The degrees of freedom of the 3-user 
\((p, p+1)\) MIMO interference channel

Master’s Final Project Dissertation

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Abstract

Data traffic demand has increased at a very high rate in recent years, whereby operators need to deliver progressively higher data rates. To overcome this, the transmission bandwidth of each user can be increased, but therefore cellular systems are forced to implement frequency subcarriers reuse due to spectrum scarcity. In this context, interference has become the major bottleneck to increase the delivered high rate as much as the traffic demand increases. As a consequence, a number of efforts has been focused on search for new more efficient communication systems, able to manage inter-cell interference by coordinating base stations transmissions.

This work deals with multi-antenna transmissions for a theoretic scenario called the interference channel, composed of 3 transmitter-receiver pairs, where each transmitter is intended to transmit messages to its associated receiver only. Currently capacity characterization for this channel remains still an open problem. In this regard, in the last few years most works in the field have chosen to study the high signal to noise ratio (SNR) regime. This can be useful to get insight about this channel properties, and understand how the general case can be tackled. The degrees of freedom (DoF) emerge as a measure to characterize capacity at high SNR, where smart linear schemes have been found achieving the optimal DoF for almost antenna configurations. Specifically, when the channel coefficients are constant and terminals are single-antenna or there are \( p, p+1 \) antennas (with \( p > 1 \)) deployed at the transmitters and the receivers, the DoF are currently not known. This work addresses the later case, for which the solution of previous works [WGJ11a] is not able to attain the optimal DoF value. In this case, it seems that schemes based on symbol extension in time and interference alignment (IA) concepts cannot attain the optimal DoF. This work proposes a transmit-receive filter design that attains the DoF outer bound with probability one by means of considering a recent approach referred to as asymmetric complex signaling. In particular, the DoF for \( p = 2, 3 \) are formally proved, and additional tools and procedures are provided to constructively prove achievability for any given value of \( p \), giving as application examples the cases \( p = 4, 5, 6 \). As a consequence, we are able to conjecture all those antenna configurations become characterized.
Acknowledgments

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## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contents</td>
<td>iii</td>
</tr>
<tr>
<td>List of Figures</td>
<td>v</td>
</tr>
<tr>
<td>List of Tables</td>
<td>v</td>
</tr>
<tr>
<td>Acronyms</td>
<td>vi</td>
</tr>
<tr>
<td>1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 State of the Art</td>
<td>3</td>
</tr>
<tr>
<td>1.2 Contributions</td>
<td>6</td>
</tr>
<tr>
<td>1.3 Organization</td>
<td>7</td>
</tr>
<tr>
<td>1.4 Notation</td>
<td>7</td>
</tr>
<tr>
<td>2 The 3-user MIMO Interference channel</td>
<td>9</td>
</tr>
<tr>
<td>2.1 Constant channel extension</td>
<td>10</td>
</tr>
<tr>
<td>3 Background</td>
<td>13</td>
</tr>
<tr>
<td>3.1 Change of basis operation</td>
<td>13</td>
</tr>
<tr>
<td>3.2 Subspace alignment chains</td>
<td>15</td>
</tr>
<tr>
<td>3.3 Signal space matrix</td>
<td>21</td>
</tr>
<tr>
<td>4 The (2,3) case</td>
<td>23</td>
</tr>
<tr>
<td>4.1 Precoding matrix design</td>
<td>23</td>
</tr>
<tr>
<td>4.2 Feasibility</td>
<td>25</td>
</tr>
<tr>
<td>5 Extension to the (p, p+1) case</td>
<td>28</td>
</tr>
<tr>
<td>5.1 Precoding matrix design</td>
<td>28</td>
</tr>
<tr>
<td>5.2 Feasibility</td>
<td>31</td>
</tr>
<tr>
<td>6 Conclusions and Future Work</td>
<td>34</td>
</tr>
</tbody>
</table>

iii
A Transformations to simplify cross-channel matrices 36

B Proof of Lemma1 38

C Results for $p = 2, 3, ..., 6$ 40
   C.1 Precoding matrices for $p = 2, 3, ..., 6$ 40
   C.2 Signal space matrix for $p = 2, 3, ..., 6$ 41

Bibliography 45
List of Figures

1.1 Channel classification ........................................... 2
1.2 DoF characterization for the 3-user MIMO IC ................... 6
2.1 The 3-user MIMO IC ............................................. 10
3.1 Change of basis for $p = 2$ .................................... 14
3.2 Alignment chain for the square case ............................ 17
3.3 General Alignment chain ....................................... 19
3.4 Subspace alignment chains example ............................ 21
5.1 Zero propagation algorithm for $p=3$ ......................... 30

List of Tables

C.1 Precoding matrices structure for $p = 2, 3, ..., 6$ ............... 42
C.2 Matrix $G^\text{des}_j$ for $p = 2, 3, ..., 6$ ........................... 43
C.3 Matrix $G^\text{int}_j$ for $p = 2, 3, ..., 6$ ........................... 44
## Acronyms

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACS</td>
<td>Asymmetric Complex Signaling</td>
</tr>
<tr>
<td>BC</td>
<td>Broadcast channel</td>
</tr>
<tr>
<td>BS</td>
<td>Base station</td>
</tr>
<tr>
<td>CSI</td>
<td>Channel State Information</td>
</tr>
<tr>
<td>CSIT</td>
<td>Channel State Information at the transmitter side</td>
</tr>
<tr>
<td>DoF</td>
<td>Degrees of Freedom</td>
</tr>
<tr>
<td>i.i.d.</td>
<td>independent and identically distributed</td>
</tr>
<tr>
<td>IA</td>
<td>Interference Alignment</td>
</tr>
<tr>
<td>IC</td>
<td>Interference Channel</td>
</tr>
<tr>
<td>MAC</td>
<td>Multiple access channel</td>
</tr>
<tr>
<td>MIMO</td>
<td>Multiple-Input Multiple-Output</td>
</tr>
<tr>
<td>pdf</td>
<td>probability density function</td>
</tr>
<tr>
<td>SISO</td>
<td>Single-Input Single-Output</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal to Noise Ratio</td>
</tr>
<tr>
<td>UE</td>
<td>User equipment</td>
</tr>
<tr>
<td>ZF</td>
<td>Zero forcing</td>
</tr>
<tr>
<td>ZP</td>
<td>Zero Propagation</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

Wireless communications research has been in significant activity during the last two decades. This is because there is a huge demand of users to be connected at anywhere and anytime, essentially for the purpose of using progressively more capable terminals, e.g. smartphones or tablets. In contrast to wired communications, wireless communications present three important drawbacks: radio spectrum scarcity, channel impairments and interference.

On the one hand, the spectrum band envisioned for the mobile communications is 400 MHz - 3.5 GHz, but other services work in some parts of the spectrum, such as the Digital Terrestrial Television or WiFi, using 470-862 MHz and 2 GHz, respectively. In this regard, the transmission coding strategy should be efficient in terms of bandwidth occupied, because it has to be shared among users.

Second, the wireless channel presents several channel impairments, such as multipath fading, shadowing, pathloss [TV05]. Those channel impairments can be tackled by means of diversity, improving the reliability of the wireless communication systems. Typically, diversity is exploited by means of time or frequency division multiplexing (TDM and FDM), transmitting the same signal through multiple and independent channels. Likewise, multiple antennas might be used for applying beamforming techniques based on the direction of arrival of the transmitted signal, when there is Channel State Information at the transmitter side (CSIT).

And finally, interference is a harsh problem, because the received signal by each user can be formulated as the linear combination of the signals transmitted by each Base station (BS). Therefore, depending on the level of interference received, the decoding process may become not possible by simply treating it as noise. In this regard, interference has been typically managed by transmitting through orthogonal
channels, that is, by using different frequency or time resource elements. But usually this approach is too harmful because it penalizes the final rate of each user by the inverse of the number of users. Therefore, one solution can be to coordinate the transmissions in order to improve the performance of the whole system.

Moreover, all those three issues can be mitigated by transmitting the same signal through multiple antennas. In fact, since mid-90s it is known that the use of multiple antennas at the source and the destination (Multiple-Input Multiple-Output (MIMO) channel) allows increasing not only the reliability of the communications (known as diversity gain), but also the transmitted rate (known as multiplexing gain) compared with the single-antenna case. These ideas were shown by Telatar in [Tel99], who proved that the capacity of the point-to-point MIMO channel increases linearly with the minimum number of transmitting and receiving antennas. In addition to this, MIMO technology has motivated the rise of a number of promising techniques to manage interference by means of coordinating the signals intended to each user in order to smartly exploit the spatial domain.

Beyond a point-to-point connection, wireless networks are built on several information sources and sinks that share the same transmission resources and pursue reliable communication of multiple information flows. A rigorous way of approaching analysis of those networks is through the study of multiuser channels borrowed from Network Information Theory [CT06], which are commonly grouped depending on the interaction and the role of the different terminals involved in the communication. Figure 1.1 depicts the principal multi-user channels: Broadcast channel (BC), Multiple access channel (MAC) and Interference Channel (IC).
The BC models the downlink transmission in a cellular communication system, where one transmitter, which is commonly a BS, sends independent messages to multiple destinations or user equipments (UEs). On the other hand, the MAC can address uplink transmissions in a cellular system where multiple sources (UEs) send independent messages to a single destination (BS). These two channels have been widely characterized during the last three decades and a summary of results can be found in [CT06]. Finally, the IC, or more precisely the $K$-user IC, models the scenario where $K$ source-destination pairs coexist in the same area, generating interference to each other. Although information-theoretic study for this channel has a long history, characterization of its capacity region still remains an open problem in general, except some special cases such as the 2-user case with strong interference [HK81].

In this regard, the analysis of this channel at the high Signal to Noise Ratio (SNR) regime is a suitable way to get insight about this channel properties. Typically, the performance at the high SNR regime is studied in terms of Degrees of Freedom (DoF)[MAMK08, CJ08], defined as

$$d = \lim_{\rho \to \infty} \frac{C(\rho)}{\log \rho}$$

where $\rho$ denotes the SNR and $C(\rho)$ is the mutual information capacity [CT06]. This measure represents the slope of the capacity curve when the SNR goes to infinity.

In this situation, the presence of noise and pathloss can be drawn, and the design is focused on combat interference. Intuitively, the DoF can be understood as the number of bitstreams that each user is able to send in the presence of other users. State of the art progress with respect to DoF characterization is reviewed in the next section.

## 1.1 State of the Art

In recent years, one of the most promising techniques to analyze interference networks in terms of DoF has been Interference Alignment (IA) concept. This technique was originally posed in the context of index coding problems, and the first known work can be found in [BK98]. Their authors present also applications of their technique for the BC channel with cognitive receivers, that is, a scenario where the receivers are able to know some of the messages intended to other users. The main contribution of their scheme is that the received interference at each receiver that cannot be removed by using the available cognitive information is aligned on its dimensional space. Therefore, it can be removed and the desired signal can be recovered without interference.
This technique follows its own way along index coding literature, and almost a decade after, the use of IA concept to study the DoF of wireless channels was introduced by Maddah-Ali et al [MAMK08] for the 2-user MIMO X-channel. Later on, this concept was extended to the $K$-user IC with $K > 2$ users and single antenna nodes by Cadambe and Jafar [CJ08]. The idea in those cases was to minimize the dimension of the received interference subspace by means of proper transmit precoders that align the interfering signals at unintended receivers, while the desired signal lies on a different subspace. In such a case, interference can be removed by applying a Zero forcing (ZF) receiver, which projects the received signal onto the orthogonal-to-interference subspace. [CJ08] introduces the idea that even though the number of users increases, each user gets "half the cake" or, equivalently, achieves 0.5 DoF. Additionally, they extend this result to the MIMO case, obtaining that equipping all nodes with $M$ antennas, each user achieves $M/2$ DoF. Their scheme is asymptotically optimal when considering infinite symbol extensions in time or frequency and assumes time-varying or frequency-selective channel coefficients. However, and only for the SISO case, it fails when considering constant channel coefficients.

Since then, many works have analyzed DoF of the IC using the IA concept, some of them providing outer bounds [YGJK10, RLL12, GJ10], and others proposing precoding schemes achieving or getting close to these outer bounds [MAMK08, GJ10, WGJ12, CJW10, BCT11]. In general, two frameworks of IA are considered: lattice level IA [MMK10] (lattice alignment) and vector space level IA [MAMK08] (vector space alignment). These two techniques arise from the choice between random and structured codes, respectively. In short, lattice alignment-based techniques are able to decode aggregate interference even when interferers are not individually decodable, and become a great candidate for interference networks. In fact, the DoF for the IC can be attained by using this approach. However, this approach is not advisable at present due to the current channel state estimation accuracy in wireless networks [HLDL11]. Moreover, it has been observed that the necessary SNR value to obtain the DoF slope in the capacity versus SNR curve is usually higher by using this approach. Because of this practical aspects, this work aims to attain the optimal DoF using vector space alignment techniques. Most relevant results of IA concept applied for the study of wireless networks is summarized in [Jaf11].

Back to the 3-user IC, [CJW10] revisited this channel with constant channel coefficients and single antenna nodes. In that work, the authors introduced Asymmetric Complex Signaling (ACS) concept, known also as improper signaling. This approach consists of signaling differently in real and imaginary parts. Using this idea, the
1.1 State of the Art

authors showed that 0.4 DoF per user can be achieved in case of constant channel coefficients. Unfortunately, no converse arguments are provided, only a proof to show that the technique used cannot afford additional DoF than those found. Hence, it is not known if any other scheme would be able to improve that value.

For the MIMO case, Wang et al show in [WGJ11a] that the DoF outer bound is achieved for almost all the cases by introducing the subspace alignment chains concept and performing symbol extensions in time or frequency (regardless the channel is constant in time or not). However, for the \((p, p + 1)\) 3-user MIMO IC case, that is, \(p\) antennas at each transmitter and \(p + 1\) antennas at each receiver, with \(p > 1^1\), these two concepts failed achieving the total DoF of the channel for the constant channel case (see section 8.3 in [WGJ11a]). In this regard, the authors claimed in later works [WGJ12, WGJ11b] that the outer bound is tight by using ACS with the methodology introduced in [WGJ11a], but without formal proof, just sustained on numerical experiments. The goal of this work is to develop a constructive proof of the DoF for those cases.

A summary of current DoF characterization for the 3-user MIMO IC with \(M\) antennas at the transmitters and \(N\) at the receivers is explicitly shown for \(M, N = 1...6\) in Figure 1.2, remarking the cases solved in this work. These results correspond to schemes based on vector space alignment since using lattice alignment all antenna configurations are completely characterized. Notice the singularity of the SISO case, being the only case where the exact DoF remain unknown after this work. For that case, the best inner and outer bounds are only able to determine that the maximum DoF are contained in the interval \([0.4,0.5]\).

Finally, the DoF for the \(K\)-user IC have been found by using lattice alignment [GMK11] and information-theoretic outer bounds [WSJ12] for a wide range of settings, i.e as a function of the number of users and antenna configuration. However, translating those results for the vector space alignment framework, as the present work does, and derive the optimal DoF for the remaining cases, remains still an open problem.

\[1^1\] The particular \(p = 1\) case can be addressed with the methodology in [WGJ11a], so this case is not tackled in this work.
1.2 Contributions

The present work addresses the design of optimal (in terms of DoF) transmit-receive filters for the 3-user MIMO IC with constant channel coefficients. First, a proposed precoding scheme based on IA concepts and symbol extension is presented. Second, an analytical proof is provided to show that the DoF outer bound can be achieved with probability one by means of the proposed linear transmitters and receivers. Consequently, we summarize here the 3 main contributions for this work:

**Contribution 1.** We prove that the 3-user MIMO IC with constant channel coefficients and \((p, p+1)\) antennas at each pair transmitter-receiver, with \(p = 2, 3, \ldots, 6\) has exactly \(p(p+1)/(2p+1)\) DoF per user, and they can be achieved by means of linear precoding at the transmitters and linear filtering at the receivers. In this work, only the cases \(p = 2, 3\) are completely addressed, due to space limitation.
Contribution 2. We conjecture that the 3-user MIMO IC with constant channel coefficients and \((p,p+1)\) antennas at each pair transmitter-receiver has exactly \(p(p+1)/(2p+1)\) DoF per user, and they can be achieved by means of linear precoding at the transmitters and linear filtering at the receivers. The conjecture is based on our results for cases \(p = 2, 3, ..., 6\) shown in Appendices.

1.3 Organization

The present work is organized as follows:

- Chapter 2 introduces the system model considered in this work, presenting the 3-user MIMO IC.
- Chapter 3 reviews the necessary background, revisiting the change of basis operation, subspace alignment chains, and signal space matrix concepts.
- We begin with the \(p = 2\) case in chapter 4 because some notation is slightly different from the general case, and in order to give some insights about the general case for the feasibility proof (appendix B).
- Chapter 5 presents the resolution of the \(p = 3\) case, similarly to Chapter 4, but in this case, we emphasize which of those procedures are generalizable to \(p > 3\) and present the zero propagation algorithm, valid for any value of \(p\), as well as the methodology to deal with cases with \(p > 3\). Additionally, some results for cases \(p = 4, 5, 6\) can be found in Appendix C, but the complete proof is omitted to avoid redundancy.
- Finally, some conclusions are drawn and some future work directions are posed in Chapter 6.

In order to facilitate the reader’s understanding of the text, a notation section is next provided, where some of the mathematical conventions and symbols used in the work are briefly described.

1.4 Notation

We use bold font and lower case for vectors \((\mathbf{x})\) and bold font and upper case for matrices \((\mathbf{X})\). For vectors and matrices, \((\cdot)^T\), \((\cdot)^H\), and \(\otimes\) stand for the transpose, transpose and conjugate, and Kronecker product operators, respectively, and we define

\[
\text{stack} \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}
\]  
\[ (1.2) \]
Furthermore, for any $N$-column vector $\mathbf{x} = [x(1), x(2), ..., x(N)]$ and a $M$-column matrix $\mathbf{Y} = [y_1, y_2, ..., y_M]$, we define $\mathbf{x}(a : b) = [x(a), x(a+1), ..., x(b)]$ and $\mathbf{y}_a^b = [y_a, y_{a+1}, ..., y_b]$. Additionally, $\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$, and $\langle \cdot \rangle_3$ stand for the ceiling, floor, and modulo-3 operators, respectively. Unless otherwise stated, it is assumed $a = \langle a - 1 \rangle_3 + 1$ for all indexes labeling the 3 transmitters or receivers in this work. This operation is similar to the modulo-3 operator. For instance, the equivalent indexes for $\{4,5,6\}$ are $\{1,2,3\}$.

Regarding vector spaces, $\text{span} (\mathbf{A})$ and $\text{null} (\mathbf{A})$ define the subspaces generated by all linear combinations of the columns of $\mathbf{A}$, and the vectors $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \mathbf{0}$, respectively, and $\text{rank} (\mathbf{A})$ denotes the dimension of $\text{span} (\mathbf{A})$. Finally, $\mathbb{R}, \mathbb{C}$ stand for the real and complex sets of numbers, respectively, and $\tau$ denotes the complex imaginary unit.
Chapter 2

The 3-user MIMO Interference channel

We consider the 3-user MIMO IC depicted in Fig. 2.1. Three source-destination pairs coexist in the system, where each transmitter (equipped with \( p \) antennas) transmits one message to the receiver (equipped \( p + 1 \) antennas) labeled with the same index. The input-output relationship of the system is described by

\[
\bar{y}_j = \bar{H}_{j,j}\bar{s}_j + \sum_{i=1,i\neq j}^3 \bar{H}_{j,i}\bar{s}_i + \bar{n}_j = \bar{H}_{j,j}\bar{V}_j\bar{x}_j + \sum_{i=1,i\neq j}^3 \bar{H}_{j,i}\bar{V}_i\bar{x}_i + \bar{n}_j
\]  

(2.1)

where \( \bar{y}_j \in \mathbb{C}^{(p+1)\times 1} \) is the received signal vector at the \( j \)th receiver, \( \bar{x}_j \in \mathbb{C}^{d_j\times 1} \) is the Gaussian signal with uncorrelated components to be decoded at the \( j \)th receiver, \( d_j \) is the number of bitstreams transmitted by the \( j \)th transmitter, \( \bar{V}_j \in \mathbb{C}^{p\times d_j} \) is the precoding matrix of the \( j \)th transmitter, \( \bar{s}_j = \bar{V}_j\bar{x}_j \in \mathbb{C}^{p\times 1} \) is the encoded signal transmitted by the \( j \)th transmitter, and \( \bar{H}_{j,i} \in \mathbb{C}^{(p+1)\times p} \) is the channel matrix representing the channel from the \( i \)th transmitter to the \( j \)th receiver where each element is defined by \( h_{j_i}^{q,r} \) which denotes the complex channel gain between the \( r \)th antenna of transmitter \( i \) and the \( q \)th antenna of receiver \( j \). Finally, the additive white Gaussian noise vector at the \( j \)th receiver \( \bar{n}_j \in \mathbb{C}^{(p+1)\times 1} \) is a vector whose components are independent and identically distributed (i.i.d.) as zero-mean, unit-variance, complex Gaussian random variables. Moreover, the channel coefficients are constant along the transmission time and perfect Channel State Information (CSI) is assumed at the transmitter and receiver sides.

**Remark 1.** We assume that channel coefficients are given from a continuous pdf. Therefore, it can be ensured that the probability that any channel coefficient takes a
value within a subset of measure zero is zero. An example of subset of measure zero is, for instance, a countable subset or a set containing one element only (See [CK99]).

Finally, the received signal is processed as follows

\[ z_j = \hat{W}_j \bar{y}_j = \bar{W}_j \bar{H}_{j,j} \bar{V}_j \bar{x}_j + \bar{W}_j \bar{n}_j \in \mathbb{C}^{d_j \times 1} \]  

(2.2)

where \( \bar{W}_j \in \mathbb{C}^{d_j \times (p+1)} \) is the linear receive filter for receiver \( j \).

## 2.1 Constant channel extension

The model considered in (2.1) can be accommodated to the case where the transmission is performed separately along different domains. By channel extension, we denote the equivalent channel for that case. Two ways of extending the channel are considered throughout this work: symbol extension in time/frequency and asymmetric complex signaling [CJW10].

### 2.1.1 Symbol extensions in time/frequency

This approach considers simply grouping the signals of \( T \) consecutive channel uses, either if multiple time instants or frequency channels are used. For simplicity, we consider symbol extension in time, while the same approach can be applied by using multiple subcarriers. In any case, since channel coefficients are considered constant...
2.1 Constant channel extension

along those different channel uses, the equivalent extended channel \( H_{j,i} \in \mathbb{C}^{(p+1)T \times pT} \) can be formulated as follows:

\[
H_{j,i} = I \otimes \overline{H}_{j,i}
\]  

(2.3)

where \( I \in \mathbb{R}^{T \times T} \) is the identity matrix. Consequently, the received signal in (2.1) can be written using this extended channel. The final expression of the received signal is not shown here in order to avoid redundancy, and it is shown at the end of this section for the case when both extensions (symbol extension in time and ACS) are used. This is the typical form of extending the channel when multiple time instants are considered, but this extension can be alternatively applied to each element of the original channel, thus obtaining

\[
H_{j,i} = \begin{bmatrix}
I \otimes \overline{h}_{1,1}^{1,1} & \ldots & I \otimes \overline{h}_{1,p}^{1,p} \\
\vdots & \ddots & \vdots \\
I \otimes \overline{h}_{p+1,1}^{1,1} & \ldots & I \otimes \overline{h}_{p+1,p}^{1,p}
\end{bmatrix} \in \mathbb{C}^{(p+1)T \times pT}
\]  

(2.4)

where \( I \otimes \overline{h}_{q,r}^{q,r} \in \mathbb{C}^{T \times T} \) is the extended version of the channel element \( \overline{h}_{q,r}^{q,r} \).

2.1.2 Asymmetric complex signaling

When considering this approach, the transmitted signals travel separately through the real and imaginary parts of the channel. As a result, a real MIMO channel with twice the number of antennas can be considered instead of a complex one. Consequently, each channel coefficient \( \overline{h}_{q,r}^{q,r} \) is reformulated as follows:

\[
\overline{H}_{j,i}^{q,r} = \left| \overline{h}_{j,i}^{q,r} \right| \mathbf{U} \left( \overline{\phi}_{j,i}^{q,r} \right)
\]  

(2.5)

where \( \overline{\phi}_{j,i}^{q,r} \) is the phase of \( \overline{h}_{j,i}^{q,r} \), and \( \mathbf{U} \left( \overline{\phi}_{j,i}^{q,r} \right) \in \mathbb{R}^{2 \times 2} \) is a unitary matrix given by:

\[
\mathbf{U} \left( \overline{\phi}_{j,i}^{q,r} \right) = \begin{bmatrix}
\cos \left( \overline{\phi}_{j,i}^{q,r} \right) & -\sin \left( \overline{\phi}_{j,i}^{q,r} \right) \\
\sin \left( \overline{\phi}_{j,i}^{q,r} \right) & \cos \left( \overline{\phi}_{j,i}^{q,r} \right)
\end{bmatrix}
\]  

(2.6)

In the sequel, we will write equivalently \( C \left( h_{j,i}^{q,r} \right) = \left| \overline{h}_{j,i}^{q,r} \right| \mathbf{U} \left( \overline{\phi}_{j,i}^{q,r} \right) \) so as to simplify notation. Finally, some properties of this matrix that will be used throughout this thesis are

\[
C(a)C(b) = C(ab)
\]

\[
C(a)^{-1} = C(a^{-1})
\]  

(2.7)

for arbitrary complex values \( a, b \).
2.1 Constant channel extension

2.1.3 Channel extension used in this work

Actually, the channel matrices used in this work are the result of the combination of symbols extensions in time in addition to asymmetric complex signaling. Here, we show the extended channel matrices and how the received signal is accommodated to that case. First, the equivalent extended channel matrix for each link is given by:

\[
H_{j,i} = \begin{bmatrix}
|h_{j,i,1}| & I \otimes \bar{U}(\bar{\phi}_{j,i}^{1,1}) & \cdots & |h_{j,i,1}| & I \otimes \bar{U}(\bar{\phi}_{j,i}^{1,p}) \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
|h_{j,i,p+1,1}| & I \otimes \bar{U}(\bar{\phi}_{j,i}^{p+1,1}) & \cdots & |h_{j,i,p+1,1}| & I \otimes \bar{U}(\bar{\phi}_{j,i}^{p+1,p})
\end{bmatrix}
\] (2.8)

Therefore, the received and processed signals at each receiver become

\[
y_j = \sum_{i=1}^{3} H_{j,i} V_i x_i + n_j \\
z_j = W_j y_j
\] (2.9)

where \( V_i \in \mathbb{R}^{2T \times d_j}, \ x_i \in \mathbb{R}^{d_j \times 1}, \ y_j, n_j \in \mathbb{R}^{2T(p+1) \times 1}, \ z_j \in \mathbb{R}^{d_j \times 1}, \) and \( W_j \in \mathbb{R}^{d_j \times 2T(p+1)} \) are equivalent magnitudes for the extended system model.
Chapter 3

Background

This chapter reviews the change of basis operation and subspace alignment chains concepts in [WGJ11a]. Additionally, the signal space matrix is reviewed, which is the cornerstone of this work.

3.1 Change of basis operation

Consider the MIMO IC model in (2.9). The change of basis operation consists on applying invertible linear transformations at each transmitter and each receiver, so as to simplify the channel structure (e.g. forcing zeros in some a priori known positions). They are used to prove the achievability of the outer bound although they are not necessary in the practical sense. Notice that since full-rank transformations are applied, the DoF remain the same.

Using the change of basis operation, the following equivalent channels are defined,

\[ H_{j,i}^{\text{eq}} = R_j H_{j,i} T_i \in \mathbb{R}^{p \times p+1} \]  

(3.1)

where \( T_i \in \mathbb{R}^{p \times p} \) and \( R_i \in \mathbb{R}^{(p+1) \times (p+1)} \) are the transformations at the \( i \)th transmitter and at the \( j \)th receiver, respectively. For the sake of notation simplicity, from now on \( H_{j,i} \) denotes the equivalent channel, whose elements will be denoted as \( h_{j,i}^{q,r} \), maintaining the random nature of \( \tilde{h}_{j,i}^{q,r} \). In this regard, note that the final transmit precoding matrices and receiving filters will be the product of the change of basis matrices, \( T_i \) and \( R_i \), with the design derived for IA (see next section). Hence, the final transmit and receive filters are given by:

\[ V_{i}^{\text{final}} = T_i V_i \quad \quad R_i^{\text{final}} = W_i R_i \]  

(3.2)
3.1 Change of basis operation

The aim of the change of basis operation is to remove the maximum number of links among antennas of the interferers and the corresponding receiver. For example, Figure 3.1 shows how $T_i$ (a) and $R_i$ (b) are designed for $p = 2$:

![Figure 3.1: Change of basis operation for $p = 2$. Links with a cross are removed, while other links are forced to be non-nulled. The change of basis operation is performed by means of using the corresponding filters $T_i$ and $R_j$ at each transmitter (a) and each receiver (b).](image)

It can be seen that $T_i$ and $R_i$ are designed to force zeros at some specific positions of $H_{j,i}^{eq}$, and ensuring that some links are preserved. This is achieved by designing the change of basis matrices rows/columns as elements of the null subspace of the columns/rows of $H_{j,i}$. The reader can refer to [WGJ11a] for details on the design and extension of those ideas for $p > 2$. The resulting matrix by applying the change of basis operation for the $p = 3$ case is shown below:

$$
H_{j,j-1} = \begin{bmatrix}
C \left( \hat{h}_{j,j-1}^{11} \right) & 0 & 0 \\
C \left( \hat{h}_{j,j-1}^{21} \right) & C \left( \hat{h}_{j,j-1}^{22} \right) & C \left( \hat{h}_{j,j-1}^{23} \right) \\
C \left( \hat{h}_{j,j-1}^{31} \right) & 0 & C \left( \hat{h}_{j,j-1}^{32} \right)
\end{bmatrix},
$$

$$
H_{j,j+1} = \begin{bmatrix}
0 & 0 & 0 \\
C \left( \hat{h}_{j,j+1}^{21} \right) & 0 & C \left( \hat{h}_{j,j+1}^{23} \right)
\end{bmatrix}
$$

(3.3)

In this work some additional transformations to the results derived in [WGJ11a] are performed by properly combining the received signals at each receiver (see Appendix A). As an example, the cross-channel matrices for the $p = 3$ case are shown below:
\[ H_{j,j-1} = \begin{bmatrix} C(h_{j,j-1}^{11}) & 0 & 0 \\ 0 & C(h_{j,j-1}^{22}) & C(h_{j,j-1}^{23}) \\ 0 & 0 & C(h_{j,j-1}^{33}) \end{bmatrix}, \quad H_{j,j+1} = \begin{bmatrix} 0 & 0 & 0 \\ C(h_{j,j+1}^{21}) & 0 & 0 \\ C(h_{j,j+1}^{31}) & C(h_{j,j+1}^{32}) & 0 \end{bmatrix} \]

(3.4)

3.2 Subspace alignment chains

This section reviews the principles of subspace alignment chains. In order to do so, the initial IA conditions are presented first, and next the subspace alignment chains.

3.2.1 Original IA conditions

IA pursues to remove all the received interference while preserving the rank of the desired signal. This can be formulated as follows:

\[ \text{rank} (W_j H_{j,j} V_j) = \bar{d}_j \quad (3.5) \]

\[ W_j H_{j,i} V_i = 0, \quad i \neq j \quad (3.6) \]

To this end, the transmit precoding matrices can be designed to achieve null-steering, thus transmitting on the null space of the channel matrices \( H_{j,i} \). Or, alternatively, arbitrary precoding matrices can be used by the transmitters and make use of ZF filtering at the receiver side. IA is useful for those cases where the dimensions of channel matrices do not allow to attain the optimal DoF by means of these techniques. In particular, IA principle aims to design the transmit precoding matrices \( V_i \) in order to reduce the dimension of the subspace spanned by the received interference. Therefore, it is possible to obtain a solution for the receiving filters to zero-force the interference while maintaining the desired signal dimension. In other words, IA proposes a design for the transmit precoding matrices \( V_i \) such that the solution of \( W_j \) for (3.6) is compatible with (3.5).

IA was first developed for scenarios with equal number of antennas at the transmit and the receiver side. In such a case, since the channel matrices are square, it can be assumed that the left and right null space do not exist with high probability. In this regard, it seems that the only way to combat the interference is by aligning the interference. Therefore, the original IA conditions are given by:

\[ \text{span} (H_{j,j+1} V_{j+1}) = \text{span} (H_{j,j-1} V_{j-1}), \quad j = 1, 2, 3 \quad (3.7) \]
3.2 Subspace alignment chains

Each of those equations corresponds to one of the receivers, and forces the 2 unintended signals received at that receiver lie on the same signal subspace. Therefore, the number of dimensions occupied by interference is enough small to project the received signal onto the orthogonal-to-interference subspace, where the desired signals are assumed to lie. As a consequence, the receiving filters are obtained directly from the precoding matrices design.

On the other hand, it is assumed that the desired signal lies on a subspace not contained on the interference subspace. This is reasonable because the direct channels $H_{11}, H_{22}$ and $H_{33}$ do not participate in the precoding matrix design. thus the rank condition in (3.5) is ensured by giving each receiver enough dimensions to allocate interference plus desired signals. However, this is not completely (or always) true, thus this discussion will be continued in section 3.3.

Finally, notice that the DoF for an extended channel are given by the rank of the equivalent direct channel, as in (3.5), but normalized by the channel extension length:

$$d_j = \frac{1}{2T} \text{rank}(W_j H_{j,j} V_j) = \frac{\bar{d}_j}{2T}$$

(3.8)

3.2.2 Different number of antennas at tx and rx sides

Once the fully symmetric MIMO case is resolved, the next step is to deal with the case where transmitters and receivers are equipped with different number of antennas. In such a case, null steering at the transmitters or zero-forcing at the receivers are possible, thus a wide range of possibilities appears. After some attempts [GJ10, BCT11], the solution that fully attained the outer bound for any antenna configuration was found by Wang et al in [WGJ11a]. The optimal transmission scheme was based on what they called subspace alignment chains, and it is the focus of this section.

We review the subspace alignment chains concepts for the particular $(p,p+1)$ case, although most of the formulation is applicable to all antenna configurations. For our case, the DoF outer bound is given by:

$$d_j = \frac{p(p+1)}{2p+1}$$

(3.9)

and we will see next how the subspace alignment chains attains to this outer bound.

The novel contribution of alignment chains with respect to previous interference alignment techniques is that the precoding matrix of each user is divided in blocks, where each block is designed according to different IA constraints (similar to those in (3.7)).
3.2 Subspace alignment chains

Specifically, each block is designed to be aligned with other specific blocks of the other precoding matrices. The concept of alignment chain comes from the fact that those alignments are connected among receivers. In order to better understand this, we start explaining how the square case can be tackled by means of alignment chains. To this end, consider Fig. 3.2 and the following diagram:

\[
\begin{align*}
&\mathbf{V}_2 \xrightarrow{\text{Rx 1}} \mathbf{V}_3 \xrightarrow{\text{Rx 2}} \mathbf{V}_1 \xrightarrow{\text{Rx 3}} \mathbf{V}_2
\end{align*}
\] (3.10)

where each oval denotes a dimensional subspace and arrows connect transmit and receive subspaces. An alignment occurs when two arrows end at the same oval. Therefore, we can say that the design of the two corresponding matrices is connected through the alignment chain. For instance, the design of \(\mathbf{V}_2\) is connected with the design of \(\mathbf{V}_3\) through the IA condition at receiver 1. Likewise, \(\mathbf{V}_3\) and \(\mathbf{V}_1\) are connected because they should align at receiver 2. This intuition encourages the idea of chain among precoding matrices, translating the concept in Fig. 3.2 to the diagram in (3.10). Further, it is worth pointing out that in this case the chain is recursive, since \(\mathbf{V}_2\) can be seen as the beginning and the end of the same chain. Therefore, all the transmitted signals are aligned at all the unintended receivers. As we will see later, this fact is particular for the square case.

In order to guide the reader throughout the general alignment chains, firstly the precoding matrices structure is introduced and next the alignment chains.
3.2 Subspace alignment chains

Precoding matrix structure

Each of the precoding matrices is divided in several blocks, each constrained by different IA conditions. Defining different blocks allows to better control or *granulate* the interference alignment, which is necessary to achieve the maximum DoF in (3.9). Consequently, let define

\[
V_i = \begin{bmatrix}
V^1_{i,(1)} & \cdots & V^1_{i,(s_{\text{max}}(i,1))} & | & V^2_{i,(1)} & \cdots & V^2_{i,(s_{\text{max}}(i,2))} & | & V^3_{i,(1)} & \cdots & V^3_{i,(s_{\text{max}}(i,3))}
\end{bmatrix} P_i \tag{3.11}
\]

where \( V_i \in \mathbb{R}^{p \times 2(p+1)} \) is the precoding matrix of user \( i \) and \( P_i \in \mathbb{R}^{2p(p+1) \times 2p(p+1)} \) is an arbitrary and unitary permutation matrix. It can be seen that there are three main blocks, separated by dashed lines, and identified by the supraindex \( k = 1, 2, 3 \), corresponding to the alignment chain from which they are designed. At the same time, each of these three blocks consists of \( s_{\text{max}}(i,k) \) sub-precoding matrix blocks \( V^k_{i,(s)} \in \mathbb{R}^{p \times 2(p+1)} \), where notation means that this sub-block belongs to user \( i \) and it is designed using alignment chain \( k \). On the other hand, index \( s = 1, \ldots, s_{\text{max}}(i,k) \) denotes that it is the \( s \)th block of user \( i \) that participates in the \( k \)th alignment chain.

A general expression for \( s_{\text{max}}(i,k) \) can be found as:

\[
s_{\text{max}}(i,k) = \left\lceil \frac{p - (k-1)}{3} \right\rceil \tag{3.12}
\]

Notice that each precoding matrix is computed by appending together

\[
N_s = \sum_k s_{\text{max}}(i,k) = \left( \left\lceil \frac{p}{3} \right\rceil + \left\lceil \frac{p-1}{3} \right\rceil + \left\lceil \frac{p-2}{3} \right\rceil \right) = p \tag{3.13}
\]

sub-precoding matrix blocks, being consistent with the defined dimensions.

On the other hand, we additionally divide by rows \( V^k_{i,(s)} \) in \( p \) blocks, as follows:

\[
V^k_{i,(s)} = \text{stack} \left( V^{k,\text{ant.1}}_{i,(s)}, V^{k,\text{ant.2}}_{i,(s)}, \ldots, V^{k,\text{ant.p}}_{i,(s)} \right) \tag{3.14}
\]

where each block correspond to one of the \( p \) transmit antennas. This is useful to prove the achievability of the proposed scheme, because it allows to take benefit of the change of basis operation when designing with the alignment chain.

Finally, it is worth pointing out that \( p \) sub-precoding matrices blocks of dimension \( 2Tp \times 2(p+1) \) are used for each user. Hence, \( \tilde{d}_j = 2p(p+1) \) bitstreams are transmitted along \( 2T \) channel uses. As a consequence, the maximum number of DoF achieved using this scheme is equal to:

\[
d_j = \frac{2p(p+1)}{2T} \tag{3.15}
\]
3.2 Subspace alignment chains

This work formally proves the achievability of exactly those DoF when taking $T = 2p + 1$ symbol extensions in time and, therefore, our scheme attains the DoF outer bound in (3.9).

Alignment chains formulation

Three alignment chains are built, giving the alignment conditions for each sub-precoding block matrix. In general, and in contrast to the square case, the chain is not recursive. Therefore, it is defined the length of the alignment chain $\kappa$ as the number of alignments carried out using each alignment chain plus one. The conditions to define this alignment chain length depend on each antenna configuration (see [WGJ11a] for details), with $\kappa = p$ for the $(p, p + 1)$ case. Following lines explain the concept behind each alignment chain for the case $p = 3$ by using Fig. 3.3 and the following diagram:

$$V^k_{k,(i)} \xleftrightarrow{\text{Rx } k+1} V^k_{k-1,(1)} \xleftrightarrow{\text{Rx } k} V^k_{k+1,(1)}$$

First, the block $V^k_{k,(1)}$ is aligned with $V^k_{k-1,(1)}$ at the $(k + 1)$th receiver. Next, the same block $V^k_{k-1,(1)}$ is aligned with $V^k_{k+1,(1)}$ at the $k$th receiver. Those alignments

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.3.png}
\caption{Occupation of the receivers for the signals designed using alignment chain $k$ for the $p = 3$ case. In this case, the beginning and the end of the chain are not aligned at all receivers. In particular, the interference generated at receiver $k - 1$ is not aligned, thus occupies twice the number of dimensions as compared to the case of aligned interference.}
\end{figure}
are graphically shown in Fig. 3.3, where it is represented how the received signals are allocated at each receiver. Notice that the blocks $V^k_{k,(1)}$ and $V^k_{k+1,(1)}$ are not aligned at receiver $k - 1$, thus they occupy twice the number of dimensions at that receiver. This is because they are the beginning and the end of the chain.

Next, we present how each alignment chain is mathematically expressed. For the $p = 2$ case, the notation of alignment chains is slightly different and will be presented in chapter 4. Otherwise, for all $p > 2$ cases the $k$th alignment chain with $k = 1, 2, 3$ is formulated as follows:

\[
\begin{bmatrix}
H_{k+1,k} & -H_{k+1,k-1} & 0 & \cdots & 0 \\
0 & H_{k,k-1} & -H_{k,k-2} & \vdots & \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & H_{n_k,n_k-1} - H_{n_k,n_k+1} & \\
\end{bmatrix}
\begin{bmatrix}
V^k_{k,(1)} \\
V^k_{k-1,(1)} \\
V^k_{k-2,(1)} \\
V^k_{k,(2)} \\
\vdots \\
V^k_{n_k+1,(\sigma_{\text{max}}(n_k+1,k))}
\end{bmatrix} = \mathbf{0} \quad (3.17)
\]

where $n_k = k - p$ denotes the last receiver of the chain. Notice that setting $p = 3$ each row imposes the same alignments described in (3.16) and graphically shown in Fig. 3.3.

Fig. 3.4 illustrates the occupation of the $j$th receiver when considering the signals involved in the 3 alignments chains. Notice that all indexes are written from the point of view of the $j$th receiver.

In this example, each transmitter employs 3 sub-precoding matrices. As commented before, out of those 3 blocks, only one is aligned at all the receivers. Therefore, at the $j$th receiver $V^{j+1}_{j-1,(1)}$ and $V^{j-1}_{j+1,(1)}$ are not aligned. In general, each user’s precoding matrix contains one block matrix that it is aligned only at one of the unintended receivers, one that is only aligned at the other, and one block matrix being aligned at all the unintended receivers. This is related with the fact that they are the beginning or the end of a chain. In this regard, we alternatively divide the precoding matrix in (3.11) as three block matrices:

\[
V_i = \begin{bmatrix} V^{IA}_i & V^{IA,i+1}_i & V^{IA,i-1}_i \end{bmatrix} \tilde{P}_i \quad (3.18)
\]

where $\tilde{P}_i$ stands for one permutation matrix that can be different from $P_i$ defined in (3.11), and each block from left to right contains a set of sub-precoding matrices which are aligned at both unintended receivers, the $(i + 1)$th receiver only and the $(i - 1)$th receiver only, respectively.
3.3 Signal space matrix

In order to decode all the transmitted messages by using linear filters at the receiver, the desired signal should be linearly independent of the received interference. This is equivalently expressed in (3.5). To analytically prove such condition, we define the signal space matrix, defined as the matrix generating the sum space of desired and interference subspaces:

\[ G_j = \begin{bmatrix} G_{\text{des}}^j & G_{\text{int}}^j \end{bmatrix} \]

\[ G_{\text{des}}^j = H_{j,j} V_j \]

\[ G_{\text{int}}^j = \begin{bmatrix} H_{j,j-1} V_{j-1} & H_{j,j+1} V_{j+1}^{/A,j-1} \end{bmatrix} \]

(3.19)

where \( H_{j,j} V_j \) denotes the desired received signal at the \( j \)th receiver, \( H_{j,j-1} V_{j-1} \) denotes the interference produced by transmitter \( j-1 \), and \( H_{j,j+1} V_{j+1}^{/A,j-1} \) (following notation introduced in (3.18)) denotes the columns of \( H_{j,j+1} V_{j+1} \) which are not aligned at this receiver. By proving that this matrix is full rank, we are able to ensure that the desired signal and interference terms are linearly independent.

Figure 3.4: Occupation of the \( j \)th receiver using subspace alignment chains scheme for \( p=3 \). Ovals represent dimensional subspaces at each transmitter or receiver.
The full rank signal space matrix condition can be interpreted graphically from Fig. 3.4 as follows. Whenever $G_j$ is a deficient rank matrix, there are some interference signals overlapping ovals occupied by desired signals at the receiver. Therefore, it would not be possible to separate interference from desired signals by using linear filtering at the receiver side.

Unfortunately, it is not easy to predict if the signal space matrix would be full rank when the channel presents a certain structure as when channel extensions are addressed. Actually, in [WGJ11a] the authors take transmit precoding matrices as a random linear combination of the null space of (3.17), and refer to simulation results to ensure that $G_j$ is full-rank. In this work, we design that linear combination properly in order to analytically prove with probability one that the full-rank property can be ensured.
Chapter 4

The (2, 3) case

This chapter shows that the DoF outer bound for the (2,3) MIMO IC with constant channel coefficients can be achieved using ACS together with symbol extensions in time, IA precoding at the transmitters and ZF filtering at the receivers. Firstly, section 4.1 shows how the precoding matrices are obtained from the alignment chain (3.17). Next, the signal space matrix $G_j$ in (3.19) is obtained for that case. Finally, this matrix is shown to be full-rank, validating the feasibility of the proposed scheme.

4.1 Precoding matrix design

For the $p=2$ case, the DoF per user outer bound as provided by (3.9) is 6/5. Taking $T=5$ so as to obtain an integer DoF value, channel extension is equal to $2T = 10$.

As commented before, the precoding matrices structure and the alignment chains notation is slightly different from (3.17) for the $p = 2$ case. In the first place, the precoding matrix of each user $V_j = [V_1^j, V_2^j] \in \mathbb{R}^{20 \times 12}$ is divided into two $20 \times 6$ blocks. And such blocks are designed using the alignment chains, as follows:

$$
\begin{bmatrix}
H_{21} & -H_{23}
\end{bmatrix}
\begin{bmatrix}
V_1^1 \\
V_3^1
\end{bmatrix} = 0
$$

$$
\begin{bmatrix}
H_{32} & -H_{31}
\end{bmatrix} 
\begin{bmatrix}
V_1^2 \\
V_2^1
\end{bmatrix} = 0
$$

$$
\begin{bmatrix}
H_{13} & -H_{12}
\end{bmatrix}
\begin{bmatrix}
V_3^1 \\
V_2^2
\end{bmatrix} = 0
$$

(4.1)

Now we derive the solution for $V_1^1$ and $V_3^1$ as null ($[H_{21} , -H_{23}]$) by taking into account the special structure of the cross-channel matrices (section 3.1). As a consequence,
the alignment chain at top of (4.1) can be written as:

\[
\begin{bmatrix}
C(h_{21}^{11}) & 0 & 0 & 0 \\
0 & C(h_{22}^{21}) & C(h_{23}^{21}) & 0 \\
0 & 0 & 0 & C(h_{23}^{31})
\end{bmatrix}
\begin{bmatrix}
V_{1,ant.1}^{1,ant.1} \\
V_{1,ant.2}^{1,ant.1} \\
V_{3,ant.1}^{1,ant.1} \\
V_{3,ant.2}^{1,ant.2}
\end{bmatrix} = 0
\] (4.2)

where the sub-precoding matrices use the notation in (3.14), with a 10 × 6 matrix for each transmitting antenna. In (4.2) we can assume that each matrix C(a) is full rank with high probability. This is because the only cases where those matrices are singular is for \( \text{Im}(a) = 0 \). Therefore, by using Remark 1, this event has very low probability, thus together with properties in (2.7), we get the following solution:

\[
\begin{align*}
V_{1,ant.1}^{1,ant.1} &= 0 \\
V_{3,ant.1}^{1,ant.1} &= C\left(\frac{h_{22}^{21}}{h_{23}^{23}}\right) V_{1,ant.2}^{1,ant.2} \\
V_{3,ant.2}^{1,ant.2} &= 0
\end{align*}
\] (4.3)

Similar arguments can be applied to solve the other alignment equations. Consequently, the precoding matrices for each user are given by:

\[
\begin{align*}
V_1 &= \begin{bmatrix} V_{1,ant.1}^{1,ant.1} & V_{2,ant.1}^{1,ant.1} \\ V_{1,ant.2}^{1,ant.1} & V_{2,ant.2}^{1,ant.1} \end{bmatrix} P_1 = \begin{bmatrix} 0 & C\left(\frac{h_{22}^{21}}{h_{23}^{23}}\right) V_{1,ant.2}^{1,ant.2} \\ V_{1,ant.2}^{1,ant.2} & 0 \end{bmatrix} P_1 \\
V_2 &= \begin{bmatrix} V_{1,ant.1}^{2,ant.1} & V_{2,ant.1}^{2,ant.1} \\ V_{1,ant.2}^{2,ant.1} & V_{2,ant.2}^{2,ant.1} \end{bmatrix} P_2 = \begin{bmatrix} 0 & C\left(\frac{h_{22}^{23}}{h_{23}^{21}}\right) V_{3,ant.2}^{2,ant.2} \\ V_{2,ant.2}^{2,ant.2} & 0 \end{bmatrix} P_2 \\
V_3 &= \begin{bmatrix} V_{3,ant.1}^{3,ant.1} & V_{2,ant.1}^{3,ant.1} \\ V_{3,ant.2}^{3,ant.1} & V_{2,ant.2}^{3,ant.1} \end{bmatrix} P_3 = \begin{bmatrix} C\left(\frac{h_{22}^{23}}{h_{23}^{21}}\right) V_{1,ant.2}^{3,ant.2} & 0 \\ 0 & V_{3,ant.2}^{3,ant.2} \end{bmatrix} P_3
\end{align*}
\] (4.4)

Recall on the fact that each column of each precoding matrix is multiplied by a bitstream of the corresponding user. Therefore, by properly reordering each \( x_j \), we can exchange the columns of \( V_j \) while maintaining the alignment conditions (this will be checked in the next section). Therefore, if we relabel \( V_{2,ant.2}^{1,ant.2} \rightarrow V_{2,ant.2}^{2,ant.2}, V_{3,ant.2}^{2,ant.2} \rightarrow V_{3,ant.2}^{3,ant.2} \), the following general expression can be found:

\[
V_j = \begin{bmatrix} C\left(\frac{h_{22}^{j+1,j+1}}{h_{j-1,j}^{j+1}}\right) V_{j+1,ant.p}^{j,ant.p} & 0 \\ 0 & V_{j,ant.p}^{j,ant.p} \end{bmatrix}, j = 1, 2, 3
\] (4.5)

where \( V_{j,ant.p}^{j,ant.p} \in \mathbb{R}^{10 \times 6} \) remain to be designed yet. Since \( V_j \) can be written only in terms of \( V_{j,ant.p}^{j,ant.p}, j = 1, 2, 3 \), we denote the three remaining matrices

\[
A_i = V_{i,ant.p}^{i,ant.p}
\] (4.6)
4.2 Feasibility

In the last section, we obtained that the IA conditions force the precoding matrices to have a particular structure, where the final precoding matrices in (4.5) depend only on 3 block matrices. We will design them to prove achievability, that is, to ensure that matrix \( G \) is full rank. In this regard, this section is devoted to prove that choosing a suitable design for \( A_j, j=1,2,3 \), \( G \) is full rank. In order to verify this, two steps are necessary: to construct the signal space matrix (section 4.2.1) and then to prove that this matrix is full-rank (section 4.2.2).

4.2.1 Signal space matrix

In order to construct the matrix \( G \) in (3.19), we should first compute the product \( H_{j,j+1} V_{j+1} \) for \( i \neq j \):

\[
H_{j,j+1} V_{j+1} = \begin{bmatrix}
0 & 0 \\
C \left( h_{j,j-1}^{22} \right) A_{j-1} & 0 \\
0 & C \left( h_{j,j+1}^{32} \right) A_{j+1}
\end{bmatrix}
\]

\[\text{and} \quad H_{j,j-1} V_{j-1} = \begin{bmatrix}
0 & 0 \\
C \left( h_{j,j-1}^{11} h_{j+1,j-1}^{22} \right) A_j & 0 \\
0 & C \left( h_{j,j-1}^{22} \right) A_{j-1} \\
0 & 0
\end{bmatrix}
\]

(4.7)

where it can be seen that the second block column of \( H_{j,j-1} V_{j-1} \) is aligned with the first block column of \( H_{j,j+1} V_{j+1} \). These results can be used to write \( G \) (see (3.19)) for the \( p=2 \) case, described by:

\[
G_{\text{des}}^j = \begin{bmatrix}
C \left( h_{j,j}^{11} \right) A_{j+1} & C \left( h_{j,j}^{12} \right) A_j \\
C \left( h_{j,j}^{21} \right) A_{j+1} & C \left( h_{j,j}^{22} \right) A_j \\
C \left( h_{j,j}^{31} \right) A_{j+1} & C \left( h_{j,j}^{32} \right) A_j
\end{bmatrix}
\]

\[
G_{\text{int}}^j = \begin{bmatrix}
C \left( h_{j,j-1}^{11} h_{j+1,j-1}^{22} \right) A_j & 0 & 0 \\
0 & C \left( h_{j,j-1}^{22} \right) A_{j-1} & 0 \\
0 & 0 & C \left( h_{j,j+1}^{32} \right) A_{j+1}
\end{bmatrix}
\]

(4.8)

The signal space matrix obtained here is similar to the one obtained in equation (16) of [WGJ11b] and it has a certain structure that can be exploited to prove the
full-rank condition. Although for this particular case of \( p = 2 \), the full-rank condition can be proved by picking entries of \( A_j \) randomly [WGJ11b], the next section presents an alternative proof that is also useful for \( p > 2 \), where it is more difficult to ensure the full-rank condition.

### 4.2.2 Rank of signal space matrix

Feasibility implies proving that the signal space matrix is full rank, i.e., it is necessary to prove that the desired signals and interference terms are linearly independent. The full-rank property can be analytically proved by showing that all \( \lambda^j_i \in \mathbb{R}^{6 \times 1}, i = 1 \ldots 5, j = 1, 2, 3 \) constrained by

\[
G_j \begin{bmatrix} \lambda^j_1 \\ \vdots \\ \lambda^j_5 \end{bmatrix} = 0
\]  

must be zero. Conversely, if there exists at least one non-zero solution for \( \lambda^j_i, i = 1..5, j = 1, 2, 3 \) then \( G_j \) is rank deficient.

Next, we describe how matrices \( A_j \in \mathbb{R}^{10 \times 6}, j = 1, 2, 3 \) are designed in order to achieve our purposes. In this regard, let define an orthonormal basis \( B = [b_1 \ b_2 \ \ldots \ b_{10}] \) in \( \mathbb{R}^{10} \). \( A_j \) are then chosen as follows:

\[
A_1 = \begin{bmatrix} b_{1:2} & b_{3:5} & b_6 \end{bmatrix} \\
A_2 = \begin{bmatrix} b_{1:2} & b_{7:9} & b_{10} \end{bmatrix} \\
A_3 = \begin{bmatrix} b_{3:5} & b_{7:9} \end{bmatrix}
\]  

and whereby, the following lemma can be used to state feasibility:

**Lemma 1** \((G_j \text{ full-rank for } p = 2)\). Consider (4.8) and (4.9) with \( A_j, j=1,2,3 \) chosen as in (4.10). Therefore, the only possible solution is \( \lambda^j_i = 0, i = 1, \ldots 5, j = 1, 2, 3 \).

**Proof:** See Appendix B.

Finally, the channel optimal DoF are settled by means of the following theorem:

**Theorem 1** \((\text{DoF for the } (2,3) \text{ case})\). The 3-user MIMO IC with constant channel coefficients and (2,3) antennas at each pair transmitter-receiver has exactly 6/5 DoF per user, and they can be achieved by means of linear precoding at the transmitters and linear filtering at the receivers.
Proof: Each user transmits 12 symbol streams along 5 symbol extensions in time, and considering ACS. Therefore, according to Lemma 1, the signal space matrix $G_j$ becomes full rank, thus interference and desired signals become linearly independent, and the desired message can be recovered. Since the DoF outer bound and the DoF achieved by this scheme match, this value corresponds to the optimal DoF. □
Chapter 5

Extension to the \((p, p+1)\) case

The number of equations to be resolved for each alignment chain increases with \(p^2\). Therefore, the complexity of the matrix equation system in (3.17) becomes intractable as \(p\) increases. This section introduces a new methodology to obtain the structure of the transmit and receive filters applicable to any value of \(p\). The core of this methodology is the zero-propagation algorithm, that allows to reduce the complexity of analytically obtaining a null space basis for (3.17) by identifying which blocks of the resulting precoding matrices are actually equal to zero. As an application example, the \(p=3\) case is resolved, and some application examples are shown in Appendices.

5.1 Precoding matrix design

We first obtain the precoding matrices forced by the alignment chains conditions. In this regard, let particularize (3.17) to the \(p=3, k=1\) case:

\[
\begin{bmatrix}
C(h_{21}^{11}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C(h_{22}^{22}) & C(h_{21}^{21}) & C(h_{22}^{22}) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C(h_{22}^{22}) & C(h_{21}^{21}) & C(h_{22}^{22}) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C(h_{43}^{43}) & 0 & 0 \\
0 & 0 & 0 & 0 & C(h_{43}^{43}) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & C(h_{43}^{43}) & C(h_{43}^{43}) & C(h_{43}^{43}) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C(h_{43}^{43}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}\]

\[E \cdot v = 0 \quad (5.1)\]

Let us explicitly write the second and fifth row of the product \(E \cdot v\) in (5.1), given by:

\[
\begin{bmatrix}
\sqrt{v_{1,ant.1}^{1,1}} \\
\sqrt{v_{1,ant.1}^{1,1}} \\
\sqrt{v_{1,ant.1}^{1,1}} \\
\sqrt{v_{1,ant.1}^{1,1}} \\
\sqrt{v_{1,ant.1}^{1,1}} \\
\sqrt{v_{1,ant.1}^{1,1}} \\
\sqrt{v_{1,ant.1}^{1,1}} \\
\sqrt{v_{1,ant.1}^{1,1}} \\
\end{bmatrix} = 0
\]
5.1 Precoding matrix design

\[ \mathbf{C} \left( h_{22}^{1} \right) \mathbf{V}_{1,3}^{\text{ant.2}} + \mathbf{C} \left( h_{21}^{23} \right) \mathbf{V}_{1,1}^{\text{ant.3}} = \mathbf{C} \left( h_{21}^{23} \right) \mathbf{V}_{3,1}^{\text{ant.1}} \]  \hspace{1cm} (5.2)

\[ \mathbf{C} \left( h_{13}^{1} \right) \mathbf{V}_{1,3}^{\text{ant.1}} = 0 \]  \hspace{1cm} (5.3)

Again, it can be argued that the only solution for (5.3) is \( \mathbf{V}_{1,3}^{\text{ant.1}} = 0 \) as a consequence of Remark 1. Therefore, \( \mathbf{V}_{1,3}^{\text{ant.1}} \) in (5.2) can be set to zero, simplifying this equation. Likewise, note that the entire 4th column in (5.1) can be ignored, thus the equation corresponding to the third row is also simplified. This procedure is denoted as a Zero Propagation (ZP) and gives the possibility to simplify the solution of \( \mathbf{v} \) in (5.1). In order to exploit these properties for the proposed scheme, we present the ZP algorithm next:

\textbf{Algorithm 1} Zero propagation algorithm

Let us define the matrix equation system defined by

\[ \mathbf{E} : \mathbf{v} = 0 \]

with \( \mathbf{v} \in \mathbb{R}^{s_v \times 1} \) and \( \mathbf{E} \in \mathbb{R}^{s_E \times s_v} \). Given an input matrix \( \mathbf{E} \), this algorithm provides a methodology to obtain the position of zeros in \( \mathbf{v} \). This can be achieved by computing the following steps:

1. Initialize \( \mathbf{E} \) as follows:

\[ \mathbf{E} (r, c) = \begin{cases} 1 & \text{if } \mathbf{E} (r, c) \neq 0 \\ 0 & \text{if } \mathbf{E} (r, c) = 0 \end{cases} \]

2. For \( r = 1 \ldots s_E \), find one row in \( \mathbf{E} \) containing only one non-zero element, located at the \( (r, c) \)th position.

3. Set the \( r \)th row and \( c \)th column of \( \mathbf{E} \) to zero. In Matlab notation one can write:

\[ \mathbf{E} = (r, :) = \text{zeros} \left( 1, s_v \right) \]
\[ \mathbf{E} = (:, c) = \text{zeros} \left( s_D, 1 \right) \]

4. Repeat step1 and step2 until there are no more rows in \( \mathbf{E} \) containing only one non-zero element.

5. The zeros in \( \mathbf{v} \) can be found from the all-zero columns in \( \mathbf{E} \). In other words, if the all the \( c \)th column of \( \mathbf{E} \) is zero, then the \( c \)th element in \( \mathbf{v} \) is set to zero, by means of Remark 1.
5.1 Precoding matrix design

\[
E = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
v = \begin{bmatrix}
v(2) \\
v(3) \\
v(5) \\
v(7) \\
v(8) \\
\end{bmatrix}
\]

**Figure 5.1**: Output of Algorithm 1 for \( p=3 \). Elements set to zero by means of step 3 and step 5 of each iteration are highlighted using a blue square.

The algorithm for the \( p=3 \) case results in Figure 5.1 with \( E \) and \( v \) defined as in (5.1), where it is remarkable that columns 1,4,6,9 are all-zero. Therefore, Algorithm 1 provides which blocks (in this case the matrix blocks 1,4,6,9 of \( v \) in (5.1) ) would be zero when solving each alignment chain. This can be obtained by inspection of the algorithm output. Moreover, by writing all the remaining equations, it can be seen that all precoding matrices can be written as a function of \( A_{j,j}=1,2,3 \). This way, the following expression of each precoding matrix is obtained:

\[
V_i = \begin{bmatrix}
C\left(\theta_{i,1,\text{ant.1}}^i \right) A_{i-1} & 0 \\
C\left(\theta_{i,1,\text{ant.2}}^i \right) A_{i-1} & C\left(\theta_{i,1,\text{ant.2}}^{i+1} \right) A_{i+1} & 0 \\
0 & 0 & A_i
\end{bmatrix}
\] (5.4)

where \( \theta_{i,1,\text{ant.1}}^q \) is the element resulting from the \( q \)th alignment chain and located at the \( r \)th block row of \( V_{i,1,\text{ant.1}}^q \). Since it is not relevant for the purposes of this work, the exact expression of \( \theta_{i,1,\text{ant.1}}^q \) in terms of channel coefficients is not derived.

Note that the number of unknown matrices is reduced from 27 in (5.1) to 3 in (5.4). We conjecture that, for any value of \( p \), the \( 3p^2 \) variables (block matrices) involved in all alignment chains can be written as a function of 3 matrices of dimension \( 2(2p + 1) \times 2(p + 1) \). This fact is due to the zero propagation, which allows us to neglect variables or to write them as a function of other variables. We run the algorithm for \( p = 4,...,6 \) obtaining the precoding matrices for each case and tightening these conjectures for more values of \( p \). Results are found in Appendix C.

Interestingly, the ZP algorithm can also be used to construct the signal space matrix which is described in the next section. Hence, this algorithm is able to reduce the
5.2 Feasibility

The feasibility proof should be argued using two steps: to construct the signal space matrix (as in section 4.2.1) for a given value of $p$, and, to prove that this matrix is full-rank (as in section 4.2.2).

5.2.1 Signal space matrix

The ZP algorithm presented in the last section helps us to construct the signal space matrix for each receiver. In order to understand this, consider those equations represented in (5.1) where, after setting the precoding matrix structure obtained in the last section, there are signals corresponding to only one of the receivers. For instance, consider (5.2), that, after applying results obtained in the last section can be written as:

$$C(h_{22}^{21})V_{1,1}^{1,ant.2} + C(h_{23}^{21})V_{1,1}^{1,ant.3} = 0$$

The left-hand side of this equation corresponds to one of the block elements of the product $H_{21}V_1$. Therefore, as the equation states, this elements would be equal to zero. This fact can be used to alleviate the computation of matrix $G_j$. In general, it can be seen from the algorithm output that the rows containing signals transmitted only by one of the users would be a zero at this position when computing the corresponding product $H_{j,i}V_i, i \neq j$. Therefore, those products are given by:

$$H_{j,j-1}V_{j-1} = \begin{bmatrix}
C(\bar{\theta}_{j,j-1})A_{j+1} & 0 & 0 \\
C(\bar{\theta}_{j,j-1})A_{j+1} & C(\bar{\theta}_{j,j-1})A_j & 0 \\
0 & 0 & C(\bar{\theta}_{j,j-1})A_{j-1} \\
0 & 0 & 0
\end{bmatrix}$$

$$H_{j,j+1}V_{j+1} = \begin{bmatrix}
0 & 0 & 0 \\
0 & C(\bar{\theta}_{j,j+1})A_j & 0 \\
0 & C(\bar{\theta}_{j,j+1})A_{j-1} & C(\bar{\theta}_{j,j+1})A_{j+1} \\
0 & 0 & C(\bar{\theta}_{j,j+1})A_{j+1}
\end{bmatrix}$$

(5.5)

where $\bar{\theta}_{j,i}^{q,r}$ is the corresponding complex number for the $(q,r)$th position of $H_{j,i}V_i, i \neq j$. Due to alignment conditions, we will have $\bar{\theta}_{j,j-1}^{q,r} = \bar{\theta}_{j,j+1}^{q-1,r}$ with $q=2,3$, thus columns
5.2 Feasibility

2,3 of $H_{j,j+1}V_{j+1}$ are aligned with columns 1,2 of $H_{j,j-1}V_{j-1}$. Thereby, from (5.5) one obtains that only one column of $H_{j,j+1}V_{j+1}$ is not aligned with $H_{j,j-1}V_{j-1}$.

Now, by using the definition of $G_j$ and applying some full-rank transformations, it is easy to obtain:

$$G_{j}^{\text{des}} = \begin{bmatrix} C(\hat{\theta}_{11}^j)A_{j-1} & C(\hat{\theta}_{12}^j)A_{j+1} & C(\hat{\theta}_{13}^j)A_j \\ C(\hat{\theta}_{21}^j)A_{j-1} & C(\hat{\theta}_{22}^j)A_{j+1} & C(\hat{\theta}_{23}^j)A_j \\ C(\hat{\theta}_{31}^j)A_{j-1} & C(\hat{\theta}_{32}^j)A_{j+1} & C(\hat{\theta}_{33}^j)A_j \\ C(\hat{\theta}_{41}^j)A_{j-1} & C(\hat{\theta}_{42}^j)A_{j+1} & C(\hat{\theta}_{43}^j)A_j \end{bmatrix}$$

(5.6)

where $\hat{\theta}_{q,r}^j$ is the complex number corresponding to the $(q,r)$th position of $G_j$. Since $\hat{h}_{q,r}^j$ is a combination of channel coefficients, the same properties (see Remark 1) than for $\bar{h}_{q,r}^j,i$ can be assumed and $h_{q,r}^j,i$. The reader can refer to Appendix C.2 to find the key ideas to generalize matrix $G_j$ for any value of $p$ using the results for $p = 2, 3, ..., 6$. Additionally, notice that for that case it is not so clear if the signal space matrix is full-rank by simply taking $A_j$ with random entries. In this regard, the tools shown for the $p = 2$ case can be used again to elaborate the proof, which is the aim of the next section.

5.2.2 Rank of signal space matrix

Once $G_j$ has been formulated, now it is necessary to prove that the desired signals and interference terms are linearly independent. In (4.10) (section 4.2.2), it is described how the remaining matrices are chosen for the particular $p = 2$ case. Here we show how $A_j \in \mathbb{R}^{2(2p+1) \times 2(p+1)}$ are chosen for a general value of $p$ in order to achieve optimal DoF. In this regard, let define an orthonormal basis $B = \{b_1, b_2, ..., b_{2(2p+1)}\} \in \mathbb{R}^{2(2p+1)}$ and the sets

$$X_1 = \{3, 4, ..., p + 3\}, X_2 = \{p + 4, p + 5, ..., 2p + 2\} \quad Y_1 = \{2p + 3, ..., 3p + 3\}, Y_2 = \{3p + 4, ..., 4p + 2\} \quad Z = \{1, 2\}$$

(5.7)
5.2 Feasibility

Those sets are used to arrange columns of $B$, e.g. $b_{X_2} = b_{p+4:2p+2}$. Using this simplified notation, we take:

$$A_1 = \begin{bmatrix} b_Z & b_{X_1} & b_{X_2} \end{bmatrix}$$
$$A_2 = \begin{bmatrix} b_Z & b_{Y_1} & b_{Y_2} \end{bmatrix}$$
$$A_3 = \begin{bmatrix} b_{X_1} & b_{Y_1} \end{bmatrix}$$

(5.8)

At this point, the problem reduces to prove the following lemma:

**Lemma 2.** For the $p=3$ case, the signal space matrix in (5.6) with $A_1, A_2$ and $A_3$ chosen as in (5.9) is full rank with probability one.

**Proof:** The proof is analogous to the proof of Lemma 1 (see Appendix B), thus it is omitted.

Consequently, the characterization of this channel for $p=3$ is next formalized:

**Theorem 2** (DoF for the (3,4) case). The 3-user MIMO IC with constant channel coefficients and (3,4) antennas at each pair transmitter-receiver has exactly $12/7$ DoF per user, and they can be achieved by means of linear precoding at the transmitters and linear filtering at the receivers.

**Proof:** The proof is analogous to the proof of Theorem 1, but by means of using Lemma 2. In this case, the DoF are achieved by using the proposed transmitting scheme delivering 24 symbol streams to each user along 14 symbol extensions in time, and considering ACS.

Additionally, using the tools introduced throughout this work, we elaborate the proof for $p = 2, 3...6$, omitted to avoid redundancy. Hence, we conjecture the characterization of the 3-user MIMO IC with $(p, p+1)$ antennas at each pair transmitter-receiver:

**Proposition 1.** The 3-user MIMO IC with constant channel coefficients and $(p, p+1)$ antennas at each pair transmitter-receiver has exactly $p(p+1)/(2p+1)$ DoF per user for $p = 2, 3, \ldots, 6$.

**Proof:** The proof follows using the same arguments and tools used for the $p = 3$ case. Some results of the proof for $p = 2, 3, \ldots, 6$ are shown in Appendix C, where a trend analysis for each value of $p$ can be found. Finally, using the tools presented in this work we are able to conjecture that the outer bound is tight for any $p > 6$. 

\qed
Chapter 6

Conclusions and Future Work

Except for the SISO case, the DoF of the 3-user MIMO IC can be achieved by using linear filters at the transmitters and receivers (IA in the vector space framework) for any antenna configuration both for time-varying or constant channel coefficients. This contribution deals with the constant channel case, being solved before this work only by using structured codes. Therefore, this work alleviates the necessity of lattice alignment techniques to achieve optimal DoF for the \((p,p+1)\) case, i.e. \(p\) antennas at the transmitters and \(p + 1\) at the receivers, when channel coefficients are constant in time. This is useful not only because of the complexity of lattice alignment or the CSI accuracy requirements, but also because the high SNR assumption for those schemes is more demanding. Hence, translating those results to the vector space alignment framework allows to obtain the optimal DoF at a reduced transmitted power value.

The proposed scheme is based on vector space IA, ACS and symbol extensions in time. The novelty of this work is the use of ACS together with the previous state-of-the-art approach in [WGJ11a]. This is useful for the \((p,p+1)\) case with constant channel coefficients. For that case, two important aspects take importance:

- **The specific antenna setting.** Since channel matrices have \(p\) columns and \(p + 1\) rows, their columns span a \(p\) dimensional subspace in the \(p + 1\) receiver dimensional space. Therefore, all channel matrices corresponding to that receiver, also the direct channels, can generate almost all the dimensional subspace.

- **The channel is constant along the transmission.** When considering constant channel coefficients, channel matrices have a specific structure. Therefore, intuitively the diversity or randomness of the channel is lower for that case with respect to the time-varying case.
These two issues impact directly on the rank of desired signals after projecting on the orthogonal to received interference subspace. In this context, use of ACS ables to manage better how the received signals are allocated at each receiver, and allows the desired signals to be linearly independent of the interference.

Further work aims to complete the characterization of the MIMO IC for more than 3 users. This was started in [WSJ12], where the authors rely on previous results based on lattice alignment to attain the proposed outer bound for some antenna configurations. In this regard, research can be focused on translating those results for the vector space alignment framework, as the present work does, and/or to derive the optimal DoF for the remaining cases.

Finally, practical aspects in order to get closer to more realistic implementation need to be addressed. On the one hand, notice that the design explained here aims to maximize the number of DoF, thus it is optimal only for the high SNR regime. A design of transmitters and receivers also optimal for the low-medium SNR regime is a good candidate to continue this work, and a good initial point can be to optimize the chosen basis for $A_j$ in (5.9).

On the other hand, this approach assumes that perfect and instantaneous CSI is available at the transmitters. A more realistic approach can be to consider imperfect or delayed CSI at transmitter side, and continue the recent work initiated in [MAT12, AGK11].
Appendix A

Transformations to simplify cross-channel matrices

In addition to the change of basis operation, we can combine the received signals to simplify the cross channel matrices. We show these transformations with an example for $p=3$. We first show the structure for the cross-channel matrices obtained by applying the original change of basis operation explained in [WGJ11a], being the same equation as in (3.3), but repeated here for reader’s convenience:

$$
H_{j,j-1} = \begin{bmatrix}
C\left(\bar{h}_{j,j}^{11}\right) & 0 & 0 \\
C\left(\bar{h}_{j,j}^{21}\right) & C\left(\bar{h}_{j,j}^{22}\right) & C\left(\bar{h}_{j,j}^{23}\right) \\
C\left(\bar{h}_{j,j}^{31}\right) & 0 & C\left(\bar{h}_{j,j}^{33}\right) \\
0 & 0 & 0
\end{bmatrix}
$$

$$
H_{j,j+1} = \begin{bmatrix}
0 & 0 & 0 \\
C\left(\bar{h}_{j,j}^{21}\right) & 0 & C\left(\bar{h}_{j,j}^{23}\right) \\
C\left(\bar{h}_{j,j}^{31}\right) & C\left(\bar{h}_{j,j}^{32}\right) & C\left(\bar{h}_{j,j}^{33}\right) \\
0 & 0 & C\left(\bar{h}_{j,j}^{43}\right)
\end{bmatrix}
$$

Now, let us define $\bar{H}_{j,j}^{i} \in \mathbb{C}^{1 \times p}$ as the $i$th block-row of $H_{j,j}$ and $\bar{s}_{j,q} \in \mathbb{C}^{p \times 1}$ as the $q$th block of the transmitted signal $s_j$ of the $j$th user. Using these definitions, the received signals at each antenna of the $j$th receiver are given by:
\begin{align*}
y_j^{\text{ant.1}} &= \left( H_{j,j}^{(1)} \right)^T s_j + C \left( \bar{h}_{j,j-1}^{11} \right) s_{j-1,1} \\
y_j^{\text{ant.2}} &= \left( H_{j,j}^{(2)} \right)^T s_j + C \left( \bar{h}_{j,j-1}^{21} \right) s_{j-1,1} + C \left( \bar{h}_{j,j-1}^{22} \right) s_{j-1,2} + C \left( \bar{h}_{j,j-1}^{23} \right) s_{j-1,3} + \\
&\quad + C \left( \bar{h}_{j,j+1}^{21} \right) s_{j+1,1} + C \left( \bar{h}_{j,j+1}^{23} \right) s_{j+1,3} \\
y_j^{\text{ant.3}} &= \left( H_{j,j}^{(3)} \right)^T s_j + C \left( \bar{h}_{j,j+1}^{31} \right) s_{j+1,1} + C \left( \bar{h}_{j,j+1}^{32} \right) s_{j+1,2} + C \left( \bar{h}_{j,j+1}^{33} \right) s_{j+1,3} + \\
&\quad + C \left( \bar{h}_{j,j-1}^{31} \right) s_{j-1,1} + C \left( \bar{h}_{j,j-1}^{33} \right) s_{j-1,3} \\
y_j^{\text{ant.4}} &= \left( H_{j,j}^{(4)} \right)^T s_j + C \left( \bar{h}_{j,j+1}^{43} \right) s_{j+1,3}
\end{align*}

(A.1)

Since there is full CSI available at the receivers, we can properly combine the received signals in order to force some zeros at the cross-channel matrices. For instance, we can obtain a transformed received signal at the second antenna as

\begin{align*}
\hat{y}_j^{\text{ant.2}} &= y_j^{\text{ant.2}} - C \left( \bar{h}_{j,j-1}^{21} \right) C \left( \bar{h}_{j,j-1}^{11} \right)^{-1} y_j^{\text{ant.1}} 
\end{align*}

(A.2)

After this operation, the bitstream \( s_{j-1,1} \) is no longer an interference at the second antenna, which can be interpreted as forcing a zero at the block position (2,1) of the partitioned matrix \( H_{j,j-1} \) while \( H_{j,j+1} \) maintains the same structure. Using this idea repeatedly, it is easy to obtain the cross-channel matrices in (3.4).
Appendix B

Proof of Lemma 1

We will prove the lemma for the equation system defined for \( j = 1 \). For \( j = 2, 3 \) it can easily be seen that our choice for \( A_j, j = 1, 2, 3 \) works, due to symmetry of the problem. Moreover, to simplify notation, we write \( \lambda_i, i = 1, \ldots, 5 \) instead of \( \lambda_1^i \). Additionally, since we are interested in determining the rank of \( G_j \) only, rank preserving transformations can be applied to this matrix. Consequently, the matrix equation system in (4.9) for \( j = 1 \) can be written as follows:

\[
\begin{align*}
\alpha_{11}^{11} U (\phi_{11}^{11}) A_2 \lambda_5 + A_1 \lambda_3 &= 0 \\
\alpha_{11}^{21} U (\phi_{11}^{21}) A_3 \lambda_5 + \alpha_{11}^{22} U (\phi_{11}^{22}) A_2 \lambda_4 + A_3 \lambda_2 &= 0 \\
\alpha_{11}^{22} U (\phi_{11}^{22}) A_1 \lambda_4 + A_2 \lambda_1 &= 0
\end{align*}
\]  

(B.1)

and introducing (4.10) in (B.1), we get:

\[
\begin{align*}
\alpha_{11}^{11} U (\phi_{11}^{11}) [b_{1,2} \ b_{7,9} \ b_{10}] \lambda_5 + [b_{1,2} \ b_{3,5} \ b_6] \lambda_3 &= 0 \\
\alpha_{11}^{21} U (\phi_{11}^{21}) [b_{1,2} \ b_{7,9} \ b_{10}] \lambda_5 + \alpha_{11}^{22} U (\phi_{11}^{22}) [b_{1,2} \ b_{3,5} \ b_6] \lambda_4 + [b_{3,5} \ b_{7,9}] \lambda_2 &= 0 \\
\alpha_{11}^{22} U (\phi_{11}^{22}) [b_{1,2} \ b_{3,5} \ b_6] \lambda_4 + [b_{1,2} \ b_{7,9} \ b_{10}] \lambda_1 &= 0
\end{align*}
\]  

(B.2)

Now, by using linear independence among \( b_i \) we can simplify (B.2). For instance, the top equation becomes:
\[ \alpha_1^{11} U (\phi_{11}^{11}) b_q \lambda_5 (q) + b_q \lambda_3 (q) = 0, \quad q = 1, 2 \quad (B.3) \]

\[
\begin{align*}
\begin{bmatrix}
\alpha_1^{11} U (\phi_{11}^{11}) b_{7:9} \lambda_5 (3:5) = 0 \\
\alpha_1^{11} U (\phi_{11}^{11}) b_{10} \lambda_5 (6) = 0 \\
\lambda_5 (3:5) = \lambda_5 (3:5) = 0 \\
\lambda_5 (6) = \lambda_3 (6) = 0 \\
\end{bmatrix}
\end{align*}
\]

\[ \lambda_5 (3:5) = \lambda_3 (6) = 0 \quad (B.4) \]

where \( \lambda_i = \begin{bmatrix} \lambda_i (1) & \ldots & \lambda_i (6) \end{bmatrix}^T \). Notice that in (B.3) and (B.4) we benefit from Remark 1. Further, (B.3) can be solved as follows. Let us define:

\[ \tilde{b}_q = \begin{bmatrix} b_q (1) + \tau b_q (2) \\
 b_q (3) + \tau b_q (4) \\
 \vdots \\
 b_q (9) + \tau b_q (10) \end{bmatrix} \quad \text{with} \quad \tilde{b}_q = \begin{bmatrix} b_q (1) \\
 b_q (2) \\
 \vdots \\
 b_q (10) \end{bmatrix}, \quad q = 1, 2 \quad (B.5) \]

Then, as in [CJW10], we write the first equation in (B.2) as follows:

\[ \alpha_1^{11} e^{\tau \phi_{11}^{11}} \tilde{b}_q \lambda_5 (q) + \tilde{b}_q \lambda_3 (q) = 0, \quad q = 1, 2 \]

Now, equating real and imaginary parts of each equation to zero, we have:

\[ \alpha_1^{11} \sin (\phi_{11}^{11}) \lambda_5 (q) = 0, \quad q = 1, 2 \]

\[ \alpha_1^{11} \cos (\phi_{11}^{11}) \lambda_3 (q) + \lambda_3 (q) = 0, \quad q = 1, 2 \quad (B.7) \]

Since the set containing all the possible values for \( \phi_{11}^{11} \) such that \( \sin (\phi_{11}^{11}) = 0 \) is a countable set, it has zero measure. Thus using Remark 1 we obtain \( \lambda_r (q) = 0, \quad r = 3, 5, \quad q = 1, 2 \). Overall, from the first equation in (B.2), we have \( \lambda_3 = \lambda_5 = 0 \). Using the same procedure for all equations in (B.2), we conclude that the only solution for (B.1) is \( \lambda_i = 0, \quad i = 1, 2, \ldots, 5 \), thus the proof becomes complete.
Appendix C

Results for $p = 2, 3, \ldots, 6$

We run the zero propagation algorithm, compute the signal space matrix and elaborate the proof for $p = 2, 3, \ldots, 6$. This section shows the precoding matrices and signal space matrix structure for each of those values of $p$. Using all those particular cases, we can predict the behavior of each matrix for any given value of $p$.

C.1 Precoding matrices for $p = 2, 3, \ldots, 6$

The results are shown in Table C.1. Using all these results, it is easy to elucidate the evolution of precoding matrix structure for any value of $p$, which is summarized next:

- All the non-zero elements can be described by the product of two matrices: $C(a)$, where $a$ is a combination of some channel coefficients, multiplied by one of the three remaining variables $A_j$, $j = 1, 2, 3$.
- The complex number $a$ is random, except for the element located in the last row and last column, where $a = 0$.
- The second matrix is the same for each column, and from right to left, we have $A_j, \ldots, A_{j+p-1}$.

When the value of $p$ is odd, the $(p + 1)/2$ column is the symmetry axis and contains only one non-zero element. From the center to the right, each column contains one additional non-zero element above the diagonal until the last column, which contains only one zero element. From the center to the left, a similar behavior is observed with elements growing below the main diagonal. When the value of $p$ is even, the structure is the same but there are two central columns with only one non-zero element.
C.2 Signal space matrix for \( p = 2, 3, ..., 6 \)

We show matrices \( G_j \) obtained by using the ZP algorithm as for the cases \( p = 2, 3 \). Given the lack of space, we will show \( G_j^{\text{des}} \) and \( G_j^{\text{int}} \) separately in Table II and Table III, respectively.

We observe that all non-zero elements \( G_j \) have an specific structure, that is, all of them are the product of one matrix \( C(a) \), where \( a \) is a combination of some channel coefficients, multiplied by one of the three remaining variables \( A_j, j = 1, 2, 3 \). The coefficient \( a \) is different for all the elements, while the matrix \( A_j \) is the same for each column. Notice that we label the argument of each matrix \( C(\cdot) \) as the position in the whole matrix \( G_j \).

We observe some similarities among the matrices in Table C.2, which can be used to describe the behavior and structure for an arbitrary value of \( p \):

- All the elements in \( G_j^{\text{des}} \) are different from zero.
- All the elements have the same structure as commented for \( V_i \) (see Appendix C.1) with \( A_j, ..., A_{j+p-1} \) from right to left.

From Table C.3, we observe the following behavior for \( G_j^{\text{int}} \), which is quite similar to the one described for the precoding matrix structure:

- From right to left, we have \( A_{j+1}, A_{j-1}, ..., A_{j+p-2} \).
- The structure of \( G_j^{\text{int}} \) is persymmetric.
- The main diagonal does not contain any non-zero element. When the value of \( p \) is even, there are three central columns with one non-zero element only whereas for the odd case there are only two of those columns. From those columns to the right, each column contains one additional non-zero element above the diagonal until the last column, which contains the two first rows equal to zero. From the center to the left, the same behavior is observed with elements below the main diagonal.
Table C.1: Precoding matrices structure for $p = 2, 3, ..., 6$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Precoding Matrices Structure</th>
</tr>
</thead>
</table>
| 2   | $\begin{bmatrix} C (b_{t+1,i+1}^{1,ant.1}) A_{i+1} & 0 \\
0 & A_i \end{bmatrix}$ |
| 3   | $\begin{bmatrix} C (\theta_{i+1,ant.1}^{1,ant.1}) A_{i-1} & 0 & 0 & 0 \\
C (\theta_{i+1,ant.2}^{1,ant.2}) A_{i-1} & C (\theta_{i+1,ant.3}) A_i & C (\theta_{i+1,ant.2}^{1,ant.2}) A_i \\
0 & 0 & 0 & A_i \end{bmatrix}$ |
| 4   | $\begin{bmatrix} C (\theta_{i+1,ant.1}^{1,ant.1}) A_{i+1} & 0 & 0 & 0 & 0 \\
C (\theta_{i+1,ant.2}^{1,ant.2}) A_{i+1} & C (\theta_{i+1,ant.3}) A_i & 0 & 0 & C (\theta_{i+1,ant.2}^{1,ant.2}) A_i \\
C (\theta_{i+1,ant.3}^{1,ant.3}) A_{i+1} & C (\theta_{i+1,ant.3}^{1,ant.3}) A_{i+1} & C (\theta_{i+1,ant.3}) A_i & C (\theta_{i+1,ant.3}^{1,ant.3}) A_{i+1} & C (\theta_{i+1,ant.3}) A_i \\
0 & 0 & 0 & C (\theta_{i+1,ant.4}^{1,ant.4}) A_{i+1} & C (\theta_{i+1,ant.4}) A_i \end{bmatrix}$ |
| 5   | $\begin{bmatrix} C (\theta_{i+1,ant.1}^{1,ant.1}) A_{i-1} & 0 & 0 & 0 & 0 & 0 \\
C (\theta_{i+1,ant.2}^{1,ant.2}) A_{i-1} & C (\theta_{i+1,ant.2}^{1,ant.2}) A_{i+1} & 0 & 0 & 0 & C (\theta_{i+1,ant.2}^{1,ant.2}) A_i \\
C (\theta_{i+1,ant.3}^{1,ant.3}) A_{i-1} & C (\theta_{i+1,ant.3}) A_i & 0 & 0 & C (\theta_{i+1,ant.3}) A_i & C (\theta_{i+1,ant.3}) A_i \\
C (\theta_{i+1,ant.4}^{1,ant.4}) A_{i-1} & C (\theta_{i+1,ant.4}) A_i & 0 & 0 & C (\theta_{i+1,ant.4}) A_i & C (\theta_{i+1,ant.4}) A_i \\
0 & 0 & 0 & 0 & 0 & C (\theta_{i+1,ant.5}^{1,ant.5}) A_{i-1} & C (\theta_{i+1,ant.5}) A_i \end{bmatrix}$ |
| 6   | $\begin{bmatrix} C (\theta_{i+1,ant.1}^{1,ant.1}) A_{i-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
C (\theta_{i+1,ant.2}^{1,ant.2}) A_{i-1} & C (\theta_{i+1,ant.2}^{1,ant.2}) A_{i+1} & 0 & 0 & 0 & C (\theta_{i+1,ant.2}^{1,ant.2}) A_i \\
C (\theta_{i+1,ant.3}^{1,ant.3}) A_{i-1} & C (\theta_{i+1,ant.3}) A_i & 0 & 0 & C (\theta_{i+1,ant.3}) A_i & C (\theta_{i+1,ant.3}) A_i \\
C (\theta_{i+1,ant.4}^{1,ant.4}) A_{i-1} & C (\theta_{i+1,ant.4}) A_i & 0 & 0 & C (\theta_{i+1,ant.4}) A_i & C (\theta_{i+1,ant.4}) A_i \\
0 & 0 & 0 & 0 & 0 & C (\theta_{i+1,ant.5}^{1,ant.5}) A_{i-1} & C (\theta_{i+1,ant.5}) A_i \end{bmatrix}$ |
### Table C.2: Matrix $G^\text{des}_j$ for $p = 2, 3, ..., 6$

<table>
<thead>
<tr>
<th>p</th>
<th>Matrix $G^\text{des}_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\begin{bmatrix} C \left( h_{j1} \right) A_{j+1} &amp; C \left( h_{j2} \right) A_j \ C \left( h_{j1} \right) A_{j+1} &amp; C \left( h_{j2} \right) A_j \ C \left( h_{j1} \right) A_{j+1} &amp; C \left( h_{j2} \right) A_j \end{bmatrix}$</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{bmatrix} C \left( \hat{\theta}<em>{j1} \right) A</em>{j-1} &amp; C \left( \hat{\theta}<em>{j2} \right) A</em>{j+1} &amp; C \left( \hat{\theta}<em>{j3} \right) A_j \ C \left( \hat{\theta}</em>{j1} \right) A_{j-1} &amp; C \left( \hat{\theta}<em>{j2} \right) A</em>{j+1} &amp; C \left( \hat{\theta}<em>{j3} \right) A_j \ C \left( \hat{\theta}</em>{j1} \right) A_{j-1} &amp; C \left( \hat{\theta}<em>{j2} \right) A</em>{j+1} &amp; C \left( \hat{\theta}_{j3} \right) A_j \end{bmatrix}$</td>
</tr>
<tr>
<td>4</td>
<td>$\begin{bmatrix} C \left( \hat{\theta}<em>{j1} \right) A</em>{j+1} &amp; C \left( \hat{\theta}<em>{j2} \right) A_j &amp; C \left( \hat{\theta}</em>{j3} \right) A_{j-1} &amp; C \left( \hat{\theta}<em>{j4} \right) A</em>{j+1} &amp; C \left( \hat{\theta}<em>{j5} \right) A_j \ C \left( \hat{\theta}</em>{j1} \right) A_{j+1} &amp; C \left( \hat{\theta}<em>{j2} \right) A_j &amp; C \left( \hat{\theta}</em>{j3} \right) A_{j-1} &amp; C \left( \hat{\theta}<em>{j4} \right) A</em>{j+1} &amp; C \left( \hat{\theta}<em>{j5} \right) A_j \ C \left( \hat{\theta}</em>{j1} \right) A_{j+1} &amp; C \left( \hat{\theta}<em>{j2} \right) A_j &amp; C \left( \hat{\theta}</em>{j3} \right) A_{j-1} &amp; C \left( \hat{\theta}<em>{j4} \right) A</em>{j+1} &amp; C \left( \hat{\theta}<em>{j5} \right) A_j \ C \left( \hat{\theta}</em>{j1} \right) A_{j+1} &amp; C \left( \hat{\theta}<em>{j2} \right) A_j &amp; C \left( \hat{\theta}</em>{j3} \right) A_{j-1} &amp; C \left( \hat{\theta}<em>{j4} \right) A</em>{j+1} &amp; C \left( \hat{\theta}<em>{j5} \right) A_j \ C \left( \hat{\theta}</em>{j1} \right) A_{j+1} &amp; C \left( \hat{\theta}<em>{j2} \right) A_j &amp; C \left( \hat{\theta}</em>{j3} \right) A_{j-1} &amp; C \left( \hat{\theta}<em>{j4} \right) A</em>{j+1} &amp; C \left( \hat{\theta}_{j5} \right) A_j \end{bmatrix}$</td>
</tr>
<tr>
<td>5</td>
<td>$\begin{bmatrix} C \left( \hat{\theta}<em>{j1} \right) A</em>{j+1} &amp; C \left( \hat{\theta}<em>{j2} \right) A_j &amp; C \left( \hat{\theta}</em>{j3} \right) A_{j-1} &amp; C \left( \hat{\theta}<em>{j4} \right) A</em>{j+1} &amp; C \left( \hat{\theta}<em>{j5} \right) A_j &amp; C \left( \hat{\theta}</em>{j6} \right) A_j \end{bmatrix}$</td>
</tr>
<tr>
<td>6</td>
<td>$\begin{bmatrix} C \left( \hat{\theta}<em>{j1} \right) A</em>{j+1} &amp; C \left( \hat{\theta}<em>{j2} \right) A_j &amp; C \left( \hat{\theta}</em>{j3} \right) A_{j-1} &amp; C \left( \hat{\theta}<em>{j4} \right) A</em>{j+1} &amp; C \left( \hat{\theta}<em>{j5} \right) A_j &amp; C \left( \hat{\theta}</em>{j6} \right) A_j \end{bmatrix}$</td>
</tr>
</tbody>
</table>
Table C.3: Matrix $G^\text{int}_j$ for $p = 2, 3, ..., 6$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Matrix $G^\text{int}_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\begin{bmatrix} A_j &amp; 0 &amp; 0 \ 0 &amp; A_{j-1} &amp; 0 \ 0 &amp; 0 &amp; A_{j+1} \end{bmatrix}$</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{bmatrix} C(\hat{\theta}<em>{j}^{27}) A</em>{j+1} &amp; 0 &amp; 0 &amp; 0 \ C(\hat{\theta}<em>{j}^{24}) A</em>{j+1} &amp; A_j &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; A_{j-1} &amp; C(\hat{\theta}<em>{j}^{37}) A</em>{j+1} \ 0 &amp; 0 &amp; 0 &amp; C(\hat{\theta}<em>{j}^{47}) A</em>{j+1} \end{bmatrix}$</td>
</tr>
<tr>
<td>4</td>
<td>$\begin{bmatrix} C(\hat{\theta}<em>{j}^{15}) A</em>{j-1} &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ C(\hat{\theta}<em>{j}^{25}) A</em>{j-1} &amp; A_{j+1} &amp; 0 &amp; 0 &amp; 0 \ C(\hat{\theta}<em>{j}^{35}) A</em>{j-1} &amp; 0 &amp; A_j &amp; 0 &amp; C(\hat{\theta}<em>{j}^{39}) A</em>{j+1} \ 0 &amp; 0 &amp; 0 &amp; A_{j-1} &amp; C(\hat{\theta}<em>{j}^{49}) A</em>{j+1} \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; C(\hat{\theta}<em>{j}^{59}) A</em>{j+1} \end{bmatrix}$</td>
</tr>
<tr>
<td>5</td>
<td>$\begin{bmatrix} C(\hat{\theta}<em>{j}^{16}) A_j &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ C(\hat{\theta}</em>{j}^{26}) A_j &amp; C(\hat{\theta}<em>{j}^{27}) A</em>{j-1} &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ C(\hat{\theta}<em>{j}^{36}) A_j &amp; C(\hat{\theta}</em>{j}^{37}) A_{j-1} &amp; A_{j+1} &amp; 0 &amp; 0 &amp; C(\hat{\theta}<em>{j}^{39}) A</em>{j+1} \ C(\hat{\theta}<em>{j}^{46}) A_j &amp; 0 &amp; 0 &amp; A_j &amp; C(\hat{\theta}</em>{j}^{410}) A_{j-1} &amp; C(\hat{\theta}<em>{j}^{411}) A</em>{j+1} \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; C(\hat{\theta}<em>{j}^{510}) A</em>{j-1} &amp; C(\hat{\theta}<em>{j}^{511}) A</em>{j+1} \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; C(\hat{\theta}<em>{j}^{611}) A</em>{j+1} \end{bmatrix}$</td>
</tr>
<tr>
<td>6</td>
<td>$\begin{bmatrix} C(\hat{\theta}<em>{j}^{17}) A</em>{j+1} &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ C(\hat{\theta}<em>{j}^{27}) A</em>{j+1} &amp; C(\hat{\theta}<em>{j}^{28}) A_j &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ C(\hat{\theta}</em>{j}^{37}) A_{j+1} &amp; C(\hat{\theta}<em>{j}^{38}) A_j &amp; A</em>{j-1} &amp; 0 &amp; 0 &amp; 0 \ C(\hat{\theta}<em>{j}^{47}) A</em>{j+1} &amp; C(\hat{\theta}<em>{j}^{48}) A_j &amp; 0 &amp; A</em>{j+1} &amp; 0 &amp; C(\hat{\theta}<em>{j}^{412}) A</em>{j-1} &amp; C(\hat{\theta}<em>{j}^{413}) A</em>{j+1} \ C(\hat{\theta}<em>{j}^{57}) A</em>{j+1} &amp; 0 &amp; 0 &amp; A_j &amp; C(\hat{\theta}<em>{j}^{512}) A</em>{j-1} &amp; C(\hat{\theta}<em>{j}^{513}) A</em>{j+1} \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; C(\hat{\theta}<em>{j}^{612}) A</em>{j-1} &amp; C(\hat{\theta}<em>{j}^{613}) A</em>{j+1} \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; C(\hat{\theta}<em>{j}^{711}) A</em>{j+1} \end{bmatrix}$</td>
</tr>
</tbody>
</table>
Bibliography


C.2 Signal space matrix for $p = 2, 3, ..., 6$


C.2 Signal space matrix for $p = 2, 3, \ldots, 6$


