4 + 2\dot{q}_1^2 \dot{q}_2^2 - \dot{q}_2^4 + 2\dot{q}_1^2 \dot{q}_2^2 =
\]

\[
= \frac{1}{8\dot{p}_2^4} (\dot{q}_1^2 + \dot{q}_2^2) (\dot{p}_1^2 + \dot{p}_2^2) - \frac{1}{\dot{p}_2^2} + \frac{1}{2\dot{p}_2^2} (\dot{q}_1^2 \dot{q}_2 \dot{p}_1 - \dot{q}_1^2 \dot{q}_2^2 \dot{p}_2 - \dot{q}_1^2 \dot{p}_2 + \dot{q}_2^2 \dot{p}_1) - \dot{q}_1^4 + 4\dot{q}_1^2 \dot{q}_2^2 - \dot{q}_2^4 =
\]
Hill’s Lunar Problem

\begin{equation}
\frac{1}{8r^2} (\hat{p}_1^2 + \hat{p}_2^2) - \frac{1}{r^2} + \frac{1}{2r^2} (\hat{q}_1^2 + \hat{q}_2^2) \left( \hat{q}_2 \hat{p}_1 - \hat{q}_1 \hat{p}_2 \right) - \hat{q}_1^4 + 4\hat{q}_1^2 \hat{q}_2^2 - \hat{q}_2^4 = \\
= \frac{1}{8r^2} (\hat{p}_1^2 + \hat{p}_2^2) - \frac{1}{r^2} + \frac{1}{2} \left( \hat{q}_2 \hat{p}_1 - \hat{q}_1 \hat{p}_2 \right) - \left( \hat{q}_1^2 - \hat{q}_2^2 \right)^2 - 2\hat{q}_1^2 \hat{q}_2^2
\end{equation}

where \( \hat{r} = \sqrt{\hat{q}_1^2 + \hat{q}_2^2}. \)

**step 2: introduction of the energy as an artificial variable**

Now consider a particular value of the Hamiltonian \( \bar{H}_H (\hat{p}, \hat{q}) = -p_0 \), and a change in time \( D(\hat{q}, \hat{p}) d\tau = dt \). The variables \( \hat{q}, \hat{p}, t, p_0 \) correspond to the extended phase space, because \( p_0 \) and \( t \) are conjugated variables. The extended phase space is introduced to find a canonical change that involves also time. To see that, the Hamiltonian \( \bar{H}_H^{\text{ext}} (\hat{p}, \hat{q}, t, p_0) = \bar{H}_H + p_0 \) will be considered. As this Hamiltonian has two new variables, there are two new equations of motion:

\[
\begin{align*}
\frac{\partial \bar{H}_H^{\text{ext}}}{\partial \hat{t}} &= 0 = \dot{p_0} \\
\frac{\partial \bar{H}_H^{\text{ext}}}{\partial p_0} &= 1 = \dot{t}
\end{align*}
\]

And note that the value of the new Hamiltonian will be always zero. Also see that the variables \( \hat{p}, \hat{q} \) are still conjugated after the introduction of the new ones in the Hamiltonian. With the Hamiltonian written in the extended phase space, the change in time is a canonical change if \( \bar{H}_H^{\text{ext}} \) is multiplied by \( D \). First observe that

\[
\begin{align*}
\hat{q}_i &= \frac{d\hat{q}_i}{dt} = \frac{d\hat{q}_i}{d\tau} = \frac{1}{D} \frac{d\hat{q}_i}{d\tau} = \frac{1}{D} \hat{q}_i' \\
\hat{p}_i &= \frac{d\hat{p}_i}{dt} = \frac{d\hat{p}_i}{d\tau} = \frac{1}{D} \frac{d\hat{p}_i}{d\tau} = \frac{1}{D} \hat{p}_i'
\end{align*}
\]

So \( \bar{H}_H^{\text{ext}} = D \bar{H}_H^{\text{ext}} \) will satisfy

\[
\begin{align*}
\frac{\partial \bar{H}_H^{\text{ext}}}{\partial \hat{q}_i} &= \hat{q}_i' \quad \text{and} \quad \frac{\partial \bar{H}_H^{\text{ext}}}{\partial \hat{p}_i} = -\hat{p}_i'
\end{align*}
\]

as we verify

\[
\begin{align*}
\frac{\partial \bar{H}_H^{\text{ext}}}{\partial \hat{q}_i} &= \frac{\partial D}{\partial \hat{q}_i} \bar{H}_H^{\text{ext}} + D \frac{\partial \bar{H}_H^{\text{ext}}}{\partial \hat{q}_i} = \begin{cases} \text{because along a} \\
\text{solution } \bar{H}_H^{\text{ext}} = 0 \end{cases} = D \frac{\partial \bar{H}_H^{\text{ext}}}{\partial \hat{q}_i} = -\dot{p}_i' \\
\frac{\partial \bar{H}_H^{\text{ext}}}{\partial \hat{p}_i} &= \frac{\partial \bar{H}_H}{\partial \hat{p}_i} + D \frac{\partial \bar{H}_H^{\text{ext}}}{\partial \hat{p}_i} = D \frac{\partial \bar{H}_H^{\text{ext}}}{\partial \hat{p}_i} = \dot{q}_i'
\end{align*}
\]
To apply this result to $\bar{H}$, it is needed to choose a function $D(\hat{q}_1, \hat{q}_2, \hat{p}_1, \hat{p}_2)$. To vanish the singularity of $\bar{H}$, it is clear that a convenient choice is $D(\hat{q}_1, \hat{q}_2) = \hat{q}_1 + \hat{q}_2$. Then,

$$\hat{H}_\text{ext}^\text{ext} = \hat{r}^2(\hat{H}_\text{ext}^\text{ext} + p_0) = \frac{1}{8} (\hat{p}_1^2 + \hat{p}_2^2) + p_0 (\hat{q}_1^2 + \hat{q}_2^2) - 1 + \frac{1}{2} (\hat{q}_1^2 + \hat{q}_2^2) (\hat{p}_1 \hat{p}_2 - \hat{q}_1 \hat{q}_2) - (\hat{q}_1^2 + \hat{q}_2^2) (\hat{q}_1^4 - 4\hat{q}_1^2 \hat{q}_2^2 + \hat{q}_2^4) =$$

$$= \frac{1}{8} (\hat{p}_1^2 + \hat{p}_2^2) + p_0 (\hat{q}_1^2 + \hat{q}_2^2) - 1 + \frac{1}{2} (\hat{q}_1^2 + \hat{q}_2^2) (\hat{p}_1 \hat{p}_2 - \hat{q}_1 \hat{q}_2) - (\hat{q}_1^4 - 3\hat{q}_1^2 \hat{q}_2^2 - 3\hat{q}_2^4 + \hat{q}_2^6).$$

And we have to study the solutions of $\hat{H}_\text{ext}^\text{ext}$ along the energy level $\hat{H}_\text{ext}^\text{ext} = 0$ (for each $p_0$). So this is a Hamiltonian only in this energy level, but by choosing $p_0$ actually we are working on different energy levels of the previous Hamiltonian $\bar{H}^\text{ext}_\text{ext}$.

**step 3: elimination of the artificial variable $p_0$**

Defining $c := \frac{p_0}{2}$, the change of variables

$$\begin{cases}
\hat{q}_1 = 2c^{3/4}Q_1 & \hat{p}_1 = 8c^{3/4}P_1 \\
\hat{q}_2 = 2c^{3/4}Q_2 & \hat{p}_2 = 8c^{3/4}P_2
\end{cases}$$

is symplectic with multiplier $\zeta = \frac{1}{16}c^{-1}$, because

$$\frac{1}{16}c^{-1} \begin{bmatrix}
c^{-1/4} & 0 & 0 & 0 \\
0 & c^{-1/4} & 0 & 0 \\
0 & 0 & c^{-3/4} & 0 \\
0 & 0 & 0 & c^{-3/4}
\end{bmatrix} J \begin{bmatrix}
c^{-1/4} & 0 & 0 & 0 \\
0 & c^{-1/4} & 0 & 0 \\
0 & 0 & c^{-3/4} & 0 \\
0 & 0 & 0 & c^{-3/4}
\end{bmatrix}^T = J$$

So executing this change into the previous Hamiltonian

$$0 = \zeta \hat{H}_\text{ext}^\text{ext} = c^{-1}c^{3/2} \left[ \frac{1}{2} (P_1^2 + P_2^2) + \frac{1}{2} (Q_1^2 + Q_2^2) + 2 (Q_1^2 + Q_2^2) (Q_2 P_1 - Q_1 P_2) - 4 (Q_1^6 - 3Q_1^4 Q_2^2 - 3Q_2^4 Q_1^2 + Q_2^6) \right] - \frac{1}{16}c^{-1}.$$
or
\[
\frac{1}{16}c^{-3/2} = \frac{1}{2} \left( P_1^2 + P_2^2 \right) + \frac{1}{2} \left( Q_1^2 + Q_2^2 \right) + 2 \left( Q_1^2 + Q_2^2 \right) (Q_2P_1 - Q_1P_2) - 4 \left( Q_1^6 - 3Q_1^4Q_2^2 - 3Q_1^2Q_2^4 + Q_2^6 \right).
\]

Now we can define a new Hamiltonian in the regularized variables
\[
\mathcal{H} = \frac{1}{2} \left( Q_1^2 + Q_2^2 + P_1^2 + P_2^2 \right) + 2 \left( Q_1^2 + Q_2^2 \right) (Q_2P_1 - Q_1P_2) - 4 \left( Q_1^6 - 3Q_1^4Q_2^2 - 3Q_1^2Q_2^4 + Q_2^6 \right) \quad (2.17)
\]
and we will study \( \mathcal{H} = \frac{1}{16c^{3/2}} \).

Note that the energy values of this Hamiltonian have to be positive, because square roots of \( c \) have been taken during the transformation. So this Hamiltonian works only for positive energy values \( C_H \). Where \( C_H \) is the Jacobi constant for Hill’s problem in synodical coordinates we already saw. And also we knew that \( C_H = \frac{H_H}{2} \). If we want to operate with negative values of \( C_H \) a different Hamiltonian must be obtained (taking \( |c| \) just results in changing the sign of the terms or order 2 in \( \mathcal{H} \)).

As \( p_0 = 2c \), and \( p_0 \) is energy level of \( H_H \), \( c = \frac{1}{4}C_H \). So
\[
\frac{1}{2}C_H^{-3/2} = \frac{1}{2} (4c)^{-3/2} = \frac{1}{16}c^{-3/2}
\]
which means that a particular value \( h \) of \( \mathcal{H} \) satisfies \( h = \frac{1}{2}C_H^{-3/2} \).

### 2.5.1 Effect of the Regularization:

If we express the position variables of the original coordinates \( q_1, q_2 \), and of the regularized variables \( \hat{q}_1, \hat{q}_2 \) before the symplectic change, in polar coordinates:

\[
\begin{align*}
q_1 &= r \cos \theta & \Rightarrow \quad \hat{q}_1 &= \hat{r} \cos \varphi \\
q_2 &= r \sin \theta & \Rightarrow \quad \hat{q}_2 &= \hat{r} \sin \varphi
\end{align*}
\]
as we have \( \hat{r}^2 = (\hat{q}_1^2 + \hat{q}_2^2) = (q_1^2 + q_2^2)^{1/2} = r \), we can compute
\[
q_1 + iq_2 = r (\cos \theta + i \sin \theta) = re^{i\theta} = \begin{cases} q_1 = \hat{q}_1^2 - q_2^2 \\ q_2 = 2\hat{q}_1\hat{q}_2 \end{cases} = (\hat{q}_1 + i\hat{q}_2)^2 = (\hat{r}e^{i\varphi})^2 = \hat{r}^2 e^{2i\varphi}.
\]
2.6 Equilibria of the Regularized Problem

So the effect of the regularization is that of doubling the angles (we need half the angle in the regularized variables to reach to the same position), and of squaring the distance to the origin (in the new variables the distance is the square of the previous distance). The first thing implies that the number of equilibria is doubled in the regularized problem. Each of the two equilibria has a twin counterpart after applying Levi-Civita.

\[ Q_1^1 = \frac{1}{2} + 36 Q_1^1 - 24 Q_1^1 Q_2^2 - 24 Q_2^1 = 0 \]
\[ Q_2^1 = \frac{1}{2} - 12 Q_1^1 - 24 Q_1^1 Q_2^2 + 36 Q_2^1 = 0 \]

and now we distinguish these cases:

![Fig. 2.2 Effect of the Regularization](image-url)
i. $Q_1 = Q_2 = 0$:

If $Q_1 = Q_2 = 0$ then $P_1 = P_2 = 0$ gives an equilibrium, the origin. As we said, the regularization transforms the singularity into an equilibrium.

ii. $Q_1 \neq 0$ and $Q_2 = 0 (\Rightarrow P_1 = 0)$:

Then $-1 + 36Q_1^4 = 0$ and $Q_1 = \pm 6^{-1/2}$. If $Q_1 = 6^{-1/2}$, then $P_2 = 2Q_1^3 = \frac{1}{9}6^{-1/2}$. This point corresponds to $L_1$ in the new coordinates. Because of the effect of doubling the angle, it has a twin point $Q_1 = -6^{-1/2}$ and $P_2 = \frac{-1}{3}6^{-1/2}$. We call this point $L_1'$.

iii. $Q_1 = 0$ and $Q_2 \neq 0 (\Rightarrow P_2 = 0)$:

Now $-1 + 36Q_2^4$ and $Q_2 = \pm 6^{-1/2}$. For $Q_2$ positive we get $P_1 = \frac{-1}{3}6^{-1/2}$. This point is $L_2$. And for $Q_2 < 0$ we get the twin point $L_2'$ with $P_1 = \frac{1}{3}6^{-1/2}$.

So the regularized problem has five equilibria:

\[
\begin{align*}
 L_1 &= \left(6^{-1/2}, 0, 0, \frac{1}{3}6^{-1/2}\right) \\
 L_1' &= \left(-6^{-1/2}, 0, 0, \frac{1}{3}6^{-1/2}\right) \\
 L_2 &= \left(0, 6^{-1/2}, -\frac{1}{3}6^{-1/2}, 0\right) \\
 L_2' &= \left(0, -6^{-1/2}, \frac{1}{3}6^{-1/2}, 0\right) \\
 O &= (0, 0, 0, 0)
\end{align*}
\]

The Jacobian of system (2.18) is:

\[
Df = \begin{bmatrix}
4Q_1Q_2 & 2Q_1^2 + 6Q_2^2 & 1 & 0 \\
-6Q_1^2 - 2Q_2^2 & -4Q_1Q_2 & 0 & 1 \\
-1 - 4Q_1P_1 - 24Q_1^2 + 12Q_1^3 + 12Q_1^3Q_2 + 4Q_2P_2 - 96Q_1^2Q_2 - 6Q_1P_3 & 4Q_2P_2 - 96Q_1^2Q_2 - 6Q_1P_3 & -4Q_1Q_2 & 6Q_1^2 + 2Q_2^2 \\
4Q_2P_2 - 96Q_1^2Q_2 - 6Q_1P_3 & -1 + 4Q_1P_2 - 24Q_1^3 - 12Q_2P_1 - 144Q_1^2Q_2^2 + 120Q_1^3 & -2Q_1^2 - 6Q_2^2 & 4Q_1Q_2
\end{bmatrix}
\]

Evaluating the matrix at $L_1$ or $L_1'$ gives,

\[
Df(L_1) = \begin{bmatrix}
0 & 1/3 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
3 & 0 & 0 & 1 \\
0 & -13/9 & -1/3 & 0
\end{bmatrix}
\]
that has as characteristic polynomial $9\lambda^4 - 8\lambda^2 - 48 = 0$, with roots

$$
\begin{align*}
\lambda_1 &= \frac{2}{3}i\sqrt{2\sqrt{7} - 1} \\
\lambda_2 &= -\frac{2}{3}i\sqrt{2\sqrt{7} - 1} \\
\lambda_2 &= \frac{2}{3}\sqrt{2\sqrt{7} + 1} \\
\lambda_2 &= -\frac{2}{3}\sqrt{2\sqrt{7} + 1}
\end{align*}
$$

So $L_1$ is a saddle center, as we saw before. But also its twin $L_1'$ is a saddle center.

Evaluating now the matrix at $L_2$ or $L'_2$ gives

$$
Df(L_2) = \begin{bmatrix}
0 & 1 & 1 & 0 \\
-\frac{1}{3} & 0 & 0 & 1 \\
-\frac{13}{9} & 0 & 0 & \frac{1}{3} \\
0 & 3 & -1 & 0
\end{bmatrix}
$$

and it can be checked that has the same characteristic polynomial $9\lambda^4 - 8\lambda^2 - 48 = 0$, and therefore the same eigenvalues.

And finally, evaluating the matrix at the origin:

$$
Df(O) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix} = J
$$

that has characteristic polynomial $(\lambda^2 + 1)^2 = 0$ and double eigenvalues $\lambda = \pm i$. Observe that we can apply Lyapunov Center Theorem to $L_1$ and $L_2$ to prove the existence of a family of periodic orbits. But we cannot do that for the origin as it does not satisfy the hypothesis of the same theorem. However, in the next section we prove the existence of two families of periodic orbits around the origin.

### 2.7 Existence of Periodic Orbits Around the Origin

From the regularized Hamiltonian of Hill’s problem,

$$
\mathcal{H} = \frac{1}{2}(Q_1^2 + Q_2^2 + P_1^2 + P_2^2) + 2(Q_1^2 + Q_2^2)(Q_2 P_1 - Q_1 P_2) - 4(Q_1^2 + Q_2^2 - 3Q_1^2 Q_2^2 - 3Q_1^2 Q_2^2)
$$
a small parameter $\epsilon$ is introduced, through a change of variables: $Q_i = \xi_i \epsilon$ and $P_i = \eta_i \epsilon$. Note that this change of variables is not strictly speaking symplectic, because it can be checked easily that

$$\begin{bmatrix}
\epsilon^{-1} & 0 & 0 & 0 \\
0 & \epsilon^{-1} & 0 & 0 \\
0 & 0 & \epsilon^{-1} & 0 \\
0 & 0 & 0 & \epsilon^{-1}
\end{bmatrix} = J$$

with $\sigma = \epsilon^{-2}$. Therefore the new Hamiltonian will be $\tilde{H} = \sigma H$, so

$$\tilde{H} = \epsilon^{-2} \left[ \frac{1}{2} (\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2) + \epsilon^4 (\xi_1^2 + \xi_2^2)(\xi_2\eta_1 - \xi_1\eta_2) - 4\epsilon^6 (\xi_1^6 + \xi_2^6 - 3\xi_1^4\xi_2^2 - 3\xi_1^2\xi_2^4) \right] = \frac{1}{2} (\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2) + 2\epsilon^2 (\xi_1^2 + \xi_2^2)(\xi_2\eta_1 - \xi_1\eta_2) - 4\epsilon^4 (\xi_1^6 + \xi_2^6 - 3\xi_1^4\xi_2^2 - 3\xi_1^2\xi_2^4) = \frac{1}{2} (\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2) + 2\epsilon^2 (\xi_1^2 + \xi_2^2)(\xi_2\eta_1 - \xi_1\eta_2) + \mathcal{O} (\epsilon^4).$$

To make it easier to locate periodic orbits, this system will be written in polar canonical variables (1.33). In the new variables, because $\eta_1^2 + \eta_2^2 = R^2 + \frac{\Theta^2}{r^2}$, the Hamiltonian becomes

$$\tilde{H} = \frac{1}{2} (r^2 + R^2 + \frac{\Theta^2}{r^2}) - 2\epsilon^2 r^2 \Theta + \mathcal{O} (\epsilon^4).$$

(2.19)

And from here the equations of motion follow, for the system up to third order resulting in an integrable Hamiltonian system

$$\begin{cases}
\frac{\partial \hat{H}}{\partial R} = R & \Rightarrow \quad \dot{R} = R \\
\frac{\partial \hat{H}}{\partial \Theta} = \frac{\Theta}{r^2} - 2\epsilon^2 r^2 & \Rightarrow \quad \dot{\Theta} = \frac{\Theta}{r^2} - 2\epsilon^2 r^2 \\
-\frac{\partial \hat{H}}{\partial r} = -r + \frac{\Theta^2}{r^2} + 4\epsilon^2 r \Theta & \Rightarrow \quad \dot{R} = -r + \frac{\Theta^2}{r^2} + 4\epsilon^2 r \Theta \\
-\frac{\partial \hat{H}}{\partial \Theta} = 0 & \Rightarrow \quad \dot{\Theta} = 0.
\end{cases}$$

(2.20)

Two periodic orbits can be now found, one direct and the other retrograde. Take $R = 0$, $r = 1$, and from the third equation of (2.20),

$$\Theta^2 + 4\epsilon^2 \Theta - 1 = 0$$
which gives

\[
\Theta = \frac{-4\epsilon^2 \pm \sqrt{16\epsilon^4 + 4}}{2} = \begin{cases} 
1 + \mathcal{O}(\epsilon^4) = \Theta_+ \\
-1 - 4\epsilon^2 + \mathcal{O}(\epsilon^4) = \Theta_-.
\end{cases}
\]

Now from the second equation,

\[
\dot{\theta} = \frac{\Theta}{r^2} - 2\epsilon^2 r^2|_{r=\Theta_{+,-}} = \begin{cases} 
1 - 2\epsilon^2 + \mathcal{O}(\epsilon^4) \\
-1 - 6\epsilon^2 + \mathcal{O}(\epsilon^4)
\end{cases}.
\]

As \(1 + \mathcal{O}(\epsilon^2) > 0\) for \(\epsilon\) small enough, we have a direct orbit of period

\[
T = \frac{2\pi}{1 - 2\epsilon^2 + \mathcal{O}(\epsilon^4)}.
\]

And as \(-1 + \mathcal{O}(\epsilon^2) < 0\) for \(\epsilon\) small enough, we have retrograde orbit of period

\[
T = \frac{2\pi}{1 + 6\epsilon^2 + \mathcal{O}(\epsilon^4)}.
\]

Besides, if \(\epsilon \to 0\) then \(T \to 2\pi\) in both cases.

The orbits for \(\epsilon = 0\), can be continued in a neighborhood and therefore the full Hamiltonian system has also the direct and retrograde periodic orbits. To see this, with \(\epsilon = 0\) the system, neglecting the first integral \(\Theta\) and the angle, is

\[
\begin{cases}
\dot{r} = R \\
\dot{R} = -r + \frac{\Theta^2}{r^2}
\end{cases}
\]

and linearizing over the solution \(\dot{r} = R, \dot{R} = -4r\), or

\[
\begin{bmatrix} \dot{r} \\ \dot{R} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} r \\ R \end{bmatrix}.
\]

The matrix of the system diagonalizes to \(\begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}\). So in the basis of eigenvalues, the system has as fundamental solution \(\begin{bmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{bmatrix}\). And evaluating at each of the periods, the multipliers of the monodromy matrix result in each case (direct or retrograde)

\[
\begin{cases}
e^{\pm 2i(\frac{2\pi}{1 - 2\epsilon^2} + \mathcal{O}(\epsilon^4))} = 1 \pm 8i\pi\epsilon^2 + \mathcal{O}(\epsilon^4) \\
e^{\pm 2i(\frac{2\pi}{1 + 6\epsilon^2} + \mathcal{O}(\epsilon^4))} = 1 \pm 24i\pi\epsilon^2 + \mathcal{O}(\epsilon^4).
\end{cases}
\]

In any case we can write \(1 \pm K\epsilon^2 + \mathcal{O}(\epsilon^4)\) for a constant \(K \in \mathbb{R} (K \neq 0)\) that depends on the direction of the orbit. On the level surface of the Hamiltonian for the given periodic orbit we can compute the Poincaré map. Considering new coordinates \(u = (u_1, u_2)^T\) such that the periodic orbit passes through \(u = (0,0)^T\) in the chosen Poincaré section,
the Poincaré map satisfies $P\left((0,0)^T\right) = (0,0)^T$. Also, as $DP(u)$ coincides with the sub-matrix of the monodromy matrix obtained by removing the unitary eigenvalues,

$$DP(u) = \begin{bmatrix} 1 + K\epsilon^2 + \mathcal{O}(\epsilon^4) & 0 \\ 0 & 1 - K\epsilon^2 + \mathcal{O}(\epsilon^4) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$ 

Define $G(u, \epsilon) = \frac{(P(u) - u)}{\epsilon^2}$ and we have the required conditions to apply the implicit function theorem. $G(0, 0) = 0$ and

$$\left.\frac{\partial G}{\partial u}\right|_{u=(0,0)} = \begin{vmatrix} Ki & 0 \\ 0 & -Ki \end{vmatrix} = K^2 \neq 0 \Rightarrow \exists \bar{u}(\epsilon) \text{ s.t } G(\bar{u}(\epsilon), \epsilon) = 0 \text{ for } \epsilon \text{ small enough. So we can continue the periodic solution for the full system. And } \bar{u}(\epsilon) = \mathcal{O}(\epsilon). \text{ This implies } r = 1 + \mathcal{O}(\epsilon) \text{ or } \xi_1 = \pm 1 + \mathcal{O}(\epsilon) \text{ over the Poincaré section } \Sigma = \{Q_2 = 0\}.$$

Also, in this Poincaré section we can deduce the values of the other variables.

$$\Theta = \pm 1 + \mathcal{O}(\epsilon^2) = \xi_1\eta_2 - \xi_2\eta_1 = \xi_1\eta_2 \Rightarrow \eta_2 = \frac{\pm 1 + \mathcal{O}(\epsilon^2)}{\pm 1 + \mathcal{O}(\epsilon^2)} = 1 + \mathcal{O}(\epsilon).$$

The signs have canceled because they coincide according to the direction of the orbit (direct or retrograde).

$$0 = \dot{R} = \frac{\xi_1\eta_1 + \xi_2\eta_2}{r} = \frac{\xi_1\eta_1}{r} = \eta_1 \Rightarrow \eta_1 = 0.$$ 

And going back to the original variables:

$$\begin{cases} 
\xi_1 = \pm 1 + \mathcal{O}(\epsilon) \Rightarrow Q_1 = \pm \epsilon + \mathcal{O}(\epsilon^2) \\
\xi_2 = 0 \Rightarrow Q_2 = 0 \\
\eta_1 = 0 \Rightarrow P_1 = 0 \\
\eta_2 = 1 + \mathcal{O}(\epsilon) \Rightarrow P_2 = \epsilon + \mathcal{O}(\epsilon^2)
\end{cases}$$

Because of the null variables, the Hamiltonian becomes

$$\mathcal{H}_{Q_2=0,P_1=0} = \frac{1}{2} \left( Q_1^2 + P_2^2 \right) - 2Q_1^3P_2 - 4Q_1^6.$$
and evaluating at the fixed point of the Poincaré map

\[ h = \epsilon^2 + O(\epsilon^3). \]

As \( Q_1 = \pm \epsilon + O(\epsilon^3) \) then \( Q_1^2 = \epsilon^2 + O(\epsilon^3) \). Using that \( \epsilon^2 = h + O(\epsilon^3) \) we can write

\[ Q_1^2 = h + O(\epsilon^3) \Rightarrow Q_1 = \pm \sqrt{h} + O(\epsilon^3). \]

And as \( O(h) = O(\epsilon^2) \) we have \( O(h^{3/2}) = O(\epsilon^3) \) so

\[ Q_1 = \pm \sqrt{h} + O(h^{3/2}). \]

We saw that for \( \Theta > 0 \) the orbit was direct and that \( \Theta = \xi_1 \eta_2 \). So \( \xi_1 \eta_2 > 0 \). If we consider that the Poincaré section is being crossed in the direction \( \dot{Q}_2 > 0 \) (direct orbit), then as \( P_2 = \dot{Q}_2 + 2Q_1 \eta \) and \( Q_1 \) is small for \( h \) small, then \( P_2 \) has the same sign as \( \dot{Q}_2 \). So for \( h \) small \( P_2 > 0 \) and \( \eta_2 > 0 \), therefore \( \xi_1 > 0 \). This means that \( Q_1 = +\sqrt{h}(1 + O(h)) \) corresponds to the direct orbit. The same argument can be done for the retrograde orbit: \( Q_1 = -\sqrt{h}(1 + O(h)) \). So we have established:

**Proposition 2.7.1** For \( h \) small enough, Hill’s problem has two simple periodic orbits with limit period \( 2\pi \). One direct and the other retrograde. On the \( Q_2 = 0, \dot{Q}_2 > 0 \) in regularized variables, they are located in \( \dot{Q}_1 = 0 \) and \( Q_1 = \sqrt{h}(1 + O(h)) \) for the direct orbit and \( Q_1 = -\sqrt{h}(1 + O(h)) \) for the retrograde.

![Fig. 2.3 Poincaré section for \( h = 0.027 \)]
In the next section, we will see that direct orbits bifurcate with a pitchfork bifurcation, while retrograde orbits do not. Figure 2.3 shows the Poincaré section for a value of $h$ before the pitchfork bifurcation, while Figures 2.4 and 2.5 are for $h$ after the bifurcation.

In particular, for Figure 2.4 we have used $h = 0.0542$ and the section is crossed in the direction $\dot{Q}_2 > 0$. The lenticular shape is the projection of Hill’s region into the $Q_1, \dot{Q}_1$ plane (see section 4.2). The fixed point in the left, surrounded by invariant tori, corresponds to the retrograde periodic orbit of the proposition. For this value of the energy, the fixed point on the right (direct periodic orbit) has already bifurcated and become unstable. This region, around the fixed points, presents chaotic behavior and islands. There


is a chain of hyperbolic points whose invariant manifolds surround the islands, with an elliptic point at their center.

As it can be seen, plotting now the Poincaré section for $\dot{Q}_2 < 0$ we just get the specular image (Figure 2.5), as it is expected because of the symmetry or the p.o. In Figure 2.6 it is shown a zoom of this last image for the region of the fixed point corresponding to the direct (Moon like) orbit. These orbits are known as Hill’s orbits.

## 2.8 Continuation of Hill’s Orbits

The previous proposition (2.7.1) ensures only the existence of Hill’s orbits for small enough values of the energy $h$. From Equations 2.12, if we consider orbits at a big distance from the origin, and neglect the small terms, the equations become:

\[
\begin{align*}
\dot{q}_1 &= p_1 + q_2 \\
\dot{q}_2 &= p_2 - q_1 \\
\dot{p}_1 &= p_2 + 2q_1 \\
\dot{p}_2 &= -p_1 - q_2 = -\dot{q}_1
\end{align*}
\]  

(2.22)

And they can be written as a second order system of two equations:

\[
\begin{align*}
\ddot{q}_1 &= \dot{p}_2 - \dot{q}_1 = -2\dot{q}_1 \\
\ddot{q}_2 &= \dot{p}_1 + \dot{q}_2 = 2q_1 + p_2 + \dot{q}_2 = 3q_1 + p_2 - q_1 + \dot{q}_2 = 3q_1 + 2\dot{q}_2.
\end{align*}
\]

or

\[
\begin{align*}
\ddot{q}_1 &= 3q_1 + 2\dot{q}_2 \\
\ddot{q}_2 &= -2\dot{q}_1.
\end{align*}
\]  

(2.23)

Considering initial conditions $\dot{q}_1 = q_2 = 0$ at $t = 0$, the following solution is obtained:

\[
\begin{align*}
q_1 &= d + a \cos t \\
q_2 &= -2a \sin t - \frac{3}{2} dt
\end{align*}
\]  

(2.24)

This orbit is only periodic if $d = 0$, and is a retrograde p.o. So maybe Hill’s retrograde orbits can be continued up to this orbit near the infinite. But observe that if we want direct orbits ($\dot{q}_2 > 0$) then we need $d \neq 0$, and they are not periodic.
Using the method of continuation described in Section 4.4, the retrograde orbits, have been continued until very large values of the energy \((h = 10)\) see Figure 2.7. The orbits can be continued after the equilibrium up to any desired value.

**Fig. 2.7 Continuation of retrograde orbits**

Direct orbits behave differently. They stay close to the Earth, and that is the source of the chaotic behavior of these orbits. In figure 2.8 we see the continuation of these orbits before and after their bifurcation at \(h \approx 0.05237\).

**Fig. 2.8 Continuation of direct orbits**
Before the bifurcation, the eigenvalues of $DP$ have unitary modulus. After the pitchfork bifurcation, the eigenvalues of the orbit in the center of the fork have modulus more and less than 1 respectively. So it is a saddle (from the point of view of the Poincaré section $\Sigma$). Before the bifurcation they were complex eigenvalues and after they become real. All these has been found numerically.

Regarding the upper branch of the pitchfork in Figure 2.8, after the bifurcation, both non trivial eigenvalues of $DP$ are smaller than 1. Remember that the trivial eigenvalues are unitary. Also notice that the monodromy matrix is not symplectic, because to integrate numerically we are using the cheaper system 4.3 (see all the details in Section 4.1). One of the trivial eigenvalues appears because there is a first integral, and the other because we are over a periodic orbit. So in this case the fixed point continues to be stable after the bifurcation. Indeed now it is asymptotically stable. But not for long, because after $h \approx 0.0524195$ both eigenvalues reach modulus greater than one and the p.o becomes unstable. The lower branch shows the same behavior as the upper for $h \approx 0.05238035$.

### 2.9 Invariant Manifolds and Chaotic Behavior

For values of $h$ after the bifurcation ($h \approx 0.05327$) of the periodic orbits of equations (2.18), we have plotted the invariant manifolds of the fixed point in the center to infer their behavior.

In Figure 2.10, a little after the bifurcation, we see that the manifolds appear to connect
into a homoclinic connection in which the manifolds coincide. Using the implemented code for the computation of the angle of intersection between two manifolds (see Section 4.6), we get the value $\alpha \approx \pi$ radians (Table 2.1).

However, the coincidence of the manifolds from the intersection onward is unlikely. Integrating the manifolds for $h = 0.05356$ it apparently happens again, but zooming in near the hyperbolic fixed point shows otherwise.

And for even greater values it is obvious that there is not such coincidence (see Figure 2.13 for $h = 0.05455$ and next).
2.9 Invariant Manifolds and Chaotic Behavior

As the invariant manifolds must be analytic functions, because their origin is in the solution of a differential equation $\dot{x} = f(x)$ with $f$ analytic, and because we are dealing with a hyperbolic fixed point (this is a consequence of the stable manifold theorem - Theorem 7 in the appendix), the angle function $\alpha(h)$ is also analytic. This implies that $\alpha(h) - \pi$ can have only a countable number of zeros unless it is zero always. As it is not always zero (see all the figures in this section and Table 2.1), then there is at most a
countable number of zeros, and therefore a countable number of values of $h$, for which the homoclinic connection happens in such a way that the stable and unstable manifolds coincide. For the rest of the values there is a transverse homoclinic connection. This also means that the probability of finding one of such coincident connections is zero. But sometimes the angle of intersection is so close to be tangent, that numerically it appears to be so, as in Figure 2.10, that even zooming in did not show otherwise.

Anyway, for the transverse homoclinic connections, we can apply the Theorem of Smale-Birkhof to state that there is chaotic behavior in the system. The horseshoe effect of the figures is topologically equivalent to that of the Smale horseshoe (see the third section of the appendix for all the theoretical details).

What happens is that when the stable and unstable manifolds intersect transversely, they will intersect again an infinite number of times. The reason being that we are dealing with a discrete dynamical system (over the Poincaré section), and if a point belongs to both of the manifolds, as these are invariant the next iterate must belong to both of the manifolds too. This fact, combined with the expansion and contraction close to the hyperbolic fixed point, creates the horseshoe effect. Different plots of this situation for different values of $h$ are shown below.

![Fig. 2.15 Inv. manifolds for $h = 0.054811$](image)
2.9 Invariant Manifolds and Chaotic Behavior

Fig. 2.16 Inv. manifolds for $h = 0.055242$

Fig. 2.17 Inv. manifolds for $h = 0.0556$
For any value of $h$ a code has been implemented that outputs the angle of intersection. As the manifolds are symmetric, the code finds first the angle with the horizontal, as in Figure 2.18, that shows the angle for the manifolds of Figure 2.17. For this is needed just to work with one manifold. By doubling this angle, the angle of intersection $\alpha$ is found (Figure 2.21).

![Figure 2.18 Angle with the horizontal for $h = 0.0556$](image)

![Figure 2.19 Inv. manifolds for $h = 0.056008$](image)
Angle of intersection $\alpha(h) = \pi$ would correspond to a homoclinic connection where the manifolds coincide, as it has been discussed.
In the data presented, we observe that it does not appear to be really smooth (Figure 2.22), specially close to the minimum. This is caused by numerical errors in the computation of the angle. As we said, this function is analytic.
Chapter 3

Averaged Problem

In this chapter we are going to study the averaged problem of Hamiltonian (2.17). Originally, the reason for doing this, was to check if the averaged Hamiltonian approximates the dynamics of the Hamiltonian of Hill’s problem regularized (2.17). But it turned out not to be so, because the conditions to Theorem 3 of the appendix were not met. However, the study of the system obtained has its own interest in the theory of dynamical systems, as another example of the study of stability, and for this reason it is included in this chapter.

3.1 Averaging

Writing the regularized Hill Hamiltonian (2.17) in polar coordinates ($Q_1 = r \cos \theta$, $Q_2 = r \sin \theta$, $R = \frac{Q_1 P_1 + Q_2 P_2}{r}$, $\Theta = Q_1 P_2 - Q_2 P_1$), we get

\[
\tilde{H} = \frac{1}{2} \left( r^2 + R^2 + \frac{\Theta^2}{r^2} \right) + 2r^2(-\Theta) - 4r^2(Q_1^4 + 2Q_1^2Q_2^2 + Q_2^4) - 6Q_1^2Q_2^2 = \\
= \frac{1}{2} \left( r^2 + R^2 + \frac{\Theta^2}{r^2} \right) + 2r^2(-\Theta) - 4r^2 \left( (Q_1^2 + Q_2^2)^2 - 6Q_1^2Q_2^2 \right) = \\
= \frac{1}{2} \left( r^2 + R^2 + \frac{\Theta^2}{r^2} \right) + 2r^2(-\Theta) - 4r^2 \left( r^4 - 6r^4 \sin^2 \theta \cos^2 \theta \right) = \\
= \frac{1}{2} \left( r^2 + R^2 + \frac{\Theta^2}{r^2} \right) + 2r^2(-\Theta) - 4r^6 \left( 1 - 6 \sin^2 \theta \cos^2 \theta \right). 
\] (3.1)
And from the Hamiltonian the equations of motion follow. We have

\[
\begin{align*}
\frac{\partial \tilde{H}}{\partial R} &= R \\
\frac{\partial \tilde{H}}{\partial \Theta} &= \Theta r^2 - 2r^2 \\
\frac{\partial \tilde{H}}{\partial r} &= r - \frac{\Theta}{r^3} - 4\Theta r - 24r^5(1 - 6\sin^2 \theta \cos^2 \theta) \\
\frac{\partial \tilde{H}}{\partial \theta} &= 48r^6 (\sin \theta \cos^3 \theta - \sin^3 \theta \cos \theta)
\end{align*}
\]

so

\[
\begin{align*}
\dot{r} &= R \\
\dot{\Theta} &= \frac{\Theta}{r^2} - 2r^2 \\
\dot{R} &= \frac{\Theta^2}{r^2} + (4\Theta - 1)r + 24r^5(1 - 6\sin^2 \theta \cos^2 \theta) \\
\dot{\Theta} &= 48r^6 (\sin^3 \theta \cos \theta - \sin \theta \cos^3 \theta).
\end{align*}
\] (3.2)

To average the system, the following integrals must be evaluated

\[
\begin{align*}
\int_0^{2\pi} \sin^2 \theta \cos^2 \theta \, d\theta &= \int_0^{2\pi} \{\sin^2 \theta - \sin^4 \theta\} \, d\theta \\
&= \pi - \frac{3}{4} \pi = \frac{\pi}{4}
\end{align*}
\]

and

\[
\begin{align*}
\int_0^{2\pi} \{\sin^3 \theta \cos \theta - \sin \theta \cos^3 \theta\} \, d\theta &= \left[\frac{1}{4} \sin^4 \theta + \frac{1}{4} \cos^4 \theta\right]_0^{2\pi} \\
&= 0.
\end{align*}
\]

So

\[
\frac{1}{2\pi} \int_0^{2\pi} 24(1 - 6\sin^2 \theta \cos^2 \theta) \, d\theta = 6
\]

and

\[
\frac{1}{2\pi} \int_0^{2\pi} 48\{\sin^3 \theta \cos \theta - \sin \theta \cos^3 \theta\} \, d\theta = 0
\]

are the averages of the periodic functions in the system.

Replacing the periodic functions by their average, the system can be rewritten as

\[
\begin{align*}
\dot{r} &= R \\
\dot{\theta} &= \frac{\Theta}{r^2} - 2r^2 \\
\dot{R} &= r (4\Theta - 1) + \frac{\Theta^2}{r^2} + 6r^5 \\
\dot{\Theta} &= 0.
\end{align*}
\] (3.3)
This is not yet the averaged system. In the averaged system the derivatives have to be with respect to $\theta$.
So to obtain the averaged system we must do the change $(\frac{\Theta}{r^2} - 2r^2)^{-1} d\theta = dt$, obtaining system 3.4.

\[
\begin{align*}
\dot{r}' &= (\frac{\Theta}{r^2} - 2r^2)^{-1} R \\
\dot{\theta}' &= 1 \\
\dot{R}' &= (\frac{\Theta}{r^2} - 2r^2)^{-1} \left( r (4\Theta - 1) + \frac{\Theta^2}{r^2} + 6r^5 \right) \\
\Theta' &= 0
\end{align*}
\]  
(3.4)

3.2 Dynamics of the Averaged System

To do the stability analysis, it is easier to deal with equations 3.3. We will refer to this system as ‘the averaged system’ also, although the averaged system as such is (3.4). But both systems have the same orbits, although covered with a different parameter. As the averaged system has a ‘new’ first integral, we fix $\Theta$ and study

\[
\begin{align*}
\dot{r} &= R \\
\dot{R} &= r (4\Theta - 1) + \frac{\Theta^2}{r^2} + 6r^5.
\end{align*}
\]  
(3.5)

Every fixed point of this system will correspond to a periodic orbit of the whole system (3.3). Observe that by direct integration of 3.3 the Hamiltonian is obtained:

\[
H^\dagger = \frac{1}{2} \left( R^2 + r^2 + \frac{\Theta^2}{r^2} \right) - 2r^2\Theta - r^6
\]  
(3.6)

To find the equilibria, we take $R = 0$ and solve $r (4\Theta - 1) + \frac{\Theta^2}{r^2} + 6r^5 = 0$ for $r > 0$, which gives two solutions:

\[
r_{\pm} = \pm \left( 1 - 4\Theta \pm \sqrt{1 - 8\Theta - 8\Theta^2} \right)^{1/4}.
\]

These solutions have physical meaning only when $\sqrt{1 - 8\Theta - 8\Theta^2}$ is real. For that it is needed that $\Theta \in \left( \frac{-2 + \sqrt{10}}{4}, \frac{-2 - \sqrt{10}}{4} \right)$.

As we see in Figure 3.1, there are two saddle-node bifurcations. So we expect that one of the equilibria is stable for the linearized system and the other unstable. Computing the
The differential of the system gives

\[
Df = \begin{bmatrix}
0 & 1 \\
4\Theta - 1 - \frac{36\Theta^2}{1 - 4\Theta \pm \sqrt{1 - 8\Theta - 8\Theta^2}} + \frac{5}{2} \left(1 - 4\Theta \pm \sqrt{1 - 8\Theta - 8\Theta^2}\right) & 0
\end{bmatrix}
\]

and evaluating \(Df\) at the equilibria

\[
Df \left( r_{\pm} \right) = \begin{bmatrix}
0 & 1 \\
\frac{36\Theta^2}{1 - 4\Theta \pm \sqrt{1 - 8\Theta - 8\Theta^2}} + \frac{5}{2} \left(1 - 4\Theta \pm \sqrt{1 - 8\Theta - 8\Theta^2}\right) & 0
\end{bmatrix}
\]

So we can write

\[
Df \left( r_{\pm} \right) = \begin{bmatrix}
0 & 1 \\
a_{\pm}(\Theta) & 0
\end{bmatrix}
\]

with

\[
a_{\pm}(\Theta) = 4\Theta - 1 - \frac{36\Theta^2}{1 - 4\Theta \pm \sqrt{1 - 8\Theta - 8\Theta^2}} + \frac{5}{2} \left(1 - 4\Theta \pm \sqrt{1 - 8\Theta - 8\Theta^2}\right).
\]

As we see in Figure 3.2, \(a_{+}(\Theta) > 0\) for \(\Theta \in \left(\frac{-2-\sqrt{6}}{4}, \frac{-2+\sqrt{6}}{4}\right)\) and \(a_{-}(\Theta) < 0\) in the same range. Also \(a_{+}(\Theta) = a_{-}(\Theta) = 0\) when \(\Theta = \frac{-2-\sqrt{6}}{4}\) or \(\Theta = \frac{-2+\sqrt{6}}{4}\).

So \(Df \left( r_{\pm} \right)\) diagonalizes to

\[
\begin{bmatrix}
\sqrt{a_{\pm}(\Theta)} & 0 \\
0 & -\sqrt{a_{\pm}(\Theta)}
\end{bmatrix}
\]

And in the case of \(r_{+}\) we have \([>0 \quad 0 \quad <0]\). Then \(r_{+}\) is a saddle and unstable (or a hyperbolic...
3.2 Dynamics of the Averaged System

periodic orbit considering \( \theta \)). In the case of \( r_- \), \[
\begin{bmatrix}
>0 & 0 \\
0 & <0
\end{bmatrix}
\] Then \( r_- \) is an elliptic point and is stable for the linearized system (or an elliptic periodic orbit considering also \( \theta \)). When \( r_- = r_+ \), in the bifurcation points, the Jordan form of the matrix is \[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\] and the equilibrium is parabolic.

In Figures 3.3 and 3.4 the level curves for different energy levels of the Hamiltonian are plotted, and the stable and unstable equilibria are shown for values of \( \Theta \) positive and negative.

In Figures 3.5 and 3.6, we do the same for the values of \( \Theta \) where the equilibria bifurcate. It shows the parabolic equilibrium.
Now one important fact has to be mentioned. We can’t apply Theorem 3 of the appendix to infer, from the ‘averaged system’ (3.4), things about the original system. The reason is that the averaged system does not have a high angular velocity. So the $\epsilon$ of the theorem is not small.

Observing the equation for the angular velocity $\dot{\theta} = \frac{\Theta}{r^2} - 2r^2$, it may seem that when $r$ is very small the angular velocity is very high. But this is not true because when $r \to 0$ then $\Theta \to 0$ (Figure 3.1). Indeed there is a jump discontinuity for $\Theta = 0$ in the angular speed (Figure 3.7). So this system is never a good approximation of the original.

Another feature of the ‘averaged system’ is the apparition of a cycle of equilibria.
3.2 Dynamics of the Averaged System

Solving for equilibria in the system

\[
\begin{align*}
\dot{r} &= R \\
\dot{\theta} &= \frac{\Theta}{r^2} - 2r^2 \\
\dot{R} &= r(4\Theta - 1) + \frac{\Theta^2}{r^3} + 6r^5,
\end{align*}
\]

\[0 = \frac{\Theta}{r^2} - 2r^2, \quad \Rightarrow \quad r = \left(\frac{\Theta}{2}\right)^{1/4}\]

so \(\left(\frac{\Theta}{2}\right)^{1/4}(4\Theta - 1) + \frac{\Theta^2}{\left(\frac{\Theta}{2}\right)^{3/4}} + 6\left(\frac{\Theta}{2}\right)^{5/4} = 0\) \(\Rightarrow 9\Theta^2 - \Theta = 0\). And neglecting the discontinuity, there is a single solution

\[\Theta = 1/9, \quad r = \left(\frac{1}{18}\right)^{1/4}, \quad R = 0.\]

For these values of \(\Theta\) and \(r\) the equilibria are unstable \(r_+\). And near this \(\Theta\) the periodic orbits change their retrograde behavior to direct behavior or vice versa (Figure 3.7). This is different to what happens in the original system. In the averaged system there is only one retrograde and one direct p.o for \(\Theta \in (0, \frac{-2+\sqrt{6}}{4})\), and for negative values of \(\Theta\) both orbits are retrograde. In the original system there is always one direct and one retrograde p.o.
Chapter 4

Numerical Methods

This section summarizes all the numerical computations needed for the elaboration of this project. All the code has been written in C language, and most of the computations have been done using quadruple precision, by means of the gcc library quadmath.h (see [15]). The numerical integration of differential equations is done with the library Taylor (see [16]). As this library is not equipped by default to be used with quadruple precision, a script has been created that modifies the code automatically after the compilation, to make the code generated by Taylor compatible with quadmath.h. The script is included in the Makefile file, so just by typing make in the Linux terminal, everything is done (for all the codes included in the project).

Regarding resolution of systems of equations, computation of determinants, other linear algebra operations and the Least Squares Method it has used with GSL (GNU Scientific Library), which has very efficient routines for all purposes (see [14]). The computations with GSL are the only ones that have been performed in double precision.

<table>
<thead>
<tr>
<th>Arithmetic</th>
<th>name in C</th>
<th>bit precision</th>
<th>digits in base 10</th>
<th>bits exponent</th>
</tr>
</thead>
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<td>double</td>
<td>53</td>
<td>16</td>
<td>11</td>
</tr>
<tr>
<td>quadruple</td>
<td>__float128</td>
<td>113</td>
<td>35</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 4.1 Different floating point arithmetic's used

The codes can be downloaded from the link:

https://app.box.com/s/kg6p99tjmjyn8u0h58gqa

There are three folders for each of the codes:
1. folder code-poincare: the code for computing the Poincaré map;

2. folder code-manifolds: code that computes the invariant manifolds over the Poincaré section $\Sigma$;

3. folder code-angle: code that computes the angle of intersection of the manifolds.

The details of these codes are explained next.
4.1 Hill’s Problem

Equations 2.18, can be written in the variables $Q_1, Q_2, \dot{Q}_3, \dot{Q}_4$, as

$$
\dot{Q}_1 = \dot{P}_1 + 4 \left( Q_1 \dot{Q}_1 + Q_2 \dot{Q}_2 \right) Q_2 + 2 \left( Q_1^2 + Q_2^2 \right) \dot{Q}_2 =
= \dot{P}_1 + 4Q_1 Q_2 P_1 + 8 \left( Q_1^2 + Q_2^2 \right) Q_1 Q_2^2 + 4Q_2^2 P_2 - 8 \left( Q_1^4 + Q_2^4 \right) Q_1 Q_2^2 + 2 \left( Q_1^2 + Q_2^2 \right) \dot{Q}_2 =
= \dot{P}_1 + 4Q_1 Q_2 P_1 + 4Q_2^2 P_2 + 2 \left( Q_1^2 + Q_2^2 \right) \dot{Q}_2 =
= -Q_1 + 8 \left( Q_1^2 + Q_2^2 \right) \dot{Q}_2 - 6 \left( Q_1^2 + Q_2^2 \right) P_2 + 12 \left( Q_1^2 + Q_2^2 \right)^2 Q_2 + 2 \left( Q_1^2 + Q_2^2 \right) P_2 + 4 \left( Q_1^2 + Q_2^2 \right) P_2 + 12Q_1 \left( 2Q_1^4 - 4Q_1^2 Q_2^2 - 2Q_2^4 \right) =
= -Q_1 + 8 \left( Q_1^2 + Q_2^2 \right) \dot{Q}_2 + 12Q_1 \left( 3Q_1^4 - 2Q_1^2 Q_2^2 - Q_2^4 \right)
$$

(4.1)

and as $\dot{Q}_2 = P_2 - 2 \left( Q_1^2 + Q_2^2 \right) Q_1$

$$
\dot{Q}_2 = \dot{P}_2 - 4 \left( Q_1 \dot{Q}_1 + Q_2 \dot{Q}_2 \right) Q_1 - 2 \left( Q_1^2 + Q_2^2 \right) \dot{Q}_1 =
= \dot{P}_2 - 4Q_1^2 P_1 - 8Q_1^2 Q_2 \left( Q_1^2 + Q_2^2 \right) - 4Q_1 Q_2 P_2 + 8Q_1^2 Q_2 \left( Q_1^2 + Q_2^2 \right) - 2 \left( Q_1^2 + Q_2^2 \right) \dot{Q}_1 =
= \dot{P}_2 - 4Q_1^2 P_1 - 4Q_1 Q_2 P_2 + 12Q_2 \left( 2Q_2^4 - 4Q_1^2 Q_2^2 - 2Q_1^4 \right) -
\ - Q_2 \ln \left( Q_1^2 + Q_2^2 \right) P_1 + 12Q_1 \left( 2Q_2^4 - 4Q_1^2 Q_2^2 - 2Q_1^4 \right) P_1 -
\ - 4 \left( Q_1^2 + Q_2^2 \right) \left( Q_1^2 + Q_2^2 \right) P_1 + 12Q_2 \left( 2Q_2^4 - 4Q_1^2 Q_2^2 - 2Q_1^4 \right) =
= -Q_2 - 8 \left( Q_1^2 + Q_2^2 \right) \dot{Q}_1 + 12Q_2 \left( 3Q_1^4 - 2Q_1^2 Q_2^2 - Q_2^4 \right)
$$

(4.2)

And rewriting it as a first order system with $y_1 = Q_1, y_2 = Q_2, y_3 = \dot{Q}_1$, and $y_4 = \dot{Q}_2$,

$$
\begin{align*}
\dot{y}_1 &= y_3 \\
\dot{y}_2 &= y_4 \\
\dot{y}_3 &= -y_1 + 8 \left( y_1^2 + y_2^2 \right) y_4 + 12y_1 \left( 3y_4^4 - 2y_1^2 y_2^2 - y_2^4 \right) \\
\dot{y}_4 &= -y_2 - 8 \left( y_1^2 + y_2^2 \right) y_3 + 12y_2 \left( 3y_4^4 - 2y_1^2 y_2^2 - y_1^4 \right)
\end{align*}
$$

(4.3)
and observe that these equations are less expensive, computationally speaking, than (2.18). And from (2.17), as the Hamiltonian can be written in terms of the new variables
\[ H = \frac{\dot{Q}_1^2}{2} + \frac{\dot{P}_1^2}{2} + 2 (Q_1^2 + Q_2^2) (Q_2 P_1 - Q_1 P_2) + 2 (Q_1^6 + Q_2^6 + 3 Q_1^2 Q_2^2 + 3 Q_1^2 Q_2^2) \]
and replacing into the Hamiltonian with the \( y \)'s, gives
\[ H = \frac{1}{2} \left( y_1^2 + y_3^2 + y_4^2 \right) - 6 \left( y_1^2 + y_2^2 \right) \left( y_1^2 - y_2^2 \right)^2. \]

Note that the system in these variables is not Hamiltonian, but this expression is useful to find the zvc's.

### 4.2 Zero Velocity Curves and Poincaré Map

Choosing \( \Sigma = \{Q_2 = 0\} \) as Poincaré section, over this section the value of the 'energy' (4.4) is
\[ h = \frac{1}{2} \left( y_1^2 + y_3^2 + y_4^2 \right) - 6 y_1^6. \]

This allow us to calculate the zvc's for a particular energy \( h \) over the section \( \Sigma \). These curves, which are the projection of the boundary Hill’s region over \( \Sigma \), lets us know the possible regions of motion over the Poincaré section, considering now the discrete map \( P \) (Poincaré first return map).

We are interested in the curves when \( \dot{y}_1 = 0 \), because over these curves, the Poincaré region can’t be crossed. If \( 0 < h \leq \frac{1}{18} \), the zvc’s delimit a closed section. To prove this write \( y_3 = \pm \sqrt{h + 6 y_1^6 - \frac{1}{2} y_1^2} \) and note that both curves are symmetrical, so if they reach \( y_3 = 0 \) they intersect. Then we need,
\[ h + 6 y_1^6 - \frac{1}{2} y_1^2 = 0 \Rightarrow 12 y_1^6 - y_1^2 + 2 h = 0. \]

and taking \( x = y_1^2 \) it becomes a cubic \( 12 x^3 - x + 2 h = 0 \).

In general, the discriminant of a cubic \( a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0 \) is
\[ \Delta = 18 a_1 a_2 a_3 a_4 - 4 a_2^3 a_4 + a_2^2 a_3^2 - 4 a_1 a_3^3 - 27 a_1^2 a_4^2, \]
so in our case \( \Delta = 48 - 15,552 h^2 \). So \( \Delta \geq 0 \) means \( h < \frac{1}{18} \) and there are three real roots. Applying Descartes’s Rule of Signs, the polynomial has at most one negative root, so it
has two positive, and the two zvc’s intersect in two points delimiting a closed region of Figures 4.1 and 4.2. This is the lenticular shape of Figures 2.5 and 2.4.

If \( h > \frac{1}{18} \) the discriminant tells that there is only one real root. But for \( h > 0 \) there is always a negative root (see that if \( x = 0 \) the polynomial equals \( 2h > 0 \) but for \( x \) negative and big enough in absolute value, the polynomial is negative). In this case the zvc’s don’t intersect and there is not close component (Figure 4.3).

If \( h > \frac{1}{18} \), the iterations of \( P \) may escape the initial region. This only happens for the direct orbits. The tori of the retrograde orbit don’t break into chaotic behavior, so these trajectories still keep themselves confined. On the other side, the chaotic region corresponding to the direct orbit is almost empty for \( h \approx 1.024/18 \) (Figure 4.4).

Check also, that when \( h = \frac{1}{18} \), the positive roots of \( 12x^3 - x + 2h = 0 \) are \( x = \frac{1}{6} \) double. Then \( y_1 = \pm 6^{1/6} \) and the extremes of the lenticular shape coincide with \( L_1, L_1' \).

In \( \Sigma \), the fixed points (corresponding to the periodic orbits), can appear only for
\( y_3 = \dot{Q}_1 = 0 \). The reason is the symmetry of the problem already discussed. A trajectory will connect to its symmetric counterpart into a single orbit only when this happens.

### 4.3 Variational Equations

The numerical integration of system 4.3 has been done along its variational equations:

\[
\begin{bmatrix}
\dot{y}_5 & \dot{y}_6 & \dot{y}_7 & \dot{y}_8 \\
\dot{y}_9 & \dot{y}_{10} & \dot{y}_{11} & \dot{y}_{12} \\
\dot{y}_{13} & \dot{y}_{14} & \dot{y}_{15} & \dot{y}_{16} \\
\dot{y}_{17} & \dot{y}_{18} & \dot{y}_{19} & \dot{y}_{20}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial y_3} & \frac{\partial f_3}{\partial y_4} \\
\frac{\partial f_4}{\partial y_1} & \frac{\partial f_4}{\partial y_2} & \frac{\partial f_4}{\partial y_3} & \frac{\partial f_4}{\partial y_4}
\end{bmatrix}
\begin{bmatrix}
y_5 & y_6 & y_7 & y_8 \\
y_9 & y_{10} & y_{11} & y_{12} \\
y_{13} & y_{14} & y_{15} & y_{16} \\
y_{17} & y_{18} & y_{19} & y_{20}
\end{bmatrix}
\] (4.5)

The variational equations, in a Hamiltonian system, allow us to check errors of integration. In a Hamiltonian system, the determinant of the solution of the variational equations is always 1.

But the main reason to integrate them, is that we need them to compute periodic orbits with the method of continuation. Expanding (4.5) the variational equations of the problem are obtained in (4.6).
4.3 Variational Equations

\[
\begin{align*}
\dot{y}_5 &= y_{13}, \quad \dot{y}_9 = y_{17} \\
\dot{y}_6 &= y_{14}, \quad \dot{y}_{10} = y_{18} \\
\dot{y}_7 &= y_{15}, \quad \dot{y}_{11} = y_{19} \\
\dot{y}_8 &= y_{16}, \quad \dot{y}_{12} = y_{20} \\
\dot{y}_{13} &= (-1 + 16x_4x_1 + 36x_1^4 - 24x_1^2x_2^2 - 12x_2^4 + 12x_1(12x_1^3 - 4x_1x_2^2))x_5 + \\
&\quad + (16x_4x_2 + 12x_1(-4x_1^2x_2 - 4x_2^3))x_9 + (8x_1^2 + 8x_2^2)x_{17} \\
\dot{y}_{14} &= (-1 + 16x_4x_1 + 36x_1^4 - 24x_1^2x_2^2 - 12x_2^4 + 12x_1(12x_1^3 - 4x_1x_2^2))x_6 + \\
&\quad + (16x_4x_2 + 12x_1(-4x_1^2x_2 - 4x_2^3))x_{10} + (8x_1^2 + 8x_2^2)x_{18} \\
\dot{y}_{15} &= (-1 + 16x_4x_1 + 36x_1^4 - 24x_1^2x_2^2 - 12x_2^4 + 12x_1(12x_1^3 - 4x_1x_2^2))x_7 + \\
&\quad + (16x_4x_2 + 12x_1(-4x_1^2x_2 - 4x_2^3))x_{11} + (8x_1^2 + 8x_2^2)x_{19} \\
\dot{y}_{16} &= (-1 + 16x_4x_1 + 36x_1^4 - 24x_1^2x_2^2 - 12x_2^4 + 12x_1(12x_1^3 - 4x_1x_2^2))x_8 + \\
&\quad + (16x_4x_2 + 12x_1(-4x_1^2x_2 - 4x_2^3))x_{12} + (8x_1^2 + 8x_2^2)x_{20} \\
\dot{y}_{17} &= (-16x_3x_1 + 12x_2(-4x_1x_2^2 - 4x_1^3))x_5 + \\
&\quad + (-1 - 16x_3x_2 + 36x_2^2 - 24x_1^2x_2^2 - 12x_1^4 + 12x_2(12x_2^3 - 4x_1^2x_2))x_9 + \\
&\quad \quad + (-8x_1^2 - 8x_2^2)x_{13} \\
\dot{y}_{18} &= (-16x_3x_1 + 12x_2(-4x_1x_2^2 - 4x_1^3))x_6 + \\
&\quad + (-1 - 16x_3x_2 + 36x_2^4 - 24x_1^2x_2^2 - 12x_1^4 + 12x_2(12x_2^3 - 4x_1^2x_2))x_{10} + \\
&\quad \quad + (-8x_1^2 - 8x_2^2)x_{14} \\
\dot{y}_{19} &= (-16x_3x_1 + 12x_2(-4x_1x_2^2 - 4x_1^3))x_7 + \\
&\quad + (-1 - 16x_3x_2 + 36x_2^4 - 24x_1^2x_2^2 - 12x_1^4 + 12x_2(12x_2^3 - 4x_1^2x_2))x_{11} + \\
&\quad \quad + (-8x_1^2 - 8x_2^2)x_{15} \\
\dot{y}_{20} &= (-16x_3x_1 + 12x_2(-4x_1x_2^2 - 4x_1^3))x_8 + \\
&\quad + (-1 - 16x_3x_2 + 36x_2^4 - 24x_1^2x_2^2 - 12x_1^4 + 12x_2(12x_2^3 - 4x_1^2x_2))x_{12} + \\
&\quad \quad + (-8x_1^2 - 8x_2^2)x_{16}
\end{align*}
\]

(4.6)
and defining

\[
\begin{align*}
A &= (-1 + 16x_4x_1 + 36x_1^4 - 24x_1^2x_2^2 - 12x_2^4 + 12x_1(12x_1^3 - 4x_1x_2^2)) \\
B &= (16x_4x_2 + 12x_1(-4x_1^2x_2 - 4x_2^3)) \\
C &= (8x_1^2 + 8x_2^2) \\
D &= (-16x_3x_1 + 12x_2(-4x_1x_2^2 - 4x_1^3)) \\
E &= (-1 - 16x_3x_2 + 36x_2^4 - 24x_1^2x_2^2 - 12x_1^4 + 12x_2(12x_2^3 - 4x_1^2x_2))
\end{align*}
\]

they can be written in a more compact form:

\[
\begin{align*}
\dot{y}_5 &= y_{13} \\
\dot{y}_6 &= y_{14} \\
\dot{y}_7 &= y_{15} \\
\dot{y}_8 &= y_{16} \\
\dot{y}_{13} &= Ax_5 + Bx_9 + Cx_{17} \\
\dot{y}_{14} &= Ax_6 + Bx_{10} + Cx_{18} \\
\dot{y}_{15} &= Ax_7 + Bx_{11} + Cx_{19} \\
\dot{y}_{16} &= Ax_8 + Bx_{12} + Cx_{20}
\end{align*}
\]

\begin{align*}
\dot{y}_9 &= y_{17} \\
\dot{y}_{10} &= y_{18} \\
\dot{y}_{11} &= y_{19} \\
\dot{y}_{12} &= y_{20} \\
\dot{y}_{17} &= Dx_5 + Ex_9 - Cx_{13} \\
\dot{y}_{18} &= Dx_6 + Ex_{10} - Cx_{14} \\
\dot{y}_{19} &= Dx_7 + Ex_{11} - Cx_{15} \\
\dot{y}_{20} &= Dx_8 + Ex_{12} - Cx_{16}
\end{align*}

\[\text{(4.7)}\]

### 4.4 Continuation of Periodic Orbits

Continuation of periodic orbits has been performed with the method of continuation. It consists in solving the system of equations,

\[
\begin{align*}
\mathcal{H}(x) - h &= 0 \\
g(x) &= 0 \\
\Phi_T(x) - x &= 0
\end{align*}
\]

for \(h, T, x\). The method assumes that we start from a known periodic orbit. For the case of Hill’s orbits, the information of Proposition 2.7.1 has proven enough to obtain an initial seed that grants convergence of the method for direct and retrograde orbits. From a known periodic orbit, we can modify the energy a little and apply the method to obtain the next one.

The first equation ensures that the obtained orbits have the desired energy level. In the
second equation
\[ \{ g(x) = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 = 0 \} = \Sigma \]
is the chosen Poincaré section (in our case \( y_2 = 0 \)). And of course the third equation
imposes that the orbit has to be periodic (\( \Phi_t \) is the flow). So we have to solve
\[
G(Z) = \begin{bmatrix} \mathcal{H}(x) - h \\ g(x) \\ \Phi_T(x) - x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
where \( Z = (h, T, x)^T \) and \( x = (y_1, y_2, y_3, y_4) \), by means of Newton’s method:
\[
DG(Z^k) \Delta Z = G(Z^k), \quad Z^{k+1} = Z^k - \Delta Z.
\]
and
\[
DG(Z) = \begin{bmatrix} -1 & 0 & \mathcal{H}_{y_1} & \mathcal{H}_{y_2} & \mathcal{H}_{y_3} & \mathcal{H}_{y_4} \\ 0 & 0 & c_1 & c_2 & c_3 & c_4 \\ 0 & f_1(\Phi_T(x)) & \frac{\partial \Phi_1}{\partial y_1} - 1 & \frac{\partial \Phi_1}{\partial y_2} & \frac{\partial \Phi_1}{\partial y_3} & \frac{\partial \Phi_1}{\partial y_4} \\ 0 & f_2(\Phi_T(x)) & \frac{\partial \Phi_2}{\partial y_1} & \frac{\partial \Phi_2}{\partial y_2} - 1 & \frac{\partial \Phi_2}{\partial y_3} & \frac{\partial \Phi_2}{\partial y_4} \\ 0 & f_3(\Phi_T(x)) & \frac{\partial \Phi_3}{\partial y_1} & \frac{\partial \Phi_3}{\partial y_2} & \frac{\partial \Phi_3}{\partial y_3} - 1 & \frac{\partial \Phi_3}{\partial y_4} \\ 0 & f_4(\Phi_T(x)) & \frac{\partial \Phi_4}{\partial y_1} & \frac{\partial \Phi_4}{\partial y_2} & \frac{\partial \Phi_4}{\partial y_3} & \frac{\partial \Phi_4}{\partial y_4} - 1 \end{bmatrix}
\]
As we said, \( \Delta h = 0 \), so we drop this variable -and first column of (4.10)-, so we will solve
\[
DG^\sim(Y^k) \Delta Y = G(Y^k), \quad Y^{k+1} = Y^k - \Delta Y
\]
with \( Y = (T, x)^T \), and
\[
DG^\sim(Y) = \begin{bmatrix} 0 & \mathcal{H}_{y_1} & \mathcal{H}_{y_2} & \mathcal{H}_{y_3} & \mathcal{H}_{y_4} \\ 0 & c_1 & c_2 & c_3 & c_4 \\ f_1(\Phi_T(x)) & \frac{\partial \Phi_1}{\partial y_1} - 1 & \frac{\partial \Phi_1}{\partial y_2} & \frac{\partial \Phi_1}{\partial y_3} & \frac{\partial \Phi_1}{\partial y_4} \\ f_2(\Phi_T(x)) & \frac{\partial \Phi_2}{\partial y_1} & \frac{\partial \Phi_2}{\partial y_2} - 1 & \frac{\partial \Phi_2}{\partial y_3} & \frac{\partial \Phi_2}{\partial y_4} \\ f_3(\Phi_T(x)) & \frac{\partial \Phi_3}{\partial y_1} & \frac{\partial \Phi_3}{\partial y_2} & \frac{\partial \Phi_3}{\partial y_3} - 1 & \frac{\partial \Phi_3}{\partial y_4} \\ f_4(\Phi_T(x)) & \frac{\partial \Phi_4}{\partial y_1} & \frac{\partial \Phi_4}{\partial y_2} & \frac{\partial \Phi_4}{\partial y_3} & \frac{\partial \Phi_4}{\partial y_4} - 1 \end{bmatrix}
\]
As this system is over-determined, we will use the Least Squares Method to solve it at each Newton step. This implies that the solution will be approximated. But the residuals
will be zero as long as the orbit can be continued, because there is an exact solution for each value of $h$.

Fig. 4.5 Continuation of direct p.o up to the bifurcation (Hill)

4.5 Integration of Invariant Manifolds

To compute the projection of the invariant manifolds (stable and unstable) over the Poincaré section $\Sigma = \{ Q_2 = 0 \}$ the following method has been followed. We take the initial conditions of a periodic orbit for a particular value of the energy $h$ (these initial conditions were obtained by means of the method of continuation of section 4.4), and integrate the orbit. The data points of the orbit are kept in an output file. In our case, the p.o has been integrated in 5.500 steps. Each of these points is a fixed point for a given Poincaré map with a different Poincaré section. For each of these points, in the direction of the stable/unstable manifolds of the linearized system, with a separation of $s = 10^{-6}$ from the fixed point, we integrate forward or backward in time as needed until the integrated orbit hits $\Sigma$. Up to 14 hits are counted. Each of the hits is stored in another data file which is the one we plot. The implemented program, when executed will ask the user for the chosen manifold (stable/unstable) and the branch of the manifold ($Q_2 > 0$ or $Q_2 < 0$) to be computed.

4.6 Computation of the Angle of Intersection of the Manifolds

To compute the angle of intersection of the invariant manifolds, over a Poincaré section $\Sigma$ we will use the following procedure. Because of the invariance of the manifolds $P(\gamma(t)) = \gamma(\lambda t)$, where $\gamma(t)$ is a parameterization of the manifold.
So $DP(\gamma(t))\gamma'(t) = \lambda\gamma'(\lambda t)$. So if $\gamma'_0$ is the initial direction of the manifold from the fixed point given by the associated eigenvalue, then $DP(x^\dagger)\gamma'_0$, with $x^\dagger$ a point a little away of the equilibrium $x^*$, is the tangent vector of the manifold in $P(x^\dagger)$. Iterating in this manner while we iterate the Poincaré map, we can know at each of the points of the chosen manifold its tangent vector. From this the computation of the angle between two manifolds at a given point is trivial.

But to perform this computation it is needed the differential of the Poincaré map. This is the way it is computed:

From $P(Y) = \Phi(\tau(Y), Y)$ where $\Phi$ is the flux of $\dot{x} = f(x)$, $\tau$ the return time of the Poincaré map, and $Y$ an arbitrary point over $\Sigma$, differentiate

$$DP(Y) = \frac{d}{dt} \Phi(\tau(Y), Y) D\tau(Y) + D\Phi(\tau(Y), Y) =$$

$$= f(\Phi(\tau(Y), Y)) D\tau(Y) + D\Phi(\tau(Y), Y). \quad (4.12)$$

Of the above expression the only thing which is not known is $D\tau(Y)$, as $D\Phi(\tau(Y), Y)$ is obtained by integrating the variational equations (4.7) as it has already been discussed. Defining the Poincaré section as $\Sigma = \{x| g(\Phi(t, x)) = 0\}$, we have

$$0 = g(\Phi(\tau(Y), Y)) \quad \Rightarrow \quad Dg(\Phi(\tau(Y), Y)) \cdot D\Phi(\tau(Y), Y) = 0$$

and

$$0 = Dg(\Phi(\tau(Y), Y)) \cdot D\Phi(\tau(Y), Y) = Dg(\Phi(\tau(Y), Y)) \cdot DP(Y) =$$

$$= Dg(\Phi(\tau(Y), Y)) \cdot [f(\Phi(\tau(Y), Y)) D\tau(Y) + D\Phi(\tau(Y), Y)]. \quad (4.13)$$

Now

$$0 = Dg(\Phi(\tau(Y), Y)) \cdot f(\Phi(\tau(Y), Y)) D\tau(Y) + Dg(\Phi(\tau(Y), Y)) \cdot D\Phi(\tau(Y), Y),$$

and

$$D\tau(Y) = -\frac{Dg(\Phi(\tau(Y), Y)) \cdot D\Phi(\tau(Y), Y)}{Dg(\Phi(\tau(Y), Y)) \cdot f(\Phi(\tau(Y), Y))} \quad (4.14)$$

where everything is known. With (4.12) and (4.15) we can write

$$DP(Y) = -f(\Phi(\tau(Y), Y)) \frac{Dg(\Phi(\tau(Y), Y)) \cdot D\Phi(\tau(Y), Y)}{Dg(\Phi(\tau(Y), Y)) \cdot f(\Phi(\tau(Y), Y))} + D\Phi(\tau(Y), Y). \quad (4.15)$$
As we are interested in computing the angle of intersection in the $x$ axis of the Poincaré section, we need to find Poincaré iterates close enough to the axis. This is done using a bisection method in which the proper $s$ is found (see section 4.4), that after a given number of Poincaré iterations it falls close to the axis with an error of $\approx 10^{-12}$. So starting with an initial range $s_1$ small enough to guarantee a small error, using $s_1$ and integrating $P$ until the axis is crossed, and choosing $s_0$ such that for the same number of iterations the axis is not crossed, it follows that in the range $(s_0, s_1)$ there is an $s$ for which the iterate falls exactly on the axis. Through the bisection method this $s$ is approximated.
Appendix: Results & Theorems

In this appendix, some of the results used during the project are included. Some of these results are proven while others are left as known theoretical theorems of dynamical systems.

1  Lyapunov Center Theorem

Definition 1 Let \( \dot{x} = f(x) \), with \( f \) smooth, be a system with an equilibrium point \( x^* \). If \( \frac{\partial f(x^*)}{\partial x} \) is non-singular, or equivalently all the exponents are non-zero, then the equilibrium point is said to be elementary.

Definition 2 Let \( \dot{x} = f(x) \), with \( f \) smooth, be a system with a \( T \)-periodic solution \( \phi(t, x^*, \nu^*) \). The periodic solution is said to be elementary if 1 is eigenvalue of the monodromy matrix \( \frac{\partial \phi(T, x^*, \nu^*)}{\partial x} \) with multiplicity one for the general case, and with multiplicity two, in the case the system has a first integral.

Theorem 1 (Cylinder Theorem): An elementary periodic orbit of a system with integral \( \mathcal{F} \), lies in a smooth cylinder of periodic solutions parameterized by the integral \( \mathcal{F} \).

Proof:
Let \( P \) be the Poincaré map with a cross section \( \Sigma \). In some coordinates (see Theorem 8.3.2 in [9]), the Poincaré map is a function of \( e = F(y) \) and other \( m - 2 \) variables (in dimension \( m \)), say \( y_1, \ldots, y_m \). So we can write \( P(e, y) = (e, Q(e, y)) \). Fixing an energy level \( e \) the Implicit function theorem can be applied to \( m(e, y) = Q(e, y) - y \). We have \( m(e, y^*) \), where \( y^* \) is the fixed point in the Poincaré map of the corresponding periodic orbit, and \( \frac{\partial m(e, y^*)}{\partial y} \neq 0 \), because by fixing an energy level, one of the trivial eigenvalues of the monodromy matrix has vanished. So \( m(e, y(e)) = 0 \) in a neighborhood of the periodic orbit.
Proposition 1 Let $\dot{x} = f(x, \nu)$, with $f$ smooth, be a system with an elementary equilibrium point, or an elementary equilibrium periodic solution, and assume it has a first integral. $\nu$ is a parameter. Then the elementary equilibrium point or the elementary periodic solution can be continued.

Proof:
For the case of the equilibrium point $(x^*, \nu^*)$, we have $f(x^*, \nu^*) = 0$. And because of Definition 1, $\frac{\partial f(x^*)}{\partial x} \neq 0$. So the Implicit Function Theorem can be applied to obtain $f(u(\nu), \nu) = 0$ in a neighborhood of the equilibrium.

For periodic orbits, consider $P(x, \nu)$ the Poincaré map with a cross section $\Sigma$. And now consider $Q(x, \nu)$, the Poincaré map in the integral surface corresponding to the periodic solution when $\nu = \nu^*$ as in Theorem 1. Then $m(x, \nu) = Q(x, \nu) - x$ satisfies $m(x^*, \nu^*) = 0$, and $\frac{\partial m(x^*, \nu^*)}{\partial x} \neq 0$. And we can apply the Implicit function theorem to obtain $m(x(\nu), \nu) = 0$ in a neighborhood of the equilibrium.

Theorem 2 (Lyapunov Center Theorem):
Let $\dot{x} = f(x)$, with $f$ smooth, be a system with a first integral and an equilibrium $x^*$, with characteristic exponents $\pm i\omega, \lambda_3, \ldots, \lambda_m$ where $i\omega \neq 0$ is pure imaginary, and $\frac{\lambda_j}{i\omega} \notin \mathbb{Z}$ for $j = 3, \ldots, m$.

Then, there exists a one parameter family of periodic orbits, emanating from $x^*$. Moreover, when approaching $x^*$, the periods of the orbits tend to $\frac{2\pi}{\omega}$.

Proof:
Assume that $x^* = 0$. We have $f(x^*) = 0$. Taylor expanding $f$ around the equilibrium we get $f(x) = Ax + g(x)$, where $A = Df(0)$, $g(x) = \mathcal{O}(x^2)$ and $g(0) = Dg(0) = 0$. So $\dot{x} = Ax + g(x)$. Now we introduce a small parameter with the change of variables $x = \epsilon y$.

$$\epsilon \dot{y} = \epsilon Ay + g(\epsilon y)$$

or

$$\dot{y} = Ay + \frac{1}{\epsilon} g(\epsilon y) \Rightarrow \dot{y} = Ay + \epsilon \tilde{g}(\epsilon y).$$
For $\epsilon = 0$ we have the linear system $\dot{y} = Ay$. And as $A$ has eigenvalues $\pm i\omega, \lambda_3, \ldots, \lambda_m$, we can write in some basis

$$A = \begin{bmatrix} 0 & -\omega & 0^T \\ \omega & 0 & 0^T \\ 0 & 0 & \tilde{A} \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} \cos \omega t & -\sin \omega t & 0^T \\ \sin \omega t & \cos \omega t & 0^T \\ 0 & 0 & e^{\tilde{A}t} \end{bmatrix}$$

and $e^{At}y_0$ is a periodic solution of period $T = \frac{2\pi}{\omega}$ of the linear system.

The monodromy matrix $\begin{bmatrix} 1 & 0 & 0^T \\ 0 & 1 & 0^T \\ 0 & 0 & e^{-2\pi i / \omega} \end{bmatrix}$ has eigenvalues $1, 1, e^{2\pi \lambda_3 / \omega}, \ldots, e^{2\pi \lambda_m / \omega}$. If $\frac{\lambda_j}{\omega} \notin \mathbb{Z}$ for $j = 3, \ldots, m$, then $e^{2\pi \lambda_j / \omega} \neq 1$ and the periodic orbit is elementary. So it can be continued (Proposition 1). And the continued orbits will have period $\tau(\epsilon) = \frac{2\pi}{\omega} + O(\epsilon)$.

\[ \square \]

2 First Order Averaging

**Theorem 3 (First Order Averaging Theorem):**

Consider the initial value problem

$$\frac{dx}{dt} = \epsilon f(t, x) + \epsilon^2 g(t, x, \epsilon), \quad x(t_0) = x_0$$

where $f$ is $T$ periodic. And consider the averaged problem

$$\frac{dy}{dt} = \epsilon f^0(y), \quad y(t_0) = x_0,$$

with $f^0(y) = \frac{1}{T} \int_0^T f(t, y) \, dt$ and $x, y, x_0 \in D \subset \mathbb{R}^n, t \in [t_0, \infty), \epsilon \in (0, \epsilon_0]$. And suppose that the following conditions are satisfied:

a) $f$ and $g$ and $\nabla f$ are defined, continuous and bounded by a constant $M$ independent of $\epsilon$, in $[t_0, \infty) \times D$.

b) $g$ is Lipschitz-continuous with respect to $x \in D$.

c) $f$ is $T$ periodic in $t$ with $T$ a constant independent of $\epsilon$.

d) $y(t)$ belongs to an ($\epsilon$-independent) interior subset of $D$ on the time scale $\frac{1}{\epsilon}$,
then

\[ x(t) - y(t) = O(\varepsilon) \text{ as } \varepsilon \to 0 \text{ on the time scale } \frac{1}{\varepsilon}. \]

### 3 Smale-Birkhoff

**Definition 3** Let \( M \) be a compact two-manifold and \( \text{Diff}^1(M) \) be the set of all diffeomorphisms on \( M \) that are \( C^1 \). If an element of such set has the property that all the fixed points are hyperbolic, and all the intersections of the stable and unstable manifolds are transverse, we say that it is a Kupta-Smale diffeomorphism.

**Theorem 4** *(Smale-Birkhoff Theorem):* Let \( f \in \text{Diff}^1(M) \) be a Kupta-Smale diffeomorphism, and \( x^\dagger \) be a transverse homoclinic point of a periodic point \( x^\ast \) of \( f \). Then there is a closed hyperbolic invariant set \( \Lambda \) of \( f^N \), containing \( x^\dagger \) such that is topologically conjugate to a shift of two symbols and \( f^p(\Lambda) = \Lambda \) for some \( p \in \mathbb{Z}^+ \).

We will use this theorem for the following argument. Let \( \Sigma_2 \) be the set of infinite sequences of two symbols:

\[ \Sigma_2 = \{ s = (\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots), \ s_i \in \{0, 1\}, n \in \mathbb{Z} \} \]

These sequences start from \( s_0 \) to the right and to the left. And let \( \sigma \) be the shift operation (to the left)

\[ \Sigma_2 \xrightarrow{\sigma} \Xi \]

\[ \sigma: (\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots) \rightarrow (\ldots, s_{-1}, s_0, s_1, s_2, s_3, \ldots). \]

Also a distance between two sequences is defined,

**Definition 4** Let \( s, \bar{s} \in \Sigma_2 \),

\[ d(s, \bar{s}) = \sum_{i \in \mathbb{Z}} \frac{|s_i - \bar{s}_i|}{2^{|i|}} \]

It is easily verified that \((\Sigma_2, d)\) is a metric space.

**Theorem 5** The discrete-time dynamical system \( \{\mathbb{Z}, \Sigma_2, \sigma^k\} \) has

- an infinite countable dense set of periodic orbits with arbitrarily long periods;
ii an infinite uncountable set of non periodic orbits;

iii a dense orbit, i.e. an orbit that passes arbitrarily close to any given sequence according to the distance \( d \):

\[ \exists s^* \in \Sigma \quad \text{s.t.} \quad \forall s \in \Sigma_2, \quad \forall \delta > 0 \quad \exists k \in \mathbb{Z} \quad \text{s.t.} \quad d(s, \sigma^k(s^*)) < \delta. \]

Because of Theorem 4, we have the following commutative diagram,

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{f} & \Lambda \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
\Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \\
\end{array}
\]

for some homeomorphism \( \Phi \), so \( \Lambda \) also satisfies Theorem 5 (see [11]). Theorem 5 can be taken as a definition of chaos. Indeed it is easy to see that the \( \{ \mathbb{Z}, \Sigma_2, \sigma^k \} \) has sensitivity to initial conditions. Take two different sequences \( s, \bar{s} \), no matter how close they are with respect to definition 4. If we iterate \( \sigma \) enough times, they will have separated by a fixed distance.

Also, it can be proven that the shift map with sequences of two symbols is topologically mixing according to the definition:

**Definition 5** Let \( X \) be a compact metric and \( f \) a continuous map. \( f \) is said to be topologically mixing if for any two non empty sets \( U, B \subset X \) there exists \( m \geq 0 \) s.t \( \forall n \geq m, f^n(U) \cap V \neq \emptyset \)

This implies that the next well known definition of chaos also applies to \( \Lambda \):

**Definition 6** A dynamical system, to be classified as chaotic must satisfy:

i it must be sensitive to initial conditions;

ii it must be topologically mixing;

iii it must have a dense periodic orbit.

## 4 Stable Manifold Theorem

**Theorem 6** *(Stable manifold theorem for flows)* Suppose that \( \dot{x} = f(x) \) has hyperbolic fixed \( \bar{x} \). Then there exist local stable and unstable manifolds \( W_{loc}^s(\bar{x}), W_{loc}^u(\bar{x}) \) of the same
dimensions $n_s, n_u$ as those of the eigenspaces $E^s, E^u$ of the linearized system, and tangent to $E^s, E^u$ at $\bar{x}$. $W^s_{loc}, W^u_{loc}$ are as smooth as the function $f$ (see [12] for all the details).

There is also a version of this theorem for maps. What is significant is that the fact that the manifolds are as smooth as the function involved, also happens in the discrete case. This implies that the manifolds over the Poincaré map for a section $\Sigma$ regarding the flow of $\dot{x} = f(x)$ are also as smooth as $f$. By ‘as smooth as $f$’ it is meant that if $f \in C^k$ then the manifolds will also be $C^k$, but even more, if $f$ is analytic, then the manifolds will be analytic (in the case of flows and over the Poincaré section as well):

**Theorem 7 (Stable manifold theorem for maps)** Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with a hyperbolic fixed point $\bar{x}$. Then there are local stable and unstable manifolds $W^s_{loc}(\bar{x}), W^u_{loc}(\bar{x})$, tangent to the eigenspaces $E^s_{\bar{x}}, E^u_{\bar{x}}$ of $DG(\bar{x})$ at $\bar{x}$ and of corresponding dimensions. $E^s_{\bar{x}}, E^u_{\bar{x}}$ of $DG(\bar{x})$ are as smooth as the map $G$. (see [12] for all the details).
References


[5] Szebehely & Zare *Time Transformations in the Extended Phase Space*, University of Texas, Austin 1974


