Title: Quantifying Optimal Capital Allocation Principles based on Risk Measures.

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Abstract

In this thesis we address the issue of covering risks by allocating capital and solving the so-called allocation problem. For this purpose, we provide functional closed-forms representations for each allocation principle built under the general framework developed by Dhaene et al. (2012). Furthermore, we assess the correlation effect which is considered to be the effect of changes in the allocated capital when changing the correlation between the losses, this effect arises when the sources of risk have different variances, otherwise correlations does not play any role in capital allocation results. We develop an R package called OCA which computes optimal Capital Allocations based on some standard principles such as Haircut, Overbeck type II and the Covariance Allocation Principle. Also it provides some functionalities for estimating two of the most popular risk measures: Value at Risk and Expectation Shortfall.

Keywords: Risk, Risk management, Optimal Capital Allocations, Allocation Principles, Value-at-Risk, Expected Shortfall

Notation

\( E(\cdot) \)  \( E(\cdot) \) Expected Value operator.
\( X_i \) Random variables with finite mean representing individual losses.
\( S \) Aggregate loss defined as \( \sum_{i=1}^{N} X_i \).
\( \rho(\cdot) \) A mapping function representing a risk measure.
\( K_i \) Non-negative real numbers representing individual capital to be allocated to the \( i \)-th business unit, \( i = 1, \ldots, N \).
\( K \) Aggregate capital to be split into \( K_i \) individual parts based on different forms of \( \rho(\cdot) \).
\( \text{VaR} \) Value at Risk.
\( \text{ES} \) Expected Shorfall.
\( \text{CTE} \) Conditional Tail Expectation.
\( \text{r.v.} \) Random variable.
\( \Gamma \) A real-valued random variable defined on a probability space \( (\Omega, F, \mathbb{P}) \).
\( \Omega \) Sample space.
\( F \) Set of all possible events (a \( \sigma \)-algebra).
\( \mathbb{P} \) Probability.
\( \inf \) Infimum.
\( p \) Probability such that \( p \in (0, 1) \).
\( F_{X_i}(x) \) Probability distribution function of a r.v. \( X \) defined as \( \mathbb{P}(X_i \leq x) \).
\( F_{X_i}^{-1}(p) \) Inverse of the distribution function, it is called the quantile function.
\( \Phi \) Standard normal distribution function.
\( \Phi^{-1}(p) \) The \( p \)-th quantile of \( \Phi \).
\( \phi \) Density of a standard normal distribution.
\( \mu \) Expected value of a r.v.
\( \sigma^2 \) Variance of a r.v.
\( \sigma \) Standard deviation of a r.v.
\( \text{Cov}(x,y) \) Covariance between two r.v.’s.
\( t_v \) Distribution function of a standard t with \( v \) degrees of freedom.
\( t_v^{-1}(p) \) The \( p \)-th quantile of \( t_v \).
\( g_v(\cdot) \) Density of a standard t-distribution with \( v \) degrees of freedom.
\( \mathbb{I}(\cdot) \) Indicator function that takes the value 1 if condition in (\cdot) is met, and takes the value zero otherwise.
\( v_i \) Measure of risk exposure.
\( \zeta_i \) Deviations of the losses from their respective allocated capital levels \( K_i \).
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1 Introduction

Risk management, among other tasks, evaluates the total capital requirements of a company and allocates it to its various business units. A natural question is: how should a given a mount of capital be allocated among the different business lines belonging to a company? To accomplish this, several allocation principles have been developed in the literature, most of them are based on risk measures sourced either by internal or external factors of the company.

Companies wish to allocate capital to their business units for solvency reasons, i.e. banks and insurance companies are legally required to set aside some amount of capital in order to remain solvent. Also capital allocation can be a useful tool for performance measurement and designing incentives schemes as managers’ performance can be assessed by the amount of capital allocated to their business units. Profit-and-loss analysis under loan pricing context and under general investment purposes are another reasons that motivate companies to carry out capital allocations.

Covering risks by allocating capital is the target of this thesis and the main problem to be solved is the so-called allocation problem. Based on the general framework proposed by Dhaene et al. (2012) we provide explicit formulations for the proportion of capitals the manager should allocate on different risk sources based on a wide variety of risk measures.

In this thesis we are particularly interested in providing the exact functional forms of each allocation principle and also paying carefully attention to the numerical part, we analyse the “correlation effect” on the allocation principles. Correlation effect is considered to be the effect of changes in the allocated capital suggested by each principle when changing the correlation between the losses.

Our findings suggest that correlation effect does not play any role when losses are characterized by the same two distributional moments (mean and variance), nevertheless when variances differ while means remain the same some important differences arise when correlation goes from 0 to 1 leading to a clear correlation effect.

This thesis has entirely developed an R package, which has been called OCA package. This package computes Optimal Capital Allocations (OCA) based on some standard principles such as Haircut, Overbeck type II and the Covariance Allocation Principle. Also it provides some short-cuts for obtaining two of the most popular risk measures: Value at Risk and Expectation Shortfall.

The remainder of this thesis is arranged as follows: section 2 discusses formally what the allocation problem is, section 3 characterizes coherent risk measures by providing its properties, then section 4 presents some well-know risk measures. Allocation principles are presented in section 5 while the general framework for capital allocation, based on Dhaene et al. (2012), is discussed in section 6. Numerical applications and simulations are reported in section 7 and section 8. Some concluding remarks are in section 9. The manual for OCA package as well as the R codes for reproducibility of this work are presented in section 10.

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1Note that capital allocation is the purpose of this work and we do not attempt going into details on how to determine the economic capital to be allocated. We assume this capital is known and given, we are after a way to determine the optimal proportions of this given capital for allocating them among different risk sources of the enterprise.
2 The Allocation Problem

Capital Allocation is a term referring to the subdivision of the aggregate capital held by the firm across its various constituents, for example, business lines, type of exposure, territories, or even individual products in a portfolio of insurance policies. This capital is often referred to as Economic Capital (EC) and is defined as the \( p \)-quantile of the loss distribution minus the expected value of the loss distribution (Overbeck, 2000), formally this is:

\[
EC(p) = F_{S}^{-1}(p) - E(S) \quad \text{with,}
\]

\[
F_{S}^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_{S}(s) \geq p\}, \quad p \in (0, 1).
\]

Since this definition of \( EC(P) \) does not account for “bad times” episodes, then it is viewed as an “all or nothing” rule for capital definition. An alternative definition, according to Overbeck (2000), tries to incorporate such “bad times” in its formulation and treat it as a more “optimistic” event, this definition states that \( EC \) must be:

\[
EC_{K} = E(S|S > K),
\]

where this definition considers Economic Capital in average also enough to cushion losses even in bad times. Note that capital allocations in subsubsection 6.2.2 are based on this capital definition.

Once the capital is defined, we have to define its counterpart, the loss. Consider a portfolio of \( n \) individual losses (random variables) \( X_1, X_2, \ldots, X_n \) materializing at a fixed future date \( T \). Assume that \((X_1, X_2, \ldots, X_n)\) is a random vector on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Throughout this thesis, we will always assume that any loss \( X_i \) has a finite mean. The distribution function \( \mathbb{P}(X_i \leq x) \) of \( X_i \) will be denoted by \( F_{X_i}(x) \).

The aggregate loss is defined by the sum of the individual losses:

\[
S = \sum_{i=1}^{n} X_i, \quad (1)
\]

where this aggregate loss can be interpreted as:

1. the total loss of a corporation, for example, an insurance company, with the individual losses corresponding to the losses of the respective business unit,

2. the loss from an insurance portfolio, with the individual losses being those arising from the different policies; or

3. the loss by a financial conglomerate, while the different individual losses correspond to the losses suffered by its subsidiaries.

Following Dhaene et al. (2012) it is the first of these interpretations we will use throughout this thesis.

Hence, \( S \) is the aggregate loss faced by an insurance company and \( X_i \) is the loss of business unit \( i \).
In order to clarify what the allocation problem is, one can view the problem from another perspective, namely, consider an investor who can invest in a fixed set of \( n \) different investment possibilities with losses represented by the random variables \( X_1, X_2, \ldots, X_n \). We have the following economic interpretations depending on the area of application (McNeil et al., 2005):

1. **Performance measurement.** Here the investor is a financial institution and the \( X_i \) represent the Profit-and-Loss distribution of \( n \) different lines of business.

2. **Loan pricing.** In this situation the investor is a loan book manager responsible for a portfolio of \( n \) loans.

3. **General investment.** Here we consider either an individual or institutional investor and the standard interpretation of \( X_i \) are profit-and-loss corresponding to a set of investments in various assets.

\( S \) is random, so usually we assume that the company has already determined the aggregate level of capital safely to face those losses and denote this total risk capital by \( K \). The company now wishes to allocate this exogenously given total risk capital \( K \) across its various business units, that is, to determine non-negative real numbers \( K_1, \ldots, K_n \) satisfying the full allocation requirement:

\[
\sum_{i=1}^{n} K_i = K. \tag{2}
\]

This allocation is in some sense a notional exercise; it does not mean that capital is physically shifted across the various units, as the company’s assets and liabilities continue to be pooled. The allocation exercise could be made in order to rank the business units according to levels of profitability. This task can be performed, for example, by determining the returns on the allocated capital for the respective business units.

The general approach of capital allocation raises the question of what the appropriate risk capital for an individual investment opportunity might be. Thus the question of performance of the investment is intimately connected with the risk measurement chosen. A two-step procedure is used in practice (McNeil et al., 2005).

1. Compute the overall risk capital \( \rho(S) \), where \( S \) is defined in Equation 1 and \( \rho \) is a particular risk measurement such as VaR or ES (see section 4 for detailed explanation on these and other measures). Coherent measures will be more appropriate than non-coherent ones.

2. Compute \( K \) as \( \rho(S) \) and allocate the capital \( K \) to the individual investment possibilities according to some mathematical capital allocation principle such that, if \( (K_i) \) denotes the capital allocated to the investment with potential loss \( X_i \). The sum of \( K_i \) fulfils the requirement in Equation 2.

\(^2\)See section 3 for a definition of what a coherent measure implies.
Throughout this study we are interested in the second step of the procedure pointed out above; roughly speaking we require a mapping that takes as input the individual losses $X_1, X_2, \ldots, X_n$ and the risk measure $\rho$ and yields as output the vector $(K_1, K_2, \ldots, K_n)$ such that:

$$\rho(S) = \sum_{i=1}^{n} K_i = K. \quad (3)$$

Such a mapping will be called a capital allocation principle. The relation Equation 3 is sometimes called the full allocation property (McNeil et al., 2005) since all of the overall risk capital $\rho(S)$ (not more, not less) is allocated to the investment possibilities; McNeil et al. (2005) consider this property to be an integral part of the definition of an allocation principle.

Given that a capital allocation can be carried out in a countless number of ways, additional criteria must be set up in order to determine the most suitable. A reasonable start is to require the allocated capital amounts $K_i$ to be “close” to their corresponding losses $X_i$ in some appropriately defined sense. This underlies the approach proposed in the present thesis. Prior to introducing the idea of “closeness” between individual loss and allocated capital, we revisit some well-known capital allocation methods. But before going into capital allocation methods it is worth to discuss about what a coherent risk measure is.

### 3 Coherent Risk Measures: Definition and Properties

A risk measure is a mapping $\rho$ from a set $\Gamma$ of real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the real line $\mathbb{R}$:

$$\rho : \Gamma \to \mathbb{R} : X \in \Gamma \to \rho[X]. \quad (4)$$

The random variable $X$ refers to the loss associated with conducting a business and $\rho[X]$ represents the amount of capital to be set aside in order to make the loss $X$ an acceptable risk.

Some well-know properties that risk measures may or may not satisfy are law invariance, monotonicity, positive homogeneity, translation invariance (or equivalence), and subadditivity, these axioms were proposed for applications in financial risk management in the seminal paper by Artzner et al. (1999). These axioms are formally defined as:

1. **Law invariance**: For any $X_1, X_2 \in \Gamma$ with $\mathbb{P}[X_1 \leq x] = \mathbb{P}[X_2 \leq x]$ for all $x \in \mathbb{R}, \rho[X_1] = \rho[X_2]$

2. **Subadditivity**: For any $X_1, X_2 \in \Gamma$, $\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_1]$

   The rational behind this is summarized by Artzner et al. (1999) in the statement that “a merger does not create extra risk”. McNeil et al. (2005) refer to subadditivity as the most debated of the axioms for a risk measure to be considered coherent, but also they provide some reasons why this axiom is indeed a reasonable requirement:

   - Subadditivity reflects the idea that risk can be reduced by diversification, a time-honoured in finance and economics.

   ³“probably because it rules out VaR as a risk measure in certain situations” (McNeil et al., 2005, p.239)
If a regulator uses a non-subadditive risk measure in determining the regulatory capital for a financial institution, that institution has an incentive to legally break up into various subsidiaries in order to reduce its regulatory capital requirements. Similarly, if the risk measure used by an organized exchange in determining the margin requirements of investors is non-subadditive, an investor could reduce the margin he has to pay by opening a different account for every position in his portfolio.

Subadditivity makes decentralization of risk-management system possible. Consider as an example two trading desks with portfolios leading to losses $X_1$ and $X_2$. Imagine that a risk manager wants to ensure that $\rho(X)$, the risk of the overall loss $X = X_1 + X_2$ is smaller than some number $M$. If he uses a risk measure $\rho$, which is subadditive, he may simply choose bounds $M_1$ and $M_2$ such that $M_1 + M_2 \leq M$ and impose on each of the desks the constraint that $\rho(X_i) \leq M_i$; subadditivity of $\rho$ then ensures automatically that $\rho(X) \leq M_1 + M_2 \leq M$.

3. **Positive homogeneity**: For any $X \in \Gamma$ and $a > 0$, $\rho[aX] = a\rho[X]$.

This axiom is easily justified if we assume that *Subadditivity* holds. Subadditivity implies that, for $n \in \mathbb{N}$,

$$\rho(nX) = \rho(X + \ldots + X) \leq n\rho(X).$$

Since there is no netting or diversification between the losses in this portfolio, it is natural to require that equality should hold in Equation 5, which leads to positive homogeneity.

4. **Monotonicity**: For any $X_1, X_2 \in \Gamma$, $X_1 \leq X_2$ implies $\rho[X_1] \leq \rho[X_2]$.

From an economic point of view this axiom is obvious: positions that lead to higher losses in every state of the world require more risk capital.

For a risk measure satisfying Axioms 2 and 4, the *Monotonicity* axiom is equivalent to the requirement that $\rho(X) \leq 0$ for all $X \leq 0$. To see this, observe that monotonicity implies that if $X \leq 0$, then $\rho(X) \leq \rho(0) = 0$; the latter equality follows from Axiom 4 since $\rho(0) = \rho(\lambda 0) = \lambda \rho(0)$ for all $\lambda > 0$. Conversely, if $X_1 \leq X_2$ and we assume that $\rho(X_1 - X_2) \leq 0$, then $\rho(X_1) = \rho(X_1 - X_2 + X_2) \leq \rho(X_1 - X_2) + \rho(X_2)$ by Axiom 2 which implies that $\rho(X_1) \leq \rho(X_2)$.

5. **Translation invariance**: For any $X_1, X_2 \in \Gamma$ and $b \in \mathbb{R}$, $\rho[X + b] = \rho[X] + b$ Axiom 5 states that by adding or subtracting a deterministic quantity $b$ to a position leading to the loss $X$ we alter our capital requirement by exactly that amount. The axiom is in fact necessary for the risk-capital interpretation of $\rho$ to make sense. Consider a position with loss $X$ and $\rho(X) > 0$. Adding the amount of capital $\rho(b)$ to the position leads to the adjusted loss $\tilde{X} = X - \rho(X)$, with $\tilde{X} = \rho(X) - \rho(X) = 0$, so that the position $\tilde{X}$ is acceptable without further injection of capital.

### 4 Some Known Risk Measures

This section is intended to briefly define two well-known risk measures: Value-at-Risk (VaR) and Conditional Tail Expectation (CTE) each of them constructed under the normality assumption.
of the vector of losses and on assuming a t-student distribution with $v$ degrees of freedom.

4.1 Value at Risk (VaR)

Value at Risk (VaR) is probably the most widely used risk measure in financial institutions (McNeil et al., 2005) and has also made its way into the Basel II capital-adequacy framework.

Definition 1 Value at Risk: For a given probability level $p \in (0, 1)$, following Dhaene et al. (2012), we denote the VaR or quantile of the loss random variable $X$ by $F_X^{-1}(p)^4$. As usual, it is defined by

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \quad p \in (0, 1) \quad (6)$$

with $\inf\{\emptyset\} = +\infty$ by convention.

In probabilistic terms, VaR is thus simply a quantile of the loss distribution. Typical values for $p$ are $p = 0.95$ or $p = 0.99$ (McNeil et al., 2005).

One important aspect the reader must to take into account, from Definition 1 is that losses will be considered as a positive value, hence if $X_i > 0$ it is a loss, otherwise it is not.

4.1.1 VaR for normal and $t$ loss distributions

Since Gaussian and t-student distributions are the most popular to assess risks we provide some explicit expressions for risk measures assuming either a normal or a t-student distribution with $v$ degrees of freedom.

Suppose that the loss distribution $F_X$ is normal with mean $\mu$ and variance $\sigma^2$. Fix $p \in (0, 1)$. Then

$$VaR_p = \mu + \sigma \Phi^{-1}(p), \quad (7)$$

where $\Phi$ denotes the standard normal distribution function and the $\Phi^{-1}(p)$ is the $p$-quantile of $\Phi$. A proof of this result can be found in (McNeil et al., 2005, p.39-40)

A similar result is obtained for any location-scale family and another useful example is the Student $t$ loss distribution. Suppose our loss $X$ is such that $(X - \mu)/\sigma$ has a standard $t$ distribution with $v$ degrees of freedom; following McNeil et al. (2005) we also denote this model by $X \sim t(v, \mu, \sigma^2)$ and note that the moments are given by $E(X) = \mu$ and $var(X) = v\sigma^2/(v - 2)$ when $v > 2$, so that $\sigma$ is not the standard deviation of the distribution. We get

$$VaR_p = \mu + \sigma t_v^{-1}(p), \quad (8)$$

where $t_v^{-1}(p)$ denotes the $p$-th quantile function of standard $t$ with $v$ degrees of freedom.

In spite of the fact that VaR is quite intuitive and yet elegant risk measure it has its own Achilles’ heel since Artzner et al. (1999) do not consider it to be a coherent risk measure due to VaR is not a subadditivity measure, as mentioned in McNeil et al. (2005):

$^4F_X^{-1}(p)$ is the quantile function which is defined as the inverse of the Distribution Function $F_X$. 
“VaR has been fundamentally criticized as a risk measure on the grounds that is has poor aggregation properties. This critique has its origins in the work of Artzner et al. (1999), who showed that VaR is not a coherent risk measure, since it violates the property of subadditivity that they believe reasonable risk measure should have.”

4.2 Conditional Tail Expectation (CTE)

Conditional Tail Expectation (CTE) is also known as the Expected Shortfall (ES) which is closely related to VaR, we will be using either one or other the term interchangeably. Note that for non-continuous random variables these concepts are not equivalent, see Denuit et al. (2005).

Definition 2 Conditional Tail Expectation: For a loss $X$ with $E(|X|) < \infty$ and distribution function denoted by $F_X$ the expected shortfall at confidence level $p \in (0, 1)$ is defined as:

$$ES_p = \frac{1}{1-p} \int_p^1 q_u(F_X)du,$$

where $q_u(F_X) = F_X^{-1}(u)$ is the quantile function of $F_X$.

Expected shortfall is thus related to VaR by

$$ES_p = \frac{1}{1-p} \int_p^1 VaR_u(X)du = E[X|X > F_X^{-1}(p)].$$

The meaning of ES is the following: instead of fixing a particular confidence level $p$ we average VaR over all levels $u \geq p$ and thus we “look further into the tail” of the loss distribution. Furthermore, ES can be interpreted as the expected loss that is incurred in the event that VaR is exceeded.

Another important relationship is $ES_p \geq VaR_p$; This follows from averaging all the losses without fixing any confidence level for the ES as we already pointed out before, since VaR requires the confidence to be fixed is always smaller than the Conditional Tail Expectation.

One clear advantage that CTE has over VaR is that CTE is a subadditive risk measure under continuous distributions for losses\(^5\) (Acerbi and Tasche, 2002; Dhaene et al., 2006). CTE meet all axioms described in section 3 that is the reason why McNeil et al. (2005) claims that currently CTE “is now preferred to VaR by many risk managers in practice”. See proposition 6.9 in (McNeil et al., 2005, p. 243) for a proof why ES is a coherent risk measure.

In order to provide explicit expressions of ES for the normal and t-student distributions we have to use the following lemma:

Lemma 1 For an integrable loss $X$ with continuous distribution function, $F_X$ and any $p \in (0, 1)$ we have

$$ES_p = \frac{1}{1-p} E(X : X \geq q_p(X)) = E(X|X \geq VaR_p)$$

\(^5\)In general, the CTE as a risk measure does not necessarily satisfy the subadditivity axiom (McNeil et al., 2005). However, it is known to be a coherent risk measure in case we restrict to random variables with continuous distribution functions (Acerbi and Tasche, 2002; Dhaene et al., 2006).
Once Lemma 1 is stated we can rely on it to calculate the ES for two common continuous distributions. A proof can be found in Dhaene et al. (2012)

4.2.1 CTE for normal and Student t loss distribution

Suppose that the loss distribution $F_X$ is normal with mean $\mu$ and variance $\sigma^2$ and we have a fixed $p \in (0, 1)$. Then

$$ES_p = \mu + \sigma \frac{\phi(\Phi^{-1}(p))}{1 - p},$$

where $\phi$ is the density of the standard normal distribution and $\Phi^{-1}$ is the inverse of the normal distribution function.

Now, suppose that the loss $X$ is such that $\tilde{X} = (X - \mu)/\sigma$ has a standard t-distribution with $v$ degrees of freedom. Suppose further that $v > 1$, since we have a location-scale family we can write $ES_p = \mu + \sigma ES_p(\tilde{X})$. The ES of the standard t-distribution is:

$$ES_p(\tilde{X}) = g_v(t_v^{-1}(p)) \frac{v + (t^{-1}_v(p))^2}{v - 1}.$$

In this last expression $g_v$ is the density of a t-Student distribution with $v$ degrees of freedom.

5 Allocation Principles

A capital allocation principle is a general rule that assigns a capital $K$ to any given risk $S$. Firms want their total capital to be allocated for several reasons as Dhaene et al. (2012) pointed out:

1. There is a need to redistribute the total (frictional or opportunity) cost associated with holding capital across various business lines so that this cost is equitably transferred back to the depositors or policyholders in the form of charges.

2. The allocation of expenses across lines of business is a necessary activity for financial reporting purposes.

3. Capital allocation provides for a useful device of assessing and comparing the performance of the different lines of business by determining the return on allocated capital for each line.

Comparing these returns allows one to distinguish the most profitable business lines and hence may assist in remunerating the business line managers or in making decisions concerning business expansions, reductions or even eliminations.

Allocation principles are methods aimed to solve the allocation problem by providing capital to each business unit for them to face their losses. This means that allocation principles gives those $K_i$ shown in Equation 2 as solution to our main problem.

Risk measures presented in section 4 give rise to some allocation principles:

1. Haircut allocation principle based on VaR.
2. Conditional Tail Expectation allocation principle, as its name suggests it relies on CTE.

3. Covariance allocation principle.

5.1 Haircut allocation principle

This a is straightforward allocation method consisting of allocating the capital \( K_i = \gamma F^{-1}_X(p), \quad i = 1, \ldots, n \) to business unit \( i \), where the factor \( \gamma \) is chosen such that the full allocation requirement Equation 2 is satisfied. This gives rise to the haircut allocation principle:

\[
K_i = \frac{K}{\sum_{i=1}^{n} F^{-1}_{X_i}(p)}, \quad i = 1, \ldots, n. \tag{12}
\]

Haircut principle is based on the idea of measuring stand-alone losses using a VaR for a given (fixed) probability level \( p \) that is why it is a very common technique among banks and insurance companies. It boils down to a principle of single proportionality.

It should be noted that \( K \) is exogenously determined, it is considered as a given value. The capital allocated by this principle does not rely on the structure dependence of the losses \( X_i \) of the different business units. Dhaene et al. (2012) consider haircut as a method which is independent of the portfolio context within which the individual losses are embedded, clearly this fact highlights the non-subadditivity property of the VaR.

The two more immediately consequences derived from non-subadditivity in the haircut principle context are: i) The portfolios does not benefit from a pooling effect (this is true even beyond haircut scope) and ii) It may happen that the allocated capitals \( K_i \) exceed the respective stand-alone capitals \( F^{-1}_X(p) \).

5.2 CTE allocation principle (Overbeck type II allocation principle)

CTE principle is based on CTE presented in subsection 4.2, we call this kind of allocation Overbeck type II allocation principle\(^6\). For a given probability level \( p \in (0,1) \), the CTE of the aggregate loss is defined as:

\[
CTE_p[S] = E \left[ S | S > F^{-1}_{X_S}(p) \right]. \tag{13}
\]

Equation 13 is just a concise version of writing Equation 9 in terms of conditional expectations. As we pointed out before, at a fixed level \( p \), CTE gives the average of the top \((1-p)\) percent losses.

The CTE allocation principle for some fixed probability level \( p \in (0,1) \) has the form:

\[
K_i = \frac{K}{CTE_p[S]} E \left[ X_i | S > F^{-1}_{X_S}(p) \right], \quad i = 1, \ldots, n. \tag{14}
\]

Unlike the haircut allocation principle, the CTE principle takes into account the dependence structure of the random losses \((X_1, X_2, \ldots, X_n)\). Interpreting the event \( S > F^{-1}_{X_S}(p) \) as the “the

---

\(^6\)See subsubsection 6.2.2 to find out why we call this principle this way.
aggregate portfolio loss $S$ is large”, we see from Equation 14 that business units with larger conditional expected loss, given that the aggregate loss $S$ is large, will be penalized with larger amount of capital required than those with lesser conditional expected loss.

### 5.3 Covariance allocation principle

The Covariance allocation principle takes the following form:

$$K_i = \frac{K}{\text{Var}[S]} \text{Cov}(X_i, S), \quad i = 1, \ldots, n,$$

(15)

where $\text{Cov}(X_i, S)$ is the covariance between the individual loss $X_i$ and the aggregate loss $S$ and $\text{Var}(S)$ is the variance of the aggregate loss. Because clearly the sum of the individual covariances is equal to the variance of the aggregate loss, the full allocation requirement in Equation 2 is automatically satisfied in this case.

The Covariance allocation principle as well as the CTE allocation principle takes into account the dependence structure of the random losses. A nice interpretation arises from the Covariance principle is “business units with a loss that is more correlated with the aggregate portfolio loss $S$ are penalized by requiring them to hold a larger amount of capital than those that are less correlated” (Dhaene et al., 2012).

### 5.4 Proportional allocations

McNeil et al. (2005) summarizes all the allocation methods explained in the previous sections into what they call Proportional Allocations which is a more general class encompassing the allocation principles described above. Depending on which risk measure $\rho$ is chosen for attributing capital $K_i$ is the key for obtaining one of them. This idea is formalized as:

$$K_i = \omega \rho(X_i), \quad i = 1, \ldots, n,$$

(16)

where $K_i$ is the capital to be allocated to each business unit $i$, $\rho(\cdot)$ is risk measure (preferably a coherent one) and the factor $\omega$ is chosen such that the full allocation requirement in Equation 2 is satisfied, this factor takes the following form:

$$\omega = \frac{K}{n \sum_{i=1}^{n} \rho(X_i)}, \quad i = 1, \ldots, n.$$

(17)

Equation 17 can be seen as a weighting scheme for capital allocation, substituting Equation 17 into Equation 16 we have an explicit and general formulation encompassing all the allocation principles discussed above:

$$K_i = \frac{K}{n \sum_{i=1}^{n} \rho(X_i)} \rho(X_i), \quad i = 1, \ldots, n.$$

(18)

The allocation principles discussed in the previous subsections follow from Equation 18 by choosing the appropriate risk measure $\rho$ (McNeil et al., 2005).
1. Haircut allocation: $\rho(X_i) = F_{X_i}^{-1}(p)$.
2. CTE allocation: $\rho(X_i) = E\left[ X_i | S > F_{X_i}^{-1}(p) \right]$.
3. Covariance allocation: $\rho(X_i) = \text{Cov}(X_i, S)$.

6 Optimal Capital Allocations

As we have pointed out above, $K$ is considered to be exogenous; because there are several allocation principles to aggregate capital $K$ to $n$ parts $K_1, \ldots, K_n$ corresponding to the different subportfolios or business units. As one can realize right away such allocation can be carried out in an infinite number of ways, some of them were illustrated in section 5, at this point Dhaene et al. (2012) claims that “there seems to be a lack of a clear motivation for preferring to choose one over another, although it appears obvious that different capital allocations must in some sense correspond to different questions that can be asked within the context of risk management” and this is the main focus of the Dhaene et al. (2012) becomes a key reference for systematizing capital allocation methods by viewing them as solutions to a particular decision problem. In order to achieve this goal they formulate a decision criterion, such as:

Capital should be allocated such that for each business unit the allocated capital and the loss are sufficiently close to each other (Dhaene et al., 2012).

In order to cast this statement in a more formal setting, consider the aggregate portfolio loss $S = X_1 + \ldots + X_n$ with aggregate capital $K$. Once the aggregate capital is allocated, the difference between the aggregate loss and the aggregate capital can be expressed as:

$$ S - K = \sum_{i=1}^{n} (X_i - K_i), \quad (19) $$

where the quantity $(X_i - K_i)$ expresses the loss minus the allocated capital for subportfolio $i$. It is important to notice that in this setting, the subportfolios are crosssubsiding each other, in the sense that the occurrence of the event “$X_k > K_k$” does not necessarily lead to “ruin”; such unfavorable performance of subportfolio $k$ may be compensated by a favorable outcome for one or more values $(X_i - K_i)$ of the other subportfolios.

Dhaene et al. (2012) propose to determine the appropriate allocation by the following optimization problem:

**Definition 3 Optimal Capital Allocation Problem** Given the aggregate capital $K > 0$, determine the allocated capitals $K_i$, $i=1, \ldots, n$, from the following optimization problem:

$$ \min_{K_1, \ldots, K_n} \sum_{i=1}^{n} v_i E \left[ \zeta_i D \left( \frac{X_i - K_i}{v_i} \right) \right], \quad \text{such that}, \quad \sum_{i=1}^{n} K_i = K, \quad (20) $$

where the $v_i$ are non-negative real numbers such that $\sum_{i=1}^{n} v_i = 1$, the $\zeta_i$ are non-negative random variables such that $E(\zeta_i) = 1$, and $D$ is a non-negative function.
Each of the component in the general optimal capital allocation problem in Equation 20 are defined as follows:

\( v_i \): The non-negative real number \( v_i \) is a measure of exposure or business volume of the \( i \)th unit, such as revenue, insurance premium, etc. These scalar quantities are chosen such that they sum to 1. Their inclusion in the expression \( D \left( \frac{X_i - K_i}{v_i} \right) \) normalizes the deviations of loss from allocated capital across business units to make them relatively more comparable. At the same time, the \( v_i \)s are used as weights attached to the different values of \( E \left[ \zeta_i D \left( \frac{X_i - K_i}{v_i} \right) \right] \) in the minimization problem in Equation 20, in order to reflect the relative importance of the different business units.

\( D \left( \frac{X_i - K_i}{v_i} \right) \): For simplicity, it is first assumed that \( v_i = 1 \) and also that \( \zeta_i \equiv 1 \). The terms \( D(X_i - K_i) \) quantify the deviations of the outcomes of the losses \( X_i \) from their allocated capital \( K_i \). Minimizing the sum of the expectations of these quantities essentially reflects the requirement that the allocated capitals should be “as close as possible” to the losses they are allocated to. Examples of distance measures are “squared or quadratic deviations” and “absolute deviations”.

\( \zeta_i \) The deviations of the losses \( X_i \) from their respective allocated capital levels \( K_i \) are measured by the terms \( E \left[ \zeta_i D(X_i - K_i) \right] \). These expectations involve non-negative random variables \( \zeta_i \) with \( E(\zeta_i) = 1 \) that are used as weight factors to the different possible outcomes of \( D(X_i - K_i) \). One possible choice for the \( \zeta_i \) could be \( \zeta_i = h(X_i) \) for some non-negative and non-decreasing function \( h \). In this case, the heaviest weights are attached to deviations that correspond to states of the world leading to the largest outcomes of \( X_i \). We will call allocations based on such a choice for the \( \zeta_i \) business unit driven allocations.

Another choice is to let \( \zeta_i = h(S) \) for some non-negative and non-decreasing function \( h \), such that the outcomes of the deviations are weighted with respect to the aggregate portfolio performance. In this case, heavier weights are attached to deviations that correspond to states of the world leading to larger outcomes of \( S \). Allocations based on such a choice for the random variables \( \zeta_i \) will be called aggregate portfolio driven allocations.

A yet different approach is to let \( \zeta_i = \zeta_M \) for all \( i \), where \( \zeta_M \) can be interpreted as the loss on a reference (or market) portfolio. In this case, the weighting is market driven and the corresponding allocation is said to be a market-driven allocation.

The Quadratic Optimization Criterion is proposed by Dhaene et al. (2012) as the General Solution of the Quadratic Allocation Problem by letting

\[
D(x) = x^2. \tag{21}
\]

This leads to Equation 20 to

\[
\min_{K_1, \ldots, K_n} \sum_{i=1}^{n} E \left[ \frac{\zeta_i (X_i - K_i)^2}{v_i} \right], \quad \text{such that,} \quad \sum_{i=1}^{n} K_i = K. \tag{22}
\]
The solution to this minimization problem is given in the following theorem.

**Theorem 1** The optimal allocation problem in Equation 22 has the following unique solution:

$$K_i = E(\zeta_i X_i) + v_i \left( K - \sum_{i=1}^{n} E(\zeta_i X_j) \right), \quad i = 1, \ldots, n.$$  \hspace{1cm} (23)

A detailed proof of the solution for this minimization problem can be found in Dhaene et al. (2012).

### 6.1 Business unit driven allocations

Following Dhaene et al. (2012), in this subsection, we consider the case where the weighting random variables $\zeta_i$ in the quadratic allocation problem in Equation 22 are given by

$$\zeta_i = h_i(X_i),$$ \hspace{1cm} (24)

with $h_i$ being a non-negative and non-decreasing function such that $E[h_i(X_i)] = 1$, for $i = 1, \ldots, n$. Hence, for each business unit $i$, the states of the world to which we want to assign the heaviest weights are those under which the business unit performs the worst. As earlier pointed out, we call allocations based on Equation 24 business unit driven allocations. In this case, the allocation rule in Equation 23 can be rewritten as

$$K_i = E[X_i h_i(X_i)] + v_i \left( K - \sum_{i=1}^{n} E[X_i h_i(X_i)] \right), \quad i = 1, \ldots, n.$$ \hspace{1cm} (25)

For an exogenously given value of $K$, the allocations $K_i$ are not influenced by the mutual dependence structure between the losses $X_i$ of the different business units. In this sense, one can say that the allocation principle (25) is independent of the portfolio context within which the $X_i$s are embedded and, hence, is indeed business unit driven. Such allocations might be a useful instrument for determining the performance bonuses of the business unit managers, in case one assumes that each manager should be rewarded for the performance of his own business unit but not extra rewarded (or penalized) for the interrelationship that exists between the performance of his business unit and that of the other units of the company. One should however note that disregarding in this way diversification between business units, the allocation may give incentives to managers that are at odds with overall portfolio optimization criteria.

The law invariant risk measure $E[X_i h_i(X_i)]$ assigns to any loss $X_i$ the expected value of the weighted outcomes of this loss, where higher weights correspond to larger outcomes of the loss, that is, to more adverse scenarios. Risk measures and premium principles of this general type are proposed and investigated in Heilmann (1989), Tsanakas (2007), and Furman and Zitikis (2008).

Defining the volumes $v_i$ by

$$v_i = \frac{E[X_i h_i(X_i)]}{\sum_{i=1}^{n} E[X_i h_i(X_i)]}. \hspace{1cm} (26)$$

The allocation principle could be found by substituting Equation 26 in Equation 25 and simplifying the expression as in:
\[ K_i = E[X_i h_i(X_i)] + \frac{E[X_i h_i(X_i)]}{\sum_{i=1}^{n} E[X_i h_i(X_i)]} \left( K - \sum_{i=1}^{n} E[X_i h_i(X_i)] \right) \]

\[ = E[X_i h_i(X_i)] + K \frac{E[X_i h_i(X_i)]}{\sum_{i=1}^{n} E[X_i h_i(X_i)]} - E[X_i h_i(X_i)] \]

\[ = \frac{E[X_i h_i(X_i)] \sum_{i=1}^{n} E[X_i h_i(X_i)] + KE[X_i h_i(X_i)] - E[X_i h_i(X_i)] \sum_{i=1}^{n} E[X_i h_i(X_i)]}{\sum_{i=1}^{n} E[X_i h_i(X_i)]}. \]

Now it can be easily seen from this last expression the allocation principle based on the business unit driven idea is given by:

\[ K_i = \frac{K}{\sum_{i=1}^{n} E[X_i h_i(X_i)]} E[X_i h_i(X_i)]. \quad (27) \]

Once we got to know the general form of the business unit driven allocation principle we are now able to choose different forms for \( h_i(X_i) \) in order to achieve several capital allocation principles based upon the business unit driven allocation framework, this is exactly the main purpose of the subsequent sections.

### 6.1.1 (Pure) Conditional Tail Expectation principle

Once we know the allocation principle for allocating \( K_i \) using business unit driven principle we can set specific forms for \( h_i(X_i) \), we can obtain several explicitly functional forms for \( K_i \), for instance by choosing \( h_i(X_i) = \frac{\mathbb{I}(X_i > F^{-1}_X(p))}{1 - F_X(F^{-1}_X(p))} \), then \( K_i \) will result in the (Pure) Conditional Tail Expectation principle.

We call this principle (Pure) Conditional Tail Expectation because both the aggregate loss and each individual business unit losses are taken conditional expectation based on the average of the top \((1 - p)\) loss. Since \( \text{CTE}() \) is applied to \( S \) and \( X_i \) then we call it (Pure) Conditional Tail Expectation so that we can distinguish it from the Conditional Tail Expectation principle based on (Overbeck, 2000) which we call Overbeck type II allocation principle which is a special case of the Aggregate Portfolio Driven Allocations, see subsection 6.2.

By choosing \( h_i(X_i) = \frac{\mathbb{I}(X_i > F^{-1}_X(p))}{1 - F_X(F^{-1}_X(p))} \) multiplying by \( X_i \) and taking expectations will lead us to:

\[ E[X_i h_i(X_i)] = E \left[ \frac{\mathbb{I}(X_i > F^{-1}_X(p))}{1 - p} \right] = \frac{1}{1 - p} E[X_i | X_i > F^{-1}_X(p)]. \]

From Lemma 1 the previous expression reduces to the Conditional Tail Expectation:

\[ E[X_i h_i(X_i)] = \text{CTE}_p[X_i]. \]
Now replacing $E[X_i h_i(X_i)]$ by $CTE_p[X_i]$ in Equation 27 we have:

$$K_i = \frac{K}{\sum_{i=1}^{n} CTE_p(X_i)} CTE_p(X_i) = \frac{K}{CTE_p(\sum_{i=1}^{n} X_i)} CTE_p(X_i).$$

\[ \sum_{i=1}^{n} CTE_p(X_i) = CTE_p(\sum_{i=1}^{n} X_i) \] follows from the additivity property of CTE.

Hence $K_i$ takes the following form:

$$K_i = \frac{K}{CTE_p(S)} CTE_p(X_i). \quad (28)$$

### 6.1.2 Standard deviation principle

The standard deviation principle (Bühlmann, 1970) can be easily obtained by choosing $h_i(X_i) = 1 + a\frac{X_i - E(X_i)}{\sigma X_i}$, $a \geq 0$, so that replacing it into $E[X_i h_i(X_i)]$ and then plug it into Equation 27 will have the so-called standard deviation principle.

In order to get an expression for $K_i$ based upon the standard deviation principle we proceed as follows:

$$E[X_i h_i(X_i)] = E \left[ X_i + a\frac{X_i^2 - X_i E(X_i)}{\sigma X_i} \right]$$

$$= E(X_i) + \frac{a}{\sigma X_i} \{ E(X_i^2) - [E(X_i)]^2 \}$$

$$= E(X_i) + \frac{a \sigma^2 X_i}{\sigma X_i}$$

$$= E(X_i) + a \sigma X_i.$$ 

For \( \sum_{i=1}^{n} E[X_i h_i(X_i)] \) to be explicitly found we proceed as follows:

$$\sum_{i=1}^{n} E[X_i h_i(X_i)] = \sum_{i=1}^{n} \{ E(X_i) + a \sigma X_i \}$$

$$= \sum_{i=1}^{n} E(X_i) + a \sum_{i=1}^{n} \sigma X_i. \quad (29)$$

Equation 29 can be simplified to Equation 30 if and only if $Cov(X_i, X_j) = 0 \ \forall i \neq j$

$$E(S) + a \sigma S,$$

this follows from the following operations:
\[
\sum_{i=1}^{n} E(X_i) + a \sum_{i=1}^{n} \sigma_{X_i} = E\left(\sum_{i=1}^{n} X_i\right) + a \sqrt{Var\left(\sum_{i=1}^{n} X_i\right)} \Leftrightarrow Cov(X_i, X_j) = 0 \quad \forall i \neq j
\]

Consequently the form taken by \( K_i \) based upon the standard deviation principle is:

\[
K_i = \frac{K}{E(S) + a\sigma_S} (E(X_i) + a\sigma_{X_i}). \tag{31}
\]

A very interesting relationship between Overbeck type I allocation principle which we will be studied in subsubsection 6.2.2 and the Standard deviation allocation principle, Equation 40 and Equation 31, respectively, is given by:

\[
K_i = \frac{K}{E(S) + a\phi} \left( E(X_i) + \frac{a}{\sigma}\gamma \right). \tag{32}
\]

Overbeck type I is retrieved by Equation 32 when choosing \( \phi = \sigma_S^2 \) and \( \gamma = Cov(X_i, S) \), in so far as the standard deviation principle is recovered when setting \( \phi = \sigma_S \) and \( \gamma = Cov(X_i, X_i) = Var(X_i) = \sigma_{X_i}^2 \).

6.1.3 Esscher principle

If we let \( h_i(X_i) \) be \( \frac{e^{aX_i}}{E[e^{aX_i}]} \) with \( a > 0 \) then \( K \) will be allocated accordingly by the Esscher Principle (Gerber, 1981), as we shall see below:

\[
E[X_i h_i(X_i)] = E\left[ \frac{X_i e^{aX_i}}{E[e^{aX_i}]} \right].
\]

\[
\sum_{i=1}^{n} E[X_i h_i(X_i)] = \sum_{i=1}^{n} E\left[ \frac{X_i e^{aX_i}}{E[e^{aX_i}]} \right].
\]

Thus, the optimal \( K_i \) will look like as Equation 33:

\[
K_i = \frac{K}{\sum_{i=1}^{n} E\left[ \frac{X_i e^{aX_i}}{E[e^{aX_i}]} \right]} E\left[ \frac{X_i e^{aX_i}}{E[e^{aX_i}]} \right]. \tag{33}
\]

6.2 Aggregate portfolio driven allocations

Unlike from the Business Unit Driven Allocation rule, this time Dhaene et al. (2012) consider the case where
Table 1: Business Unit Driven Capital Allocation

<table>
<thead>
<tr>
<th>Reference</th>
<th>$h_i(X_i)$</th>
<th>$K_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Pure) Conditional Tail Expectation Overbeck (2000)</td>
<td>$\frac{\mathbb{P}(X_i &gt; F_{X_i}^{-1}(p))}{1 - F_{X_i}(F_{X_i}^{-1}(p))}$, $p \in (0, 1)$</td>
<td>$\frac{K}{\text{CTE}_p(S)} \text{CTE}_p(X_i)$</td>
</tr>
<tr>
<td>Standard deviation principle Bühlmann (1970)</td>
<td>$1 + a \frac{X_i - E(X_i)}{\sigma_{X_i}}$, $a \geq 0$</td>
<td>$K \frac{E(S) + a\sigma_S}{E(X_i) + a\sigma_{X_i}}$</td>
</tr>
<tr>
<td>Esscher principle Gerber (1981)</td>
<td>$\frac{e^{aX_i}}{E(e^{aX_i})}$, $a &gt; 0$</td>
<td>$K_i = \frac{K}{\sum_{i=1}^{n} E\left[\frac{X_i e^{aX_i}}{E(e^{aX_i})}\right]}$</td>
</tr>
</tbody>
</table>

$\zeta_i = h(S), \quad i = 1, \ldots, n,$ \hspace{1cm} (34)

with $h$ being a non-negative and non-decreasing function such that $E[h(S)] = 1$. In this case, the states of the world to which we assign the heaviest weights are those under which the aggregate portfolio performs worst. Therefore, we call such allocations aggregate portfolio driven allocations. The allocation rule (23) can now be rewritten as:

$$K_i = E[X_i h(S)] + v_i (K - E[Sh(S)]), \quad i = 1, \ldots, n.$$ \hspace{1cm} (35)

Hence, the capital $K_i$ allocated to unit $i$ is determined using a weighted expectation of the loss $X_i$, with higher weights attached to states of the world that involve a large aggregate loss $S$. Notice that the allocation principle (35) can be reformulated as:

$$K_i = E(X_i) + \text{Cov}[X_i, h(S)] + v_i (K - E[Sh(S)]), \quad i = 1, \ldots, n.$$ \hspace{1cm} (36)

This means that the capital allocated to the $i$th business unit is given by the sum of the expected loss $E[X_i]$, a loading that depends on the covariance between the individual and aggregate losses $X_i$ and $h(S)$, plus a term proportional to the volume of the business unit. A strong positive correlation between $X_i$ and $h(S)$, which reflects that $X_i$ could be a substantial driver of the aggregate loss $S$, produces a higher allocated capital $K_i$.

Using aggregate portfolio driven allocations might be appropriate when one wants to investigate each individual portfolio’s contribution to the aggregate loss of the entire company. In other words, the company wishes to evaluate the subportfolio performances, for example, the returns on the allocated capitals, in the presence of the other subportfolios. This can provide relevant information to the company within which it can further be used to evaluate either business expansions or reductions.

Defining the volumes $v_i$ by

$$v_i = \frac{E[X_i h(S)]}{E[Sh(S)]}, \quad i = 1, \ldots, n.$$ \hspace{1cm} (37)

Plugging Equation 37 into Equation 35 we have:

---

8This follows from the fact that $\text{Cov}(X_i, h(S)) = E(X_i h(S)) - E(X_i)E(h(S))$ solving for $E(X_i h(S))$ we end up with $E(X_i) + \text{Cov}[X_i, h(S)]$ since $E(h(S)) = 1$
\[ K_i = E[X_i h(S)] + \frac{E[X_i h(S)]}{E[Sh(S)]}(K - E[Sh(S)]) \]

\[ = E[X_i h(S)] + K \frac{E[X_i h(S)]}{E[Sh(S)]} - E[X_i h(S)] \]

\[ = \frac{E[X_i h(S)]E[Sh(S)] + KE[X_i h(S)] - E[X_i h(S)]E[Sh(S)]}{E[Sh(S)]}. \]

Simplifying this last expression we end up with a proportional allocation rule:

\[ K_i = \frac{K}{E[Sh(S)]} E[X_i h(S)]. \] (38)

Using the proportional allocation principle shown in Equation 38 and choosing some structure for \( h(S) \) one can be allowed to construct several ways for allocating \( K \). For instance let us consider a particular choice for \( h(S) \) to be \( h(S) = S - E(S) \) this yields to the covariance allocation principle introduced in section 5 by means of determining the expression for both \( E[X_i h(S)] \) and \( E[Sh(S)] \) and then plug them into Equation 38 as it is shown below.

### 6.2.1 Covariance allocation principle

This subsection is intended to derive the Covariance allocation principle from the general setting presented in the previous section by setting \( h(S) = S - E(S) \) and using the philosophy of the plug-in principle.

Setting \( h(S) = S - E(S) \) the aim is to determine \( E[X_i h(S)] \) and \( E[Sh(S)] \).

For \( E[X_i h(S)] \) the way to go is:

\[ E[X_i h(S)] = E[X_i S - E(S)] \]

\[ = E[X_i S - X_i E(S)] \]

\[ = E(X_i S) - E(X_i)E(S) \]

\[ = Cov(X_i, S). \]

For \( E[Sh(S)] \) to be explicitly found we proceed as follows:

\[ E[Sh(S)] = E[S(S - E(S))] \]

\[ = E[S^2 - SE(S)] \]

\[ = E(S^2) - E(S)E(S) \]

\[ = E(S^2) - [E(S)]^2 \]

\[ = Var(S). \]

Once we have the expressions for \( E[X_i h(S)] \) and \( E[Sh(S)] \) we can now plug them into Equation 38 in order to have the expression for allocating capital \( K \) among the different business units.
\((X_i \text{ with } i = 1, \ldots, n)\) based on the Aggregate Portfolio Driven idea. So the allocation principle has the form:

\[ K_i = \frac{K}{\text{Var}[S]} \text{Cov}(X_i, S), \quad i = 1, \ldots, n. \]  \hspace{1cm} (39)

Precisely this is exactly the expression shown in Equation 15 from this fact one can notice that Covariance Principle is a special case of the Aggregate Portfolio Driven Allocation when choosing \(h(S) = S - E(S)\).

### 6.2.2 Overbeck allocation principles

Within this subsection we provide an explicit expression for the Aggregate Portfolio Driven Allocation principle based on Overbeck (2000). We call Overbeck Type I allocation principle to the principle obtained by setting \(h(S) = 1 + a\frac{S - E(S)}{\sigma_S}, a \geq 0\). And we will call Overbeck Type II allocation principle to that when using \(h(S) = \frac{1}{1 - p} \mathbb{I}(S > F_S^{-1}(p)), p \in (0, 1)\).

As in the previous sections we now proceed to find an explicit expression for \(K_i\) by setting \(h(S) = 1 + a\frac{S - E(S)}{\sigma_S}, a \geq 0\).

For \(E[X_i h(S)]\) we have:

\[
E[X_i h(S)] = E \left[ X_i \left( 1 + a\frac{S - E(S)}{\sigma_S} \right) \right] \\
= E \left[ X_i \sigma_S + aX_i - aX_i E(S) \right] \\
= \frac{1}{\sigma_S} E [X_i \sigma_S + aX_i - aX_i E(S)] \\
= E(X_i) + \frac{a}{\sigma_S} E(X_i S) - \frac{a}{\sigma_S} E(X_i) E(S) \\
= E(X_i) + \frac{a}{\sigma_S} [E(X_i S) - E(X_i) E(S)] \\
= E(X_i) + \frac{a}{\sigma_S} \text{Cov}(X_i, S).
\]

Working a little on \(E[S h(S)]\) we find:

\[
E[S h(S)] = E \left[ S + \frac{aS(S - E(S))}{\sigma_S} \right] \\
= E(S) + \frac{a}{\sigma_S} E [S(S - S(S))] \\
= E(S) + \frac{a}{\sigma_S} E [S^2 - S E(S)] \\
= E(S) + \frac{a}{\sigma_S} E [E(S^2) - E(S) E(S)] \\
= E(S) + \frac{a}{\sigma_S} \sigma_S^2 \\
= E(S) + a \sigma_S.
\]
Applying the plug-in principle and substituting the respective expressions of $E[X_i h(S)]$ and $E[Sh(S)]$ into the general framework presented in Equation 38 we get the allocation principle we’ve just called Overbeck Type I allocation principle whose form is:

$$K_i = \frac{K}{E(S) + a\sigma_S} \left[ E(X_i) + \frac{a}{\sigma_S} Cov(X_i, S) \right]. \quad (40)$$

Overbeck Type II allocation principle is determined by letting $h(S)$ be $\frac{1}{1-p} \mathbb{I}(S > F^{-1}_S(p))$ with $p \in (0, 1)$:

$$E[X_i h(S)] = \frac{1}{1-p} E[X_i \mathbb{I}(S > F^{-1}_S(p))]$$

$$E[Sh(S)] = \frac{1}{1-p} E[S \mathbb{I}(S > F^{-1}_S(p))] = CTE_p(S).$$

Therefore, $K_i$ could be written as:

$$K_i = \frac{K}{CTE_p(S)} E[X_i \mathbb{I}(S > F^{-1}_S(p))]. \quad (41)$$

Note this principle is exactly the same one presented in Equation 14 in subsection 5.2

6.2.3 Wang allocation principle

Let us consider $h(S) = \frac{e^{aS}}{E[e^{aS}]}$ with $a > 0$, we can construct an allocation principle based on Wang (2007) and give an expression for $K_i$. In order to achieve our goal the procedure is similar to the ones used in previous sections.

Once we consider $h(S) = \frac{e^{aS}}{E[e^{aS}]}$, the expression for $E[X_i h(S)]$ is found in the following way:

$$E[X_i h(S)] = E \left[ Sh(S) = \frac{e^{aS}}{E[e^{aS}]} \right] = \frac{1}{E[e^{aS}]} E[X_i e^{aS}] = \frac{E(X_i e^{aS})}{E[e^{aS}]].$$
Then $E[Sh(S)]$ is:

$$E[Sh(S)] = E \left[ X_i h(S) = \frac{e^{aS}}{E[e^{aS}]} \right] = \frac{E(Se^{aS})}{E(e^{aS})}.$$ 

Therefore, the allocation of the exogenously given aggregate capital $K$ to $n$ parts $K_1, \ldots, K_n$ corresponding to the different business units can be carried out using:

$$K_i = \frac{K}{E(Se^{aS})} E(X_i e^{aS}). \quad (42)$$

6.2.4 Tsanaka allocation principle

If we let $\int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma$ be $h(S)$ with $a > 0$, then this leads us to the Tsanakas (2009) principle.

Expressions for constructing the $K_i$ are as follow:

$$E[X_i h(S)] = E \left[ X_i \int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma \right],$$

$$E[Sh(S)] = E \left[ S \int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma \right],$$

where the $K_i$ to be allocated takes the following form:

$$K_i = \frac{K}{E(S\int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma)} E \left[ X_i \int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma \right]. \quad (43)$$

Letting $\Psi$ be $\int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma$, then $K_i$ could be rewritten as:

$$K_i = \frac{K}{E(S\Psi)} E(X_i \Psi). \quad (44)$$

Table 2 summarizes the Aggregate Portfolio Driven Allocations by providing expressions for $K_i$.

7 Numerical Examples

Previously we introduced some well-known capital allocation principles, now we give the practical examples of these approaches and their impact on amounts of allocated capital. For this purpose we use Public data risk no. 1 and Public data risk no. 2 from Bolancé et al. (2012), these data consist of 1000 and 400 observed loss amounts for categories 1 and 2, respectively.

Let us consider these data as operative losses in a banking environment. For Public data risk no. 1 to have some sense in this context we consider it as bank transfer mistakes which means
that a bank teller transfers more money than the required to a client’s bank account and Public data risk no. 2 is to be considered as fraudulent transactions, for instance, a client loses her credit card and another person uses it, if the bank’s client reports this situation to bank then the non-authorized use of the credit card will charge some losses to bank.

In this section we quantify individual capital requirements based on risk measures over each operative losses. Given an exogenous amount of total capital, $K$ calculated as the empirical Value at Risk at 99% of the aggregate loss ($VaR_{0.99}(S)$), the goal is allocating to each loss source an optimal portion of this capital and comparing three well-known allocation principles: Haircut, Covariance and Overbeck type II allocation principles, all of them belonging to the proportional allocations, note that both Covariance and Overbeck type II allocation principles belong to the Aggregate portfolio allocation principle described in subsection 6.2.

The reason why we decide to use aggregate portfolio allocations is knowing the overall portfolio performance taking into account the dependence structure, meanwhile, on the other hand, we choose using Haircut allocation principle in order to make comparisons against a stand-alone risk measure which does leave out the dependence structure of the risk factors.

Some descriptive insights are provided in Table 3 where one eye-catching fact is the difference in the number of observations in each vector of losses, Public data risk no. 1 has 1000 observations and Public data risk no. 2 has 400 which represents a drawback for the configuration of the allocation principles where all of them implicitly assume identical length for vector of losses, we overcome this inconvenient by using two different re-sampling techniques: bootstrapping and an uniformly pairwise random extraction. Another important characteristic of these data is the strong non-normality suggested by the skewness and the kurtosis coefficients, also this data show a strong right asymmetry since the mean is larger than the median for both vectors.

The numerical exercises presented below consists of two cases: the first one where the dependence structure is removed by the simulation procedure and in the second one a strong dependence structure between individual losses is artificially created. The aim of it is checking the performance of the allocations principles when two extreme situations might happen.

### 7.1 Case I: Lack of dependence structure

In this subsection, we assess the performance of allocation principles we are interested in when losses exhibit a low degree of linear dependence, this means that the correlation coefficient between
Table 3: Descriptive statistics for numerical example data

<table>
<thead>
<tr>
<th></th>
<th>Public data risk no. 1</th>
<th>Public data risk no. 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>nobs</td>
<td>1000.00</td>
<td>400.00</td>
</tr>
<tr>
<td>NAs</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Maximum</td>
<td>5122.14</td>
<td>1027.53</td>
</tr>
<tr>
<td>1. Quartile</td>
<td>2.24</td>
<td>2.67</td>
</tr>
<tr>
<td>3. Quartile</td>
<td>8.46</td>
<td>8.62</td>
</tr>
<tr>
<td>Mean</td>
<td>42.06</td>
<td>20.89</td>
</tr>
<tr>
<td>Median</td>
<td>3.47</td>
<td>4.29</td>
</tr>
<tr>
<td>Sum</td>
<td>42059.41</td>
<td>8357.32</td>
</tr>
<tr>
<td>SE Mean</td>
<td>9.23</td>
<td>4.80</td>
</tr>
<tr>
<td>Variance</td>
<td>85242.64</td>
<td>9199.45</td>
</tr>
<tr>
<td>Stdev</td>
<td>291.96</td>
<td>95.91</td>
</tr>
<tr>
<td>Skewness</td>
<td>13.61</td>
<td>9.10</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>210.87</td>
<td>89.20</td>
</tr>
</tbody>
</table>

the losses is close enough to zero.

Let \( X \) and \( Y \) be vectors consisting of 1000 and 400 observations on individual losses, moreover \textit{Public data risk no. 1} is now denoted by \( X \) and \textit{Public data risk no. 2} is denoted by \( Y \). Recalling the fact that all the allocation principles presented in this thesis require the vectors to have the same length, nevertheless \( X \) and \( Y \) have not that same length, this situation might be the more common one in practice, to overcome this drawback and compute the allocation principles we proceed as:\footnote{See the implementation of this procedure using R code in the appendix section.}

1. Draw 1000 observations from \( X \) and 400 from \( Y \) using re-sampling with replacement and obtain \( X_1 \) and \( Y_1 \).
2. Generate \( x^*_1 = \sum_{i=1}^{1000} X_{1,i} \) and \( y^*_1 = \sum_{i=1}^{400} Y_{1,i} \).
3. Repeat steps 1) and 2) 10000 times to obtain two vectors of equal lengths: \( X^* \) and \( Y^* \) with \( X^* = \{x^*_i\}_{i=1}^{10000} \) and \( Y^* = \{y^*_i\}_{i=1}^{10000} \).

Once we have \( X^* \) and \( Y^* \) having the same length we can now compute the allocations based on the principles previously discussed.

Summarizing we generate for both vectors of losses 10000 replications of size 1000 and 400 for \textit{Public data risk no. 1} and for \textit{Public data risk no. 2}, respectively in order to obtain two vectors of length 10000 over which we can apply the allocation principle we are interested in. In the \( i-th \) iteration we sum up all the re-sampled points to get the \( i-th \) element of each vector and we repeat this procedure 10000 times as \( i \) moves from 1 to 10000. This way the non-identical length problem of the vectors is overcome. Now we have to allocate a total amount of exogenous capital.
which we estimate using an empirical $\text{VaR}_{99}(S)$, this means, the aggregate capital is chosen to be the empirical Value at Risk of the aggregate loss. Note that we now have two risk sources and 10 000 observations associated to each risk source.

An aggregate capital amount of 50 416.73 monetary units would be enough for facing the total loss for this particular sample comprised by $X$ and $Y$ (Public data risk no. 1 and Public data risk no. 2, respectively). Nevertheless, in order to guarantee a coverage even when large deviations might occur we use the empirical $\text{VaR}_{99}(S) = 75 573.96$ that ensures 99% coverage of potential losses and this is why we set the exogenous capital to be this value. Aggregate capital to be allocated is 75 573.96 monetary units.

Table 4 shows the allocated capital to each vector of losses based on different capital allocation principles, these results show the amount of capital to be set aside for each risk source. Note that Haircut allocation principle (HAP) boils down to a simple proportion when there is not any dependence structure (in a linear sense) between the losses, this happens when the correlation coefficient between $X^*$ and $Y^*$ is close to zero and in this particular case such correlation is $\approx 0.00014$, therefore results obtained from HAP will be identical to those obtained using:

$$K_i = \frac{K}{\sum_{i=1}^{n} X_i} X_i,$$

(45)

recalling the fact that $\sum_{i=1}^{n} X_i = S$, this "simple proportional" allocation principle (SPA) reduces to $(K/S)X_i$. When $K = 1$ and multiplying the result by 100 gives us the percentage of $X_i$ as a portion of the aggregate loss $S$ as it is shown in Table 5.

According to Table 3 the losses seem to be non-normal, therefore both Haircut and Overbeck type II allocations are computed using the normal and the t-student distribution, for the t-student we used several degrees of freedom and results do not differ from those ones reported when using a normal distribution, so in Table 4 only normal results are reported.

For us to assess how well the allocations fit, we now calculate the proportions of capital to be set aside instead of the amount of capital, we reach this goal by choosing $K = 1$ and the new results are reported in Table 5.

As it was expected, the Haircut allocation principle is a good choice since it does not take into account the dependence structure and since the correlation between $X^*$ and $Y^*$ is almost zero the best choice for this case is using Equation 45 as the allocation principle, because its results are a good enough approximation for HAP and its calculation is enormously simplified, furthermore it does not rely on any distributional assumption. Table 5 shows how the approximation to HAP using Equation 45 performs compared to HAP results.

---

Table 4: Case I. Capital allocation based on different principles

<table>
<thead>
<tr>
<th></th>
<th>HAP</th>
<th>CAP</th>
<th>Overbeck II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^*$ (dat1.boot)</td>
<td>62953.00</td>
<td>72497.53</td>
<td>66070.96</td>
</tr>
<tr>
<td>$Y^*$ (dat2.boot)</td>
<td>12620.96</td>
<td>3076.43</td>
<td>9503.00</td>
</tr>
<tr>
<td>Total</td>
<td>75573.96</td>
<td>75573.96</td>
<td>75573.96</td>
</tr>
</tbody>
</table>

---

10This aggregate capital comes from summing 42059.41 and 8357.32, see row labelled Sum in Table 3.
Table 5: Case I. Proportions of capital allocation based on different principles

<table>
<thead>
<tr>
<th></th>
<th>SPA</th>
<th>HAP</th>
<th>CAP</th>
<th>Overbeck II</th>
</tr>
</thead>
<tbody>
<tr>
<td>dat1.boot</td>
<td>0.8344</td>
<td>0.833</td>
<td>0.9593</td>
<td>0.8743</td>
</tr>
<tr>
<td>dat2.boot</td>
<td>0.1656</td>
<td>0.167</td>
<td>0.0407</td>
<td>0.1257</td>
</tr>
</tbody>
</table>

In Table 5 there is an additional column: SPA which is the approximation to the HAP when correlation tends to zero, we present this information in order to compare the proportions based on each principle. We can see that HAP is identical to SPA, nevertheless the Covariance allocation principle overestimates the contribution of the first vector and underestimates the second one in a stronger way than Overbeck II does. In a rough sense we can see that in absence of correlation between losses, the estimates of the Covariance allocation principle are more biased than those of Overbeck II.

Clearly in this part of the exercise we conclude that Covariance allocation principle performs the worst compared to the other two principles.

In the next section we introduce a strong dependence structure in order to assess the performance of the allocations which account for correlation among losses.

7.2 Case II: Strong dependence structure

This section can be seen as the counterpart of the previous one as now we go to the other extreme case where a strong dependence framework is involved.

In order to create two vectors of losses strongly correlated we base the sampling scheme on quantiles-based extractions, this means for each probability $p_i$ with $i = 1, \ldots, 10000$, which is common for both vectors $X$ and $Y$, recall that $X$ is the label for Public data risk no. 1 and $Y$ is the label for Public data risk no. 2, we take the value located at quantile given by $F_X^{-1}(p_i)$ and $F_Y^{-1}(p_i)$, each $p_i$ was randomly drawn from a $U(0,1)$, to make this point clear, we go through the following steps:\footnote{See the implementation of this procedure using R code in the appendix section.}

1. Draw randomly 10000 values from a $U(0,1)$ for probabilities such that $p_1$ is one realization of $U(0,1)$, $p_2$ is another, and so on until $p_{10000}$.

2. Generate $W$ and $Z$ such that both are vectors of dimension $10000 \times 1$ holding $F_X^{-1}(p_i)$ and $F_Y^{-1}(p_i)$.

3. Constructing $W$ and $Z$ this way guarantees that when we have a small value for $W$ we also have a small value for $Z$ and when we have a large for one $W$ we also have a large one for $Z$. We store $W$ and $Z$ into a matrix $M$ of dimension $10000 \times 2$ so that $W$ and $Z$ are now matched (pairwise).

4. Resample row-wise with replacement from $M$ and draw 10000 pairs of observations, sum them colwise and get $m_1$ which is a $1 \times 2$ vector, repeat this step 10000 times in order to get $m_i$ with $i = 1, \ldots, 10000$.\footnote{See the implementation of this procedure using R code in the appendix section.}
5. The data set we are going to work with is the matrix $M^*$ consisting of the colwise concatenation of $m_i$ with $i = 1, \ldots, 10000$. $M^*$ should look like:

$$M^* = \begin{pmatrix}
  m_{1,1} & m_{1,2} \\
  \vdots & \vdots \\
  m_{10000,1} & m_{10000,2}
\end{pmatrix}$$

6. We call the first column of $M^*$ as $X'$ and the second one is called $Y'$ where $X'$ is the resampled observations of the transfer mistakes (Public data risk no. 1) and $Y'$ is the resampled associated to the fraudulent transactions (Public data risk no. 2). Here the apostrophe does not mean transpose, it is just a way to name $X$ and $Y$ in order to distinguish them from the originals $X$ and $Y$.

Given that we suffer from different lengths for vectors of losses, we base this part of the exercise on a resampling technique using a uniform distribution as described above, this consists of generating 10000 random numbers from a uniform distribution, $U(0,1)$, then we use this numbers to extract the empirical quantiles from each vectors, this way we obtain two vector of length 10000 with a strong dependence structure since each time we draw a “small” value from the first vector we also get a “small” value from the second one, the same happens with “big” values, this is because we are using the 10000 uniform number as index for the inverse distribution function to retrieve those numbers.

The correlation coefficient enrolled in this case is $\approx 0.8875$, this is the correlation between $X'$ and $Y'$, which is the “strong” dependence structure giving name to this section.

Following the same idea from the previous section, we consider the total capital to be allocated as exogenously determined and taken as given, so we consider this capital to be the empirical Value at Risk at 99% which is 628 724.6 monetary units.

<table>
<thead>
<tr>
<th></th>
<th>HAP</th>
<th>CAP</th>
<th>Overbeck II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X'$</td>
<td>412897.2</td>
<td>464021.7</td>
<td>414842.6</td>
</tr>
<tr>
<td>$Y'$</td>
<td>215827.3</td>
<td>164702.9</td>
<td>213882.0</td>
</tr>
<tr>
<td>Total</td>
<td>628724.6</td>
<td>628724.6</td>
<td>628724.6</td>
</tr>
</tbody>
</table>

Table 6 presents the total capital and the amounts to be allocated to each business units. Note that the first business unit, called $X'$ is again the riskiest one, so more capital is allocated to it. One important point, when linear dependence between these two business unit becomes higher, is that all allocation principles are very close to each other, we were aware of this fact for both CAP and Overbeck II since they takes into account the dependence structure. Looking at the Haircut allocation principle (HAP) we can see that when correlation between losses is close to one then its results quietly differ from those obtained with Equation 45, it is clearly seen since now risks are dependent each other and this is the key reason why allocations based on (the approximation to HAP) surely leads us to misleading allocations. Note that approximation provided by Equation 45, when correlation is high, becomes biased.
Table 7: Case II. Proportions of capital allocation based on different principles

<table>
<thead>
<tr>
<th></th>
<th>SPA</th>
<th>HAP</th>
<th>CAP</th>
<th>OverbeckII</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X'$</td>
<td>0.6503</td>
<td>0.6567</td>
<td>0.7380</td>
<td>0.6598</td>
</tr>
<tr>
<td>$Y'$</td>
<td>0.3497</td>
<td>0.3433</td>
<td>0.2620</td>
<td>0.3402</td>
</tr>
</tbody>
</table>

In terms of proportions, Table 7 gives a picture of how the principles distribute the total capital between the business units. The first column represents the results using Equation 45, this would be the allocation if correlation between risk sources were zero, in this case the optimal distribution of the total capital should be 65.03% allocated to the first business line (bank transfer mistakes) and 34.97% to the loss caused by fraudulent transactions. Since correlation between risk sources is 0.8910, then allocation based on Equation 45 is biased, so principles that includes the linear dependence in its calculations are needed.

In spite of the fact that HAP is based on the idea of measuring stand-alone losses using a VaR (normal VaR in this case) it performs well enough even if the correlation is high, but one has to have in mind that VaR is not a coherent risk measure so in this case it is better off using a coherent risk measure for capital allocation, from this point we can choose either Covariance allocation principle or Overbeck type II allocation principle, but in practice HAP and CAP results are not so different.

We perform a set of simulations in order to find evidence of differences among these three allocation principles when varying the correlations and running thousands of random drawings.

8 A simulation study: Assessing the correlation effect

This section is intended to examine how sensible the allocation principles under study are when changing the linear dependence between the losses. The aim of this section is to answer, through a simulation study, the following question: How much does the allocated capital based on each principle differ when changing the correlation between the losses? In order to answer this question we consider two cases which are explained below.

The simulation study consists of drawing 100 random number from a multivariate normal distribution for two business lines with vector of means $\mu$ and covariance matrix $\Sigma$ for different correlation coefficients: 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1, where we can see the linear relationship goes from non-relationship at all until a perfect correlation. We apply the three capital allocation principles to these losses, and we repeat this step 1000 times, then we report the mean value of the allocations over these 1000 results.

8.1 Case I: Risk sources with same mean and variance

In this case we consider both business lines are equally risky (in terms of variances) and also they have the same mean.$^{12}$

$^{12}$Note that these means correspond to the mean of Public data risk no. 1 presented in Table 3
\[ \mu = \begin{pmatrix} 42.05941 \\ 42.05941 \end{pmatrix}, \]

and the variance-covariance matrix takes the form:

\[ \Sigma = \begin{pmatrix} 94442.09 & \sigma_{1,2} \\ \sigma_{2,1} & 94442.09 \end{pmatrix}, \]

where \( \sigma_{1,2} = \sigma_{2,1} \) and it is such that we can obtain the following correlation coefficients:

\[ \rho = 0, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, 7, 0, 8, 0, 9 \text{ and } 1. \]

Within this case we have two business lines with equal variances and equal means, the only difference in each replication of the simulation is the covariance (correlation) so that we can disentangle the "correlation effect" for each allocation based on the different principles which are the target of this study.

The reason why we set both, the means and the variances to be the same for both business lines, is that we expect the allocated capital to be one half in expected value for each vector, so this is a rough method to look into the robustness of each allocation principle when only the correlation is allowed to change.

Table 8: Simulation results Case I.

<table>
<thead>
<tr>
<th>Business / correlation</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line1 (Haircut)</td>
<td>0.4982</td>
<td>0.4999</td>
<td>0.5009</td>
<td>0.5003</td>
<td>0.5003</td>
<td>0.5009</td>
<td>0.501</td>
<td>0.5007</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>Line2 (Haircut)</td>
<td>0.5018</td>
<td>0.5001</td>
<td>0.4991</td>
<td>0.4997</td>
<td>0.4994</td>
<td>0.4997</td>
<td>0.4991</td>
<td>0.499</td>
<td>0.4993</td>
<td>0.4993</td>
<td>0.4993</td>
</tr>
<tr>
<td>Line1 (Covariance)</td>
<td>0.4975</td>
<td>0.5007</td>
<td>0.5015</td>
<td>0.5007</td>
<td>0.5007</td>
<td>0.5004</td>
<td>0.5009</td>
<td>0.5011</td>
<td>0.5007</td>
<td>0.4996</td>
<td>0.5</td>
</tr>
<tr>
<td>Line2 (Covariance)</td>
<td>0.5025</td>
<td>0.4993</td>
<td>0.4985</td>
<td>0.4993</td>
<td>0.4993</td>
<td>0.4996</td>
<td>0.4991</td>
<td>0.4989</td>
<td>0.4993</td>
<td>0.5004</td>
<td>0.5</td>
</tr>
<tr>
<td>Line1 (Overbeck II)</td>
<td>0.5025</td>
<td>0.4997</td>
<td>0.5064</td>
<td>0.4991</td>
<td>0.5013</td>
<td>0.4987</td>
<td>0.5008</td>
<td>0.5005</td>
<td>0.5005</td>
<td>0.5003</td>
<td>0.5</td>
</tr>
<tr>
<td>Line2 (Overbeck II)</td>
<td>0.4975</td>
<td>0.5003</td>
<td>0.4936</td>
<td>0.5009</td>
<td>0.4987</td>
<td>0.5013</td>
<td>0.4992</td>
<td>0.4995</td>
<td>0.4995</td>
<td>0.4997</td>
<td>0.5</td>
</tr>
</tbody>
</table>

From Table 8 the main conclusion is that the linear relationship between the losses does not play any important role for the allocation principles to be good enough and whose results are almost identical, this is true (based on the simulation study) if and only if the vector of losses are characterized by the same mean and variance. This result leads us to think that there is not reason to choose one or other principle, under these circumstances (identical mean and variance) any principle gives almost the same answer.

### 8.2 Case II: Risk sources with same mean but different variances

The second case under study in the simulation is keeping the means equal for both business lines and allowing different variances for them, and again we study the effect of correlations on the estimation of the allocated capital using the three principles which are the scope of this thesis.

The vector of means is the same as that used in the previous section and the covariance matrix now is:

\[ \text{Note that these variances correspond to the mean of Public data risk no. 2 presented in Table 3} \]
\[ \Sigma = \begin{pmatrix} 85242.64 & \sigma_{1,2} \\ \sigma_{2,1} & 94442.09 \end{pmatrix}, \]

here \( \sigma_{1,2} = \sigma_{2,1} \) and it is chosen such that we can obtain the following correlation coefficients:
\[ \rho = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 \text{ and } 1. \]
Note that these variances are the sample variances of the Public data risk 1 and 2, respectively.

Table 9: Simulation results Case II.

<table>
<thead>
<tr>
<th>Business / correlation</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line1 (Haircut)</td>
<td>0.721</td>
<td>0.7222</td>
<td>0.7215</td>
<td>0.7223</td>
<td>0.7228</td>
<td>0.7232</td>
<td>0.7229</td>
<td>0.7227</td>
<td>0.7231</td>
<td>0.7235</td>
<td>0.723</td>
</tr>
<tr>
<td>Line2 (Haircut)</td>
<td>0.279</td>
<td>0.2778</td>
<td>0.2785</td>
<td>0.2777</td>
<td>0.2772</td>
<td>0.2768</td>
<td>0.2771</td>
<td>0.2773</td>
<td>0.2769</td>
<td>0.2765</td>
<td>0.277</td>
</tr>
<tr>
<td>Line1 (Covariance)</td>
<td>0.9022</td>
<td>0.8796</td>
<td>0.859</td>
<td>0.8414</td>
<td>0.825</td>
<td>0.8105</td>
<td>0.7965</td>
<td>0.7838</td>
<td>0.7727</td>
<td>0.7628</td>
<td>0.7527</td>
</tr>
<tr>
<td>Line2 (Covariance)</td>
<td>0.0978</td>
<td>0.1204</td>
<td>0.141</td>
<td>0.1586</td>
<td>0.175</td>
<td>0.1895</td>
<td>0.2035</td>
<td>0.2162</td>
<td>0.2273</td>
<td>0.2372</td>
<td>0.2473</td>
</tr>
<tr>
<td>Line1 (Overbeck II)</td>
<td>0.8545</td>
<td>0.834</td>
<td>0.8187</td>
<td>0.801</td>
<td>0.7916</td>
<td>0.7792</td>
<td>0.7666</td>
<td>0.7558</td>
<td>0.7438</td>
<td>0.7373</td>
<td>0.7284</td>
</tr>
<tr>
<td>Line2 (Overbeck II)</td>
<td>0.1455</td>
<td>0.166</td>
<td>0.1813</td>
<td>0.199</td>
<td>0.2084</td>
<td>0.2208</td>
<td>0.2334</td>
<td>0.2442</td>
<td>0.2562</td>
<td>0.2627</td>
<td>0.2716</td>
</tr>
</tbody>
</table>

Some interesting pointers from this simulation are highlighted below:

- One interesting result stems1 from Table 9 is the allocation suggested by the Covariance principle when the correlation is zero, in general, when the variances are different keeping the means equal for both business lines, reduces to \( \frac{\text{Var}(X_i)}{\sum_{i=1}^{n} \text{Var}(X_i)}, \quad i = 1, \ldots, n. \) When correlation is different from zero the Covariance principle incorporates this information for the allocation estimates.

- The Haircut allocation principle is not affected by the linear dependence of the losses, this is because it is based on the Value at Risk and this is well-known as “stand-alone risk measure” see subsection 5.1.

- The larger the correlation the closer is the Overbeck type II allocation estimates to the Haircut allocation estimates.

- The larger the correlation the closer is the Covariance allocation estimates to the Haircut allocation estimates, but Overbeck type II converges faster than the Covariance allocation principle to the Haircut allocations.
9 Conclusions and Future Research

In this study we present the allocation problem and based on Dhaene et al. (2012) we provide explicit formulation for $K_i$ when using different specifications for business unit driven principles as well as aggregate portfolio driven allocations.

The numerical exercise carried out shows that the configuration of the allocations depends on the degree of linear dependence, this result motivates a simulation study to investigate the “correlation effect”. We find that if losses have same mean and variance, then correlation plays no important role on the allocations. On the other hand, non-identical variance gives rise to the correlation effect.

Changes in the correlation structure of the losses are important when losses have different second-moment, the more correlated the losses are, the bigger the amount of money required for facing risk.

Haircut allocation principle, even being a principle based on a non-coherent risk measure, experiences a good performance and it is less affected by the “correlation effect” (changes in the correlations). Haircut allocations are very similar to those suggested by Overbeck type II principle when correlation is high, this confirms the good performance of Haircut allocation principle.

This thesis is companied by an R package named OCA to allow other researchers to reproduce the results presented here and widespread the use of different allocation principle under R programming language. The package has been entirely developed by the author.

OCA package computes optimal capital allocation based on some standard principles such as Haircut, Overbeck type II and the Covariance Allocation Principle. Also it provides some shortcuts for obtaining the Value at Risk and the Expectation Shortfall, using both the normal and the $t$-student distribution.

As future research we propose to expand the package functionality by adding some other allocation principles discussed in this work. Once OCA is completed we can enrich both the numerical example and the simulation study.
References


10 Appendix

10.1 R codes for estimations of the numerical examples: Case I and Case II

# Installing and loading OCA package
# install.packages("OCA_0.1.tar.gz", repos = NULL, type = "source")
library(OCA)

# Loading the datasets
data(dat1)
data(dat2)

#------------------------------------------------------------------------------------
# Descriptive Statistics
library("fBasics")
library("Hmisc")
Descript <- cbind( basicStats(dat1), basicStats(dat2))
colnames(Descript) <- c("X", "Y")
#latex(round(Descript, 2), file="")

#------------------------------------------------------------------------------------
# Numerical example: Case I.
#------------------------------------------------------------------------------------
# Overcoming the non-equal length of vectors
set.seed(1)
Replications <- 10000
dat1.boot <- colSums(replicate(Replications, sample(dat1[,1], nrow(dat1), replace=TRUE)))
dat2.boot <- colSums(replicate(Replications, sample(dat2[,1], nrow(dat2), replace=TRUE)))

# Both sampled vectors now have the same length
length(dat1.boot)
length(dat2.boot)

# Building the Loss matrix
Lboot <- cbind(dat1.boot, dat2.boot)
cor(Lboot)

# Building S_i
S_i <- rowSums(Lboot)

# Setting the aggregate capital K so be the 99\% empirical VaR
K <- quantile(S_i, probs = 0.99) # alternative: sort(S_i)[length(S_i)*.99]
# Getting $K_i$ based on different capital allocation principles:
# Haircut Allocation Principle
H <- hap(Loss=Lboot, Capital=K)
Hprop <- hap(Loss=Lboot, Capital=1)

# Covariance Allocation Principle
C <- cap(Loss=Lboot, Capital=K)
Cprop <- cap(Loss=Lboot, Capital=1)

# Overbeck type II Allocation Principle
Cte <- Overbeck2(Loss=Lboot, Capital=K)
Cteprop <- Overbeck2(Loss=Lboot, Capital=1)

# Comparing all results
Allboot <- data.frame( H, C, Cte)
colnames(Allboot) <- c( "HAP", "CAP", "Overbeck_II")
Allboot <- rbind(Allboot, Total=colSums(Allboot))
Allboot
latex(round(Allboot, 2), file="")

Allprop <- data.frame(colSums(Lboot)/sum(Lboot), Hprop, Cprop, Cteprop)
colnames(Allprop) <- c("SPA", "HAP", "CAP", "CTEAP")
Allprop
latex(round(Allprop, 4), file="")

# Numerical example: Case II.

# Overcoming the non-equal length of vectors
set.seed(1)
unif <- runif(10000)
dat1.unif <- quantile(dat1[,1], unif)
dat2.unif <- quantile(dat2[,1], unif)
Lunif <- cbind(dat1.unif, dat2.unif)

# Creating a boot sampling
set.seed(1)
Lunif.boot <- t(replicate(Replications,
colSums(Lunif[sample(1:Replications,
Replications, TRUE), ])))

# Verifying the correlation
cor(Lunif.boot)
# Building S_i
S_iunif <- rowSums(Lunif.boot)

# Setting the aggregate capital K so be the 99\% empirical VaR
K <- quantile(S_iunif, probs = 0.99)

### Getting K_i based on different capital allocation principles:

### Haircut Allocation Principle
H <- hap(Loss=Lunif, Capital=K)
Hprop <- hap(Loss=Lunif, Capital=1)

### Covariance Allocation Principle
C <- cap(Loss=Lunif, Capital=K)
Cprop <- cap(Loss=Lunif, Capital=1)

### Overbeck type II Allocation Principle
Cte <- Overbeck2(Loss=Lunif, Capital=K)
Cteprop <- Overbeck2(Loss=Lunif, Capital=1)

### Comparing all results
AlLunif <- data.frame( H, C, Cte)
colnames(AlLunif) <- c("HAP", "CAP", "CTEAP")
AlLunif

Allprop <- data.frame(colSums(Lunif)/sum(Lunif), Hprop, Cprop, Cteprop)
colnames(Allprop) <- c("SPA", "HAP", "CAP", "CTEAP")
Allprop

10.2 R codes for the simulation study: Case I and Case II

# A simulation study

# "getCov" takes a vector of correlation as argument and two variances to return
# a covariance matrix for each rho and var1 and var1, it solves this equation
# cov=rho*sqrt(var1, var2)
getCov <- function(rho, var1, var2){
  rho <- rho
  prodsigmas <- sqrt(var1) * sqrt(var2)
  Cov <- rho * prodsigmas
Sigmas <- lapply(1:length(Cov), function(x,i) {
  matrix(c(var1, x[i], x[i], var2), ncol=2)
}, x=Cov)
return(Sigmas)

library(MASS) # to use "mvrnorm" function

# CASE I

# Setting initial parameters
mu1 <- rep(mean(dat1[,1]), 2)
variance1 <- var(dat1[,1]) + var(dat2[,1])
sigmas <- getCov(rho=seq(0,1,.1), var1=variance1, var2=variance1)
N <- 1000    # Number of simulations to run used in "replicate" funcion
n <- 100     # sample size to be passed to "mvrnorm" function
DimNames <- list(c("Line 1", "Line 2"), paste(seq(0,1,.1)))

# HAP
set.seed(1)
case1.sim.hap <- replicate(N, sapply(lapply(sigmas, mvrnorm, n=n, mu=mu1), hap, Capital=1))
case1.hap <- apply(case1.sim.hap, c(1,2), mean)
dimnames(case1.hap) <- DimNames
round(case1.hap, 4)

# CAP
set.seed(1)
case1.sim.cap <- replicate(N, sapply(lapply(sigmas, mvrnorm, n=n, mu=mu1), cap, Capital=1))
case1.cap <- apply(case1.sim.cap, c(1,2), mean)
dimnames(case1.cap) <- DimNames
round(case1.cap, 4)

# Overbeck II
set.seed(1)
case1.sim.Overbeck2 <- replicate(N, sapply(lapply(sigmas, mvrnorm, n=n, mu=mu1), Overbeck2, Capital=1))
case1.Overbeck2 <- apply(case1.sim.Overbeck2, c(1,2), mean, na.rm=TRUE)
dimnames(case1.Overbeck2) <- DimNames
round(case1.Overbeck2, 4)
# CASE II

# Setting initial parameters
mu2 <- rep(mean(dat1[,1]), 2)
sigmas2 <- getCov(rho=seq(0,1,.1), var1=var(dat1[,1]), var2=var(dat2[,1]))

# HAP
set.seed(1)
case2.sim.hap <- replicate(N, sapply(lapply(sigmas2, mvrnorm, n=n, mu=mu2), hap, Capital=1))
case2.hap <- apply(case2.sim.hap, c(1,2), mean)
dimnames(case2.hap) <- DimNames
round(case2.hap, 4)

# CAP
set.seed(1)
case2.sim.cap <- replicate(N, sapply(lapply(sigmas2, mvrnorm, n=n, mu=mu2), cap, Capital=1))
case2.cap <- apply(case2.sim.cap, c(1,2), mean)
dimnames(case2.cap) <- DimNames
round(case2.cap, 4)

# Overbeck II
set.seed(1)
case2.sim.Overbeck2 <- replicate(N, sapply(lapply(sigmas2, mvrnorm, n=n, mu=mu2), Overbeck2, Capital=1))
case2.Overbeck2 <- apply(case2.sim.Overbeck2, c(1,2), mean, na.rm=TRUE)
dimnames(case2.Overbeck2) <- DimNames
round(case2.Overbeck2, 4)
Package ‘OCA’

OCA stands for Optimal Capital Allocations

Description:
OCA computes optimal capital allocation based on some standard principles such as Haircut, Overbeck type II and the Covariance Allocation Principle. It also provides some shortcuts for obtaining the Value at Risk and the Expectation Shortfall, under both normality and student $t$ distributional assumptions for the vector of losses.

Details:

Package: OCA
Type: Package
Version: 0.1
Date: 2013-04-14
License: LGPL (>= 2)

Author:
Jilber Urbina

Maintainer:
Jilber Urbina <jilberurbina@gmail.com>
Package ‘OCA’

June 10, 2013

Type Package

Title Optimal Capital Allocations

Version 0.1

Date 2013-04-14

Author Jilber Urbina

Maintainer Jilber Urbina <jilberurbina@gmail.com>

Description Computes optimal capital allocations based on different risk measures

License GPL2.0

Encoding latin1

R topics documented:

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OCA-package

Optimal Capital Allocation Principles

Description

OCA computes optimal capital allocation based on some standard principles such as Haircut, Overbeck type II and the Covariance Allocation Principle. Also it provides some shortcuts for obtaining the Value at Risk and the Expectation Shortfall, using both the normal and the t-student distribution.

Details

Package: OCA
Type: Package
Version: 0.1
Date: 2013-04-14
License: LGPL (>= 2)

Author(s)

Jilber Urbina
Maintainer: Jilber Urbina <jilberurbina@gmail.com>

References


cap

Covariance Allocation Principle

Description

This function implements the covariance allocation principle for optimal capital allocation.
Usage

cap(Loss, Capital)

Arguments

Loss A matrix containing the individual losses in each column
Capital A scalar representing the capital to be allocated to each loss.

Details

The Covariance Allocation Principle correspond to the following expression:

\[ K_i = \frac{K}{\text{Var}[S]} \text{Cov}(X_i, S), \quad i = 1, \ldots, n, \]

where \( K_i \) is the capital to be allocated to the \( i \)th loss, \( K \) is the total capital to be allocated, \( X_i \) is the individual unit loss and \( S \) is the total (aggregate) loss, this comes from \( \sum_i X_i \). \( \text{Cov}(X_i, S) \) is the covariance between the individual loss \( X_i \) and the aggregate loss \( S \); and \( \text{Var}(S) \) is the variance of the aggregate loss.

Value

A \( n \times 1 \) matrix containing each asset and the corresponding capital allocation. If Capital=1, then the returned value will be the proportions of capital required by each loss to be faced.

Author(s)

Jilber Urbina

References


See Also

Overbeck2, hap

Examples

data(dat1, dat2)
Loss <- cbind(Loss1=dat1[1:400, ], Loss2=unname(dat2))
# Proportions of capital to be allocated to each business unit
cap(Loss, Capital=1)

# Capital allocation,
# capital is determined as the empirical VaR of the losses at 99%
K <- quantile(rowSums(Loss), probs = 0.99)
cap(Loss, Capital=K)
Description

Dataset named Public data risk no. 1 consisting in 1000 of simulated data.

Usage

data(dat1)

Format

A data frame with 1000 observations on the following variable.

y  a numeric vector

References


Examples

data(dat1)

---

Description

Dataset named Public data risk no. 2 consisting in 400 of simulated data.

Usage

data(dat2)

Format

A data frame with 400 observations on the following variable.

y  a numeric vector

References

**Examples**

```r
data(dat2)
```
where $\alpha$ is the significance level, $\text{VaR}_\alpha(X)$ is the Value-at-Risk of $X$.

ES for the normal case is based on the following expression:

$$ES_\alpha = \mu + \sigma \frac{\Phi^{-1}(\alpha)}{1 - \alpha}$$

Meanwhile, ES for the t-student distribution takes comes from:

$$ES_\alpha(\tilde{X}) = \frac{g_\upsilon(t^{-1}_\upsilon(\alpha))}{1 - \alpha} \left( \frac{\upsilon + (t^{-1}_\upsilon(\alpha))^2}{\upsilon - 1} \right)$$

Value

A data.frame containing the ES for each significance level specified.

Author(s)

Jilber Urbina

References


See Also

VaR, Risk

Examples

# Exercise 2.21, page 46 in McNeil et al (2/zero.noslash/zero.noslash5)
alpha <- c(.9/zero.noslash, .95, .975, .99, .995)
(ES(Loss=1, varcov=(/zero.noslash.2/sqrt(25/zero.noslash))^2, alpha=alpha, model='normal')-1)*1/zero.noslash/zero.noslash/zero.noslash/zero.noslash
(ES(Loss=1, varcov=(/zero.noslash.2/sqrt(25/zero.noslash))^2, alpha=alpha, model='t-student', df=4)-1)*1/zero.noslash/zero.noslash/zero.noslash/zero.noslash

# Both type of models at once.
(ES(Loss=1, varcov=(/zero.noslash.2/sqrt(25/zero.noslash))^2, alpha=alpha, model='both', df=4)-1)*1/zero.noslash/zero.noslash/zero.noslash/zero.noslash

# A vector of losses
L <- c(10,40) # a vector of two (mean) losses
varcov <- matrix(c(100,150,150,900), 2) # covariance matrix
w <- c(0.5, 0.5) # a vector weights
ES(Loss=L, varcov=varcov, weights=w, alpha=0.95)
Description

Capital allocation based on the Haircut Allocation Principle.

Usage

```r
hap(Loss, Capital, alpha = 0.95,
    model = "normal", df = NULL)
```

Arguments

- **Loss**: Either a scalar or a vector of size $N$ containing the mean losses.
- **Capital**: A scalar representing the capital to be allocated to each loss.
- **alpha**: A numeric value (either a single one or a vector) consisting of the significance level at which ES has to be computed, it can either be a single numeric value or a vector of numeric values.
- **model**: A character string indicating which distribution is to be used for computing the VaR underlying the Haircut Allocation Principle (HAP), the default value is the normal distribution, the other alternative is t-student distribution with $\upsilon$ degrees of freedom. When `model='both' 'normal'` as well as `model='t-student'` are used when computing the HAP, see examples.
- **df**: An integer indicating the degrees of freedom for the t-student distribution when setting `model='t-student'` and `model='both'`. $df$ must be greater than 2.

Details

This function computes the capital allocation based on the so-called Haircut Allocation Principle whose expression is as follows:

$$ K_i = \frac{K}{\sum_{j=1}^{n} F_{X_j}^{-1}(p) F_{X_i}^{-1}(p)} $$

For $i = 1, \ldots, n$, where $K_i$ represents the optimal capital to be allocated to each individual loss for the $i$-th business unit, $K$ is the total capital to be allocated, $F_{X_j}^{-1}(p)$ is the quantile function (VaR) for the $i$-th loss.

Value

A real-valued $n \times 1$ matrix containing the optimal capital allocation, if `Capital` is set to 1, then the returned matrix will consist of the proportions of capital each individual loss needs to be optimally faced.
Author(s)

Jilber Urbina

References


See Also

Overbeck2, cap

Examples

data(dat1, dat2)
Loss <- cbind(Loss1=dat1[1:400, ], Loss2=unname(dat2))
# Proportions of capital to be allocated to each business unit
hap(Loss, Capital=1)

# Capital allocation,
# capital is determined as the empirical VaR of the losses at 99%
K <- quantile(rowSums(Loss), probs = 0.99)
hap(Loss, Capital=K)

Overbeck2

*Overbeck type II Allocation Principle*

Description

This function implements the Overbeck type II allocation principle for optimal capital allocation.

Usage

```r
Overbeck2(Loss, Capital, alpha = 0.95,
model = c("normal", "t-student", "both"), df = NULL)
```

Arguments

<table>
<thead>
<tr>
<th>Loss</th>
<th>A scalar or a vector of size $N$ containing the mean losses.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital</td>
<td>A scalar representing the capital to be allocated to each loss.</td>
</tr>
<tr>
<td>alpha</td>
<td>A numeric value (either a single one or a vector) consisting of the significance level at which the allocation has to be computed, it can either be a single numeric value or a vector of numeric values.</td>
</tr>
</tbody>
</table>
model A character string indicating which distribution is to be used for computing the VaR underlying the Overbeck type II principle, the default value is the normal distribution, the other alternative is t-student distribution with $\nu$ degrees of freedom. When model='both' 'normal' as well as 't-student' are used when computing the allocations, see examples.

df An integer indicating the degrees of freedom for the t-student distribution when setting model='t-student' and model='both'. df must be greater than 2.

Details

Overbeck2 computes the capital allocation based on the following formulation:

$$K_i = \frac{K}{CTE_p[S]} E \left[ X_i | S > F_{X_S}^{-1}(p) \right], \quad i = 1, \ldots, n.$$  

Where $K$ is the aggregate capital to be allocated, $CTE_p[S]$ is the Conditional Tail Expectation of the aggregate loss at level $p$, $X_i$ is the individual loss, $S$ is the aggregate loss and $F_{X_S}^{-1}(p)$ is the quantile function of $X$ at level $p$.

Value

A real-valued $n \times 1$ matrix containing the optimal capital allocation, if Capital is set to 1, then the returned matrix will consist of the proportions of capital each individual loss needs to be optimally faced.

Author(s)

Jilber Urbina

References


See Also

hap, cap

Examples

data(dat1, dat2)
Loss <- cbind(Loss1=dat1[1:400, ], Loss2=unname(dat2))
# Proportions of capital to be allocated to each business unit
Overbeck2(Loss, Capital=1)

# Capital allocation,
# capital is determined as the empirical VaR of the losses at 99%
K <- quantile(rowSums(Loss), probs = 0.99)
Overbeck2(Loss, Capital=K)
Risk measures such as Value at Risk (VaR) and Expected Shortfall (ES) with normal and t-student distributions.

Description

Standard risk measures such VaR and ES are provided by Risk. Both VaR and ES can be computed using either the normal or t-student distribution.

Usage

Risk(Loss, varcov, alpha = 0.95, measure = c("VaR", "ES", "both"), weights = NULL, model = c("normal", "t-student", "both"), df = NULL)

Arguments

Loss It could be either a scalar or a $m \times 1$ matrix containing the mean losses.
varcov A scalar corresponding to the variance of the loss, if Loss is a $m \times 1$ matrix, then varcov must be a $m \times m$ matrix containing the variances and covariances of the losses.
alpha The confidence level at which either the VaR or the ES will be computed, by default alpha is set to 0.95.
measure An optional character string giving a measure for computing the risk. "VaR" stands for Value at Risk, "ES" stands for Expected Shortfall, and if both is chosen, then the function returns both the VaR and the ES as a result. By default measure is set to be "VaR".
weights A vector containing the weights. It is only needed if Loss is a matrix, if it is not then weights is set to 1.
model A character string indicating which probability model has to be used for computing the risk measures, it could only be a normal distribution or a t-student distribution with $\nu$ degrees of freedom. The normal distribution is the default model for this function. model also allows the user to set ‘both’ if she wishes both normal and t-student VaR or ES depending on what she choses in measure. See example below.

Value

A data.frame containing each risk measure at its corresponding confidence level.
Author(s)
Jilber Urbina.

References

See Also
VaR

Examples

```r
# Reproducing Table 2.1 in page 47 of
alpha <- c(.9, .95, .975, .99, .995)
(Risk(Loss=1, varcov=(.2/sqrt(25))^2, alpha=alpha,
    measure='both', model='both', df=4)-1)*10000

# only VaR results
(Risk(Loss=1, varcov=(.2/sqrt(25))^2, alpha=alpha,
    measure='VaR', model='both', df=4)-1)*10000

# only normal VaR results
(Risk(Loss=1, varcov=(.2/sqrt(25))^2, alpha=alpha)-1)*10000

# only SE based on a 4 degrees t-student.
(Risk(Loss=1, varcov=(.2/sqrt(25))^2, alpha=alpha,
    measure='ES', model='t-student', df=4)-1)*10000
```

<table>
<thead>
<tr>
<th>VaR</th>
<th>Value at Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Description
Computes Value at Risk based on both normal and t-student distribution.

Usage

```
VaR(Loss, varcov, alpha = 0.95, weights = NULL,
    model = c("normal", "t-student", "both"),
    df = NULL)
```
Arguments

Loss               It could be either a scalar or a $m \times 1$ matrix containing the mean losses.
varcov            A scalar corresponding to the variance of the loss, if Loss is a $m \times 1$ matrix,
                   then varcov must be a $m \times m$ matrix containing the variances and covariances
                   of the losses.
alpha             The confidence level at which either the VaR or the ES will be computed, by
default alpha is set to 0.95.
weights           A vector of weights of size $N$ for computing both the mean and the variance
                   of the vector of Losses, it is applicable only when Loss is a vector. When
                   weights=NULL, mean and variances used to compute ES are the original values
                   supplied to Losses and varcov.
model             A character string indicating which probability model has to be used for
                   computing the risk measures, it could only be a normal distribution or a t-student
                   distribution with $v$ degrees of freedom. The normal distribution is the default
                   model for this function. model also allows the user to set 'both' if she wishes
                   both normal and t-student VaR or ES depending on what she choses in measure.
                   See example below.
df                An integer (df>2) denoting the degrees of freedom, only required if model='t-
                   student'. Otherwise it has to be NULL.

Value

A data.frame containing each risk measure at its corresponding confidence level

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References


See Also

Risk

Examples

# Reproducing VaR from Table 2.1 in page 47 of

alpha <- c(.90, .95, .975, .99, .995)
(VaR(Loss=1, varcov=(.2/sqrt(250))^2, alpha=alpha,
model='both', df=4)-1)*10000

# only normal VaR results
(VaR(Loss=1, varcov=(.2/sqrt(250))^2, alpha=alpha)-1)*10000
# Same result using
(Risk(Loss=1, varcov=(0.2/sqrt(250))^2, alpha=alpha)-1)*10000
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