Master Erasmus Mundus in Photonics Engineering, Nanophotonics and Biophotonics

Master Thesis Work

Quantum Non-Linear Optics with Single Atom

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In this thesis we tried to investigate the non-linear effects introduced by a generalized interaction of a two-level atom coupled to two photons. Scattering matrices were used to calculate the non-linear effects introduced and the dependence of outgoing frequencies of quantum light on these effects was found. It was seen as quite opposed to typical classical theories, that the non-linear susceptibility for an atom dictates very precise output frequencies from it.

As a next step this work probes into quantum computing in which the two-photon bound state problem was mathematically established to show that a single two-level atom cannot be used in realization of non-linear logical operations in quantum computations. In the end a possible intuitive solution, involving dissimilar decay rates in three level systems, was thought of and shown that it can be effectively applied to quantum logic.
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INTRODUCTION

In modern day technology, the application of classical non-linear optics is quite ubiquitous. Concepts of nonlinear optics are applied in areas like spectroscopy, frequency generators, optical modulators etc. It is some of the limitations of classical non-linear optics and some inherent advantages, that give rise to Quantum Non-Linear Optics.

Quantum non-linear optics essentially deals with non-linear effects at the level of few-photons. This enables the production of low-power non-linear devices like switches[1] and transistors[2], generation and manipulation of non-classical optical fields, quantum information processing etc. As the interactions of single photons are very well defined it is possible to go beyond the shot-noise limit suffered by its classical counterpart. It is possible to create states of light which have sub-shot-noise signatures and can be used for various high precision applications. One such example is the application of ”squeezed light” in LIGO II (Laser Interferometer Gravitational-Wave Observatory) or advanced LIGO [3], for the observation of gravity waves by measuring fluctuations in interference patterns, where precision of the highest degree is necessary. Classical light is used in LIGO [4] but for improved results they plan on using squeezed light [5] in LIGO II.

In this thesis, we are going to explore the very basics of all these complex and precise futuristic technology: the mathematics of the interaction between few photons and a simple 2-level atom. We are going to use the scattering matrix approach for solving the problem. In this method the atomic scattering region is characterized by a scattering matrix depending on the frequencies of impinging photons on it. Firstly, we are going to see a simple example of the problem: the theory of single-photon interacting with a two-level atom. The theory is extensively developed from the paper on Strongly Correlated Two-Photon Transport in a One-Dimensional Waveguide Coupled to a Two-Level System(Shanhui Fan et al)[6]. The theory is extensively reproduced in the successive chapters with rigorous mathematical description. In the problem the atom is coupled to a waveguide. For the situation to look more realistic I added an extra decay channel outside the waveguide to the entire problem.
Then, I went into a analogical description to the real problem at hand: the two-photon single-atom interaction scheme. After calculation it is shown that there exists a non-linear part along with the regular linear scattering field. This interaction is the the very basis for Quantum Non-Linear optics.

For this kind of interactions to occur, the system dynamics must be dominated by the interaction of individual atoms to individual photons. This is not the case when normal laser light falls on a medium, where the dynamics arise from the many atom-many photon interactions(the so called weak coupling regime). So for our situation what we need is the so called strong coupling between light and matter. Now, it is quite intuitive that coupling a single atom to a few photons is a very difficult task. In the background section of the thesis I have included some practical schemes suggested by various researchers for increasing the probability of such interactions by attaining the strong-coupling regime.

Afterwards, I made a small simulation to show that the nonlinear response of an atom is highly correlated with the color of the outgoing light. In previous literature, third order nonlinearity has been treated as a constant for the output processes. During the calculation we noticed that for a particular frequency difference generation in the output, the non-linear effect dominates over all other by leaps and bounds. For any excitation frequency, the atom-field system has a large probability to produce frequencies at resonance with the atomic frequency and another frequency which results from the conservation of energy of the system. This effect is very peculiar in nature and typical non-linear modelling doesn’t suggest this. So this effect is stated as a conjecture and can be verified in future.

As a further extension, we wanted to try our system in the field of quantum computing which is probably the biggest potential area for the application of QNLO up to this date. The various quantum logic gates are non-linear in nature. So we wanted to investigate the effectiveness of our results with one of them (the C-Phase gate [7]). Quantum gates require the output photons (or bits) to remain isolated in time or propagation vectors from each other as they were in the input state. This sounds really trivial, but not all systems can do that. So we took our system to be the same two-level system interacting with two photons and tried to calculate the eigen states from the scratch. It is seen that there exists "bound states" where photons seem to stick to one another, thereby un-isolated from each other. Hence it can be concluded that two-level atom is not suitable for quantum computing. So the natural question arises: which kind of atoms are necessary or what is the defining characteristics of an atom which will make the output photons remain isolated from one another? In the final part of this project I’ll try to explain intuitively the basic requirement in an atom to exhibit this kind of behaviour. This is stated as yet another conjecture and requires to be considered later on.
CHAPTER ONE

CLASSICAL AND QUANTUM THEORY OF NON-LINEAR OPTICS

In this chapter the basics of classical non-linear optics are introduced. A short comparison between classical and quantum nonlinear theory is made. Thus we will be in a situation where we understand the very necessity of Quantum Non-linear Optics: it’s advantages over its classical counterpart as well as some inherent properties of it.

1.1 The Classical Theory

Non-linear optics is the study of light-matter interaction when the optical properties of the matter in question has been modified by the impinged light. Classically, only laser light is intense enough to modify such properties in matter. Let’s go into a bit more qualitative details of the problem at hand. We start from the Maxwell’s equation for a dielectric media without free charge or current. They are,

\[ \nabla \cdot \vec{D}(\vec{r}, t) = 0 \]  
\[ \nabla \cdot \vec{B}(\vec{r}, t) = 0 \]  
\[ \nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t} \]  
\[ \nabla \times \vec{H}(\vec{r}, t) = -\frac{\partial \vec{D}(\vec{r}, t)}{\partial t} \]

where \( \vec{D} \) is the electric displacement, \( \vec{B} \) and \( \vec{E} \) are the magnetic and electric field respectively. Our concern is with the electric displacement. It is related to the electric field as,

\[ \vec{D}(\vec{r}, t) = \varepsilon_0 \vec{E}(\vec{r}, t) + \vec{P}(\vec{r}, t) \]
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This is valid for instantaneous medium response to the electric field and no dispersion. The more general equation would be,

\[ \vec{D}(\vec{r}, \omega) = \epsilon_0(\omega)\vec{E}(\vec{r}, \omega) + \vec{P}(\vec{r}, \omega) \]

where \( \vec{P} \) is the dipole moment per unit volume, or polarization of the material. Now we analyze the dependence of this quantity with respect to the field strength \( \vec{E} \) of an applied field. In case of linear optics the general response of the system can be summarized by the following equation.

\[ \vec{P}_L(\vec{r}, t) = \epsilon_0 \chi^{(1)} \vec{E}(\vec{r}, t) \]  

where \( \chi^{(1)} \), the constant of proportionality is known as the linear susceptibility and \( \epsilon_0 \) is the permittivity of free space. Now, in non-linear optics this response is described by a generalized power series expansion in field strength as

\[ \vec{P}(\vec{r}, t) = \epsilon_0 \left[ \chi^{(1)} \vec{E}(\vec{r}, t) + \chi^{(2)} \vec{E}^2(\vec{r}, t) + \chi^{(3)} \vec{E}^3(\vec{r}, t) + ... \right] \]

\[ \equiv \vec{P}_L(\vec{r}, t) + \vec{P}^{(2)}(\vec{r}, t) + \vec{P}^{(3)}(\vec{r}, t) + ... \]  

\[ (1.7) \]

the constants \( \chi^{(2)} \) and \( \chi^{(3)} \) are known as second and third order non-linear optical susceptibilities respectively. Here we assume that the polarization depends instantaneously on the electric field which also leads to the assumption that the medium is lossless and dispersion-free. In general the non-linear susceptibilities depends on the frequencies of the applied field.

In the following Figure 1.1 we use a model of simple atom to illustrate the effect of electric field on it. If the externally applied field is weak the dependence of the induced dipole moment in the atom and the external field is linear, whereas, in case of strong external field, the dependence is non-linear in most cases. Hence from the model of a simple atom we can extend this analogy to dipole moment per unit volume or polarization, that is induced in the

\[ Figure 1.1: \text{Atom-Field Interaction. Courtesy: Prof. Dr. J. Leuthold [8]} \]
material. Thus in presence of strong external field the induced polarization in the material is non-linear with the driving electric field.

If we derive the wave equation from the Maxwell’s equation using the polarization, we will arrive at a equation as the following.

$$-\nabla^2 \vec{E}(\vec{r},t) + \frac{n(\omega)^2}{c^2} \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = -\mu_0 \frac{\partial^2 \vec{P}_{NL}(\vec{r},t)}{\partial t^2}$$

(1.8)

where the linear refractive index, $n^2(\omega) = \epsilon(\omega) = 1 + \chi^{(1)}$ and $\mu_0$ is the permeability of the medium.

Hence from the wave equation we can see that non-linear polarization acts as a source term for a driven wave equation. The radiation propagates as a free wave with the speed $v = c/n$.

Most nonlinear can be described by this equation and are relates to the $\chi^{(j)}$ tensor. Some phenomena arising from second order non-linear effects(real part of $\chi^{(2)}$) are electro-optic effect, sum/difference frequency generation etc. From third order non-linear effect(real part of $\chi^{(3)}$) we get processes like intensity dependent refractive index, third harmonic generation, self-phase modulation, self-focussing and four-wave fixing.

### 1.2 Challenges for Classical Theory

In the classical theory the system dynamics is dominated by multi-photon interacting with multi-atoms and the natural coupling between the light and matter in such a case is weak in nature. Hence a large amount of energy is required to establish non-linearities in such materials. Therefore energy requirements for classical non-linear optics is very high.

Classical light has a Poissonian distribution. This sets a minimum noise level and hence for classical light the best possible situation is being shot-noise limited. Going beyond that limit is not possible with classical light.

Quantum light on other hand requires strong coupling hence non-linear effects are possible at very low power levels(few photons). Hence this is a much power-efficient method. Also, quantum states of light can sub-Poissoninan in nature implying the ability to go beyond shot noise [9]. Thus noise being no longer a matter of concern, it is possible to make high precision measurements.
1.3 The Quantum Theory

As opposed to the classical theory, the non-linear phenomenon in quantum picture is the interaction of quantum light with quantum matter. Thus the light is characterized by a few photons. So the interaction probability of such a system is extremely low in normal scenarios.

The natural system of choice in quantum non-linear optics is the atom. Photons couple to the atoms and give rise to non-linear effects. But the chance of coupling is intrinsically very low owing to the infinitesimally small scattering cross-section of single atoms. This coefficient is also limited by the diffraction limit of the optical beam. The following Figure 1.2 illustrates the relation.

\[ P_{sc} \sim \frac{\lambda^2}{A} \]

Figure 1.2: The relation between diffraction limit, scattering cross-section and scattering probability

Here A is the beam area, which can be in the best case scenarios be pushed to the diffraction limit, \( \sigma_{sc} \) is the scattering cross-section of the atom and \( P_{sc} \) is the probability that the light couples to the atom. This is given by,

So it is necessary to increase the coupling between the photons and the atoms to get practically realizable results. Conventional systems using lasers typically operate in the weak-coupling regime which involves large number of atoms and photons. As a result of which individual quanta have negligible effect on the system dynamics. But when the system approaches the so called strong-coupling regime the above statement is not true. It involves a small number of atoms and photons and their effect is strong enough to dominate the system dynamics. Different schemes for performing the desired coupling between the atoms and light have been proposed. The techniques can be broadly generalized as:

1. Cavity QED[10]
2. Many atom schemes using quantum EIT (Electromagnetically Induced Transparencies)[11]
3. Free space optics(without the need for a cavity)[12]
Theoretically these methods have been verified and reports of experimental verifications also exist.

1.3.1 Cavity QED

Cavity QED is a theory of electromagnetic interaction between charged particles and electromagnetic fields in the presence of macroscopic bodies like metallic or dielectric mirrors or cavity walls. In the cavity the electromagnetic field is quantized and the charges that are free or bound in atoms and molecules are described by non-relativistic quantum mechanics. The macroscopic bodies such as mirrors are introduced into the picture by assuming proper boundary conditions that are satisfied by the electromagnetic fields at their surfaces. This scheme is basically isolating a single atom in an optical cavity in a regime for which one photon saturates the atomic transition. The observed characteristics of the atom-cavity system are qualitatively different from the case of conventional many-atoms many-photon cases.

Here $\gamma$ denotes the decay rate of the atom into free space, $\kappa$ represents the cavity decay rate and $g$ is basically the atom-cavity coupling rate. The strong coupling regime is basically achieved when the following condition is fulfilled i.e. coupling is greater than both the decay rates.

$$g > \kappa, \gamma \quad (1.10)$$

When this condition is met we get Rabi oscillations in which the populations of the two states flip in regular cycles at the precise Rabi frequency. If the condition is not met we get an exponentially decaying pattern in the population vs time curve, which is what we expect classically when an atom depopulates exponentially from the excited state.

The figure 1.4 shows the comparison between classical and quantum atomic excited level population probabilities vs time.
1.3.2 Many Atom scheme using Quantum EIT

In contrast to the single-atom single-photon scheme, it is easy to realise intuitively that more the number of atoms present, more the coupling probability is. In fact coupling probability is directly proportional to the number of atoms in the same cross-sectional area. The cavity QED method faces several technical challenges which paved the way for a easier and more efficient way of coupling light to a large collection of atoms, an atomic ensemble. Various schemes can be used to make strong coupling between photons and atoms in an ensemble. We will discuss a bit about one of the methods: Electromagnetically Induced Transparency (EIT)[14],[15],[11]. In this scheme the states of an atom are coupled via several(let us assume two) possible alternative transition processes. The amplitudes of these processes may interfere destructively thereby canceling out completely the total transition probability.

The linear response of an atom to light at resonance is described by the first order susceptibility $\chi$. The imaginary part describes the dissipation of the field(i.e absorption), while the real part is responsible for the refractive index. The $\text{Im}[\chi]$ for an allowed transition is Lorentzian in shape with broadening due to the damping factors. The $\text{Re}[\chi]$ has a dispersion profile as shown in the figure 1.5. We see that this profile has an anomalous nature(negative slope with frequency) in the central part of the absorption profile within the linewidth. Figure 1.5 shows this plot of susceptibility for conventional atomic system as compared to an EIT systems.
Figure 1.5: Susceptibility as a function of the frequency $\omega_p$ of the applied field relative to the atomic resonance frequency $\omega_{31}$, for a radiatively broadened two-level system with radiative decays (dashed line) and an EIT system with resonant coupling field (solid line): left, imaginary part of $\chi$ characterizing absorption; right, real part of $\chi$ determining the refractive properties of the medium. Courtesy M. Fleischhauer [11]

Solid lines in Figure 1.5 shows the imaginary and the real parts of linear susceptibility of a $\Lambda$–type three level system against a probe laser frequency as in Figure 1.6 driven by a strong-coupling laser field.

Figure 1.6: Three-level $\Lambda$ type medium resonantly coupled to a laser field with Rabi frequency $\Omega_c$ and a probe field $\hat{E}(z,t)$ with a Rabi frequency $\Omega_p$. $\Gamma_{ik}$ represents the radiative decay rates from $|i\rangle$ to $|k\rangle$. Detunings of both field are taken to be zero for simplicity.

We are going to analyse as to how this scheme works. We consider the interaction Hamiltonian from [11] with coupling laser Rabi frequency $\Omega_c$ and the probe laser Rabi frequency $\Omega_p$. For simplified result both the coupling and probe detunings are taken to be zero.

$$ H_{\text{int}} = -\frac{\hbar}{2} \begin{bmatrix} 0 & 0 & \Omega_p \\ 0 & \Omega_c & 0 \\ \Omega_p & \Omega_c & 0 \end{bmatrix} $$ (1.11)

From (1.11) we calculate the eigenvalues and the eigenstates of the interaction Hamiltonian. The eigenvalues are: 0 and $\pm \sqrt{\Omega_c^2 + \Omega_p^2}$. We also see that the eigenstate associated with the
zero eigenvalue doesn’t have contributions from the excited state. This implies there is no dissipation and hence the dip in the $\text{Im}[\chi]$ curve giving rise to the transparency phenomenon. However the other two eigenstates from the eigenvalues have the excited state contribution and hence are absorptive in nature, thus the absorption peaks occurs. Since the probe field is usually much weak compared to the coupling field the eigen-values can be approximated to $\pm \Omega_c$. The negative and the positive signs in the eigenvalues imply that the peaks are to be located on both sides of the transparency region. Also, it can be easily seen that if we change the value of $\Omega_c$ the peaks shift, hence increasing or decreasing the transparency window which also implies ability to make the slope of the $\text{Re}[\chi]$ extremely sharp in the zero detuning area i.e. attaining ultra-slow group velocities. In Figure 1.5, we see the result of the same analysis. So an EIT medium can be rendered transparent to the probe beam($\hat{E}(z,t)$) by tuning the coupling beam. Thus for the probe beam, due to suppression of absorption, the nonlinear effects are enhanced compared to linear effects in the medium, which by itself without the driving laser field, would have been optically thick(absorptive). Furthermore, there is a normal dispersion in the region of low absorption from the plot of $\text{Re}[\chi]$, the slope of which is controlled by the coupling-laser strength. Thus, despite being transparent the medium can offer strong dispersion to the laser pulse. So, despite EIT being a linear effect, the absence of absorption or group-velocity dispersion on resonance makes it a useful tool from which quantum nonlinearities might be realised.

### 1.3.3 Free-space Optics

Controlling the interaction between single photon and single atom can be quite fascinating as well as challenging. It means ultimate control over light quanta, and has potential applications in single-photon switching and transistors, long range optical coupling of quantum bits etc. In quantum information theory, for efficient transfer of quantum information from a single photon to a quantum system like atom, the probability of absorption has to be close to unity. In the previous sections we saw how different tricks were involved to increase this coupling probability. In this section we talk about free-space photon-atom coupling models.

Protocols of quantum information theory require localized carriers or nodes such as atoms or ensembles that exchanges the information via a quantum channel with the help of 'flying' qubits or photons. These quantum channels are highly well defined photonic modes coupled strongly to the atoms. Cavities are effective couplers but when many such modes are to be coupled, the process becomes experimentally very difficult. An alternative is free-space strong coupling between atoms and photons. Let’s go into one of the coupling schemes. We consider the experimental scheme shown in [12].
The heart of the setup is basically two identical aspheric lens in a confocal formation inside a ultra-high vacuum chamber. A Gaussian mode acts as the quantum channel. The Gaussian probe beam from a single-mode fiber is focussed and collected by the lens system and coupled into another single-mode fiber again which goes to a avalanche photodiode. There is a trapped atom($^{87}$Rb) at the focus between the two lens. The trap used is know as far-off-resonant optical dipole trap (FORT). This trap allows no more than one atom occupancy at any time in the trap by a mechanism known as collisional blockade mechanism[16]. There are other associated factors and manipulations associated with this mechanism but this is the basic idea behind the method. So applying a tighter focus it can be assumed that the scattering probability goes up. Thus we can achieve a strong coupling regime. So simplicity is the main advantages of this mechanism, which makes it appealing for many applications.

### 1.3.4 Hybrid Methods

In recent times there has been a lot of research on the efficiency of coupling and many researchers came up with a lot of schemes which can be thought of as hybrid schemes as they include mixed features of the basic schemes described above. These new schemes can be effective in the sense that they seek to extract the advantages from several schemes and put them into use to the problem at hand.

One of these schemes is cavity QED with atomic mirrors[17], where a lattice of atoms forms a high finesse cavity within a fiber. Special impurity atoms were designed in the cavity which experiences strong coupling with light in the fiber.

As another example there exists a scheme where highly focussed laser field on a single ion
in front of a far-distant dielectric mirror gives rise to cavity effects and the system behaves as a Fabry-Pérot cavity: a strong coupling around the atom is established[18].

1.4 Summary

In this chapter we had a quick glance at the classical theory of non-linear optics in general. In the quantum theory we found that strong coupling between atom and photon is necessary for any significant effect to be perceived. Hence we went into a few mechanisms by which this so called strong coupling is achieved. This is just to present a practical perspective to the theoretical details of quantum nonlinear optics to follow.
In this chapter the mathematical tools necessary for the thesis are introduced. Firstly, the light-matter interaction will be treated using the Jaynes-Cummings model[19][20]. This being a fully quantized model will be the basis for the calculations to follow. After being able to set up the interaction Hamiltonian, the elements of scattering theory will discussed in details thereby enabling us to make way for the calculations in the next chapters.

2.1 The Jaynes-Cummings model

We consider a simple two level atom with ground state $|g\rangle$ and excited state $|e\rangle$. This atom interacts with a single mode cavity field of the form,

$$\hat{E} = \varepsilon \sqrt{\frac{\hbar \omega}{\epsilon_0 V}} (\hat{a} + \hat{a}^\dagger) \sin kz$$

2.1

$$\varepsilon \rightarrow$$ arbitrarily oriented polarization vector.

Interaction Hamiltonian,

$$\hat{H}^{(I)} = -\hat{d}.\hat{E}$$

$$= \hat{d}g(\hat{a} + \hat{a}^\dagger)$$

where $g = -\sqrt{\frac{\hbar \omega}{\epsilon_0 V}} \sin kz$ and $\hat{d} = \hat{d}.e$
Now we consider atomic transition operators, $\hat{\sigma}_+ = |e\rangle \langle g|$ and $\hat{\sigma}_- = |g\rangle \langle e| = \hat{\sigma}_+^\dagger$
and, inversion operator $\hat{\sigma}_z = |e\rangle \langle e| - |g\rangle \langle g|$
These operators obey Pauli spin algebra i.e.

$$\begin{align*}
[\hat{\sigma}_+, \hat{\sigma}_-] &= \hat{\sigma}_z \\
[\hat{\sigma}_z, \hat{\sigma}_\pm] &= 2\hat{\sigma}_\pm
\end{align*}$$

(2.3)

From parity considerations ($\langle e|\hat{d}|e\rangle = 0 = \langle g|\hat{d}|g\rangle$), so

$$\hat{d} = d|g\rangle \langle e| + d^*|e\rangle \langle g|$$

$$= d\hat{\sigma}_- + d^*\hat{\sigma}_+$$

$$= d(\hat{\sigma}_+ + \hat{\sigma}_-) \quad d \in \mathbb{R}$$

(2.4)

Hence Interaction Hamiltonian,

$$\hat{H}^{(I)} = \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger) \quad \lambda = dg/\hbar$$

(2.5)

If we define zero energy level halfway between the states $|e\rangle$ and $|g\rangle$ then the free atomic Hamiltonian becomes,

$$\hat{H}^{(A)} = \frac{1}{2}\hbar\Omega [ |e\rangle \langle e| - |g\rangle \langle g| ]$$

$$= \frac{1}{2}\hbar\Omega\hat{\sigma}_z \quad \Omega = \text{Atomic Transition Frequency}$$

(2.6)

The free field Hamiltonian, after dropping the zero-point energy term since it doesn’t contribute to the dynamics of the system,

$$\hat{H}^{(F)} = \hbar\omega\hat{a}^\dagger\hat{a}$$

(2.7)

Therefore, total Hamiltonian of the system,

$$\hat{H} = \hat{H}^{(A)} + \hat{H}^{(F)} + \hat{H}^{(I)}$$

$$= \frac{1}{2}\hbar\Omega\hat{\sigma}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger)$$

From the interaction Hamiltonian we isolate the fast oscillating terms which doesn’t conserve energy and drop them from the total Hamiltonian. This is basically the Rotating Wave Approximation (RWA). Hence the final Hamiltonian assumes the form,

$$\hat{H} = \frac{1}{2}\hbar\Omega\hat{\sigma}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger)$$

(2.8)

In the next section we extend the free-field Hamiltonian to infinite forward propagating modes and call the field Hamiltonian as $\hat{H}_0$. The rest of the Hamiltonian is named as $\hat{H}_1$. 
2.2 Hamiltonian

We take into account a 2-level atom coupled to a single-mode waveguide. But we use a
generalized treatment with infinite forward propagating modes. Using and manipulating the
formula derived in the previous section,

\[ \tilde{H} = \tilde{H}_0 + \tilde{H}_1 \]  \hspace{1cm} (2.9)

\[ \tilde{H}_0 = \int_0^\infty d\beta \tilde{\omega}(\beta) \tilde{a}_\beta \tilde{a}^\dagger_\beta \]  \hspace{1cm} (2.10)

\[ \tilde{H}_1 = \frac{1}{2} \tilde{\Omega} \sigma_z + V \int_0^\infty d\beta (\sigma_\pm \tilde{a}_\beta + \tilde{a}^\dagger_\beta \sigma_-) \]  \hspace{1cm} (2.11)

\[ \tilde{H}_0 \rightarrow \text{Hamiltonian for one way waveguide} \]
\[ \tilde{H}_0 \rightarrow \text{Hamiltonian for atom and atom-waveguide interaction} \]
\[ \tilde{\Omega} \rightarrow \text{Atomic Transition frequency} \]
\[ \sigma_\pm \rightarrow \text{raising and lowering operator} \]
\[ \sigma_z = 2\sigma_+ \sigma_- - 1 \]
\[ V = \lambda \rightarrow \text{coupling strength} \]
\[ \hbar \text{ is set to unity for convenience} \]

We linearize the waveguide dispersion around \((\beta_0, \omega_0)\) i.e. \[ \tilde{\omega}(\beta) = \omega_0 + v_g (\beta - \beta_0) \] (to
have \(H\) in terms of frequencies).

Since we deal with states with wave-vector around \(\beta_0\) extension of limits to \(-\infty\) is justified
in order to define Fourier Transform.
Finally we have,

\[ H_0 = \int_{-\infty}^{\infty} d\omega \omega a_\omega^\dagger a_\omega \]  
\[ H_1 = \frac{1}{2} \Omega \sigma_z + \frac{V}{\sqrt{v_g}} \int_{-\infty}^{\infty} d\omega (\sigma_+ a_\omega + a_\omega^\dagger \sigma_-) \]  

where
\[ \Omega = \tilde{\Omega} - \omega_0 \]
\[ \omega \equiv v_g \beta; \quad a_\omega \equiv \frac{\tilde{a}_\beta + a_0}{\sqrt{v_g}}; \quad [a_\omega, a_\omega^\dagger] = \delta(\omega - \omega') \]

### 2.3 Scattering Theory

In this formalism, we prepare an input state and let it pass through the scattering region. In our detection system we measure the output state. From the knowledge of the output state we derive information about the interaction in the scattering region with the help of a characteristic matrix known as the **Scattering Matrix**.

Mathamatical formulation of scattering matrix:

\[ S_{p,k} = \langle p|S|k \rangle \]

|\(k\rangle\) and |\(p\rangle\), the input and output states respectively are considered free states long before (\(t_0 \to -\infty\)) and long after (\(t_1 \to \infty\)). The scattered states could be phase-shifted or frequency converted due to the presence of the atom. \(S|k\rangle\) basically defines the scattered state and the term \(\langle p|S|k \rangle\) compares the overlap between the scattered state and state |\(p\rangle\) which propagates in absence of the atom. So \(\langle p|S|k \rangle\) can be thought of as a measure of overlap between the the output |\(p\rangle\) and the scattered \(S|k\rangle\) states.

Thus the S operator is basically the evolution operator \(U_I\) from time \(-\infty\) to \(\infty\)

\[ S = \lim_{(t_0, t_1) \to (-\infty, \infty)} U_I(t_1, t_0) \]

An equivalent way of describing is using the scattering eigen-states |\(k^\pm\rangle\). Thus dropping limits,

\[ |k^+\rangle = U_I(0, t_0)|k\rangle = \Omega_+|k\rangle \]

Similarly,

\[ |k^-\rangle = U_I(0, t_1)|k\rangle = \Omega_-|k\rangle \]
\( \Omega_{\pm} \): Møller operators (relating scattering states and free states)

\[ \Omega_{+} \quad \Omega_{-} \]

\[ t_{0} = -\infty \quad 0 \quad t_{0} = \infty \]

\[ \Omega^{+} \]

**Figure 2.2:** Schematic Diagram of Møller operators in time axis

\[ S = \Omega^\dagger_+ \Omega_+ \quad (2.14) \]

So the scattering matrix elements can be written as,

\[ \langle p | S | k \rangle = \langle p^- | k^+ \rangle \quad (2.15) \]

The energy of the scattering eigen-states are same as the free states with same quantum numbers.

### 2.3.1 Input and Output operators

Now we define operators which signifies the quantum fields: the incoming field immediately before the interacting with the atom and the outgoing field just after the interaction. The input operator characterizes the input field and the output operator defines the outgoing field from the atom They are defined as [21],

\[ a^\dagger_{\text{in}}(k)|0\rangle = |k^+\rangle \quad (2.16) \]

\[ a^\dagger_{\text{out}}(p)|0\rangle = |p^-\rangle \quad (2.17) \]

Therefore,

\[ a_{\text{in}}(k) \equiv \Omega_{+} a_k \Omega_{+}^\dagger \quad (2.18) \]

\[ a_{\text{out}}(k) \equiv \Omega_{-} a_k \Omega_{-}^\dagger \quad (2.19) \]

have the property of creating input and output scattering eigen-states. Also,

\[ [a_{\text{in}}(k), a^\dagger_{\text{in}}(p)] = \delta(k - p) \]
\[ [a_{\text{out}}(k), a_{\text{out}}^\dagger(p)] = \delta(k - p) \]

Now using the definition of input field operator,

\[ a_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int dk a_k(t_0)e^{-ik(t-t_0)} \]  \hspace{1cm} (2.20)

where

\[ a_k(t_0) = e^{iH_0 t_0} a_k e^{-iH_0 t_0} \]  \hspace{1cm} (2.21)

Thus,

\[ a_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int dk e^{iH_0 t_0} a_k e^{-iH_0 t_0} e^{-ik(t-t_0)} \]

Therefore,

\[ a_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_{\lim t_0 \to -\infty} dk a_{\text{in}}(k)e^{-ikt} \quad \text{as} \quad [H_0, a_k] = -ka_k \]  \hspace{1cm} (2.22)

and similarly,

\[ a_{\text{out}}(t) = \frac{1}{\sqrt{2\pi}} \int_{\lim t_1 \to \infty} dk a_{\text{out}}(k)e^{-ikt} \]  \hspace{1cm} (2.23)

Thus \( a_{\text{in/out}}(k) \) provides a spectral representation of \( a_{\text{in/out}}(t) \) in the corresponding limits.

### 2.4 Summary

In this chapter we have got acquainted with the mathematical tools necessary for the calculations to follow. We saw the Jaynes-Cummings model and how it was modified to describe the two-level atom-photon coupling Hamiltonian. We also saw how we are going to describe our system by using Scattering Theory with the evolution of input and output states by input and output operators. A real scenario will be treated in the following section with the help of the mathematics in this chapter.
In this chapter we are basically interested in studying the linear responses from an atom, characterized by the transmission and excitation coefficients. So we will consider a single atom coupled to a single mode waveguide. Apart from decaying into the waveguide, the atom also has an additional decay channel by which it decays out of the guide. We consider the additional decay rate as $\Gamma'$.

First from [6] we write the Heisenberg equations of motions without the extra decay channel outside using the equations (2.9), (2.10) and (2.11). Also we use $\tilde{V} = \frac{V}{\sqrt{v_g}}$

\begin{align}
    i \frac{da_k}{dt} &= ka_k + \tilde{V}\sigma_- \tag{3.1} \\
    i \frac{d\sigma_-}{dt} &= \Omega\sigma_- - \tilde{V} \int dk \sigma_z a_k \tag{3.2} \\
    i \frac{d\sigma_z}{dt} &= 2 \tilde{V} \int dk (-a_k^\dagger \sigma_+ + \sigma_+ a_k) \tag{3.3}
\end{align}

Now, plugging in the definition of Input and Output operators from the previous section and establishing a relationship between them, we come to the Input-Output equations,

\begin{align}
    \frac{dN}{dt} &= -i \sqrt{\frac{2}{\tau}} (\sigma_+ a_{in} - a_{in}^\dagger \sigma_-) - \frac{2}{\tau} N \tag{3.4} \\
    \frac{d\sigma_-}{dt} &= -i \sqrt{\frac{2}{\tau}} \sigma_z a_{in} - \frac{1}{\tau} \sigma_- - i \Omega \sigma_- \tag{3.5} \\
    a_{out} &= a_{in} - i \sqrt{\frac{2}{\tau}} \sigma_- \tag{3.6}
\end{align}

where
\[ \tau^{-1} = \frac{\pi V^2}{v_g} \to \text{proportional to spontaneous emission rate}\left(\Gamma = \frac{2}{\tau}\right) \]

\[ N = \sigma_+ \sigma_- = (\sigma_z + 1)/2 \to \text{probability of atom in excited state} \]

Physically this implies output field is the sum of input field with atomic scattering field.

### 3.1 Decay Rates

Now we consider an independent decay rate\((\Gamma')\) outside the waveguide. This decay out of the guide is a non-unitary process and thus an evolution term has to be added to the Hamiltonian previously described. This takes the form of a Liouville’s operator \(L[\rho]\) as shown below,

\[ \dot{\rho} = L[\rho] = -\frac{\Gamma'}{2}(\sigma_+ \rho + \rho \sigma_+ - 2\rho \sigma_\rho \sigma_+) \]

Therefore,

\[ \langle \dot{\sigma}_- \rangle = Tr(\sigma_- \dot{\rho}) \]

\[ = Tr\left[-\frac{\Gamma'}{2}(\sigma_- \sigma_+ \rho + \rho \sigma_+ \sigma_- - \sigma_- \rho \sigma_\rho \sigma_+)\right] \]

\[ = Tr\left[-\frac{\Gamma'}{2} \sigma_- \rho\right] \]

\[ \therefore \quad Tr[\sigma_- \dot{\rho}] = Tr[-\sigma_- \frac{\Gamma'}{2} \rho] \quad (3.7) \]

and,

\[ \langle \dot{\sigma}_z \rangle = Tr(\sigma_z \dot{\rho}) \]

\[ = Tr[-2\Gamma' \sigma_+ \sigma_- \rho - \frac{\Gamma'}{2} \rho] = Tr[-\Gamma' (\sigma_z + 1) \rho] \]

\[ \therefore \quad Tr[\sigma_z \dot{\rho}] = Tr[-\Gamma' (\sigma_z + 1) \rho] \quad (3.8) \]
3.2 Transport Equations

Now plugging the external decay terms from (3.7) and (3.8), into the Input-Output Equations (3.4), (3.5) & (3.6), we get the transport equations for a 2 level atom with external decay channel coupled to a single photon in a waveguide

\[
\frac{dN}{dt} = -i\sqrt{2}\tau (\sigma_+ a_{in} - a_{in}^\dagger \sigma_-) - \frac{2}{\tau}N - \Gamma' N (3.9)
\]

\[
\frac{d\sigma_-}{dt} = i\sqrt{2}\tau \sigma_z a_{in} - \frac{1}{\tau}\sigma_- - i\Omega \sigma_- - \frac{\Gamma'}{2} \sigma_- (3.10)
\]

\[
a_{out} = a_{in} - i\sqrt{2}\tau \sigma_- (3.11)
\]

3.3 Transmission Coefficient

Now, from (3.10)

\[
\frac{d}{dt} \langle 0|\sigma_-|k^+ \rangle = i\sqrt{2}\tau \langle 0|\sigma_z a_{in}|k^+ \rangle - \left( \frac{1}{\tau} + i\Omega + \frac{\Gamma'}{2} \right) \langle 0|\sigma_-|k^+ \rangle (3.12)
\]

and from (3.11)

\[
\langle 0|a_{out}|k^+ \rangle = \langle 0|a_{in}|k^+ \rangle - i\sqrt{2}\tau \langle 0|\sigma_-|k^+ \rangle (3.13)
\]

Now,

\[
\langle 0|a_{in}|k^+ \rangle = \langle 0|a_{in}a_{in}^\dagger|0 \rangle = \frac{1}{\sqrt{2\pi}} e^{-ikt} (3.14)
\]

and

\[
\langle 0|\sigma_z a_{in}|k^+ \rangle = \langle 0|a_{in}|0 \rangle (3.15)
\]

this is because of the single-excitation nature of input state. The result is similar to the approximation for weak excitation limit when atom is assumed to be in ground state for most of the time (when single photon pulse has a duration much longer than spontaneous lifetime of the atom). This approximation is not valid in general. However, this formalism removes the need for this assumption.
Combining the four equations, (3.12),(3.13),(3.14) and (3.15) we get a differential equation of the first order,

\[
\frac{d}{dt}\langle 0|\sigma_-|k^+ \rangle = i\sqrt{\frac{2}{\pi}}\tau \langle 0|a_{\text{in}}|k^+ \rangle - \left(\frac{1}{\tau} + i\Omega + \frac{\Gamma'}{2}\right)\langle 0|\sigma_-|k^+ \rangle
\]

\[
= \frac{i}{\sqrt{2\pi}}\sqrt{\frac{2}{\tau}} e^{-ikt} - \left(\frac{1}{\tau} + i\Omega + \frac{\Gamma'}{2}\right)\langle 0|\sigma_-|k^+ \rangle
\]

solving which we get,

\[
\langle 0|\sigma_-|k^+ \rangle = \frac{i}{\sqrt{2\pi}} e^{-ikt} \frac{\sqrt{2/\tau}}{k - \Omega + i/\tau + i\Gamma'/2}
\]  \hspace{1cm} (3.16)

Similarly,

\[
\langle 0|a_{\text{out}}|k^+ \rangle = \frac{i}{\sqrt{2\pi}} e^{-ikt} - \frac{i}{\sqrt{2\pi}}\tau \frac{i}{\sqrt{2\pi}} e^{-ikt} \frac{\sqrt{2/\tau}}{k - \Omega + i/\tau + i\Gamma'/2}
\]

\[
= \frac{i}{\sqrt{2\pi}} e^{-ikt} \frac{k - \Omega + i/\tau - 2i/\tau + i\Gamma'/2}{k - \Omega + i/\tau + i\Gamma'/2}
\]

\[
= \frac{i}{\sqrt{2\pi}} e^{-ikt} \frac{k - \Omega - i/\tau + i\Gamma'/2}{k - \Omega + i/\tau + i\Gamma'/2}
\]  \hspace{1cm} (3.17)

Now from the relation of input and output operator with the scattering matrix for single photon,

\[
\langle p|S|k \rangle = \langle 0|a_{\text{out}}(p)a_{\text{in}}^\dagger(k)|0 \rangle
\]

\[
= \frac{1}{\sqrt{2\pi}} \int dt \langle 0|a_{\text{out}}(t)|k^+ \rangle e^{ipt}
\]

\[
= t_k \delta(k - p)
\]  \hspace{1cm} (3.18)

\[
t_k \rightarrow \text{single-photon transmission coefficient}
\]

Hence Inverse Fourier Transform of \(\langle 0|a_{\text{out}}(t)|k^+ \rangle\) yields the Single-Photon Scattering Matrix

So from (3.17) and (3.19),

\[
IFT(\langle 0|a_{\text{out}}(t)|k^+ \rangle) = \frac{k - \Omega - i/\tau + i\Gamma'/2}{k - \Omega + i/\tau + i\Gamma'/2} \delta(k - p)
\]  \hspace{1cm} (3.20)

As a result of the above calculation from (3.20) and (3.19) in accordance with the results
from [6]

\[
t_k = \frac{k - \Omega - i/\tau + i\Gamma'/2}{k - \Omega + i/\tau + i\Gamma'/2}
\]

\[
= \frac{k - \Omega - i(\Gamma/2 - \Gamma'/2)}{k - \Omega + i(\Gamma/2 + \Gamma'/2)}
\]

(3.21)

\(\Gamma\) \(\rightarrow\) rate of spontaneous emission into wave-guide

We make a simple plot of the transmittance vs detuning\( (\Delta = k - \Omega) \) taking \(\Gamma'/\Gamma = 5\)

![Graph showing transmittance vs detuning](image)

**Figure 3.1:** The transmittance vs detuning \((k - \Omega)\) showing distinct absorption in the atomic transition frequency as anticipated for linear transmission

We see from Figure 3.1 that there is an transmission minima at the atomic resonance frequency as expected for a linear transmission in a single atom.

We also define,

\[
s_k \equiv \frac{\sqrt{2/\tau}}{(k - \Omega) + i/\tau + i\Gamma'/2}
\]

(3.22)

which measures the excitation coefficient of an atom by single-photon wave when normalized against incident wave amplitude.

Therefore there exists the relation between \(t_k\) and \(s_k\),

\[
t_k = 1 - i\sqrt{\frac{2}{\tau}}s_k
\]

(3.23)
3.4 Excitation probability for scattering eigen-state

The excitation probability for the scattering eigen-state $|k^+\rangle$,

$$
\langle k^+|N|k^+\rangle \quad \text{where} \quad N = \sigma_+\sigma_-
$$

$$
= \langle k^+|\sigma_+\sigma_-|k^+\rangle
$$

$$
= \langle k^+|\sigma_+|0\rangle\langle 0|\sigma_-|k^+\rangle
$$

$$
= \frac{1}{2\pi}|s_k|^2
$$

$|k^+\rangle$ is a single excitation state, and $\sigma_+$ acting on any state except $|0\rangle$ would result in a multi-excitation state resulting in zero overlap with $\langle k^+|$.

So, generally

$$
\langle k^+|\sigma_z(t)|p^+\rangle = \langle k^+|(2\sigma_+\sigma_- - 1)|p^+\rangle
$$

$$
= 2\langle k^+|\sigma_+|0\rangle\langle 0|\sigma_-|p^+\rangle - \delta(k - p)
$$

$$
= \frac{1}{\pi}e^{-i(p-k)t}s^*_ks_p - \delta(k - p) \quad (3.24)
$$

3.5 Summary

In this chapter we took a single atom coupled to a single photon and calculated the transport equations from the Heisenberg equations of motion. From the transport equations we derived the transmission and the excitation coefficient. This chapter is basically a prelude to the two photon coupling chapter. Results from this chapter will be imported to the next one for further calculations.
In this chapter we are going to deal with the transport properties of two-photons coupled with a two-level system. We are going to see that the mathematical calculation is analogical with the single-photon case. We will be using some of the results from the previous sections and in the end get the S-matrix element which this time is going to show both linear and nonlinear contributions as anticipated.

4.1 S-Matrix Element

Now we consider the case of two-photon transport in analogy with the single photon case. So the scattering matrix element as in [6],

\[
\langle p_1 p_2 | S | k_1 k_2 \rangle = \langle p_1 p_2^- | k_1 k_2^+ \rangle = \langle 0 | a_{\text{out}}(p_1) a_{\text{out}}(p_2) a_{\text{in}}^+(k_1) a_{\text{in}}^+(k_2) | 0 \rangle
\]

This equation basically implies that we are inputting two different frequencies \( k_1 \) and \( k_2 \) and as in the one-photon case, measuring the overlap magnitude of the output state \( | p_1 p_2 \rangle \) with the scattered state \( S | k_1 k_2 \rangle \). Now inserting identity operator between \( a_{\text{out}}(p_1) \) and \( a_{\text{out}}(p_2) \)

\[
= \langle 0 | a_{\text{out}}(p_1) \left( \int dk | k^+ \rangle \langle k^+ | a_{\text{out}}(p_2) a_{\text{in}}^+(k_1) a_{\text{in}}^+(k_2) | 0 \rangle \right) \tag{4.1}
\]

\[
= \langle 0 | \left( \int dk a_{\text{out}}(p_1) a_{\text{in}}^+(k_1) | 0 \rangle \langle k^+ | a_{\text{out}}(p_2) a_{\text{in}}^+(k_1) a_{\text{in}}^+(k_2) | 0 \rangle \right) \tag{4.1}
\]

Using Fourier Transform of (3.17)

\[
= t_{p_1} | p_1^\perp \rangle a_{\text{out}}(p_2) a_{\text{in}}^+(k_1) a_{\text{in}}^+(k_2) | 0 \rangle \tag{4.2}
\]
Now using the relation between the input and the output operator from (3.11) in the above equation

\[ = t_{p_1} \langle p_1^+ | a_{in}(p_2) - i \sqrt{\frac{2}{\tau}} \sigma_-(p_2) \rangle (p_2) a_{in}^+(k_1) a_{in}^+(k_2) |0\rangle \]  

(4.3)

Fourier Transforming and using orthogonality of scattering matrix:

\[ = t_{p_1} \delta(p_1 - k_1) \delta(p_2 - k_2) + t_{p_1} \delta(p_1 - k_1) \delta(p_2 - k_1) - i \sqrt{\frac{2}{\tau}} t_{p_1} \langle p_1^+ | \sigma_-(p_2) | k_1 k_2^+ \rangle \]  

(4.4)

So for calculating 2-photon $S$-matrix, we now need to calculate $\langle p_1^+ | \sigma_-(t) | k_1 k_2^+ \rangle$ and do its Fourier transform.

Now, from the transport equation (3.10),

\[ \frac{d \sigma_-}{dt} = i \sqrt{\frac{2}{\tau}} \sigma_z a_{in} - \frac{1}{\tau} \sigma_- - i \Omega \sigma_+ - i \frac{\Gamma'}{2} \sigma_- \]

\[ \frac{d}{dt} \langle p_1^+ | \sigma_-(t) | k_1 k_2^+ \rangle = i \sqrt{\frac{2}{\tau}} \langle p_1^+ | \sigma_z a_{in} | k_1 k_2^+ \rangle - \left( \frac{1}{\tau} + i \Omega + i \frac{\Gamma'}{2} \right) \langle p_1^+ | \sigma_- | k_1 k_2^+ \rangle \]  

(4.5)

(4.6)

Now we simply need to simplify the first part of RHS.

For that we calculate $\langle p_1^+ | \sigma_z(t) a_{in}(t) | k_1 k_2^+ \rangle$ and take the Fourier Transform

As $a_{in}$ is an annihilation operator for scattering states

\[ \langle p_1^+ | \sigma_z(t) a_{in}(t) | k_1 k_2^+ \rangle = \frac{1}{\sqrt{2\pi}} \left[ \langle p_1^+ | \sigma_z(t) | k_2^+ \rangle e^{-ik_1 t} + \langle p_1^+ | \sigma_z(t) | k_1^+ \rangle e^{-ik_2 t} \right] \]

Now, using the result from (3.24)

\[ = \frac{1}{\sqrt{2\pi}} \frac{1}{\pi} e^{-i(k_1+k_2-p_1) t} s_{p_1}^* (s_{k_1} + s_{k_2}) - \frac{1}{\sqrt{2\pi}} \delta(k_2 - p_1) e^{-ik_1 t} - \delta(k_1 - p_1) e^{-ik_2 t} \]  

(4.7)

This is what we were looking for. Now we are in a position to calculate the differential equation, (4.5), plugging in the result from (4.7).

Therefore, solving and taking Fourier transform,

\[ \langle p_1^+ | \sigma_-(p_2) | k_1 k_2^+ \rangle = -\frac{1}{\pi} \delta(k_1 + k_2 - p_1 - p_2) s_{p_2}^* s_{p_1}^* (s_{k_1} + s_{k_2}) + s_{k_1} \delta(k_2 - p_1) \delta(k_1 - p_2) + s_{k_2} \delta(k_1 - p_1) \delta(k_2 - p_2) \]  

\[ + s_{k_1} \delta(k_2 - p_1) \delta(k_1 - p_2) + s_{k_2} \delta(k_1 - p_1) \delta(k_2 - p_2) \]
Lastly, using $t_{p_1}s_{p_1}^* = s_{p_1}$, we obtain

$$\begin{align*}
&\langle 0|a_{\text{out}}(p_1)a_{\text{out}}(p_2)a_{\text{in}}^\dagger(k_1)a_{\text{in}}^\dagger(k_2)|0\rangle \\
&= t_{k_1}t_{k_2}[\delta(k_2 - p_1)\delta(k_1 - p_2) + \delta(k_1 - p_1)\delta(k_2 - p_2)] \\
&+ \frac{i}{\pi}\sqrt{\frac{2}{\tau}}\delta(k_1 + k_2 - p_1 - p_2)s_{p_1}s_{p_2}(s_{k_1} + s_{k_2})
\end{align*}\quad (4.8)$$

The linear part of the S-matrix shows that the photons are transmitted keeping the frequencies conserved with an amplitude given by the product of single photon transmission coefficients for the individual frequencies. In the non-linear part however, it can be seen that the sum of frequencies are conserved, hence, there is the possibility of generation of color different from the inputs as long as the sum is conserved. In contrast to the linear response, the transmission amplitude is not directly the product of some functions of the individual frequencies which signifies the two photons are not remaining separated after the interaction.
4.2 Non-Linear Susceptibility of Atoms

For many macroscopic materials, the non-linear susceptibility is taken as constant with certain incoming frequency in classical theory. And it remains constant for the different set of outcomes from the material.

\[ \chi^{(3)}(\omega_{4}^{out}, \omega_{3}^{out}, \omega_{2}^{in}, \omega_{1}^{in}) \approx \chi^{(3)} \rightarrow \text{const.} \]  

(4.9)

The motivation of this section is to show that the dependence of the quantum non-linear susceptibility of an atom as a function of outgoing frequency is non-trivial in nature. The non-linear susceptibility which is directly related to the non-linear part of the scattering matrix element has intricate dependence with the transition frequency of the atom as a result of which gives a high probability for a specific outcome. The following calculations shows the probabilistic output from a two-level atom for a frequency of input light.

Let us consider a 2-level atom coupled to a single-mode waveguide with a monochromatic light source with detuning \( \Delta \) i.e. the difference between the light frequency and the atomic frequency.

\[ \Gamma_{tot} = \Gamma + \Gamma' \]

\( \Gamma \) → decay in waveguide

\( \Gamma' \) → decay through extra channel

\( \Delta \) → detuning (k-\(\Omega\))

\[|g\rangle \]

Figure 4.1: Interaction of single atom with quantum light with detuning \( \Delta \)

Assuming \( \Delta = k - \Omega \gg \Gamma_{tot} \)

Transmission and Excitation coefficient, from (3.21) and (3.22)

\[ t_{k} = \frac{(k - \Omega) - i(\Gamma/2 + \Gamma'/2)}{(k - \Omega) + i(\Gamma/2 + \Gamma'/2)} \approx 1 \]

\[ s_{k} = \frac{\sqrt{\Gamma}}{(k - \Omega) + i(\Gamma/2 + \Gamma'/2)} \approx \frac{\sqrt{\Gamma}}{\Delta} \rightarrow 0 \]  

(4.10)

We don’t want the photon energy to be near the absorption region of the atom, so we set large values of \( \Delta \) such that the transmission coefficient is large. For simplicity we assume that the two input photons are of same color. So we consider \( |k\rangle \) and \( |k\rangle \) as the two input states. At this instance we consider only the non-linear effect, hence we consider the conservation
of sum of frequencies. So, the output states can be thought of as $|k + \delta\rangle$ and $|k - \delta\rangle$ where $\delta$ is the assumed deviation from the input frequency as a result of non-linear effect.

Now, as we have already seen the second part of 2-photon S-matrix represents the non-linearity, so

$$S_{NL}^{II} = 2s_{p_1}s_{p_2}s_k$$

where $|p_1\rangle = |k - \delta\rangle$ and $|p_2\rangle = |k + \delta\rangle$

$$= 2 \cdot \frac{\sqrt{\Gamma}}{\Delta} \cdot \frac{\sqrt{\Gamma}}{(\Delta + \delta) + i\frac{\Gamma + \Gamma'}{2}} \cdot \frac{\sqrt{\Gamma}}{(\Delta - \delta) + i\frac{\Gamma + \Gamma'}{2}}$$  \hspace{1cm} (4.11)

And

$$\frac{S_{NL}^{II} \Gamma_3^{3/2}}{\Gamma_3^{3/2}} = \frac{2}{\Delta' (\Delta' + \delta' + i/2)(\Delta' - \delta' + i/2)}$$  \hspace{1cm} (4.12)

where, $\Delta' = \Delta/\Gamma_{tot}$, $\delta' = \delta/\Gamma_{tot}$ and $\Gamma_{tot} = \Gamma + \Gamma'$

Now we plot $S_{NL}^{II}$ vs $\delta'$

![Figure 4.2: Plot of 2-photon S-matrix element vs non-linear deviation](image)

Clearly from the Figure 4.2, there is a maxima of probability for non-linear process when $\delta = \Delta$. So one photon is expected to come out at the resonance frequency of the atom and
the other at some well defined off-resonance frequency.

The non-linear output states are $|\Omega\rangle$ and $|2k - \Omega\rangle$, which conserves the sum of frequencies. 

So we arrive at a conjecture that need to be considered in the future. Following the same logic, for an array of atoms, $|\Omega\rangle$ is absorbed in short distances, as without detuning the atom is mostly absorptive, whereas $|2k - \Omega\rangle$ emerges as the output state due to low absorption at off-resonance frequencies.

4.3 Summary

Following the calculations from the previous chapter, we established that the two-photon scattering involves linear as well as non-linear effects as opposed to the previous case. The non-linear part of the scattering matrix implies the non-linear susceptibility of the atom. So we made a dependence calculation of the non-linear effect on output from the atom for any incoming frequency. Quite fascinatingly, the dependence, far from the absorption region, was very sharp implying atomic non-linear response is very precise for any incoming frequency, which is counter-intuitive to the typical theories of non-linear optics.
Perhaps the greatest potential application of quantum non-linear optics is in the field of quantum computing. The interaction of few photons and atom forms the very basics of quantum computation. Quantum computers like ordinary uses logic gates but at quantum level. These logic gates are intrinsically non-linear in nature.

5.1 On Quantum Computing

In this section I’ll discus very briefly an example illustrating how non-linear optics is actually implemented in quantum logic. One of the very basic logical operation in quantum logic circuits is the C-Phase gate. It looks like the following,

\[
\begin{align*}
|0_A, 0_B\rangle &\rightarrow |0_A, 0_B\rangle \\
|0_A, 1_B\rangle &\rightarrow |0_A, 1_B\rangle \\
|1_A, 0_B\rangle &\rightarrow |1_A, 0_B\rangle \\
|1_A, 1_B\rangle &\rightarrow -|1_A, 1_B\rangle 
\end{align*}
\]

The inputs are in a product state of two-photonic states where 0 represents absence of photons and 1 represents presence of it. The first three statement may look trivial but in the fourth line the presence of the negative sign implies a phase of \(\pi\) being introduced when there is one photon in each of the channels. This is strictly a non-linear operation. technically speaking any phase(\(\phi \neq 0\)) in the fourth case is allowable, until and unless it is the product of the second and third phase, under which the gate behaviour is linear. Figure 5.1 shows schematically the gate behavior for unit input in both channels.
The two-qubit C-Phase and single qubit rotation forms the universal set of gates necessary for quantum computation i.e. any operation on a quantum computer can be reduced to a circuit using only these two operations. This is somewhat similar to NAND and NOR gates in the classical counterpart. So implementation of C-Phase gate is sufficient to verify our system’s ability for quantum computations.

5.2 Eigen-states of the S-Matrix and Bound States

We take the two-level atom as in the previous case and let it interact with two photons. Now we want to find the eigen-states of the S-matrix and see whether such a system is suitable for quantum logics in general. For calculating the eigen-states we take an input state, let it pass through the scattering region and see calculate the output state. Now if this output state is equal to the input state with some added phase factors, we can say the input state was an eigen-state of the system.

Let us assume the input state of the following form,

$$\Psi_{in}(x_1, x_2) = e^{iq_{cm}(x_1+x_2)} e^{-\alpha|x_1-x_2|}$$

where $q_{cm}$ and $\alpha$ are coefficients defining the positions of the two photons with respect to their centre of mass and relative separation respectively. This is just a trial solution for the eigen-state problem. So in k-space the input state looks like the following,

$$\Psi_{in}(k_1, k_2) = \int dx_1 dx_2 \, \Psi_{in}(x_1, x_2) e^{ik_1x_1+ik_2x_2}$$

Now, after the input state passes through the scattering region, we get the output state from

$$\Psi_{out}(p_1, p_2) = \int dk_1 dk_2 \, S(p_1, p_2, k_1, k_2) \Psi_{in}(k_1, k_2)$$
where $S(p_1, p_2, k_1, k_2)$ is the scattering matrix element calculated previously. All the integrations are from $-\infty$ to $\infty$. Now for $\Psi$ to be an eigen-state of the system, the following condition must hold.

$$\Psi_{\text{out}}(p_1, p_2) = \Psi_{\text{in}}(p_1, p_2) \cdot f()$$

(5.4)

where $f()$ is an arbitrary function representing the phase introduced by scattering. Since the wave-functions are normalized, the magnitude of $f$ is equal to unity. There is one more thing which we need to keep in mind before going in the calculation part. Previously we have introduced a generalized decay term outside the channel, which implies loss outside the system. In that case the magnitude of $f$ can never reach unity as the output function from the scattering region will suffer loss of amplitude due to external decay factor. So in this case we take our atom to be ideal and the external decay factor is eliminated. Hence once again we can now assume without ambiguity that, $|f()| = 1$.

Let’s now calculate the eigen-state of the $S$-matrix using the above protocol.

$$\Psi_{\text{in}}(k_1, k_2) = \int dx_1 dx_2 e^{iq_{\text{cm}}(x_1+x_2)} e^{-\alpha|x_1-x_2|} e^{ik_1 x_1 + ik_2 x_2}$$

Dividing the integration into two separate parts, $x_1 \geq x_2$ and $x_1 < x_2$

$$\Psi_{\text{in}}^{(1)}(k_1, k_2) = \int_{x_1 \geq x_2} dx_1 dx_2 e^{iq_{\text{cm}}(x_1+x_2)} e^{-\alpha(x_1-x_2)} e^{ik_1 x_1 + ik_2 x_2}$$

And

$$\Psi_{\text{in}}^{(2)}(k_1, k_2) = \int_{x_2 > x_1} dx_1 dx_2 e^{iq_{\text{cm}}(x_1+x_2)} e^{\alpha(x_1-x_2)} e^{ik_1 x_1 + ik_2 x_2}$$

Now, we make a change of variables for the first part of $\Psi_{\text{in}}$:

Let $x_1 - x_2 = x_{\text{rel}}$ and $x_1 + x_2 = x_{\text{cm}}$. So the Jacobian,

$$J(x_{\text{rel}}, x_{\text{cm}}) = \begin{vmatrix} \frac{\partial x_1}{\partial x_{\text{rel}}} & \frac{\partial x_1}{\partial x_{\text{cm}}} \\ \frac{\partial x_2}{\partial x_{\text{rel}}} & \frac{\partial x_2}{\partial x_{\text{cm}}} \end{vmatrix} = 2$$

Since the limits of $x_1 : -\infty \to \infty$ and $x_2 : -\infty \to x_1$ implies $x_{\text{cm}} : -\infty \to \infty$ and $x_{\text{rel}} : 0 \to \infty$

Therefore,

$$\Psi_{\text{in}}^{(1)}(k_1, k_2) = 2 \int_{-\infty}^{\infty} dx_{\text{rel}} dx_{\text{cm}} e^{-(\alpha-i\frac{k_1-k_2}{2})x_{\text{rel}}} e^{(q_{\text{cm}}+\frac{k_1+k_2}{2})x_{\text{cm}}}$$

$$= \frac{2\delta(q_{\text{cm}}+\frac{k_1+k_2}{2})}{\alpha-i\frac{k_1-k_2}{2}}$$

(5.5)
Similarly, for the second part we take $x_2 - x_1 = x_{\text{rel}}$ and $x_1 + x_2 = x_{\text{cm}}$. The Jacobian remains the same and so does the limits. Therefore,

$$
\Psi^{(2)}_{in}(k_1, k_2) = 2 \int_0^{\infty} \int_{-\infty}^{\infty} dx_{\text{rel}} dx_{\text{cm}} \ e^{-(\alpha - i \frac{k_2 - k_1}{2})x_{\text{rel}}} \ e^{i(q_{\text{cm}} + \frac{k_1 + k_2}{2})x_{\text{cm}}}
$$

$$
= \frac{2 \delta(q_{\text{cm}} + \frac{k_1 + k_2}{2})}{\alpha + i \frac{k_1 - k_2}{2}} \quad (5.6)
$$

Hence from (5.5) and (5.6) we have,

$$
\Psi_{in}(k_1, k_2) = \Psi^{(1)}_{in}(k_1, k_2) + \Psi^{(2)}_{in}(k_1, k_2)
$$

$$
= 2 \delta \left( q_{\text{cm}} + \frac{k_1 + k_2}{2} \right) \left[ \frac{1}{\alpha - i \frac{k_1 - k_2}{2}} + \frac{1}{\alpha + i \frac{k_1 - k_2}{2}} \right]
$$

$$
= 4 \delta \left( q_{\text{cm}} + \frac{k_1 + k_2}{2} \right) \frac{\alpha}{\alpha^2 + \left( \frac{k_1 - k_2}{2} \right)^2} \quad (5.7)
$$

Now we have the input state for the system in k-space. We will now substitute (5.7) in (5.3) to derive the output state from the scattering region. So,

$$
\Psi_{out}(p_1, p_2) = \int \int dk_1 dk_2 S(p_1, p_2, k_1, k_2) 4 \delta \left( q_{\text{cm}} + \frac{k_1 + k_2}{2} \right) \frac{\alpha}{\alpha^2 + \left( \frac{k_1 - k_2}{2} \right)^2}
$$

From (4.8) we have the value of the S-matrix element. The element has two parts: linear and a non-linear part. So for mathematical simplicity we divide the integral in two corresponding parts.

The non-linear part

$$
\Psi^{NL}_{out}(p_1, p_2)
$$

$$
= \int \int dk_1 dk_2 \ \frac{4i \alpha}{\pi} \sqrt{\frac{2}{\tau}} \delta(k_1 + k_2 - p_1 - p_2) s_{p_1} s_{p_2} \left( s_{k_1} + s_{k_2} \right) \frac{\delta \left( q_{\text{cm}} + \frac{k_1 + k_2}{2} \right)}{\alpha^2 + \left( \frac{k_1 - k_2}{2} \right)^2} \quad (5.8)
$$

And the linear part

$$
\Psi^{L}_{out}(p_1, p_2)
$$

$$
= \int \int dk_1 dk_2 \ \left[ \delta(k_2 - p_1) \delta(k_1 - p_2) + \delta(k_1 - p_1) \delta(k_2 - p_2) \right] t_{k_1} t_{k_2}.
$$

$$
4 \cdot \delta \left( q_{\text{cm}} + \frac{k_1 + k_2}{2} \right) \left[ \frac{\alpha}{\alpha^2 + \left( \frac{k_1 - k_2}{2} \right)^2} \right] \quad (5.9)
$$
Analogically from previous integral, we take $k_1 - k_2 = k_{rel}$ and $k_1 + k_2 = k_{cm}$. Thus it follows that the Jacobian $J = 2$ and limits: $k_{rel} : -\infty \rightarrow \infty$ and $k_{cm} : -\infty \rightarrow \infty$

So from (5.8) and (3.22), we have

$$
\Psi_{NL}^{out}(p_1, p_2) = \frac{16i s_p s_p \alpha}{\pi \tau} \int \int dk_{rel} dk_{cm} \delta(k_{cm} - p_1 - p_2) \cdot \delta \left( q_{cm} + \frac{k_{cm}}{2} \right) \cdot \left( \frac{1}{\alpha^2 + \left(\frac{k_{rel}}{2}\right)^2} \right)
$$

$$
\cdot \left( \frac{1}{k_{cm} + k_{rel} - \Omega + i/\tau} + \frac{1}{k_{cm} - k_{rel} - \Omega + i/\tau} \right)
$$

Now, using property of Dirac delta function, and putting in the equation

$$
\int dk \ \delta(k - A) \cdot \delta(k + B) \cdot f(k) = \delta(A + B) \int dk \ \delta(k - A) \cdot f(k) = \delta(A + B) f(A)
$$

Hence,

$$
\Psi_{NL}^{out}(p_1, p_2) = \frac{16i s_p s_p \alpha \delta \left( q_{cm} + \frac{p_1 + p_2}{2} \right)}{\pi \tau} \int dk_{rel} \left( \frac{1}{p_1 + p_2 + k_{rel} - \Omega + i/\tau} \right)
$$

$$
+ \frac{1}{p_1 + p_2 - k_{rel} - \Omega + i/\tau} \right) \cdot \left( \frac{1}{\alpha^2 + \left(\frac{k_{rel}}{2}\right)^2} \right)
$$

This integral is solved in Mathematica and the expression simplified. The result is as follows

$$
\Psi_{NL}^{out}(p_1, p_2) = \frac{64s_p s_p \delta \left( q_{cm} + \frac{p_1 + p_2}{2} \right)}{\tau \left[ \alpha - i \left( \frac{p_1 + p_2}{2} - \Omega + i/\tau \right) \right]}
$$

(5.10)
And for the linear part from (5.9) and (3.21)

\[
\Psi_{\text{out}}^{L}(p_1, p_2) = \int \int dk_1 dk_2 \left[ \delta(k_2 - p_1)\delta(k_1 - p_2) + \delta(k_1 - p_1)\delta(k_2 - p_2) \right] \cdot t_{k_1}t_{k_2} \cdot \\
4 \cdot \delta \left( q_{\text{cm}} + \frac{k_1 + k_2}{2} \right) \left[ \frac{\alpha}{\alpha^2 + \left( \frac{k_1 - k_2}{2} \right)^2} \right] \\
= 8\alpha \int \int dk_{\text{rel}}dk_{\text{cm}} \left[ \delta \left( \frac{k_{\text{cm}} - k_{\text{rel}}}{2} - p_1 \right) \cdot \delta \left( \frac{k_{\text{cm}} + k_{\text{rel}}}{2} - p_2 \right) \right. \\
\left. + \delta \left( \frac{k_{\text{cm}} + k_{\text{rel}}}{2} - p_1 \right) \cdot \delta \left( \frac{k_{\text{cm}} - k_{\text{rel}}}{2} - p_2 \right) \right] \cdot \frac{\delta \left( q_{\text{cm}} + \frac{k_{\text{cm}}}{2} \right)}{\alpha^2 + \left( \frac{k_{\text{rel}}}{2} \right)^2} \cdot t_{k_1}t_{k_2} \\
= 8\alpha \int \int dk_{\text{rel}}dk_{\text{cm}} \left[ \delta(k_{\text{rel}} - p_2 + p_1) \cdot \delta(k_{\text{cm}} - p_2 - p_1) \right. \\
\left. + \delta(k_{\text{rel}} - p_1 + p_2) \cdot \delta(k_{\text{cm}} - p_2 - p_1) \right] \cdot \frac{\delta \left( q_{\text{cm}} + \frac{k_{\text{cm}}}{2} \right)}{\alpha^2 + \left( \frac{k_{\text{rel}}}{2} \right)^2} \\
\cdot \frac{k_{\text{cm}} + k_{\text{rel}}}{2} - \frac{k_{\text{cm}} - k_{\text{rel}}}{2} - \Omega - i/\tau \\
\cdot \frac{k_{\text{cm}} + k_{\text{rel}}}{2} - \frac{k_{\text{cm}} - k_{\text{rel}}}{2} - \Omega + i/\tau \\
\cdot \frac{k_{\text{cm}} + k_{\text{rel}}}{2} - \Omega - i/\tau \\
\cdot \frac{k_{\text{cm}} + k_{\text{rel}}}{2} - \Omega + i/\tau \\
\right)
\]

Using the properties of Dirac delta function as in the previous case we get the linear part of the output state,

\[
\Psi_{\text{out}}^{L}(p_1, p_2) = 16\alpha \delta(p_1 + p_2 + 2q_{\text{cm}}) \left[ \frac{1}{\alpha^2 + \left( \frac{p_1 + p_2}{2} \right)^2} \cdot \frac{(p_1 - \Omega - i/\tau)(p_2 - \Omega - i/\tau)}{(p_1 - \Omega + i/\tau)(p_2 - \Omega + i/\tau)} \right] \\
\]

(5.11)
Therefore,

$$\Psi_{out}(p_1, p_2)$$

$$= \left[ \frac{64(2/\tau) \cdot \delta (q_{cm} + \frac{p_1 + p_2}{2})}{\tau (\alpha - i (\frac{p_1 + p_2}{2} - \Omega + i/\tau))} \right.$$  

$$+ \frac{16\alpha \delta(p_1 + p_2 + 2q_{cm})(p_1 - \Omega - i/\tau)(p_2 - \Omega - i/\tau)}{\alpha^2 + (\frac{p_1 - p_2}{2})^2} \right]$$  

$$\cdot \left( \frac{1}{(p_1 - \Omega + i/\tau)(p_2 - \Omega + i/\tau)} \right)$$

$$= \frac{32 \delta (q_{cm} + \frac{p_1 + p_2}{2})}{(p_1 - \Omega + i/\tau)(p_2 - \Omega + i/\tau)} \left[ \frac{4}{\tau^2 [\alpha - i (\frac{p_1 + p_2}{2} - \Omega + i/\tau)]} \right.$$

$$\left. + \frac{\alpha(p_1 - \Omega - i/\tau)(p_2 - \Omega - i/\tau)}{\alpha^2 + (\frac{p_1 - p_2}{2})^2} \right] \right} \right]$$

(5.12)

Now from the protocol we are using, we need to calculate $f()$ and establish that its magnitude equal to unity. From that equation we wish to find out the value of alpha, for which our assumed input state is an eigen-state of the system. Also we want to analyze the nature of $f()$, which basically defines the eigen-value, and see whether such an eigen-value is suitable for complete isolation of the two photons in output state.

Therefore,

$$|f()| = \left| \frac{\Psi_{out}}{\Psi_{in}} \right| = \left| \frac{\Psi_{out}}{|\Psi_{in}|} \right| = 1$$
Now we calculate the value of $f()$ from (5.12) and (5.7).

$$f() = \frac{\Psi_{out}}{\Psi_{in}}$$

$$= \frac{32^8 \delta (q_{cm} + \frac{p_1 + p_2}{2})}{(p_1 - \Omega + i/\tau)(p_2 - \Omega + i/\tau)} \cdot \left\{ \frac{4 \left[ \alpha^2 + \left( \frac{p_1 - p_2}{2} \right)^2 \right] + \alpha^2 (p_1 - \Omega + i/\tau)(p_2 - \Omega + i/\tau) \left[ \alpha - i \left( \frac{p_1 + p_2}{2} - \Omega + i/\tau \right) \right]}{\tau^2 \left[ \alpha - i \left( \frac{p_1 + p_2}{2} - \Omega + i/\tau \right) \right] \left[ \alpha^2 + \left( \frac{p_1 - p_2}{2} \right)^2 \right]} \right\} \cdot \frac{\alpha^2 + \left( \frac{p_1 - p_2}{2} \right)^2}{\delta (q_{cm} + \frac{p_1 + p_2}{2})}$$

$$= \frac{32 \left[ \alpha^2 + \left( \frac{p_1 - p_2}{2} \right)^2 \right] + 8 \alpha \tau^2 (p_1 - \Omega - i/\tau)(p_2 - \Omega - i/\tau) \left[ \alpha + 1/\tau - i \left( \frac{p_1 + p_2}{2} - \Omega \right) \right]}{\alpha \tau^2 (p_1 - \Omega + i/\tau)(p_2 - \Omega + i/\tau) \left[ \alpha + 1/\tau - i \left( \frac{p_1 + p_2}{2} - \Omega \right) \right]}$$

Now this expression is too complex to be solved in Mathematica or by hand, so we make a few logical assumptions to make the problem a bit simpler.

Assumptions,

$$p_1 = -q_{cm} + \delta \quad \& \quad p_2 = -q_{cm} - \delta$$

(5.13)

This is justified as $q_{cm} = -\frac{p_1 + p_2}{2}$ from the previous calculations. The $\delta$ here represents the difference in frequency between the two input photons. Then,

$$\Omega = 0$$

(5.14)

$\Omega$ is the atomic transition frequency. Here in this equation this is just an offset value, so it can be removed without disturbing the functional form. And, finally putting

$$\alpha = \Gamma$$

(5.15)

The spectral spread of the atomic levels are $\Gamma$. So we take a trial solution of $\alpha$ equal to that. we will see whether our guess gives us the desired result or not.
Using the assumptions stated earlier the value of $\Psi_{out}$ becomes (solved using Mathematica),

$$\Psi_{out} = \frac{4\Gamma}{\Gamma^2 + \delta^2} \cdot \frac{2\Gamma - iq_{cm}}{2\Gamma + iq_{cm}}$$

Now this has the same form as the $\Psi_{in}$ using the assumptions. Hence

$$f() = \frac{-q_{cm} - 2i\Gamma}{-q_{cm} + 2i\Gamma}$$

Hence it is evident that the value of $|f()|$ is equal to unity. Now this can be written as a phase factor in the form of $e^{i\phi}$. Therefore

$$e^{i\phi} = \frac{-q_{cm} - 2i\Gamma}{-q_{cm} + 2i\Gamma} \neq e^{i\phi_1} \cdot e^{i\phi_2}$$

where $\phi_1 = g(p_1)$ and $\phi_2 = g(p_2)$.  

So, it is clear that the output is in a bound state as the function can’t be isolated for the individual photons. The bound state has a spatial extent of $1/\Gamma$ and behaves as an single composite particle effectively with energy $p_1 + p_2$ and remains un-isolated after passing through the atom. Hence such a system is not suitable for the purpose of non-linear quantum computations. This is what an attempted C-Phase gate output looks like for the case of a two-level atom, Figure 5.2.

![Figure 5.2: C-Phase output with two-level atom can no longer be realised due to entangling of the output states with one another implying two-level atom is useless for quantum logic.](image)
5.3 A Possible Solution

Let us consider a three level system and see whether it can be used for C-Phase logic. The following analysis is schematic in nature. Let us consider a three-level atom with the lower transition coupling to frequency \( A \) and the upper transition to \( B \). Now we try to use the same logical inputs as in the previous case and look at the output states. Following figure shows schematically what happens.

![Diagram](image)

**Figure 5.3:** C-Phase-like logic realized with 3-level system showing the 4 possible outcomes: \( \Gamma_{32}: \text{fast and } \Gamma_{21}: \text{slow (} \Gamma_{32} \gg \Gamma_{21} \text{)} \)

In the scheme shown in Figure 5.3, the output is not exactly like the C-Phase logic, but very similar to it. If \( \Gamma_{21} \) is a slow decay with respect to \( \Gamma_{32} \), then the above results are valid. For (a) \(|00\rangle\) state the output is same as input as there is no photon to interfere. For the (b) \(|01\rangle\) input state, the atom excites and re-emits from the A transition, so a phase of \( \pi \ (t_A = -1) \) is introduced. For the (c) \(|10\rangle\) input the photon is not coupled to the atom because it is in the ground state and there is no excitation to get it to the first excited level. Lastly for the (d) \(|11\rangle\) input, the A-photon couples to the A-transition first. Now if the first excited state has a long enough lifetime, the B-photon couples with the B-transition. Hence a total phase of \( \pi + \pi = 2\pi \ (t_A \cdot t_B = 1) \) is introduced for both transitions and the output phase-factor is unity. This behaviour is similar to the ideal C-Phase output (output phase-factor for (b) \(|01\rangle\) and (d) \(|11\rangle\) are different) as shown in the beginning of this chapter. Thus we once again arrive at yet another conjecture which states that the scheme with multi-level atom having dissimilar decay rates can be used to construct a quantum logic gate. This is worth a quantitative study with S-matrix method in the future.
5.4 Summary

In this chapter we got acquainted with a very basic quantum logic using non-linear optics. Then we went on to derive the eigenstate of the two-photon-atom interaction S-matrix to show that there exists a bound state associated with the output state. The presence of such a bound state makes the system no good for the purpose of quantum logic. Hence a two level atom can’t be used in nonlinear quantum logic circuits. So we need to find other systems which can be used in quantum logics. Thus we introduce ourselves to the final part of the chapter, to a kind of scheme under which the output state is no longer a bound state and hence can very well be used in logical operation.
As a overview of this thesis, we have seen quite a few different things. Firstly, that the approximation of third order nonlinearity as a constant with output frequencies for an atom is a gross misinterpretation. Secondly, a two-level system is no good for real quantum circuits and lastly, the kind of systems we actually need for quantum logic. Although calculating transport properties of single atom seems to be impractical for realistic tasks, this thesis can be thought as a preamble to two entirely different train of future thoughts.

The calculations are very general in nature and can be used to calculate the dynamics of systems creating large, tunable optical nonlinearities by strongly correlated quantum gas of photons using one-dimensional optical systems with tight field confinement and coherent photon trapping techniques and using many-atom schemes like EIT [22].

Also, extending the calculations in the chapter regarding two-photon transport to multi-level schemes and combining the ideas for quantum logic pave the way for calculation of transport phenomenon for more systems such as in single photon transistors using three level systems [2].


[3] The Debut of LIGO II. 
http://www.ligo.caltech.edu/LIGO_web/9911news/9911two.html#Article_1.

http://www.ligo.caltech.edu/.


