Appendix: Theoretical calculation of the multimode $g^{(2)}(T)$

Collett and Gardiner [2] calculate the Bogoliubov transformation that the annihilation and creation operators undergo while passing through the double-ended cavity containing nonlinear crystal (Fig. 1) that has only one resonance. The result cannot be directly applied in case the cavity has multiple resonances, therefore here we derive a Bogoliubov transformation for the cavity that has infinite number of equally spaced spectral modes. Next we present a calculation of the $g^{(2)}(T)$ correlation function for the multimode cavity that bases on the definition 2 in the thesis and the obtained Bogoliubov transformation.

1. Bogoliubov transformations for a multi-resonance cavity

Let us consider a two-sided ring cavity as in Fig. 1 (A). Amplitude transmission and reflection coefficients are t_1, t_2, r_1 and r_2. Cavity roundtrip time is denoted by τ. Intracavity field annihilation operator just before exiting the cavity is denoted as a, the input fields are a_{in} and b_{in} and the output field a_{out}. For an empty cavity we have the following relations:

$$\begin{aligned} &a(t) = r_1 r_2 a(t-\tau) + t_1 r_2 a_{in}(t-\tau) + t_2 b_{in}(t-\tau) \\ &a_{out}(t) = r_1 a_{in}(t) + t_1 a(t) \end{aligned} \tag{1}$$

Including the transformation of the operators due to squeezing inside the cavity (r is squeezing amplitude for the single pass through the crystal):

$$\begin{aligned} &a(t) = r_1 r_2 \cosh(r) a(t-\tau) + r_1 r_2 \sinh(r) a^\dagger(t-\tau) + \\ &+ t_1 r_2 \cosh(r) a_{in}(t-\tau) + t_1 r_2 \sinh(r) a_{in}^\dagger(t-\tau) + \\ &+ t_2 \cosh(r) b_{in}(t-\tau) + t_2 \sinh(r) b_{in}^\dagger(t-\tau) \end{aligned}$$

This yields the Bogoliubov transformation:

$$a_{out}(\omega) = A(\omega) a_{in}(\omega) + B(\omega) a_{in}^\dagger(-\omega) + C(\omega) b_{in}(\omega) + D(\omega) b_{in}^\dagger(-\omega) \tag{2}$$

Where:

$$\begin{aligned} &A(\omega) = d(\omega)(t_1^2 r_2 (e^{-i\omega\tau}\cosh(r) - r_1 r_2) + r_1 d(\omega)^{-1}) \\ &B(\omega) = d(\omega)(\sinh(r) t_1^2 r_2 e^{i\omega\tau}) \\ &C(\omega) = d(\omega)(t_2 t_1 (e^{-i\omega\tau}\cosh(r) - r_1 r_2) \\ &D(\omega) = d(\omega)(\sinh(r) t_2 t_1 e^{i\omega\tau}) \end{aligned} \tag{3}$$

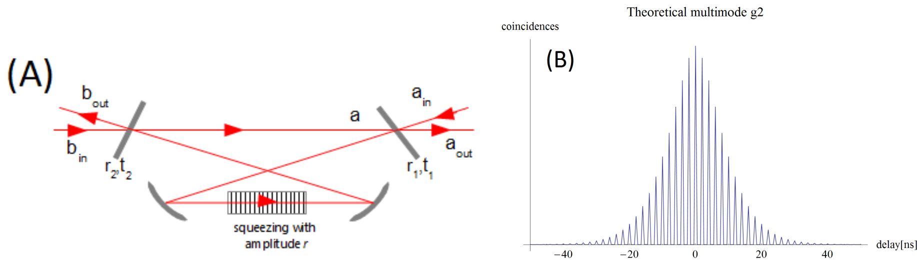

Figure 1. (A);Input, output and intracavity field operators for double-sided cavity with a nonlinear crystal inside. Operator a denotes intracavity field annihilation operator just before exiting the cavity. (B): $g^{(2)}(T)$ plotted from eq. 5

The pre-factor:

$$d(\omega) = \frac{1}{1 - 2r_1r_2\cos(\omega\tau)\cosh(r) + r_1^2r_2^2}$$

2. Multimode $g^{(2)}(T)$

The objective is to calculate $g^{(2)}(T)$ of the output field (for the vacuum inputs in both modes) which after expressing the electric field operator in eq.**??** in terms of creation and annihilation operators reduces to:

$$g^{(2)}(T) ={}_{b_{in}}\langle 0|_{a_{in}}\langle 0|a^\dagger_{out}(t)a^\dagger_{out}(t+T)a_{out}(t+T)a_{out}(t)|0\rangle_{a_{in}}|0\rangle_{b_{in}} \tag{4}$$

Knowing that:

$$a(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(\omega)e^{-i\omega t}d\omega$$
$$a^\dagger(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a^\dagger(-\omega)e^{-i\omega t}d\omega$$

We can use the operators in the Fourier domain:

$$g^{(2)}(T) = \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \int_{-\infty}^{\infty} d\omega_3 \int_{-\infty}^{\infty} d\omega_4 e^{-i(\omega_2+\omega_3)(t+T)}e^{-i(\omega_1+\omega_4)t}$$
$$\times_{b_{in}}\langle 0|_{a_{in}}\langle 0|a^\dagger_{out}(-\omega_1)a^\dagger_{out}(-\omega_2)a_{out}(\omega_3)a_{out}(\omega_4)|0\rangle_{a_{in}}|0\rangle_{b_{in}}$$

After the reduction of the operators we find:

$$\begin{aligned}
&{}_{b_{in}}\langle vac|_{a_{in}}\langle vac|a^\dagger_{out}(\omega_1)a^\dagger_{out}(\omega_2)a_{out}(\omega_3)a_{out}(\omega_4)|vac\rangle_{a_{in}}|vac\rangle_{b_{in}} = \\
&= \delta(\omega_1+\omega_2)\delta(\omega_3+\omega_4)[C^*(\omega_1)A^*(\omega_2) + D^*(\omega_1)B^*(\omega_2)][A(\omega_3)C(\omega_4) + B(\omega_3)D(\omega_4)] \\
&+\delta(\omega_2+\omega_3)\delta(\omega_1+\omega_4)[C^*(\omega_1)C(\omega_4) + D^*(\omega_1)D(\omega_4)][C^*(\omega_2)C(\omega_3) + D^*(\omega_2)D(\omega_3)] \\
&+\delta(\omega_1+\omega_3)\delta(\omega_2+\omega_4)[C^*(\omega_1)C(\omega_3) + D^*(\omega_2)D(\omega_3)][C^*(\omega_2)C(\omega_4) + D^*(\omega_2)D(\omega_4)]
\end{aligned}$$

This yields:

$$\begin{aligned}
g^{(2)}(T) &= \mathcal{F}[A(\omega)C(-\omega) + D(-\omega)B(\omega)](T)\mathcal{F}[A(\omega)C(-\omega) + D(-\omega)B(\omega)](-T) \\
&+(\mathcal{F}[C^*(\omega)C(\omega) + D^*(\omega)D(\omega)](0))^2 + (\mathcal{F}[C^*(\omega)C(\omega) + D^*(\omega)D(\omega)](T))^2
\end{aligned}$$

Let us express the pre-factor as a sum of Dirac delta functions:

$$\mathcal{F}[d(\omega)^2](T) = \sum_{k=-\infty}^{\infty} \delta(T-k\tau)F(k)$$

$$F(k) = \sum_{m=-\infty}^{\infty} \left(\frac{2r_1r_2\cosh(r)}{1+r_1^2r_2^2+\sqrt{(1+r_1^2r_2^2)^2-(2r_1r_2\cosh(r))^2}}\right)^{|k-m|}$$

$$\times\left(\frac{2r_1r_2\cosh(r)}{1+r_1^2r_2^2+\sqrt{(1+r_1^2r_2^2)^2-(2r_1r_2\cosh(r))^2}}\right)^{|m|}((1+r_1^2r_2^2)^2-(2r_1r_2\cosh(r))^2)^{-1}$$

Therefore the final expression:

$$g^{(2)}(T) = f_1^2(0) + \Sigma_{k=-\infty}^{\infty}\delta(T-k\tau)[f_1^2(k)+f_2(k)f_2(-k)] \tag{5}$$

Where

$$f_1(k) = t_1^2t_2^2([-r_1r_2\cosh(r)]F(k+1)+[-r_1r_2\cosh(r)]F(k-1)+$$
$$+[r_1^2r_2^2+\cosh(2r)]F(k))$$
$$f_2(k) = t_1t_2([-r_1^2r_2\cosh(r)^2]F(k-2)+$$
$$+[r_1(1+r_2^2(3r_1^2-1))\cosh(r)]F(k-1)+[-\frac{1}{2}r_2(r_1^2(3+2r_2^2(2r_1^2-1))+$$
$$+(3r_1^2-2)\cosh(2r)]F(k)+[r_1r_2^2(2r_1^2-1)\cosh(r)]F(k+1))$$

The $g^{(2)}(T)$ of the multimode cavity output has an envelope of the shape of double falling exponential and peaks approximately every cavity roundtrip time (plotted for our experimental conditions on Fig. 1 (B)), resulting from the interference between the modes. As opposed to that, the single mode $g^{(2)}(T)$ should also have the shape of double exponential decay (Fourier transform of a single Lorentzian peak being the power spectral density)[1].

References

[1] Y. J. Lu and Z. Y. Ou *Optical parametric oscillator far below threshold: Experiment versus theory,* Phys. Rev. A **62**, 033804 (2000).

[2] M. J. Collett and C. W. Gardiner *Squeezing of intracavity and traveling-wave light fields produced in parametric amplification,*Phys. Rev. A 30, 13861391 (1984)

[3] J. A. Zielińska, F. A. Beduini, N. Godbout, and M. W. Mitchell, *Ultranarrow Faraday rotation filter at the Rb D1 line,* arXiv:1110.2362 [physics.optics]