Title: Solution of implicit differential equations using desingularization theory

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Solution of implicit differential equations using desingularization theory

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# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contents</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>Introduction</td>
</tr>
<tr>
<td>2</td>
<td>Analytic geometry</td>
</tr>
<tr>
<td>2.1</td>
<td>Analytic manifolds</td>
</tr>
<tr>
<td>2.2</td>
<td>Vector bundles</td>
</tr>
<tr>
<td>3</td>
<td>Linearly singular systems</td>
</tr>
<tr>
<td>3.1</td>
<td>Linearly singular systems</td>
</tr>
<tr>
<td>3.2</td>
<td>Morphisms of linearly singular systems</td>
</tr>
<tr>
<td>4</td>
<td>Semialgebraic, semianalytic and subanalytic sets</td>
</tr>
<tr>
<td>4.1</td>
<td>Semialgebraic subsets</td>
</tr>
<tr>
<td>4.2</td>
<td>Semianalytic sets</td>
</tr>
<tr>
<td>4.3</td>
<td>Subanalytic sets</td>
</tr>
<tr>
<td>5</td>
<td>Desingularization theorem and applications</td>
</tr>
<tr>
<td>5.1</td>
<td>Transforming analytic functions to normal crossing using blow-ups</td>
</tr>
<tr>
<td>5.2</td>
<td>Desingularization theorem</td>
</tr>
<tr>
<td>5.3</td>
<td>Subanalytic curves</td>
</tr>
<tr>
<td>6</td>
<td>Solutions</td>
</tr>
<tr>
<td>7</td>
<td>Algorithm</td>
</tr>
<tr>
<td>7.1</td>
<td>Recursive step</td>
</tr>
<tr>
<td>7.2</td>
<td>Discussion</td>
</tr>
<tr>
<td>7.3</td>
<td>Implementation notes</td>
</tr>
<tr>
<td>8</td>
<td>Examples</td>
</tr>
<tr>
<td>A</td>
<td>Tarski–Seidenberg theorem</td>
</tr>
<tr>
<td>A.1</td>
<td>Sturm theorem</td>
</tr>
<tr>
<td>A.2</td>
<td>Real roots satisfying inequalities</td>
</tr>
<tr>
<td>A.3</td>
<td>Tarsi–Seidenberg theorem</td>
</tr>
<tr>
<td>B</td>
<td>Weierstrass’ preparation theorem</td>
</tr>
<tr>
<td>B.1</td>
<td>Analytic functions: definitions and some results</td>
</tr>
<tr>
<td>B.2</td>
<td>Weierstrass’ preparation theorem</td>
</tr>
<tr>
<td>C</td>
<td>Implementation of the algorithm</td>
</tr>
<tr>
<td>References</td>
<td>81</td>
</tr>
</tbody>
</table>
1 Introduction

Explicit and implicit differential equations

A differential equation is a relation which involves some independent variables and some depen-
dent variables (functions whose parameters are the independent variables) and their derivatives
with respect to the independent ones. Differential equations arise in many areas of science and
technology, such as physics, chemistry, biology or economics. For instance, Newton’s second
law,
\[ F = \frac{dp}{dt}, \]
states that the total net force \( F \) applied on a particle is equal to the infinitesimal change of its
linear momentum \( p \).

More notable equations are Hamilton’s equations for classical mechanics [AM78],
\[ \dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \]
where \( \dot{x} = \frac{dx}{dt} \) and the Hamiltonian \( H \) describes the energy of the system; Schrödinger equation
for quantum mechanics [GP78],
\[ i\hbar \frac{\partial}{\partial t} \Psi = \hat{H} \Psi, \]
where \( \Psi \) is the wave function of the quantum system, \( \hbar \) the reduced Plank constant and \( \hat{H} \) the
Hamiltonian operator of the system; Black-Scholes equation [BS73],
\[ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \]
which describes the evolution of price of options; or the logistic equation,
\[ \frac{d}{dt} P = P(1 - P), \]
which describes the behavior of growth of some population \( P \).

When a differential equation has only one independent variable, it is called an ordinary differ-
ential equation (ODE). In this case, using the notation \( x^{(n)} = \frac{d^n x}{dt^n} \), an ODE can be written as
an expression of the kind
\[ F(t, x, \dot{x}, \ldots, x^{(n)}) = 0, \]
where \( x \in \mathbb{R}^m \) and \( F : \mathbb{R}^{m(n+1)+1} \to \mathbb{R}^k \) for some \( k \geq 1 \). An ODE of this form is usually
called an implicit differential equation. If \( F \) does not depend on the time \( t \), then \( F \) is called an
autonomous system.

An ODE of order \( n \) can be transformed into a first order ODE
\[ G(t, y, \dot{y}) = 0, \]
by the introduction of some new dependent variables:
\[ y_i = x^{(i)}, \quad i = 0, \ldots, n - 1; \]
then (1) is equivalent to the first order system

\[
\begin{aligned}
y_1 \dot{y}_0 &= 0 \\
\vdots \\
y_{n-1} \dot{y}_{n-2} &= 0 \\
F(t, y_0, y_1, \ldots, y_{n-1}, \dot{y}_{n-1}) &= 0.
\end{aligned}
\]

If it is possible to isolate the derivatives, we can write the differential equation as an explicit ODE: for a first order equation this means

\[\dot{x} = f(t, x),\] (3)

where \(x \in \mathbb{R}^m\) and \(f : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^m\). In the autonomous case, it is an equation of the form \(\dot{x} = f(x)\).

If \(f\) is continuous, and locally Lipschitz with respect to \(x\), then Picard’s theorem \([TP85]\) says that, fixed an initial condition \((t_0, x_0)\), there exists a unique (maximal) solution of (3) such that \(x(t_0) = x_0\). This hypothesis holds, for instance, if \(f\) is a \(C^1\) function.

Implicit differential equations arise in a natural way in many problems. Very often these equations take the particular form

\[A(x)\dot{x} = b(x)\] (4)

where \(x \in \mathbb{R}^m\), \(A(x)\) is an \(n \times m\) matrix and \(b(x)\) is an \(n\)-dimensional vector. We will call such equations linearly singular systems (LSS) \([GP92]\), an these will be the main object of study in this thesis.

In fact, any autonomous implicit differential equation can be written as a system of the form (4). Let \(F(x, \dot{x}) = 0\) be any first order autonomous implicit differential equation. As we did before, we can introduce a new variable \(u = \dot{x}\), so that the implicit equation is equivalent to the linearly singular system

\[
\begin{aligned}
\dot{x} &= u \\
0 &= F(x, u).
\end{aligned}
\]

Therefore, even if we will restrict ourselves to this particular kind of systems, this theories can be applied easily to any ODE.

**Example.** Consider the system

\[
\begin{pmatrix}
1 & x_1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix}
= \begin{pmatrix}
f_1(x_1, x_2) \\
f_2(x_1, x_2)
\end{pmatrix}
\]

Since \(A(x)\) is invertible, this system is equivalent to the explicit ODE

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix}
= \begin{pmatrix}
1 & -x_1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
f_1(x_1, x_2) \\
f_2(x_1, x_2)
\end{pmatrix}
= \begin{pmatrix}
f_1(x_1, x_2) - x_1 f_2(x_1, x_2) \\
f_2(x_1, x_2)
\end{pmatrix}.
\]

**Example.** The system

\[(x_1^2 - x_2^2)\dot{x}_1 = e^{x_1 x_2}\]

cannot be written in explicit form.
As stated for instance in [GMR04], singular differential equations have many applications. For example, in theoretical physics a Lagrangian \( L(t, q^i, \dot{q}^i) \) is singular whenever the Hessian matrix \( \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \) is singular; such Lagrangians are widely used in relativistic theories. Another example can be found in control theory, where some simple models can be formulated as

\[
E \dot{x} = Ax + Bu, \quad y = Cx + Du,
\]

where \( A, B, C, D, E \) are constant matrices and \( E \) is singular. There are more applications in circuit theory, industrial engineering, biology and econometry. The existence of these many applications makes interesting to study singular differential equations.

Solving implicit differential equations

Our goal in this work is to give an algorithmic method to find solutions of a LSS. To accomplish this objective, we will give a geometric description of the problem and use some geometric tools to transform the equation and the manifold (the “ambient space”) into a new set of equations, each one being an explicit ODE. When this is accomplished, one may apply Picard’s theorem to these new systems to obtain the solutions.

It is important to notice that (as many other theoretical results) Picard’s theorem proves the existence of a solution but does not give an explicit method to find it. Usually, in problems which require to find explicit solutions, numerical methods are used. There also exist numerical methods for solving LSS, so one may think that (since we will use numerical methods anyway to find the explicit solution) we could use them directly instead of wasting time implementing a method that will still depend on numerical approximations. The main point of using alternative methods for this kind of problem is (beyond the theoretical value of the algorithm) that, while numerical methods for explicit ODEs are usually accurate (in the sense that it is easy to control the error of the calculations), numerical approximations for an arbitrary LSS require (as it is explained in [RR02]) numerical inversions of quasi-singular matrices; these operations are numerically unstable. Therefore, the desingularization procedure studied here may allow to transform a numerically unstable problem into a new stable one.

There is some previous work on this topic. For instance, we can find an algebraic approach in [KM06] or a geometrical one in [GP92]. They are, in some sense, equivalent and require some regularity conditions to be checked at each step of the algorithm and, if they are not fulfilled, the algorithms do not solve the problem everywhere. Moreover, checking these regularity conditions is not easily implementable. One can try to overcome [CE06] those problems by working in the analytic \((C^\omega)\) category instead of the smooth \((C^\infty)\) category and applying desingularization theory [BM88] to the subsets obtained in the constraint algorithm. In this way, the singular set can be “changed” by a smooth (analytic) manifold. In some sense, this algorithm may have a high practical value.

Analytic functions are a small subset of smooth functions, so it may seem a very restrictive condition. However most of the explicit functions we usually work with are analytic. Some examples of analytic functions are polynomials, rational and trigonometric functions, exponentials, roots and any linear combination, multiplication, division or composition of them. As an
example of a smooth non-analytic function, we have

\[ f(x) = \begin{cases} 
  e^{-1/x} & \text{if } x > 0 \\
  0 & \text{if } x \leq 0 \end{cases} \]

Contents

Section 2 Analytic geometry In this section we recall some basic properties of analytic differential geometry. We explain some of its similarities and differences with smooth differential geometry and give some examples. Some basic concepts about fiber bundles are also included in this section.

Section 3 Linearly singular systems Following [GP92], a geometrical description of linearly singular systems is presented using vector bundles. Moreover, it is explained how to, given a smooth map \( f : N \to M \) and a LSS on \( M \), a new LSS on \( N \) can be constructed. Also, the canonical morphisms of vector bundles induced by this construction is used to explain some basic relations between the solutions of both system.

Section 4 Semialgebraic, semianalytic and subanalytic sets Here we present some results about several kinds of sets, which can be found in [Cos92] and [BM88]. First, following [Cos92], semialgebraic sets are introduced. This category is composed by sets defined by inequalities of polynomials and is stable under union, intersection, complementary, projections and closure. After this, as in [BM88], semianalytic (defined by inequalities of analytic functions) and subanalytic sets are presented. This last kind of subsets is a generalization of semianalytic sets in such a way that the category of subanalytic sets is stable under projections and complementary operation. To explain the proof of this last property is the main goal of the section. Also, a notion of dimension for these singular sets is defined, which results to be equivalent to the local dimension of their smooth points.

Section 5 Desingularization theorem and applications This is one of the main sections of this dissertation, where desingularization theory is explained. Desingularization theory was developed mainly by Łojasiewicz, Gabrielov and Hironaka. It was greatly clarified by subsequent work by Bierstone and Milman, whose paper [BM88] we closely follow. In the first part, the notion of a blow-up, which can be thought as changing a point (or, more generally, a smooth set) by a projective space is defined. In this way, if we have a subset which is formed by two transversal lines intersecting at one point and we blow up that point, these two lines will not intersect at the blew-up space. More generally, if two curves intersect up two an order \( k \) at a point, after blowing-up this point, this intersection will be of order \( k - 1 \). In this way, the main theorem of this section (theorem 5.8) states that, locally, the zero set of a nonzero function is (after finitely many local blow-ups) many transversal hyperplanes. With this theorem and a bit of work, desingularization theorem follows: if \( X \) is a subanalytic subset of an analytic manifold \( M \), then there exists an analytic manifold \( N \) and a proper analytic map \( f : N \to M \) such that \( f(N) = X \). Finally, using this theorem, some useful properties of subanalytic subsets are proved.
Section 6 Solutions In this section we give an important result about conservation of solutions through desingularizations. Assume we have a subanalytic subset \( X \) of \( M \), and construct (using constructions explained in section 3) a lifted system on a desingularization \( N \) of \( X \). We will see that any solution on \( N \) projects to a solution of the system on \( M \) and, conversely, that any solution on \( M \) is partially lifted to \( N \). This “partially” comes from the fact that, even if \( X \) is connected, \( N \) may not be connected, and so we cannot ensure that the lift of a continuous solution is continuous. This section follows the ideas of [CE06]. The results presented in this section are slightly different (in some cases, more general) and the proofs are done using geometrical tools, which give a different appearance to the problem.

Section 7 Algorithm This is the main section of the work, where we present (simplifying the work in [CE06]) an algorithm that transforms an arbitrary LSS to a set of constant rank differential equations which can be solved, maybe not uniquely. The main idea of the algorithm is to restrict to a subset of the original system where solutions may exist. This subset will be a subanalytic subset of dimension strictly less than that of the original manifold. Using desingularization theorem, we change the original system to a family of systems on some manifolds of lower dimension. Recursively, we can solve the problem. Section 6 shows that we find all the solutions to the original system (up to some points).

Section 8 Examples In this section, we present two examples of application of the algorithm, doing them in a strict way. This means that maybe some steps would be skipped by anyone trying to solve the problem, but we did them because any implementation of the algorithm would require to do them. They are simple examples, but they are long enough to let any reader understand the behavior of the algorithm.

Appendix A Tarski–Seidenberg theorem In the first appendix, we explain (following mainly [Cos02]) the proof of Tarski–Seidenberg theorem, which is used in section 4 to prove that the projection of a semialgebraic set is a semialgebraic set, result which is widely used along the work.

Appendix B Weierstrass’ preparation theorem In the second appendix, some concepts about analytic functions are introduced, which are needed to prove Weierstrass preparation theorem. This theorem states that, in a convenient small neighborhood and coordinates, any analytic function is expressed as a monic polynomial in one of its variables, whose coefficients are analytic functions of the other variables. This result is used along the work to simplify many proofs.

Appendix C Implementation of the algorithm In the last appendix there is the implementation of a simplified version of the algorithm, which assumes that there is no need to desingularize any set. It would be interesting to implement a full version of the algorithm. However, implementing the desingularization of a subanalytic set may require some time and be somewhat involved, so we decided to assume this regularity condition.
Conclusions and outlook

In this thesis we have reviewed desingularization theory and its application to solving implicit differential equations, following [CE06]. The algorithm presented in that paper transforms an analytic linearly singular system into a (locally finite) family of locally constant rank ODEs, which can be easily solved. We have clarified this algorithm and improved some results about the solutions. The advantage of this algorithm over other ones is that it uses less condition checks in each step of the algorithm, which makes easier to implement it. As weak point, we must remark that its solutions may loose some of its points.

An interesting work for the future may be to generalize this algorithm for any kind of implicit differential equations (or inequalities), without linearizing, as well as studying the desingularization process to try to lift solutions to continuous solutions in the desingularized system.
2 Analytic geometry

As we stated in the introduction, we are going to work in the analytic category. Therefore, we will begin our work by introducing some concepts about analytic geometry, along with its similarities and differences with the well-known smooth geometry. Even if we won’t prove some results in analytic geometry, most of them are equivalent to a result on smooth geometry (but proofs commonly use sheaf theory). Therefore, any reader with a background on differential geometry of smooth manifolds ([Lee03], [Boo86], [Con01]) will have no problem to follow this work. For a more detailed review on real analytic geometry, our recommended references are [Lew11] and [Car57].

2.1 Analytic manifolds

Definition 2.1. An analytic manifold is a paracompact Hausdorff topological space $M$ equipped with an atlas of class $C^\omega$.

Since any $C^\omega$-class atlas is also a $C^\infty$-atlas, an analytic manifold can be though as a smooth manifold whose transition maps are analytic diffeomorphisms.

Example 2.2. • Any finite dimensional vector space $E$ has a trivial analytic structure.
• The $n$-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$ is an analytic manifold: Let $N = (0, \cdots, 0, 1)$ and $S = (0, \cdots, 0, -1)$. Take the stereographic projections
  \[
  \phi_N : S^n \setminus N \rightarrow \mathbb{R}^n, \quad (x_1, \cdots, x_{n+1}) \mapsto \left(\frac{x_1}{1-x_{n+1}}, \cdots, \frac{x_n}{1-x_{n+1}}\right),
  \]
  \[
  \phi_S : S^n \setminus S \rightarrow \mathbb{R}^n, \quad (x_1, \cdots, x_{n+1}) \mapsto \left(\frac{x_1}{1+x_{n+1}}, \cdots, \frac{x_n}{1+x_{n+1}}\right).
  \]

  Since the inverse map for $\phi_N$ is
  \[
  \phi_N^{-1} : \mathbb{R}^n \rightarrow S^n \setminus N, \quad (x_1, \cdots, x_n) \mapsto \left(\frac{2x_1}{1+x_1^2+\cdots+x_n^2}, \cdots, \frac{2x_n}{1+x_1^2+\cdots+x_n^2}, \frac{-1+x_1^2+\cdots+x_n^2}{1+x_1^2+\cdots+x_n^2}\right),
  \]

  the transition map $\phi_N^{-1} \circ \phi_S$ is clearly analytic.
• Any complex (holomorphic) manifold has a natural real analytic structure.
• Any open set of an analytic manifold is an analytic manifold.

As we have already stated, we can think the class of analytic manifolds as a subclass of smooth manifolds. Using that fact, many properties of smooth manifolds are also fulfilled by analytic manifolds. But we must be careful, because there are some constructions and proofs that do not preserve the analytic structure. To let our readers familiarize with analytic manifolds, we are standing out some similarities and differences between smooth and analytic geometry:
• Any $C^1$-path passing through a point $p \in M$ is tangent in $p$ to an analytic path. This implies that the tangent and cotangent bundle can be constructed and are analytic manifolds of dimension $2n$. Moreover, the tangent space is still isomorphic to the space of derivations of the genus of analytic functions. Also, fiber bundles are well defined.

• Since any analytic function vanishing on an open set $U \subset M$ vanishes identically on $M$ (Appendix, Theorem B.3), bump functions are smooth but non-analytic functions. Therefore, analytic manifolds do not have analytic partitions of unity.

• Lack of bump functions is a strong restriction at a theoretical level. As consequence, most of constructive proofs used in smooth geometry do not ensure analyticity of constructions. To overcome this problem, it is frequent to work with sheaves of functions. (An interested reader can find more information about them in [Bre97], [God58] or [Hir66]; however, knowledge of sheaf theory is not necessary to understand this thesis.)

• Whitney’s theorem do not hold for analytic manifolds. Instead, Nash’s theorem ([Nas54], [Nas56]) proves that any riemannian manifold of dimension $n$ admits an analytic embedding in $\mathbb{R}^m$, $m \leq n(n+1)(3n+11)/2$ (this covers all smooth manifolds).

• Finally, just remark that existence and uniqueness of solutions of differential equations for ordinary differential equations still holds [TP85].

2.2 Vector bundles

**Definition 2.3.** A fiber bundle consists of the data $(\pi, E, B, F)$ where

- $E, B$ and $F$ are manifolds.
- $\pi : E \to B$ is a surjective map.
- For all $b \in B$ there exists an open neighbourhood $U$ of $b$ and a diffeomorphism
  
  $\pi^{-1}(U) \cong U \times F$

  such that the following diagram commutes:

  \[
  \begin{array}{ccc}
  \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\
  \pi \downarrow & & \downarrow \text{pr}_1 \\
  B & & \\
  \end{array}
  \]

  The space $B$ is called the base space, $E$ the total space and $F$ the fiber. The map $\pi$ is called projection. If $x \in M$ is a point, the fiber of $x$ is $\pi^{-1}(b)$. If $F$ is a vector space and $\varphi$ is fiberwise linear, a fiber bundle is called a vector bundle.

**Example 2.4.** The trivial bundle $\pi : M \times \mathbb{R}^n \to M$ and the tangent and cotangent bundles.

**Definition 2.5.** Let $\pi : E \to B$ and $\pi' : E' \to B'$ be vector bundles. A vector bundle morphism is a pair $(\alpha, f)$ such that the following diagram commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\pi \downarrow & & \pi' \downarrow \\
B & \xrightarrow{f} & B'
\end{array}
\]
and is linear on each fiber. Note that \( f \) determines uniquely \( \alpha \).

**Definition 2.6.** Let \( \pi : E \to B \) be a vector bundle and \( f : C \to B \) an analytic map. The pull-back of \( E \) by \( f \) is the fiber bundle \((p_1, C \times_f E, B, F)\) where the total space is the fiber product:

\[
C \times_f E := \{(c, e) \in C \times E \mid f(c) = \pi(e)\}.
\]

Its fibers are the same as those of \( E \), but over the points of \( C \).
3 Linearly singular systems

An explicit autonomous, first order, ordinary differential equation (ODE) in $\mathbb{R}^m$ is an expression of the form

$$\dot{x} = f(x). \quad (5)$$

A solution of (5) is a map

$$\xi : I \rightarrow \mathbb{R}^m$$

such that

$$\dot{\xi}(t) = f(\xi(t)) \quad \forall t \in I.$$ 

Geometrically, an autonomous ODE is equivalent to a vector field

$$X \in \mathcal{X}(M)$$

where $M$ is a smooth or analytic manifold and a solution or integral curve of $X$ is a path

$$\xi : I \rightarrow M$$

such that the following diagram commutes:

$$\begin{array}{ccc}
TM & \xrightarrow{\dot{\xi}} & X \\
\downarrow & & \downarrow \\
I & \xrightarrow{\xi} & M
\end{array}$$

This is to say,

$$\dot{\xi} = X(\xi).$$

In this kind of differential equations (assuming that $f$ or $X$ is locally Lipschitz or $C^1$), Picard existence and uniqueness theorem [TP85] shows that, given a point $p \in M$ and a time $t_0$, there exists a unique solution $\xi$ passing through $p$ at $t_0$. This is to say that $\xi(t_0) = p$.

However, not all differential equations have this simple form. Our work is focused on finding solutions of differential equations that are (locally) of the form

$$A(x)\dot{x} = \sigma(x). \quad (7)$$

Such equations have been given a geometric description in [GP92]. In this section we summarize some of the results given in this paper.

3.1 Linearly singular systems

Definition 3.1. A linearly singular system (LSS) is a quintuple $(M,F,\pi,A,\sigma)$ where

- $M$ is a differentiable manifold.
- $\pi : F \rightarrow M$ is a vector bundle.
- $A : TM \rightarrow F$ is a morphism of vector $M$-bundles.
- $\sigma : M \rightarrow F$ is a section of $F$. 
Definition 3.2. A LSS is called homogeneous if $\sigma$ is the zero section.

Definition 3.3. Consider a LSS $(M,F,\pi,A,\sigma)$. Let $\xi : I \to M$ be a path and $I \subset \mathbb{R}$ an open interval. The equation of motion of $\xi$ is:

$$A \circ \dot{\xi} = \sigma \circ \xi$$

If $\xi$ satisfies the equation of motion, it is called a solution. That is equivalent to this diagram to be commutative:

\[
\begin{array}{ccc}
TM & \xrightarrow{A} & F \\
\downarrow \tau_M & & \downarrow \pi \\
I & \xrightarrow{\xi} & M
\end{array}
\]

Note that, if $(x,v) \to (x,A(x) \cdot v)$ is the local expression of $A$ and $x \to (x,\sigma(x))$ is the local expression of $\sigma$, then local expression of the equation of motion is $A(x) \cdot \dot{x} = \sigma(x)$

If $A$ is an isomorphism, then it can inverted and the problem is reduced to find the integral curves of the vector field $A^{-1} \circ \sigma$. However, in general $A$ will not be invertible, and the equation will not have solutions passing through every point. Moreover, even if there is a solution passing through one point, it may not be unique.

Example 3.4. Consider the diagram

\[
\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^4 \\
\downarrow \pi & & \downarrow \sigma \\
\mathbb{R}^2 & \xrightarrow{\sigma} & \mathbb{R}^4
\end{array}
\]

were

$$A_x \cdot \dot{x} = (x^1, x^2, x^1 + x^2, \dot{x}^1 - \dot{x}^2)$$

and

$$\sigma(x) = (x^1, x^2, x^1, 0).$$

Then, the local expression of the system is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} x^1 \\ 0 \end{pmatrix},$$

which is invertible. Then the only vector field which makes the diagram commutative is

$$X = \frac{1}{2} x^1 \frac{\partial}{\partial x^1} + \frac{1}{2} x^2 \frac{\partial}{\partial x^2}.$$

Example 3.5. Using the same manifold and vector bundles, let

$$A_x \cdot \dot{x} = (x^1, x^2, x^1 + x^2, \dot{x}^1 + \dot{x}^2)$$

and

$$\sigma(x) = (x^1, x^2, x^1, 0).$$

Then, the local expression of the system is

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} x^1 \\ 0 \end{pmatrix},$$

which is nowhere invertible.
This last example shows us that even if the system is not invertible, maybe there are points \( x \in M \) such that the system is solvable at that point. Therefore, it makes sense the following definition:

**Definition 3.6.** The motion set is the subset \( S \) of points \( x \in M \) such that there is a solution passing through \( x \).

Note that \( S \) may not be a manifold.

### 3.2 Morphisms of linearly singular systems

**Definition 3.7.** A morphism of linearly singular systems between two LSS \( (M, F, \pi, A, \sigma) \) and \( (M', F', \pi', A', \sigma') \) is a vector bundle morphism \((\alpha, f)\) between \( F \) and \( F' \) such that

- \( \alpha \circ \pi = \pi' \circ f \)
- \( f \circ A = A' \circ T\alpha \)
- \( f \circ \sigma = \sigma' \circ \alpha \)

This is to say that the following diagram commutes:

![Diagram](image)

**Lemma 3.8.** Let \( j \in C^\infty(N, M) \). Then, the quintuple \((N, j^*(F), j^*(\pi), j^*(A), j^*(\sigma))\), where

- \( j^*(F) = N \times_j F \)
- \( j^*(\pi) = pr_1 \)
- \( j^*(A) \cdot v_y = (y, (A \circ Tj) \cdot v_y) \)
- \( j^*(\sigma)(y) = (y, (\sigma \circ j)(y)) \)

is a LSS. Moreover, if \( J = pr_2 \) then \((j, J)\) is a LSS-morphism.

**Proof.** First, check that \((N, j^*(F), j^*(\pi), j^*(A), j^*(\sigma))\) is a LSS:

- \( N \) is a manifold by hypothesis.
- \( j^*(\pi) : N \times_j F \rightarrow N \) is a vector bundle by definition of fiber product.
- \( j^*(\pi) \circ j^*(\sigma) = j^*(\pi \circ \sigma) = j^*(\pi \circ \sigma) = j^*(Id_M) = Id_N \)
• $Tj$ is a morphism of vector bundles. Therefore, $j^*(A) = A \circ Tj$ is a morphism of vector bundles. Moreover, $\text{Im}(j^*(A)) \subset N \times_j F$.

For $(j, J)$ to be a LSS-morphism, we need to check the commutativity of the following diagram:

![Diagram]

• $j \circ j^*(\pi) = \pi \circ J$ due the definition of inverse image of $F$ by $j$.
• $(J \circ j^*(A)) \cdot v_y = J(j^*(A) \cdot v_y) = J(y, (A \circ T(j)) \cdot v_y) = (A \circ T(j)) \cdot v_y$
• $(J \circ j^*(\sigma))(y) = J(y, (\sigma \circ j)(y)) = (\sigma \circ j)(y)$

Lemma 3.9. Let $\pi_G : G \to M$ be a vector $M$-bundle and $p : F \to G$ be a vector $M$-bundle morphism. Suppose that $(M, F, \pi, A, \sigma)$ is a LSS. Then, $(M, G, \pi_G, p \circ A, p \circ \sigma)$ is a LSS and $(\text{Id}_M, p)$ is a LSS-morphism.

Proof. Similar to the proof of last lemma.

Proposition 3.10. Let $(M, F, \pi, A, \sigma)$ and $(M', F', \pi', A', \sigma')$ be LSS and $(\alpha, f)$ be a LSS-morphism where $\alpha : M \to M'$. Let $\xi$ be a solution of the equation of motion in $M$. Then $\xi' = \alpha \circ \xi$ is a solution in $M'$.

Proof. One of the implications follows from proposition (3.10). For the other one, suppose that $\xi'$ is a solution. Then

$A' \circ \dot{\xi}' = A' \circ T(\alpha) \circ \dot{\xi} = f \circ A \circ \xi = f \circ \sigma \circ \xi = \sigma' \circ \alpha \circ \xi = \sigma' \circ \xi'$

Proposition 3.11. Suppose that $F = M \times_\alpha F'$ and that, for each $x \in M$, $f_x$ is the projection onto $F$. If $\xi$ and $\xi'$ are paths related by $\xi' = \alpha \circ \xi$, then $\xi$ is a solution of the equation of motion if and only if $\xi'$ is also.

Proof. One of the implications follows from proposition (3.10). For the other one, suppose that $\xi'$ is a solution. Then

$\alpha^*(A) \circ \dot{\xi} = (\pi_M \circ \dot{\xi}, (A \circ T(\alpha)) \circ \dot{\xi}) = (\xi, A \circ \dot{\xi}') = (\xi, \sigma \circ \xi') = (\xi, \sigma \circ \alpha \circ \xi) = \alpha^*(\sigma) \circ \xi$.

Thus, $\xi$ is also a solution.
Let $(M, F, \pi, A, \sigma)$ be a LSS and consider the subset $S = \{x | \sigma(x) \in \text{Im}(A)\}$. Clearly, if there exists a solution passing through $x$ then $x \in S$. So, it makes sense to restrict our problem to $S$. However, in general this set will not be a manifold and we will not be able to use our geometric tools on it. To overcome this problem, the so called desingularization theorem is used, which allows to change this set by a manifold of the same “dimension” (the notion of dimension for non-smooth sets will be defined). The meaning of “change” is given by the following definition:

**Definition 4.1.** Let $M$ be a real analytic manifold and let $S \subset M$. A desingularization of $S$ is a proper real analytic map $f : N \to M$ such that $f(N) = S$ and $N$ is a real analytic manifold of the same dimension as $S$.

And the desingularization theorem states:

**Theorem 4.2** (Desingularization theorem). Let $M$ be a real analytic manifold and let $S$ be a closed subanalytic subset. Then, there exists a desingularization $f : N \to M$ of $S$.

A subanalytic set (in a first approach) the projection of a set defined by inequalities of analytic functions. The hypothesis of desingularization theorem and the definition of subanalytic sets show us the reason of working in the analytic category. As stated, desingularization theorem lets us change a subanalytic subset by a manifold. This change comes at a price: some solutions will lose part of their points (a discrete set).

In this section, we explain some tools and properties about subanalytic sets which will be useful to prove desingularization theorem. A more detailed review on semialgebraic sets can be found in [Cos02]. All definitions, results and proofs about semianalytic and subanalytic sets can be found in more detail in [BMS8].

### 4.1 Semialgebraic subsets

**Definition 4.3.** The class of semialgebraic subsets $\mathcal{SA}_n$ of $\mathbb{R}^n$ is the smallest collection of subsets of $\mathbb{R}^n$ such that

1. If $P \in \mathbb{R}[X_1, \ldots, X_n]$, then $\{x \in \mathbb{R}^n | P(x) = 0\} \in \mathcal{SA}_n$ and $\{x \in \mathbb{R}^n | P(x) > 0\} \in \mathcal{SA}_n$.
2. If $A, B \in \mathcal{SA}_n$, then $A \cup B, A \cap B, \mathbb{R}^n \setminus A \in \mathcal{SA}_n$.

**Proposition 4.4.** A subset $X$ is semialgebraic if and only if it can be written as a finite union of sets of the form $\{x \in \mathbb{R}^n | P_i(x) = 0, Q_j(x) > 0, j = 1, \ldots, p, i = 1, \ldots, q\}$.

**Proof.** By definition, this kind of sets are semialgebraic. The intersection of two sets of this kind is the set defined by the union of the two sets of equations defining the original sets, therefore is semialgebraic. Also, since the complementary of the union is the intersection of complementaries, the conversely is true. □
Theorem 4.5 (Tarski–Seidenberg theorem – first form). Let \( P_1(X,Y), \ldots, P_l(X,Y) \in \mathbb{R}[X,Y] \) be a finite family of polynomials where \( X = (X_1, \ldots, X_n) \). Then, given a polynomial system

\[
S(X,Y) : \begin{cases} 
  P_1(X,Y) \sigma_1 0 \\
  \vdots \\
  P_l(X,Y) \sigma_l 0
\end{cases}
\]

with \( \sigma_i \in \{<, >, =\} \) there exists a finite list \( C_1(X), \ldots, C_l(X) \) of systems of polynomial equations and inequalities in \( X \) with coefficients in \( \mathbb{R} \) such that, for every \( x \in \mathbb{R}^n \), the system \( S(x,Y) \) has a real solution if and only if at least one \( C_j(x) \) is satisfied.

Proof. Can be found in Appendix A. \( \square \)

Theorem 4.6 (Tarski–Seidenberg theorem – second form). Let \( A \) be a semialgebraic subset of \( \mathbb{R}^{n+1} \) and let \( \pi : \mathbb{R}^{n+1} \to \mathbb{R}^n \) be the projection on the first \( n \) coordinates. Then \( \pi(A) \) is a semialgebraic subset of \( \mathbb{R}^n \).

Proof. Using proposition 4.4, \( A \) is the union of finitely many subsets of the form

\[
\{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid P_1(x, y) = 0, \ldots, P_p(x, y) = 0, Q_1(x, y) > 0, \ldots, Q_q(x, y) > 0\}.
\]

Since the union of semialgebraic sets is semialgebraic, it suffices to prove Tarski–Seidenberg theorem for this type of sets. So, assume that \( A \) is itself of this form. It follows from the first form of the Tarski–Seidenberg theorem that there exists some polynomial systems \( C_1(X), \ldots, C_l(X) \) such that

\[
\pi(A) = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R} \text{ st } (x, y) \in A\}
\]

is the union of the sets \( \{x \in \mathbb{R}^n \mid C_j(x) \text{ is satisfied} \} (j = 1, \ldots, t) \), which is semialgebraic. \( \square \)

Example 4.7. Let \( P(x, y) = xy - 1 \) and let \( \pi : \mathbb{R}^2 \to \mathbb{R} \) be the projection onto the first component. Define \( C_1(x) = \{x < 0\} \) and \( C_2(x) = \{x > 0\} \). Fix \( x_0 \in \mathbb{R} \). Clearly, there exists a \( y \in \mathbb{R} \) such that \( P(x_0, y) = 0 \) if and only if \( x_0 \) satisfies \( C_1(x) \) or \( C_2(x) \).

Lemma 4.8. Let \( A \subset \mathbb{R}^n \) be a semialgebraic subset. Then \( \mathbb{R} \times A \) is a semialgebraic subset of \( \mathbb{R}^{n+1} \).

Corollary 4.9.

1. If \( A \) is a semialgebraic subset of \( \mathbb{R}^{n+k} \), its image by the projection on the space of the first \( n \) coordinates is a semialgebraic subset of \( \mathbb{R}^n \).

2. If \( A \) is a semialgebraic subset of \( \mathbb{R}^m \) and \( F : \mathbb{R}^m \to \mathbb{R}^n \) is a polynomial map, then \( F(A) \) is semialgebraic on \( \mathbb{R}^n \).

Proof. The first statement is trivial. For the second, write \( F \) as the composition of the graph function and a projection \( F : \mathbb{R}^m \xrightarrow{\Gamma_F} \mathbb{R}^m \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n \) and note that \( \Gamma_F(A) = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \mid x \in A \text{ and } y = F(x)\} \) is semialgebraic. The result follows from the first statement. \( \square \)
Another useful corollary of Tarski–Seidenberg theorem is

**Corollary 4.10.** The closure of a semialgebraic subset $A$ of $\mathbb{R}^n$ are semialgebraic subsets of $\mathbb{R}^n$.

**Proof.** It is enough to check that the exterior of a semialgebraic set is semialgebraic.

$$\text{Ext}(A) = \{ x \in \mathbb{R}^n \mid (\exists \varepsilon > 0)(\forall y \in A)||x - y||^2 \geq \varepsilon^2 \}$$

Denote

$$B = \{ (x, \varepsilon, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \mid y \in A \text{ and } \sum_{i=1}^{n} (x_i - y_i)^2 < \varepsilon^2 \}$$

$$= \mathbb{R}^n \times \mathbb{R} \times A \cap \{(x, \varepsilon, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \mid ||x - y||^2 < \varepsilon^2 \}$$

and

$$C = \{(x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R} \mid \varepsilon > 0 \}.$$

Let

$$p : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$$

$$q : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n .$$

Clearly,

$$p(B) = \{ (x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R} \mid (\exists y \in A)||x - y||^2 < \varepsilon \}$$

and, therefore,

$$q(C - p(B)) = \{ x \in \mathbb{R}^n \mid (\exists \varepsilon > 0)(\forall y \in A)||x - y||^2 \geq \varepsilon^2 \} = \text{Ext}(A).$$

Thus, the exterior and the closure of $A$ are semialgebraic subsets.

**Definition 4.11.** Let $A \subset \mathbb{R}^n$, $B \in \mathbb{R}^m$ be semialgebraic subsets. A map $f : A \to B$ is semialgebraic if its graph is semialgebraic.

**Proposition 4.12.**

1. Polynomial functions are semialgebraic.

2. The direct image and the inverse image of a semialgebraic set by a semialgebraic map are semialgebraic.

3. The composition of two semialgebraic maps is semialgebraic.

4. If $f, g : A \to \mathbb{R}^m$ are semialgebraic, then $f + g$ is semialgebraic.

5. If $f, g : A \to \mathbb{R}$ are semialgebraic, then $f \cdot g$ is semialgebraic.

6. Semialgebraic functions $f : A \to \mathbb{R}$ form a ring.

**Proof.**

1. Let $A \subset \mathbb{R}^n$ be a semialgebraic set. If $P \in \mathbb{R}[x_1, \cdots, x_n]$, then $\Gamma P(A) = \{(x, y) \in \mathbb{R}^{n+1} \mid x \in A, y = P(x)\}$, which is semialgebraic by corollary 4.9.
2. Let \( f : \mathbb{R}^m \to \mathbb{R}^n \) be a semialgebraic map and \( X \subset \mathbb{R}^m \) a semialgebraic set. Then the set \( Z \subset \mathbb{R}^m \times \mathbb{R}^n \) defined as the intersection
\[
Z = \Gamma_f \cap (X \times \mathbb{R}^n)
\]
is semialgebraic and
\[
f(X) = \pi_2(Z)
\]
where
\[
\pi_2 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n.
\]
For the inverse image, let \( Y \subset \mathbb{R}^n \) and define the set \( T \subset \mathbb{R}^m \times \mathbb{R}^n \) as
\[
T = \Gamma_f \cap (\mathbb{R}^m \times Y).
\]
Finally,
\[
f^{-1}(Y) = \pi_1(T)
\]
where
\[
\pi_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m.
\]
3. Let \( f : \mathbb{R}^m \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^l \). Denote
\[
\Gamma = \Gamma_f \times \Gamma_g \subset \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l
\]
the product of the graphs of \( f \) and \( g \), and denote
\[
K = \Gamma \cap (\mathbb{R}^m \times \Delta \times \mathbb{R}^l).
\]
\( K \) is clearly algebraic. Let
\[
\pi : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m \times \mathbb{R}^l
\]
be the projection. Then,
\[
\pi(K) = \Gamma_{gof}.
\]
4. Let \( f, g : \mathbb{R}^m \to \mathbb{R}^n \). Then
\[
(f, g) : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n.
\]
5. For the product, proceed similar to the sum.


From the proof of Tarski–Seidenberg theorem, it follows

**Theorem 4.13.** Let \( P(x, y), x = (x_1, \cdots, x_n), \) be a polynomial. Then there exists a semialgebraic partition \( \{A_1, \cdots, A_m\} \) of \( \mathbb{R}^n \) such that, for each \( k = 1, \cdots, m, \) either \( P \) has constant sign in \( A_k \times \mathbb{R}, \) or there exists continuous semialgebraic functions \( \xi_1 < \cdots < \xi_{rk} \) on \( A_k \) such that
1. For each \( x \in A_k \), \( \{ \xi_1(x), \cdots, \xi_{r_k}(x) \} \) is the set of zeros of the function \( P_x(y) = P(x,y) \).

2. The sign of \( P(x,y), x \in A_k \) depends only on the signs of \( y - \xi_i(x), i = 1, \cdots, r_k \).

**Corollary 4.14.** Let \( P_i(x,y), i = 1, \cdots, t \) be polynomials. Then there is a semialgebraic partition \( \{ A_1, \cdots, A_m \} \) of \( \mathbb{R}^n \) such that, for each \( k = 1, \cdots, m \), the zeros of \( P_1, \cdots, P_t \) on \( A_k \) are given by continuous semialgebraic functions \( \xi_1 < \cdots < \xi_{r_k} \) and the sign of each \( P_j(x,y) \) on \( A_k \) depends only on the signs of \( y - \xi_i(x), i = 1, \cdots, r_k \).

**Example 4.15.** Let \( P(x,y) = xy - 1 \) and let \( \pi : \mathbb{R}^2 \to \mathbb{R} \) be the projection onto the first component. Define \( A_1 = \{ x \mid x < 0 \} \), \( A_2 = \{ x \mid x > 0 \} \) and \( A_3 = \{ 0 \} \). Then

- **A1:** The zeros of \( P \) are given by \( \xi(x) = 1/x \) and \( P(x,y) > 0 \iff y - \xi(x) > 0 \).
- **A2:** The zeros of \( P \) are given by \( \xi(x) = 1/x \) and \( P(x,y) > 0 \iff y - \xi(x) > 0 \).
- **A3:** There are no zeros of \( P \) over \( A_3 \).

### 4.2 Semianalytic sets

**Analytic sets**

Let \( M \) be a real analytic manifold and \( U \subset M \) be an open subset. Denote by \( \mathcal{O}(U) \) the ring of real analytic functions on \( U \).

**Definition 4.16.** Let \( M \) be a connected \( m \)-dimensional real analytic manifold. A subset \( X \) is called analytic subset if for each point \( x \in M \) there exists a neighborhood \( U \) of \( x \) in \( M \) such that

\[
X \cap U = \{ x \in U \mid f_1(x) = \cdots = f_r(x) = 0 \},
\]

where \( f_i \in \mathcal{O}(U) \).

Note that an analytic submanifold is not necessary an analytic subset. For instance, the half hyperplane on \( \mathbb{R}^n \) is an analytic submanifold, but not an analytic subset. However,

**Lemma 4.17.** A closed analytic submanifold is an analytic subset.

**Semianalytic sets**

Semianalytic sets are a generalization of semialgebraic sets. The idea is to change polynomials by analytic functions. Let \( \mathcal{A} \) be a ring of real-valued functions defined on a set \( E \). Denote \( S(\mathcal{A}) \) the smallest family of subsets of \( E \) containing all sets \( \{ x \mid f(x) > 0 \}, f \in \mathcal{A}, \) which is stable under finite intersection, finite union and complement.

This definition is equivalent to say that \( X \) belongs to \( S(\mathcal{A}) \) if and only if is a finite union of sets of the form

\[
\{ x \in \mathbb{R}^n \mid f_i(x) = 0, g_j(x) > 0, j = 1, \ldots, p, i = 1, \ldots, q \}
\]

where \( f_i, g_j \in \mathcal{A} \).

Note that, if \( \mathcal{A} = \mathbb{R}[x_1, \cdots, x_n] \), then \( S(\mathcal{A}) \) is the set of semialgebraic sets of \( \mathbb{R}^n \).
Theorem 4.18. Let $\pi : E \times \mathbb{R}^k \to E$ be the projection onto the first component. Given indeterminates $t_1, \ldots, t_k$, denote $\mathcal{A}[t_1, \ldots, t_k]$ the ring of functions $f : E \times \mathbb{R}^k \to \mathbb{R}$ which are a polynomial on $t_1, \ldots, t_k$ whose coefficients belong to $\mathcal{A}$. If $X \in S(\mathcal{A}[t_1, \ldots, t_k])$, then $\pi(X) \in S(\mathcal{A})$.

Proof. Suppose $X \in S(\mathcal{A}[t_1, \ldots, t_k])$. Then $X$ is the finite union of sets, each of whom is described by functions of the form

$$f_i(x, t) = \sum_{|\alpha| \leq N} \lambda_{i,\alpha}(x) t^\alpha$$

where $\lambda_{i,\alpha} \in \mathcal{A}$. Denote $\lambda(x) = (\lambda_{i,\alpha}(x))$ and $\Lambda(x, t) = (\lambda(x), t)$. Clearly, the following diagram commutes:

$$
\begin{array}{ccc}
E \times \mathbb{R}^k & \xrightarrow{\Lambda} & \mathbb{R}^l \times \mathbb{R}^k \\
\downarrow \pi & & \downarrow \pi' \\
E & \xrightarrow{\lambda} & \mathbb{R}^l
\end{array}
$$

Then $X$ is the inverse image of a set $X'$ by the map $\Lambda(x, t)$ which is described by the polynomial $p_i(\lambda, t) = \sum_{|\alpha| \leq N} \lambda_{i,\alpha} t^\alpha$, thus $X'$ is semialgebraic. Therefore,

$$\pi(X) = \pi(\Lambda^{-1}(X')) = \{x \mid \exists t \text{ st } (\lambda(x), t) \in X' \} = \lambda^{-1}(\pi'(X'))$$

where $\pi'(\lambda(x), t) = (\lambda(x))$.

$$
\begin{array}{ccc}
X & \xrightarrow{\Lambda} & X' \\
\downarrow \pi & & \downarrow \pi' \\
\pi(X) & \xrightarrow{\lambda} & \pi'(X')
\end{array}
$$

Since $X'$ is semialgebraic, using Tarski–Seidenberg theorem, $\pi'(X')$ is also semialgebraic. Since $\pi'(X')$ is described by some polynomials $q_j$, $\pi(X)$ is described by $\lambda^*(q_j) \in \mathcal{A}$ and $\pi(X) \in S(\mathcal{A})$.

Definition 4.19. A subset $X \subset M$ is semianalytic if for every $x \in M$ there exists a neighborhood $U$ of $x$ in $M$ such that $X \cap U \in S(\mathcal{O}(U))$.

That is to say, $X$ is locally defined by inequalities of analytic functions.

Remark 4.20. This property has to be checked only around the points of the closure $\overline{X}$.

Remark 4.21. Semianalyticity is not a transitive property: let $V$ be an open set and $U \subset V$ be a semianalytic open subset. Assume $X \subset U$ is a semianalytic subset. Then, in general, it is not true that $X \subset V$ is semianalytic. However, if $\overline{X} \subset U$ then it is true.

Again, an analytic submanifold doesn’t need to be a semianalytic subset.

Example 4.22. The set

$$Y = \{(x, y) \mid x \neq 0, y = x \sin(1/x)\} \subset \mathbb{R}^2$$

is not a semianalytic subset (it does not exists an open neighborhood $U$ of $(0, 0)$ such that $X \cap U$ is defined by analytic functions).
Example 4.23. Let $X \subset \mathbb{R}^3$ be the set of zeros of
\[
p(x, y, z) = zx - 1
\]
\[
q(x, y, z) = y - x \sin(1/x), \quad x \neq 0.
\]
This set is clearly semianalytic, since the points of the plane $x = 0$ are exterior to $X$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ the projection onto the first two coordinates. Then
\[
Y = \pi(X) = \{(x, y) \mid y = x \sin(1/x), x \neq 0\},
\]
which is not semianalytic.

Definition 4.24. Let $X$ be a semianalytic subset of $M$. A function $f : X \to \mathbb{R}$ is semianalytic if its graph is semianalytic.

As stated in the examples, the projection of a semianalytic set is not necessarily semianalytic. In particular, projections and polynomials are not semianalytic maps.

Proposition 4.25. Let $f_1(x, y), \ldots, f_t(x, y) \in \mathcal{O}(M)[y]$. Then there exists a semialgebraic partition $\{A_1, \ldots, A_m\}$ of $M$ such that for each $k \in [1, m]$:  

1. The zeros of $f_1, \ldots, f_t$ on $A_k$ are given by continuous functions $\xi_1 < \cdots < \xi_{r_k}$.
2. The sign of each $f_i(x, y)$ on $A_k$ depends only on the signs of the $y - \xi_i(x)$.

Proof. Let
\[
f_i(x, t) = \sum_{|\alpha| \leq N} \lambda_{i, \alpha}(x) t^\alpha.
\]
Then, $f_i(x, y) = P_i(\lambda(x), y)$, where $P_i$ are polynomials. For these polynomials, using corollary 4.14, there exists a semialgebraic partition $\{A'_k\}$ and, for each $k$, continuous semialgebraic functions $\xi'_j(\lambda)$. Then $f_i(x, \xi'_j(\lambda(x))) = P_i(\lambda(x), \xi'_j(\lambda(x))) = 0$, when $\lambda(x) \in A'_k$. Define $A_k = \lambda^{-1}(A'_k)$ and $\xi_j = \xi'_j \circ \lambda$.

Separating families

Definition 4.26. Let $U$ be an open subset of $M$. A finite family $f_1, \ldots, f_m \in \mathcal{O}(U)$ is separating if, for any choice of the symbols $\sigma_i \in \{<, >, =\}$, the set
\[
A = \bigcap_{i=1}^m \{x \in U \mid f_i(x) \sigma_i 0\}
\]
satisfies

1. $A$ is either empty or connected.
2. If $A \neq \emptyset$, then the closure of $A$ in $U$ is given by
\[
\overline{A} = \bigcap_{i=1}^m \{x \in U \mid f_i(x) \sigma_i 0\}
\]
where $\sigma$ represents $\{\leq, \geq, =\}$ according to whether $\sigma$ was $\{<, >, =\}$.
It is easy to see that (2) is equivalent to:
(2') If \( A \neq \emptyset \) and \( B \) is also given by sign conditions on the \( f_i \), then \( B \subset \mathcal{A} \) if and only if every strict sign condition on the \( f_i \) in \( B \) is also satisfied in \( A \).

**Example 4.27.** One can check that the family
\[
\begin{align*}
p_1(x, y) &= x + y - 1 \\
p_2(x, y) &= xy - \frac{1}{4}
\end{align*}
\]
is separating.

**Example 4.28.** The family
\[
\begin{align*}
p_1(x, y) &= x^2 + y^2 - 1 \\
p_2(x, y) &= x
\end{align*}
\]
is not separating because the set defined by
\[
\{(x, y) \in \mathbb{R}^2 \mid p_1(x, y) = p_2(x, y) = 0\}
\]
is not connected.

**Lemma 4.29** (Thom’s lemma). Let \( P_1(x), \ldots, P_m(x) \in \mathbb{R}[x] \) be a finite family of polynomials in one variable, which is stable under differentiation. Let
\[
A = \bigcap_{i=1}^{m} \{ x \in \mathbb{R} \mid P_i(x) \sigma_i 0 \},
\]
where \( \sigma_i \in \{<, >, =\} \). Then:

1. \( A \) is either empty or connected (if it is connected, it must be a point or a open interval)
2. If \( A \) is nonempty, then
\[
\bar{A} = \bigcap_{i=1}^{m} \{ x \in U \mid P_i(x) \sigma_i 0 \},
\]
where \( \sigma \) represents \( \{\leq, \geq, =\} \), according \( \sigma \) was \( \{<, >, =\} \).

**Proof.** By induction on \( m \). Thom’s lemma is trivial if \( m = 0 \). Let \( m \geq 1 \). Suppose that \( P_m \) has maximal degree in this family. Then \( P_1, \ldots, P_{m-1} \) is stable under differentiation. Define
\[
A' = \bigcap_{i=1}^{m-1} \{ x \in U \mid P_i(x) \sigma_i 0 \}.
\]
Then
\[
A = A' \cap \{ x \in U \mid P_m(x) \sigma_m 0 \}.
\]
Apply induction on \( A' \) and suppose it is not empty. If it is a point, the result it is clear. Suppose that \( A' \) is an open interval. Since the family is stable under differentiation, \( P'_m \) has constant sign in \( A' \) and, therefore, \( P_m \) is monotone in \( A' \) and the condition \( \{P_m \sigma_i 0\} \) divides \( A' \) into two connected subintervals. Thus, \( A \) is empty, a point or an open set. \( \square \)
Theorem 4.30. Any finite family of analytic functions on an analytic manifold \( M \) can be completed, in a neighborhood of any point, to a separating family.

Proof. By induction on \( m = \dim(M) \). Let \( f_1, \ldots, f_p \in \mathcal{O}(M) \). By the Weierstrass preparation theorem (Appendix B.II), we can assume that for each point \( x \in M \) there exists a coordinate neighborhood \( U \) such that:

1. \( U = U' \times I \) where \( U' \subset \mathbb{R}^{m-1} \) and \( I \) is an open interval.

2. Let \( (x, y) = (x_1, \cdots, x_{m-1}, y) \) denote the coordinates of \( U \). Then each

\[
f_i(x, y) = u_i(x, y)g_i(x, y)
\]

where \( u_i \) are analytic nowhere vanishing functions and \( g_i \) are monic polynomials in \( y \) whose coefficients are analytic functions on \( U' \) such that, for each \( x \in U' \), all roots of \( g_i(x, y) \) are on \( I \).

In other words, \( g_i \in \mathcal{O}(U')[y] \subset \mathcal{O}(U' \times \mathbb{R}) \). Since \( u_i \) do not affect to the sign of the functions, we only need to prove the theorem for \( g_1, \ldots, g_p \). If \( m = 1 \), this is Thom’s lemma (to get a separating family just add all derivatives).

In general, we proceed in a similar way. Given \( g_1, \ldots, g_p \), first we add all derivatives with respect to \( y \) to get an extended family \( g_1, \cdots, g_p, g_{p+1}, \cdots, g_{p+q} \) all monic in \( y \) (except for constant factors). Using proposition 4.25 there exists a semianalytic partition \( \{ B_1, \cdots, B_s \} \) of \( U' \) such that, for each \( k = 1, \cdots, s \), the zeros of \( g_1, \cdots, g_{p+q} \) over \( B_k \) are given by a continuous semianalytic functions \( \xi_1, \cdots, \xi_k \), and the sign of each \( g_i(x, y) \) on \( B_k \) depends only on the signs of the \( y - \xi_j(x) \). Also, each \( B_k \) can be described by finitely many analytic functions on \( U' \) (shrinking it if necessary). By induction, we complete the list of functions which describe \( B_k \) to a separating family \( g_{p+q+1}(x), \cdots, g_{p+q+r}(x) \).

Claim: \( g_1, \cdots, g_{p+q+r} \) is a separating family in \( U' \times \mathbb{R} \). Suppose that

\[
A = \bigcap_{j=1}^{p+q+r} \{(x, y) \in U' \times \mathbb{R} \mid g_j(x, y) \sigma_j 0 \}
\]

and define

\[
B = \bigcap_{i=p+q+1}^{p+q+r} \{x \in U' \mid g_i(x) \sigma_i 0 \}.
\]

Let \( \xi_1 < \cdots < \xi_t \) the roots of \( g_1, \cdots, g_{p+q} \) over \( B \). Let \( \pi : U' \times \mathbb{R} \to U' \) be the projection. Fix \( x_0 \in U' \). Using Thom’s lemma, \( A \cap \pi^{-1}(x_0) \) is either empty, a point \((x_0, \xi_i(x_0))\) or an open interval \( \{(x, y) : \xi_i(x_0) < y < \xi_{i+1}(x_0)\} \) (admitting \( \pm \infty \) as values of \( \xi_j \)). Remember that the sign of the functions \( g_j(x, y), j = 1, \cdots, p + q \) depends only on the signs of \( y - \xi_j(x) \). Therefore, \( A \) is either empty, \( (x, \xi_i(x)) \) or \( \{(x, y) \mid \xi_i(x) < y < \xi_{i+1}(x)\} \). Thus, \( A \) is either empty or connected. Suppose that \( A \) is nonempty. Let

\[
A' = \bigcap_{i=1}^{p+q+r} \{(x, y) \in U' \times \mathbb{R} \mid g_i(x, y) \sigma_i 0 \}.
\]

Clearly \( \overline{A} \subset A' \). Let’s see that \( A' \subset \overline{A} \). By induction, we have that

\[
\overline{B} = \bigcap_{i=p+q+1}^{p+q+r} \{x \in U' \mid g_i(x) \sigma_i 0 \}.
\]
Let \( x_0 \in B \). Since \( g_j(x, y) \) are monic with respect to \( y \), we can find a neighborhood \( V' \) of \( x_0 \) in \( U' \) and \( K > 0 \) such that the roots \( \xi_1, \ldots, \xi_t \) are bounded by \( K \) on \( B \cap V' \). Thus, for all \( x \in B \cap V' \), \( \mathcal{A} \cap (\{x\} \times [-K, K]) \neq \emptyset \) and \( \mathcal{A} \cap \pi^{-1}(x_0) \) is nonempty. Applying Thom’s lemma, there are only two possibilities for the fiber of \( A' \) over \( x_0 \):

1. It is a point. Then, it coincides with the fiber of \( \mathcal{A} \) over \( x_0 \).

2. It is a closed interval with non-empty interior. Suppose that \( (x_0, y) \) belongs to its interior. Then \( g_j(x_0, y) \sigma_j 0 \). Therefore, \( (x_0, y) \in \mathcal{A} \) and all the interval lies in \( \mathcal{A} \).

Therefore, \( A' \subset \mathcal{A} \), which concludes the proof.

Properties of semianalytic subsets

**Corollary 4.31.** Let \( X \) be a semianalytic subset of \( M \). Then

1. Every connected component of \( X \) is semianalytic.

2. The family of connected components of \( X \) is locally finite.

3. \( X \) is locally connected.

**Proof.** First statement is trivial. For the other two, it is enough to show that each \( a \in M \) has a neighborhood \( U \) such that \( X \cap U \) has finitely many connected components, all semianalytic in \( U \). Let \( U \) be a neighborhood of \( a \) such that \( X \cap U \) can be described using finitely many elements \( f_1, \cdots, f_p \) of \( \mathcal{O}(U) \). By last theorem, we can complete them to a separating family \( f_1, \cdots, f_{p+q} \) (shrinking \( U \) if necessary). Then \( X \cap U \) is a disjoint union of finitely many connected semianalytic subsets of \( U \), each given by a sign condition on \( \{f_i\}_i \).

**Corollary 4.32.** The closure and the interior of a semianalytic set is semianalytic.

**Proof.** Assume \( X \) is a semianalytic subset of the form

\[
X = \bigcap_i \{x \in M \mid f_i(x) \sigma_i 0\}.
\]

Using last theorem, complete \( \{f_i\}_i \) to a separating family. Since \( X \) is the union of some of the sets \( A \) that appear on the definition of a separating family, \( X \) is semianalytic. For the interior note that

\[
\text{Int}(X) = \overline{X} - (M - X).
\]

**Corollary 4.33.** 1. Let \( X \) be a semianalytic subset of \( M \), and let \( U \subset X \) be a semianalytic subset of \( M \) which is open in \( X \). Then, locally \( U \) is a finite union of semianalytic sets of the form

\[
\{x \in X \mid f_1(x) > 0, \cdots, f_k(x) > 0\}
\]

where \( f_i \) are analytic functions.
2. Every closed semianalytic subset of \( M \) is, locally, a finite union of sets of the form

\[
\{ x \mid f_1(x) \geq 0, \ldots, f_k(x) \geq 0 \}
\]

where \( f_i \) are analytic functions.

**Proof.**

1. Locally, we can complete a list of analytic functions used to describe \( X \) in \( U \) to a separating family \( f_1, \ldots, f_k \). Then \( U = \bigcup_{i=1}^{p} T_i \), where \( T_i = \bigcap_{j=1}^{k} \{ x \mid f_j(x) \sigma_{ij} 0 \}, T_i \neq \emptyset \), and \( \sigma_{ij} \) is either \(<\), \(>\) or \(=\). Let \( V_i \) be the open semianalytic set given by the intersection of the sets with strict sign conditions in the preceding representation of \( T_i \). Then each \( T_i \subseteq V_i \), so that \( U \subset X \cap \bigcup_{i=1}^{p} V_i \).

To show \( X \cap V_i \subset U \), note that \( X \cap V_i \) is also a union of semianalytic sets given by sign conditions on each \( f_i \). Let \( A \) be one of these sets. By the definition of \( V_i \), every strict condition satisfied of \( T_i \) is also satisfied in \( A \). Therefore, \( T_i \subset A \) (by condition (2′) from definition of semianalytic sets). Since \( U \) is open in \( X \) then \( U \cap A \neq \emptyset \). Thus, \( A \subset U \).

2. Follows from (1).

---

**Semianalytic stratifications**

We can prove a more general version of last theorem whose proof follows from it.

**Definition 4.34.** Let \( U \subset M \) be a semianalytic open neighborhood of a point \( a \). A locally finite collection of subsets \( \{ A_k \} \) is called a semianalytic stratification of \( U \) if

1. \( U \) is the disjoint union of the \( A_k \).

2. Each \( A_k \) is a connected semianalytic subset and analytic submanifold of \( M \).

3. (Condition of the frontier) If \( A_k \cap \overline{A_l} \neq \emptyset \), then \( A_k \subset \overline{A_l} \) and \( \dim(A_k) < \dim(A_l) \).

**Definition 4.35.** Let \( U \subset M \) be a semianalytic open set and \( X \subset U \) an arbitrary set. A semianalytic stratification of \( U \) is subordinated to \( X \) if there exists a subcollection \( \{ A_k \} \) such that

\[
X = \bigcup_{k} A_k.
\]

**Theorem 4.36.** Let \( f_1, \ldots, f_p \in \mathcal{O}(M) \). Let \( a \in M \). Then there is a semianalytic open neighborhood \( U \) of \( a \), and a separating \( h_1, \ldots, h_p, h_{p+1}, \ldots, h_{p+s} \in \mathcal{O}(U) \) such that \( h_j = f_j|_U \) \((j = 1, \ldots, p)\) and the collection \( \{ A_k \} \) of subsets of \( U \) of the form

\[
\bigcap_{j=1}^{p+s} \{ x \in U \mid h_j(x) \sigma_j 0 \}
\]

is a semianalytic stratification of \( U \).

**Corollary 4.37.** Let \( X \) be a semianalytic subset of \( M \). Then, there is a locally finite semianalytic stratification \( \{ A_k \} \) of \( M \) subordinated to \( X \).
Definition 4.38. Let \( X \) be a semianalytic set and \( \{ A_k \} \) an stratification subordinated to \( X \). Define dimension of \( X \) as
\[
\dim(X) := \max_k \{ \dim(A_k) \}.
\]

Remark 4.39. This definition is independent of the stratification. In fact, \( \dim(X) = d \) if and only if \( X \) contains an open set homeomorphic to an open ball in \( \mathbb{R}^d \), but not an open set homeomorphic to an open ball in \( \mathbb{R}^{d+1} \).

Example 4.40. Consider the set defined by the zeros of \( p(x, y) = xy \).
A stratification of this set is the family of sets \( A_0 = \{ (0, 0) \} \), \( A_1 = \{ (x, y) \mid x = 0, y > 0 \} \), \( A_2 = \{ (x, y) \mid x = 0, y < 0 \} \), \( A_3 = \{ (x, y) \mid x > 0, y = 0 \} \) and \( A_4 = \{ (x, y) \mid x < 0, y = 0 \} \), and its dimension is \( 1 = \dim(A_3) \).

4.3 Subanalytic sets

As we said, the projection of a semianalytic set is not always semianalytic. To overcome this, we introduce one last kind of subsets: subanalytic sets. From now on until the end of the section, \( M \) and \( N \) will be real analytic manifolds.

Definition 4.41. A subset \( X \) of \( M \) is called subanalytic if for each point \( x \in M \) there exists a neighborhood \( U \) of \( x \) and a relatively compact semianalytic set \( Y \subset U \times \mathbb{R}^k \) such that \( \pi(Y) = X \cap U \), where \( \pi : U \times \mathbb{R}^k \to U \) is the projection.

Note that semianalytic sets are also subanalytic. Closed subanalytic sets are the kind of subsets which can be desingularized using desingularization theorem. Our goal now is to show some properties about subanalytic sets. For example, it is clear that the union of two subanalytic subsets is subanalytic and that every connected component of a subanalytic set is subanalytic. What is not quite obvious, is that the complement of a subanalytic subset is also subanalytic. This will be the subject of the theorem of complement.

Definition 4.42. Let \( X \subset M \) be a subanalytic subset. A map \( f : X \to N \) is called subanalytic if its graph is subanalytic in \( M \times N \).

Lemma 4.43. The image of a relatively compact subanalytic set by a subanalytic map is subanalytic.

Proof. Consider the map as the composition of the graph function and the projection onto the second component. \( \square \)

Definition 4.44. Let \( X \) be a subanalytic subset of \( M \). A point \( x \in X \) is called smooth of dimension \( k \) if there is a neighborhood \( U \) of \( x \) in \( M \) such that \( X \cap U \) is an smooth (analytic) submanifold of \( M \) of dimension \( k \).

Definition 4.45. Let \( X \) be a subanalytic subset of \( M \). \( X \) is called smooth if each point of \( M \) is smooth.
Note that a smooth subanalytic subset is an analytic submanifold but the converse is not true. To prove the theorem of complement, first we need to prove some technical lemmas. Let $U$ and $V$ denote finite dimensional Euclidean spaces, $W = U \oplus V$, $n = \dim(W)$, and $\pi : W \to U$ the projection.

**Lemma 4.46.** Let $X$ be a relatively compact semianalytic subset of $W$. Then $X$ is a finite union of connected smooth semianalytic subsets $A$ such that, for each one:

1. The rank of $\pi$ is constant on $A$.
2. The linear subspaces $T_x A \cap V$, $x \in A$, admit a common complement in $V$, and the subspaces $\pi(T_x A)$, $x \in A$, admit a common complement in $U$.
3. There is an analytic function $g$ in a neighborhood of $A$ such that $g > 0$ on $A$ and $g = 0$ on $A - A$.

**Proof.** We prove this lemma by induction on the dimension $k$ of $X$. The result is obvious if $k = 0$. Let $k > 0$. Then, there exists a semianalytic subset $Y \subset X$ such that $\dim Y < k$ and $X - Y$ consist of smooth points of dimension $k$. By induction, we can assume that the result is true on $Y$. So, without loss of generality, we can assume that $X$ is smooth and connected.

Define

$$X_0 = \{ x \in X \mid \text{rank}_x(\pi|_X) \text{ is maximal} \}.$$

Then $X_0$ is semianalytic and $\dim(X - X_0) < k$. Locally, $X_0$ lies in an analytic set of dimension $k$. Therefore, there are $n - k$ nearly everywhere independent analytic functions $h_1, \ldots, h_{n-k}$ defined in a neighborhood of $X_0$ such that they vanish on $X_0$. Let

$$Z = \{ x \mid \dim < dh_i(x); i = 1, \ldots, n > < n - k \}.$$

Then,

$$\dim(X_0 \cap Z) < k.$$

Again, using the induction hypothesis on this subset, we can assume that $\text{rank}_x(\pi|_X)$ is constant on $X$ and that the functions $h_i$ are all independent.

Let $G = G_k(W)$ denote the Grassmannian of $k$-dimensional linear subspaces of $W$. Given linear subspaces $E \subset U$ and $F \subset V$, define

$$G_{E,F} = \{ T \in G \mid F \oplus (T \cap V) = V \text{ and } E \oplus \pi(T) = U \}.$$

It is easy to see that $G_{E,F}$ is an open semialgebraic subset of $G$. Since Grassmannians are compact spaces, there exists finitely many pairs $(E, F)$ such that

$$G = \bigcup_{(E,F)} G_{E,F}.$$

Then,

$$X = \bigcup_{(E,F)} \{ x \in X \mid T_x X \in G_{E,F} \}$$

and each set in this union is open in $X$. To prove that they are semianalytic, define

$$\Sigma = \{ (z_1, \ldots, z_{n-k}) \in W^{n-k} \mid z_1, \ldots, z_{n-k} \text{are not linearly independent} \}.$$
Let 
\[ S : W^{n-k} - \Sigma \rightarrow G \]
be the map that sends \((z_1, \ldots, z_{n-k})\) to the \(k\)-dimensional subspace which is orthogonal to the subspace spanned by \(\{z_1, \ldots, z_{n-k}\}\). This is a continuous semialgebraic map. Therefore, 
\[ S^{-1}(G_{E,F}) \]
is semialgebraic and, denoting 
\[ H(x) = (\text{grad } h_1(x), \ldots, \text{grad } h_{n-k}(x)), \]
\[ \{x \in X \mid T_x X \in G_{E,F}\} = X \cap H^{-1}(S^{-1}(G_{E,F})) \]
is semianalytic. This proves (1) and (2). To prove (3), suppose that \(A\) satisfies (1) and (2). Locally, \(A - A\) lies inside an analytic set \(Y\) of dimension \(< \dim(A)\). Since we are using induction over the dimension of \(A\), we only need to prove it on \(A - Y\). Assume that 
\[ Y = \{x \mid g_1(x) = \cdots = g_l(x) = 0\}. \]
Then \(g = \sum g_i^2\) satisfies (3).

**Definition 4.47.** The dimension of a relatively compact subanalytic set is the highest dimension of its smooth points.

**Lemma 4.48.** [Fiber-cutting lemma] Let \(X\) be a relatively compact semianalytic subset of \(W\). Then there exist finitely many smooth semianalytic subsets \(B_i, i = 1, \ldots, r\) of \(X\) such that
1. \(\pi(X) = \pi(\bigcup_i B_i)\)
2. For each \(B\), \(\pi|_B : B \rightarrow U\) is an immersion.
3. For each \(B\), the subspaces \(\pi(T_x B), x \in B\), have a common complement in \(U\).

**Proof.** Again, this proof will be done by induction (note that the 0-dimensional case is trivial). Let \(k\) be the dimension of \(X\). First, let \(\{A_i\}\) be the sets from last lemma. Property (1) is satisfied by these sets. For \(\bigcup_{\dim A < k} A\), the result is true by induction. Let \(A\) be a set such that \(\dim(A) = k\) and \(rk(\pi|_A) = k\). Then \(A\) satisfies (2) and (3). On the other hand, if \(rk(\pi|_A) < k\), we only need to find a semianalytic subset \(B\) such that \(\pi(A) = \pi(B)\) and \(\dim(B) < k\). Once we find it, by induction hypothesis, \(B\) will satisfy the two properties and, since \(\pi(A) = \pi(B)\), it won’t break the first property. Using last lemma, \(\pi\) has constant rank on \(A\); therefore \(\pi^{-1}(\pi(x))\) is a manifold. Then for any \(x \in A\), define 
\[ A_{\pi(x)} = A \cap \pi^{-1}(\pi(x)). \]
Since \(\pi(T_x A)\) admits a common complement in \(U\) (by last lemma), then \(A_{\pi(x)}\) is a submanifold of \(\pi^{-1}(\pi(x))\). Let \(C\) be a connected component of \(A_{\pi(x)}\). Using again last lemma, \(C\) is such that \(\overline{C} - C\) is non empty. Then, there exist a function \(g\) positive on \(C\) and vanishing on \(\overline{C} - C\). Define 
\[ B = \{x \in A \mid d_x g|_{T_x A \cap V} = 0\}. \]
Since \(T_x A \cap V\) admits a common complement in \(V\) (which is semianalytic), then \(B\) is also semianalytic. Since \(g\) is not constant on \(C\) but it is constant on \(B\) (which is a closed submanifold), then \(\dim(B) < \dim(A)\). Also, since \(g\) has a maximum on \(C\), then \(B \cap C \neq 0\) and \(\pi(B) = \pi(A)\). 
\[ \square \]
Remark 4.49. Let $X$ be a subanalytic set. Then the dimension of $X$ coincides with the maximum dimension of the sets $B_i$.

Example 4.50. Consider on $\mathbb{R}^3$ the union of the sphere of radius 1 and the plane $x = 0$ and its projection onto the plane $z = 0$. To get a partition as in the lemma, choose the two caps, the circle on $z = 0$ and the set defined by the zeros of $\{x, z, |y| > 1\}$.

Lemma 4.51. Assume that, in $U$, the complement of every subanalytic set is subanalytic. Let $B$ denote a bounded smooth semianalytic subset of $W$ such that $\pi|_B : B \to U$ is a local diffeomorphism. For every $u \in U$, let $\mu(u)$ denote the number of points in the fiber $B_u = B \cap \pi^{-1}(u)$. Then $\mu(u)$ is bounded on $U$.

Proof. Since $\pi$ is a local diffeomorphism from a bounded set, $\mu(u) < \infty$ for all $u \in U$. Clearly, $\mu(u)$ is lower semicontinuous. Define

$$C = \pi(B - B).$$

Clearly $C$ is a closed subanalytic subset of $U$ whose dimension is lower than $\dim(U)$. Therefore, $C$ is nowhere dense in $U$. Thus, proving the lemma on $U - C$ is enough. But, by hypothesis, $U - C$ is subanalytic and, therefore, has finitely many connected components and $\mu(u)$ is constant on each of them.

Definition 4.52. Let $\varphi : X \to Y$ be a map between sets. For any positive integer $s$, let $X^s_\varphi$ denote the $s$-fiber product

$$X^s_\varphi := X \times \varphi \cdots \times \varphi X = \{x = (x^1, \cdots, x^s) \mid \varphi(x^1) = \cdots = \varphi(x^s)\}$$

and let $\varphi : X^s_\varphi \to Y$ denote the induced map.

Lemma 4.53. Assume that, in $U$, the complement of every subanalytic set is subanalytic. Let $X$ be a relatively compact subanalytic subset of $W$. Suppose that the number of points $\mu(u)$ in the fiber $X_u = X \cap \pi^{-1}(u)$ is bounded on $U$. Then $W - X$ is subanalytic.

Proof. Denote

$$\Lambda_s = \{x \in W^s_\pi \mid x^i = x^j \text{ for some } i \neq j\}.$$ 

Then $X^s \cap (W^s_\pi - \Lambda_s)$ is a relatively compact subanalytic subset of $W^s$. Denote

$$C_s = \{u \in U \mid \mu(u) \geq s\}$$

and

$$D_s = \{u \in U \mid \mu(u) = s\}.$$ 

Then

$$C_s = \pi(X^s \cap (W^s_\pi - \Lambda_s))$$

and

$$D_s = C_s - C_{s+1}$$

are subanalytic. Moreover, there exists a number $t$ such that $U = \bigcup_{s=0}^{t} D_s$. We can split

$$W - X = \bigcup_{s=0}^{t} (\pi^{-1}(D_s) - X).$$

28
and each one of the pieces can be decomposed as the intersection
\[ \pi^{-1}(D_s) - X = \pi^{-1}(D_s) \cap p ((W \times X^s) \cap (W^{s+1}_x - \Lambda_{s+1})), \]
where \( p : W \times W^s \to W \) is the projection. Since \((W \times X^s) \cap (W^{s+1}_x - \Lambda_{s+1})\) is subanalytic in \( W \times W^s \) and \( \text{“W-relatively compact”} \) (its intersection with \( p^{-1}(K) \) is relatively compact for any compact set \( K \in W \)), then \( \pi^{-1}(D_s) - X \) is subanalytic. Therefore, \( W - X \) is subanalytic.

**Theorem 4.54 (Theorem of the complement).** Let \( M \) be a real analytic manifold. If \( X \) is a subanalytic subset of \( M \), then \( M - X \) is also subanalytic.

**Proof.** Remember that being subanalytic is a local condition on \( M \). Thus, choosing an appropriate coordinate system, we can assume that \( M \) is the \( n \)-dimensional Euclidean space \( W \) and that \( X \) is relatively compact (since it is the projection on a relatively compact set). Again, we proceed by induction on \( n \) (note that \( n = 0 \) is a trivial case). By definition, there exist a finite-dimensional vector space \( Z \) and a relatively compact semianalytic subset \( B \) of \( W \times Z \) such that \( X = \pi(B) \), being \( \pi : W \times Z \to W \) the projection. Let \( B_i \) be one of the subsets from the fiber-cutting lemma. This means that \( \pi(B) = \pi(\bigcup B_i) \) and in each \( B_i \), \( \pi|_{B_i} \) is an immersion and the \( \pi(T_xB_i) \) have a common complement \( V \) for all \( x \in B_i \). Depending on the dimension of \( B_i \), we need to distinguish two cases:

- If \( \dim(B_i) < n \), then decompose \( W = U \oplus V \) and let \( \pi_0 : U \oplus V \to U \) be the projection. By induction, since \( \dim(U) < n \), in \( U \) the complement of a subanalytic subset is subanalytic.
  Using lemma 4.53 for any \( u \in U, |B_i \cap (\pi_0\circ \pi)^{-1}| \) is bounded on \( U \). Therefore, \( |\pi(B_i) \cap \pi_0^{-1}| \) is also bounded and, using last lemma, the complement of \( X = \pi(B_i) \) is subanalytic.

- If \( \dim(B_i) = n \), then \( \pi|_{B_i} \) is a local diffeomorphism. Let \( C = \overline{B_i} - B_i \). Then \( \pi(C) \) is a subanalytic set of dimension \( < n \), and so \( W - \pi(C) \) is subanalytic (using first case). Note that \( W - \pi(\overline{B_i}) \) is open and closed in \( W - \pi(C) \) (since \( \pi(C) \) disconnects the space). Thus, it is also subanalytic. Decomposing
  \[ W - \pi(B_i) = (W - \pi(\overline{B_i})) \cup (\pi(\overline{B_i}) - \pi(B_i)) = (W - \pi(\overline{B_i})) \cup (\pi(C) - \pi(B_i) \cap \pi(C)) \]
and using that \( \pi(B_i) \cap \pi(C) \) is subanalytic of dimension \( < n \) then, applying the first case, we conclude that \( W - \pi(B_i) \) is subanalytic.

Finally, using that \( W - \pi(B) = \bigcap_i(W - \pi(B_i)) \), we finish the proof of the theorem.

**Proposition 4.55.** Let \( M \) be a real analytic manifold and let \( X \) be a closed subanalytic subset of \( M \). Then each point of \( X \) admits a neighborhood \( U \) such that \( X \cap U = \pi(A) \), being \( A \) a closed analytic subset of \( U \times \mathbb{R}^q \), with \( \dim(A) = \dim(X \cap U) \) and such that \( \pi|_A \) is proper.

**Proof.** Since every subanalytic set is the projection of a semianalytic set, and using the Fiber-cutting Lemma, we can assume that \( X \) is semianalytic. Let \( a \in X \). Then, there is a neighborhood \( U \) of \( a \) such that

\[ X \cap U = \bigcup_i Y_i \]

where \( Y_i = \{ x \in U | f_{ij} \geq 0, j = 1, \cdots , p_i \} \) and \( f_{ij} \in \mathcal{O}(U) \). Let \( Y \) be one of this sets and define the closed analytic subset

\[ A = \{(x,y) = (x,y_1,\cdots ,y_n) \in U \times \mathbb{R}^{p+1} | f_j(x) = y^2_j, j = 1, \cdots , p \}. \]
Trivially, \( \dim(A) = \dim(Y) \), \( Y = \pi(A) \) and \( \pi|_A \) is proper. Putting together all these sets \( A \) in a big enough space will give us the set we wanted.

Next proposition gives several equivalent definitions for subanalytic sets. It is interesting to remark it, but since we do not use it, we will not prove it. An interested reader can find a proof on [BM88].

**Proposition 4.56.** Let \( M \) be a real analytic manifold and \( X \) be a subset of \( M \). The following conditions are equivalent:

1. \( X \) is subanalytic.
2. Every point of \( M \) has a neighborhood \( U \) such that
   \[
   X \cap U = \bigcup_{i=1}^{p} (f_{i1}(A_{i1}) - f_{i2}(A_{i2}))
   \]
   where \( A_{ij} \) are compact analytic subsets of real analytic manifolds \( N_{ij} \) and \( f_{ij} : N_{ij} \to U \) are real analytic maps such that \( f_{ij}|_{A_{ij}} \) is proper.
3. Every point of \( M \) has a neighborhood \( U \) such that \( X \cap U \) belongs to the class of subsets \( U \) obtained using finite intersection, finite union and complement, from the family of closed subsets of \( U \) of the form \( f(A) \), where \( A \) is a closed analytic subset of a real analytic manifold \( N \), \( f : N \to U \) is real analytic and \( f|_A \) is proper.
5 Desingularization theorem and applications

In this section a proof of desingularization theorem is given, along with some of its applications. Desingularization theorem is based on the so called blow-ups. Therefore, first it is mandatory to recall its definition and prove a theorem about normal crossing functions. The proof of desingularization theorem can be found after that. As stated, the idea is to change a subanalytic set for a manifold and lift the LSS on the original manifold to this new manifold. To prove that solutions are lifted we need some consequences of desingularization theorem, which can also be found in [BM88], as well as more consequences of the desingularization theorem.

5.1 Transforming analytic functions to normal crossing using blow-ups

Many of the definitions and proofs of this section can be applied to the complex field, but since we are only working with real numbers, we will denote $\mathbb{P}^n$ the n-dimensional real projective space.

Definition 5.1. Let $V$ be an open neighborhood of 0 in $\mathbb{R}^n$. The blow-up of $V$ with center $\{0\}$ is the projection $\pi : V' \to V$ where

$$V' := \{(x, l) \in V \times \mathbb{P}^{n-1} | x \in l\}.$$ 

With this definition, $\pi$ is proper and restricts to a homeomorphism over $V - \{0\}$. Note that $\pi^{-1}(0) = \mathbb{P}^{n-1}$. The idea of a blow-up is to add (at the center) all the derivatives or directions.

Proposition 5.2. $V'$ is an algebraic and smooth submanifold of $V \times \mathbb{P}^{n-1}$.

Proof. Let $x = (x_1, \cdots, x_n)$ denote affine coordinates of $V$ and $\xi = [\xi_1, \cdots, \xi_n]$ denote projective coordinates of $\mathbb{P}^{n-1}$. Then

$$V' = \{(x, \xi) \in V \times \mathbb{P}^{n-1} | x_i \xi_j = x_j \xi_i, i, j = 1, \cdots, n\}$$

and so it is an algebraic submanifold. For analyticity, just use the coordinate charts $(V_i, (x_{ij}))$, $i, j = 1, \cdots, n$:

$$V'_i = \{(x, \xi) \in V' | \xi_i \neq 0\}$$

$$x_{ij} = x_i$$

$$x_{ij} = \xi_j/\xi_i \quad j \neq i$$

Next, we will extend the definition of blow-up to more general centers and, using the fact that outside the center a blow-up is a homeomorphism, we will blow-up manifolds.

Definition 5.3. Suppose that $m > n$ and $W$ is an open subset of $\mathbb{R}^{m-n}$. Then the mapping $\pi \times id : V' \times W \to V \times W$ is called blow-up of $V \times W$ with center $\{0\} \times W$.

Definition 5.4. Let $M$ be an analytic manifold and $Y$ a closed analytic submanifold of $M$. Define the blow-up with center $Y$ as the pair $(M', \pi)$ where $M'$ is an analytic manifold and $\pi : M' \to M$ is a proper analytic map such that:
1. \( \pi \) restricts to an analytic isomorphism \( M' - \pi^{-1}(Y) \to M - Y \).

2. Take any \((U, \psi)\) coordinate chart of \( M \) adapted to \( Y \), \( \psi : U \to V \times W \), where \( V \) and \( W \) are open neighborhoods of the origins in \( \mathbb{R}^n \) and \( \mathbb{R}^{m-n} \). Let \( \pi_0 : V' \to V \) be the blow-up of \( V \) with center \( \{0\} \). Then, there exists an analytic isomorphism \( \psi' : \pi^{-1}(U) \to V' \times W \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\psi'} & V' \times W \\
\downarrow \pi & & \downarrow \pi_0 \times \text{id} \\
U & \xrightarrow{\psi} & V \times W
\end{array}
\]

Note that this conditions define \( \pi \) uniquely (up to isomorphism of \( M' \) commuting with \( \pi \)). Also note that the condition is for any chart of \( M \), which is related to the definition of semianalytic and subanalytic sets.

**Definition 5.5.** Let \( M \) be an analytic manifold. Let \( U \) be an open subset of \( M \) and \( Y \) a closed analytic submanifold of \( U \). The local blow-up of \( M \) over \( U \) with smooth center \( Y \) is the composition \( \pi : U' \to M \) of the blow-up \( U' \to U \) with center \( Y \) and the inclusion \( U \to M \).

**Definition 5.6.** Let \( M \) be an analytic manifold and let \( \mathcal{O}(M) \) denote the ring of analytic functions on \( M \). Let \( f \in \mathcal{O}(M) \). We say that \( f \) is locally normal crossing if, for all \( x \in M \), there exists a neighborhood \( U \) of \( x \) with coordinates \( x = (x_1, \ldots, x_n) \) such that, in \( U \),

\[
f(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n} g(x) \quad \alpha_i \in \mathbb{N}
\]

where \( g \in \mathcal{O}(U) \) is nowhere vanishing.

Let \( K\{y\}, \ y = (y_1, \ldots, y_p) \), denote the set of convergent power series.

**Lemma 5.7.** Let \( y = (y_1, \ldots, y_p) \). Let \( \alpha, \beta, \gamma \in \mathbb{N}^p \) and let \( a(y), b(y), c(y) \) be invertible elements of \( \mathbb{R}\{y\} \). If

\[
a(y)y^\alpha - b(y)y^\beta = c(y)y^\gamma
\]

then either \( \alpha \leq \beta \) or \( \beta \leq \alpha \) (where \( \alpha \leq \beta \) means \( \alpha_k \leq \beta_k, k = 1, \ldots, p \)).

**Proof.** Put \( \delta = \inf(\alpha, \beta) \) where \( \alpha = (\alpha_1, \ldots, \alpha_p) \) and \( \beta = (\beta_1, \ldots, \beta_p) \). Let

\[
\delta = (\delta_1, \ldots, \delta_p).
\]

If \( \delta = \alpha \), then \( \alpha \leq \beta \). Otherwise, choose \( k \) such that \( \delta_k \neq \alpha_k \). Then, on the set \( \{y \mid y_k = 0\} \), we have

\[
y^{\alpha-\delta} = 0
\]

and

\[
0 \neq -b(y)y^{\beta-\delta} = c(y)y^{\gamma-\delta}.
\]

Since \( b \) and \( c \) are invertible, it follows that \( \beta = \gamma \). Then

\[
a(y)y^\alpha = (b(y) + c(y))y^\beta
\]

and \( \beta \leq \alpha \). \( \square \)
Theorem 5.8. Let $M$ be a analytic manifold. Let $f \in \mathcal{O}(M)$ non-identically zero on any component of $M$. Then, there exists a countable collection of analytic mappings $\pi_j : W_j \to M$ such that:

1. Each $\pi_j$ is the composition of a finite sequence of local blow-ups with smooth centers.
2. There exists a locally finite open covering $\{U_i\}$ of $M$ such that $\pi_j(W_j) \subset U_j$ for all $j$.
3. If $K \subset M$ is compact, then there are compact subsets $L_j \subset W_j$ such that $K = \bigcup_j \pi_j(L_j)$ (finite by 2).
4. For each $j$, $f \circ \pi_j$ is locally normal crossing on $W_j$.

We will call a countable collection of mappings $\{\pi_j : W_j \to M\}$ satisfying (1) – (3) a $\Sigma$-covering of $M$. Note that $\Sigma$-coverings can be composed.

Proof. Let $a \in M$. Let $\mathcal{O}_a$ denote the local ring of germs of analytic functions on $M$ at $a$ and let $m_a$ denote the maximal ideal of $\mathcal{O}_a$ (ideal of functions vanishing at $a$). Suppose that $f$ is an analytic function on a neighborhood $U$ of $a$. Let $f_a$ denote the germ of $f$ at $a$. Assuming $f_a$ is not identically zero, put

$$
\mu_a(f) := \max\{k \in \mathbb{N} \mid f_a \in m_a^k\} \text{ (multiplicity of $a$)}
$$

$$
\nu_a(f) := \min\{\mu_a(g) \mid f_a = g \prod_{i=1}^r l_i, \text{ where } g \in \mathcal{O}_a \text{ and } l_i \in m_a - m_a^2, i = 1, \ldots, r\} \text{ ("singular" multiplicity of $a$).}
$$

Clearly, $\nu_a(f) = 0$ if an only if either $f(a) \neq 0$ or $f_a$ is a product of the “smooth factors” $l_i$. Both $\mu_a(f)$ and $\nu_a(f)$ are upper semicontinuous as functions of $a \in M$. We will prove the result by induction on $m = \dim(M)$. If $m = 1$, then $f$ is already locally normal crossing (take $g = f/(x - a)^k$ where $k = \mu_a(f)$). Suppose $m > 1$. Let $a \in M$ and suppose that $f(a) = 0$ (if $f(a) \neq 0$, it is nonvanishing on a neighborhood of $a$ and normal crossing on that neighborhood).

Put $d = \nu_a(f)$. Then, in some neighborhood $U$ of $a$, $f$ factors as $f \sim l_1^{n_1} \cdots l_r^{n_r} g$ where $\mu_a(g) = 0$ and the $l_i$ are different factors such that $\mu_a(l_i) = 1$ distinct $\mu_a(f) = d + \sum n_i$.

If $F, G \in \mathcal{O}(U)$ (or $\mathcal{O}_a$), we will say that $F \sim G$ if they are equal up to a multiplication by an invertible factor in $\mathcal{O}(U)$ (or $\mathcal{O}_a$).

There are local coordinates $x = (x_1, \cdots, x_m)$ centered at $a$ such that

$$
f_a(0, \cdots, 0, x_m) \sim x_m^e
$$

where $e = \mu_a(f)$. It follows that $g_a(0, \cdots, 0, x_m) \sim x_m^d$ and $l_i(a)(0, \cdots, 0, x_m) \sim x_m$. By Weierstrass preparation theorem (Appendix B17), we can assume that $U = V \times D$ where $V \subset \mathbb{R}^{m-1}$, $D \subset \mathbb{R}$ are open neighborhoods of $0$, $a = 0$, and

$$
f(x) \sim l_1(x)^{n_1} \cdots l_r(x)^{n_r} g(x)
$$

$x = (x_1, \cdots, x_m) \in U$, where

- $l_i(x) = x_m + a_i(x_1, \cdots, x_{m-1})$, $i = 1, \cdots, r$
- $g(x) = x_m^d + \sum_{j=1}^d c_j(x_1, \cdots, x_{m-1})x_m^{d-j}$
- The $a_i$ are different. For each $i = 1, \cdots, r$, $a_i \in \mathcal{O}(V)$ and $a_i(0) = 0$. For each $j = 1, \cdots, d$, $c_j \in \mathcal{O}(V)$ and $\mu_0(c_j) \geq j$. 

33
\[ \{ x \in U \mid f(x) = 0 \} = \{ x \in V \times \mathbb{R} \mid l_1(x)^{n_1} \cdots l_r(x)^{n_r} g(x) = 0 \} \]

Clearly, we can assume that \( M = U = V \times \mathbb{R} \). Put \( \tilde{x} = (x_1, \cdots, x_{m-1}) \). If \( d > 0 \), then, after a coordinate transformation

\[
\begin{align*}
x'_k &= x_k, \quad k = 1, \cdots, m - 1 \\
x'_m &= x_m + \frac{1}{d} c_1(\tilde{x})
\end{align*}
\]

we can further assume that \( c_1(\tilde{x}) \equiv 0 \); i.e.,

\[ g(x) = x_m^d + \sum_{j=2}^{d} c_j(\tilde{x}) x_m^{d-j} \]

The significance of this representation is that, since \( \partial^{d-1} g/\partial x_m^{d-1} = d! x_m \), then \( \mu_x(g) = d \) only if \( x_m = 0 \). If \( d = 0 \), then, after a coordinate transformation

\[
\begin{align*}
x'_k &= x_k, \quad k = 1, \cdots, m - 1 \\
x'_m &= x_m + a_1(\tilde{x})
\end{align*}
\]

we can assume that \( a_1(\tilde{x}) \equiv 0 \). Let \( A_f(\tilde{x}) \) denote the product of all nonzero functions from the following list and all of their nonzero differences:

\[ a_i^{d_l} \quad i = 1, \cdots, r c_j^{d_l/j} \quad j = 2, \cdots, d \]

By induction, there exists a \( \Sigma \)-covering \( \{ \tilde{\pi}_k : V_k \rightarrow V \} \) such that each \( A_f \circ \tilde{\pi}_k \) is locally normal crossing in \( V_k \). Then \( \{ \tilde{\pi}_k \times \text{id} : V_k \times \mathbb{R} \rightarrow V \times \mathbb{R} \} \) is a \( \Sigma \)-covering of \( U \). Using the fact that we can compose \( \Sigma \)-coverings, to simplify the proof, we will assume that \( A_f(\tilde{x}) \) is locally normal crossing in \( V \) itself. Shrinking \( V \) if necessary, we can also assume that

\[ A_f(\tilde{x}) \sim \tilde{x}^\theta = x_1^{\theta_1} \cdots x_{m-1}^{\theta_{m-1}} \]

Then, each nonzero \( a_i(\tilde{x})^{d_l} \sim \tilde{x}^{a_i} \) and each nonzero \( c_j(\tilde{x})^{d_l/j} \sim \tilde{x}^{c_j} \), where \( \alpha^i = (\alpha_1^i, \cdots, \alpha_{m-1}^i) \in \mathbb{N}^{m-1} \) and \( \gamma^j = (\gamma_1^j, \cdots, \gamma_{m-1}^j) \in \mathbb{N}^{m-1} \). Moreover, by Lemma 5.7 these exponents are totally ordered with respect to the induced partial ordering from \( \mathbb{N}^{m-1} \).

Again, using the composition of \( \Sigma \)-coverings, it is enough to prove:

Case 1. \( d > 0 \). There is a (finitely indexed) \( \Sigma \)-covering \( \{ \pi_t : W_t \rightarrow U \} \) such that, for each \( t \), \( \nu_y(f \circ \pi_t) < d \), for all \( y \in W_t \).

Case 2. \( d = 0 \). There is a (finitely indexed) \( \Sigma \)-covering \( \{ \pi_t : W_t \rightarrow U \} \) such that, for each \( t \), \( f \circ \pi_t \) is locally normal crossing on \( W_t \).

Case 1. We will use an inductive argument. To set up the induction, it is convenient to begin with \( f \) of the following more general form:

\[ f(x) \sim \tilde{x}^\alpha l_1(x)^{n_1} \cdots l_r(x)^{n_r} l_s(x)^{n_s} g(x) \]

where \( \alpha \in \mathbb{N}^{m-1} \), \( r \leq s \), the \( l_i \) are distinct smooth factors, \( g \) and \( l_1, \cdots, l_r \) are as before and \( l_{r+1}, \cdots, l_s \), vanish nowhere on \( \{ x_m = 0 \} \) (We will begin with \( \alpha = 0 \) and \( s = r \)). Of course, \( \nu_x(f) = d \) only if \( \mu_x(g) = d \).
Clearly, where each $k\nu x x x$ Since, for every $x$ where $Z J x$ As we did before, $U$ blow-up with center $Z$. Then $\pi(x, \xi) = x_l \xi_k$, for $k, l \in I \cup \{m\}$

As we did before, $U'$ is covered by coordinate charts

$$U'_k = \{(x, \xi) \in U' | \xi_k \neq 0\}, \quad k \in I \cup \{m\}$$

where $U'_k$ has coordinates $y = (y_1, \cdots, y_m)$ such that

$$x_l = y_l, \quad l \notin I \cup \{m\}$$

$$x_l = y_k, \quad l \in (I \cup \{m\}) - \{k\}$$

Since, for every $x \in Z_I$, $a_i(x) = 0, i = 1, \cdots, r$ and $\mu_x(c_j) \geq j, j = 2, \cdots, d$, it follows that $\nu_y(f \circ \pi) = 0$ at each point $y$ of $U' - \bigcup_{k \in I} U'_k$. Therefore, if suffices to consider $f \circ \pi_k$ for each $k \in I$, where $\pi_k = \pi|U'_k$. Fix $k \in I$. If $y = (y_1, \cdots, y_m) \in U'_k$, put $\bar{y} = (y_1, \cdots, y_{m-1})$ and $\pi_k(\bar{y}) = \pi_k(y)$.

Clearly, $U'_k = V'_k \times \mathbb{R}$, where $V'_k = \{y \in U'_k | y_m = 0\}$, and $\pi_k : V'_k \rightarrow V$. Then

$$(l_i \circ \pi_k)(y) = y_k(y_m + a'_i(\bar{y})), \quad i = 1, \cdots, r$$

$$(g \circ \pi_k)(y) = y_k^d g(y)$$

where

$$a'_i(\bar{y}) = \frac{1}{y_k} (a_i \circ \pi_k)(\bar{y}) \in \mathcal{O}(V'_k), \quad i = 1, \cdots, r$$

$$g'(y) = y_m^d + \sum_{j=2}^d c'_j(\bar{y}) y_m^{d-j}$$

$$c'_j(\bar{y}) = \frac{1}{y_k} (c_j \circ \pi_k)(\bar{y}) \in \mathcal{O}(V'_k), \quad j = 2, \cdots, d$$
It follows that each nonzero \( a'_i(\tilde{y})^{d_l} \sim \tilde{y}^{\beta_i} \) and each nonzero \( c'_j(\tilde{y})^{d_j/\gamma} \sim \tilde{y}^{\delta_j}, \) where \( \beta^i = (\beta_1^i, \ldots, \beta^i_{m-1}) \in \mathbb{N}^{m-1}, \delta^j = (\delta_1^j, \ldots, \delta^j_{m-1}) \in \mathbb{N}^{m-1}, \) and
\[
\beta^i_l = \alpha^i_l, \text{ if } l \neq k \\
\beta^i_k = \sum_{l \in I} \alpha^i_l - d! \\
\delta^j_l = \gamma^j_l, \text{ if } l \neq k \\
\delta^j_k = \sum_{l \in I} \gamma^j_l - d!
\]

In particular, the exponents \( \beta^i \) and \( \delta^j \) are totally ordered in the same way as the \( \alpha^i \) and \( \gamma^j \). If \( \nu_y(f \circ \pi_k) < d \) for all \( y \in U'_k \), we are done. Suppose \( \nu_y(f \circ \pi_k) = d \) for some \( y = (y_1, \ldots, y_m) \in U'_k \). It follows that \( \mu_y(g') = d \), and hence that \( y_m = 0 \). Therefore, since each \( (c'_j)^{d_j/\gamma} \sim \tilde{y}^{\delta_j} \), we have \( \mu_0(g') = d \). Likewise, for each \( i = 1, \ldots, r \), if \( a'_i(\tilde{y}) = 0 \), \( \tilde{y} \in V'_k \), then \( a'_i(0) = 0 \).

Let \( \tau = (\tau_1, \ldots, \tau_{m-1}) \) denote the smallest among the nonzero exponents \( \beta^i \) and \( \delta^j \); then \( \tau \) is associated with \( f \circ \pi_k \) in the same way as \( \sigma \) is associated with \( f \). Let \( q \) denote the number of indices \( i = 1, \ldots, r \) such that \( a'_i(0) = 0 \). If \( q = r \), then
\[
\tau_k = \sigma_k, \text{ if } l \neq k \\
\tau_k = \sum_{l \in I} \sigma_l - d!
\]
in particular, \(|\tau| < |\sigma|\). Therefore, either \( q < r \) or \( q = r \) and \(|\tau| < |\sigma|\). It follows that, after transforming \( f \) by a \( \Sigma \)-covering involving finitely many sequences of at most \(|\sigma|/d! \) local blows-up over successive coordinate charts, as above, either \( r \) or \( d \) must decrease. Case 1 follows by induction.

Case 2. To set up an appropriate induction, it is again convenient to begin with \( f \) of a more general form
\[
f(x) \sim \tilde{x}^\alpha l_1(x)^{n_1} \cdots l_r(x)^{n_r} \cdots l_s(x)^{n_s}
\]
where \( \alpha \in \mathbb{N}^{m-1} \) and the \( l_i \) are distinct smooth factors \( l_i(x) = x_m + a_i(\tilde{x}), a_i(0) = 0 \), such that \( a_1(\tilde{x}) \equiv 0 \) and \( A_f(\tilde{x}) \sim \tilde{x}^\theta, \theta \in \mathbb{N}^{m-1} \), where \( A_f \) is the product of all nonzero \( a_i \) and their differences. (At the beginning, we have \( \alpha = 0 \).) In particular, the exponents of the nonzero \( a_i(\tilde{x}) \sim \tilde{x}^\alpha \) are totally ordered. As before, let \( \sigma \) denote the smallest among these exponents. Then \(|\sigma| = \sum_{k=1}^{m-1} \sigma_k \geq 1 \). Let
\[
Z = \{ x \in U \mid l_i(x) = 0, i = 1, \ldots, r \} \\
= \{ x \in U \mid x_m = 0 \text{ and } \sum_{k \in J(x)} \sigma_k \geq 1 \}
\]
where \( J(x) = \{ k \mid x_k = 0, k = 1, \ldots, m - 1 \} \). For each \( k = 1, \ldots, m - 1 \), let
\[
Z_k = \{ x \in U \mid x_k = x_m = 0 \}.
\]
The $Z_k$ such that $\sigma_k \geq 1$ are the irreducible components of $Z$. Let $\pi : U' \to U$ be the blow-up with center $Z_k$, for some $k = 1, \ldots, m - 1$ such that $\sigma_k \geq 1$. Then

$$U' = \{(x, \xi) \in U \times \mathbb{P}^{m-1} \mid \xi_l = 0, l \neq k, m \text{ and } x_k \xi_m = x_m \xi_k\}$$

and $U' = U'_k \cup U'_m$, where, for $l = k, m, U'_l$ is the coordinate chart $\{(x, \xi) \in U' \mid \xi_l \neq 0\}$. Let $\pi_l = \pi_{U'}$. The chart $U'_m$ has coordinates $y = (y_1, \ldots, y_m)$ in which $\pi_m$ is given by

$x_k = y_k y_m, \quad x_m = y_m \text{ and } x_l = y_l$ when $l \neq k, m$. Let

$$X'_m = \bigcup_{i=1}^r \{y \in U'_m \mid 1 + a_i(\overline{\pi_m(y)})/u_m = 0\}.$$

Then $X'_m$ is a closed analytic subset of $U'_m$. Clearly, $X'_m \cap (U'_m - U'_l) = \emptyset$ and $f \circ \pi_m$ is a locally normal crossing on $U'_m - X'_m$.

The chart $U'_k$ has coordinates $y = (y_1, \ldots, y_m)$ in which $\pi_k$ is given by $x_k = y_k, x_m = y_k y_m$ and $x_l = y_l, l \neq k, m$. Let $\tilde{y} = (y_1, \ldots, y_{m-1})$ and $\tilde{\pi}_k(\tilde{y}) = \pi_k(y)$. Clearly, $U'_k = V'_k \times \mathbb{R}$, where $V'_k = \{y \in U'_k \mid y_m = 0\}$, and $\tilde{\pi}_k : V'_k \to V$. Let $f' = f \circ \tilde{\pi}_k$. Then

$$f'(y) \sim \tilde{y}^\beta l'(y)^{m_1} \cdots l'(y)^{m_r}$$

where $\beta = (\beta_1, \ldots, \beta_{m-1}) \in \mathbb{N}^{m-1}$, with $\beta_l = \alpha_l$ when $l \neq k$ and $\beta_k = \alpha_k + n_1 + \cdots + n_r$, and where

$$l'_i(y) = y_m + a'_i(y)$$

$$a'_i(y) = \frac{1}{y_k}(a_i \circ \tilde{\pi}_k)(\tilde{y}) \in \mathcal{O}(V'_k)$$

Therefore, each nonzero $a'_i(\tilde{y}) \sim \tilde{y}^{\beta_i}$, where $\beta^i = (\beta^i_1, \ldots, \beta^i_{m-1}) \in \mathbb{N}^{m-1}, \beta^i_l = \alpha^i_l - 1$, and $\beta^i_k = \alpha^i_k$ for $l \neq k$.

Suppose that $a'_i(\tilde{y}) = 0$, $i = 1, \ldots, r$. Then $A_{f'}(\tilde{y}) \sim \tilde{y}^\sigma$, where $\varphi \in \mathbb{N}^{m-1}$, and the $\beta^i$ are totally ordered in the same way as the $\alpha^i$. Let $\tau = \min \beta^i$. Then $\tau_k = \sigma_k - 1$ and $\tau_l = \sigma_l$, $l \neq k$. Thus, $1 \leq |\tau| = |\sigma| - 1$. Therefore, after repeating the process of blow-up $|\sigma|$ times, we can assume that $a'_{i_0} \neq 0$ for some $i_0 = 2, \ldots, r$.

Let $b'_m, p = 1, \ldots, s$, denote the distinct values $-a'_i(0), i = 1, \ldots, r$. Since $a'_i \equiv 0$, then $2 \leq s \leq r$. For each $p$, let

$$I(p) = \{i \in [1, r] \mid b'_m = -a'_i(0)\}$$

Choose $i(p) \in I(p)$. Put $U^p = U'_k$, with coordinates $z = (z_1, \ldots, z_m)$ centered at $b^p = (0, \ldots, 0, b^p_m)$ defined by

$$z_l = y_l \quad l = 1, \ldots, m - 1$$

$$z_m = y_m + a'_{i(p)}(\tilde{y})$$

Then, for each $i = 1, \ldots, r, l'_i(y) = l'_{i}(z)$, where $l'_{i}(z) = z_m + a'_i(z)$ and $a'_i(z) = a'_i(\tilde{z}) - a'_{i(p)}(\tilde{z})$. Put

$$X^p = \{z \in U^p \mid l'_{i}(z) = 0, \text{ for some } i \notin I(p)\}$$

37
Since each $a'_i(z) - a'_j(z) \sim z^{\gamma_{ij}}$, for some $\gamma_{ij} \in \mathbb{N}^{m-1}$, it follows that $X^p \cap \{z \mid l''_i(z) = 0\} = \emptyset$ for all $i \in I(p)$. In $U^p - X^p$,

$$f^p(z) \sim z^{\gamma} \prod_{i \in I(p)} l''_i(z)^{m_i}, \quad z \in U^p$$

where $\gamma \in \mathbb{N}^{m-1}$, $a''_i(p) = 0$, and $A_{\gamma''}(z) \sim z^\psi$ for some $\psi \in \mathbb{N}^{m-1}$. But $I(p)$ has fewer than $r$ elements.

Since the $U^p - X^p$, $p = 1, \ldots, s$, together with $U'_m - X'_m$, cover $U'$, case 2 follows by induction on $r$.

\[\square\]

**Remark 5.9.** The proof shows that there is a countable collection of analytic maps $\pi_j : W_j \to M$ satisfying conditions (1) – (4) in the theorem and having the following additional property: Write each $\pi_j = \pi_{j1} \circ \cdots \circ \pi_{jk(j)}$ where for each $k = 1, \ldots, k(j)$, $\pi_{jk} : U_{jk} \to U_{jk}$ is a local blow up of $U_{jk}$ with smooth center $Y_{jk}$ (denote $U_{j1} = M$ and $U_{jk+1} = W_j$). Let $E_k$ denote the union of the inverse images in $U_{jk}$ of $Y_{j1}, \ldots, Y_{jk-1}$. When $k = k(j) + 1$, these hypersurfaces are transverse.

### 5.2 Desingularization theorem

**Definition 5.10.** Let $M$ be a connected $m$-dimensional real analytic manifold and let $X \subset M$. A desingularization of $X$ is a proper analytic map $\psi : N \to M$ such that $\psi(N) = X$ and $N$ is a real analytic manifold of the same dimension as $X$.

**Theorem 5.11** (Uniformization theorem). Let $X$ be an analytic subset of $M$. Then, there exists a desingularization of $X$.

**Proof.** Let $a \in M$. Let $X_a$ denote the germ of $X$ at $a$. Let $f_1, \ldots, f_n$ be real analytic functions defined in a neighborhood $U$ of $a$, such that

$$X \cap U = \{x \in U \mid f_i(x) = 0, i = 1, \ldots, n\}.$$

Let $r = \dim(X_a)$. We can assume that there exists a analytic subset $Z$ of $U$ such that $\dim(Z) < r$ and $X \cap U - Z$ is pure smooth and of dimension $r$. It is sufficient to find a compact real analytic manifold $N$ such that $\dim(N) = r$, and a real analytic map $\psi : N \to M$ such that $\psi(N) \subset X \cap U$ and $\psi(N)$ is a neighborhood (maybe not open) of $a$ in $X \cap U - Z$. We will do this for any $a \in X$ and, then, we will “patch together” all these manifolds. To prove this, we proceed by induction on $\text{codim}(X_a) = m - r$. If $\text{codim}(X_a) = 1$, the result holds by theorem 5.8. Define $f = f_1 \cdots f_n$. Using theorem 5.8, there is a countable collection of real analytic $\pi_j : W_j \to U$ such that:

1. Each $\pi_j$ is the composition of a finite sequence of local blow-ups with smooth submanifolds as centers.

2. For each $j$ there is a compact subset $L_j \subset W_j$ such that $\bigcup_j \pi_j(L_j)$ is a neighborhood of $a$ in $U$.

3. For each $j$, $f \circ \pi_j$ is locally normal crossing on $W_j$.

38
Denote \( U_1 = U \) and \( U_{j,k+1} = W_j \). Then, for each \( j \), write \( \pi_j = \pi_{j,1} \circ \cdots \circ \pi_{j,k_j} \), where, for each \( k = 1, \ldots, k_j \), \( \pi_{j,k} : U_{j,k+1} \to U_{j,k} \) is a local blow-up of \( U_{j,k} \) over an open subset \( V_{j,k} \subset U_{j,k} \), with center a closed analytic submanifold \( Y_{j,k} \subset V_{j,k} \). Denote, for each \( i = 2, \ldots, k_j + 1, E_{j,i} \) the union of the inverse images in \( U_{j,i} \) is \( Y_{j,1}, \ldots, Y_{j,i-1} \). We can assume that \( E_{j,i} \) is the union of smooth hypersurfaces in \( U_{j,i} \) and, when, \( i = k_j + 1 \), these hypersurfaces are transverse \([5,9]\). Choosing \( U \) small enough, suppose that \( V_{j,i} = U_{j,i} = U \). For each \( j \) and \( i \), denote \( X_{j,i} = X \cap U \) and define

\[
X_{j,i+1} = \pi_{j,i}^{-1}(X_{j,i} \cup Y_{j,i}).
\]

We can assume that, for each \( j \) and \( i \), there exists \( a_{ji} \in U_{ji} \) such that \( V_{ji} \) is an open neighborhood of \( a_{ji} \), small enough so that:

1. \( X_{ji} \cap V_{ji} \) is a finite union of irreducible analytic subsets \( X_{ji,l} \) of \( V_{ji} \).
2. For each \( l \), every connected component of the smooth points of \( X_{ji,l} \) is adherent to \( a_{ji} \).

For each \( j \) and \( i \), let \( L(j, i) \) denote the set of those \( l \) such that \( X_{ji,l} \) is not an irreducible component of \( E_{ji} \). If \( l \in L(j, i) \), then \( \dim(X_{ji,l}) \leq r \). Suppose that \( X_{ji,l} \subset Y_{ji} \), where \( l \in L(j, i) \) and \( \dim(X_{ji,l}) = r \). Since \( \dim(Y_{ji}) = m \), the codimension of \( X_{ji,l} \) in \( Y_{ji} \) is less than codimension of \( X_{j1} \) in \( U \). By induction, there is a real analytic manifold \( N'_{ji,l} \) of dimension \( r \), and a proper real analytic map \( \psi'_{ji,l} : N'_{ji,l} \to Y_{kl} \) such that \( \psi'_{ji,l}(N'_{ji,l}) \subset X_{ji,l} \) and \( \psi'_{ji,l}(N'_{ji,l}) \) includes the smooth points of dimension \( r \) of \( X_{ji,l} \). Therefore, there is a compact real analytic manifold \( N_{ji,l} \) of dimension \( r \), and a real analytic map \( \psi_{ji,l} : N_{ji,l} \subset Y_{jl} \) such that \( \psi_{ji,l}(N_{ji,l}) \subset X_{ji,l} \) and \( \psi_{ji,l}(N_{ji,l}) \) includes the smooth points of dimension \( r \) of \( X_{ji,l} \) within some neighborhood of the image of \( L_j \) in \( U_{ji} \). Now, for each \( j \), \( \prod_{i=1}^{n}(f_i \circ \pi_j)(x) \) is locally normal crossing in \( W_j \); therefore, we can find finitely many points \( a_{jp} \) of \( L_j \) such that:

1. For each \( p \), there is a neighborhood \( W_{jp} \) of \( a_{jp} \), with coordinates \( x = (x_1, \ldots, x_n) \) centered at \( a_{jp} \) in which

\[
\prod_{i=1}^{n}(f_i \circ \pi_j)(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n} u(x)
\]

where \( u \) is an analytic function vanishing nowhere in \( W_{jp} \).
2. There is a positive number \( \varepsilon_{jp} \) such that the balls \( B_{jp}(0, \varepsilon_{jp}) \) cover a neighborhood of \( L_j \) in \( W_j \).

Define \( X_{jp} \) as the zeros of \( \prod_{i=1}^{n}(f_i \circ \pi_j)(x) \). It is clear that \( X_{jp} \) is the union of the coordinate subspaces of \( W_{jp} \). Write \( X_{jp} = X'_{jp} \cup E'_{jp} \), where \( E'_{jp} \) is the union of the irreducible components of \( X_{jp} \) lying in \( E_{jp} \). Denote \( X_{j,pq} \) the irreducible components of \( X'_{jp} \) of dimension \( r \). For each \( p \) and \( q \), denote \( S_{j,pq} \) the standard \( r \)-dimensional sphere of radius \( \varepsilon_{jp} \) and \( \phi_{j,pq} : S_{j,pq} \to W_j \) the standard map onto the ball \( B_{jp} \cap X_{j,pq} \). Finally, take \( N \) as the disjoint union of all \( N_{ji,l} \) and \( S_{j,pq} \), and \( \psi : N \to M \) the map defined by \( \pi_{j,1} \circ \cdots \circ \pi_{jmi-1} \circ \psi_{ji,l} \) on each \( N_{ji,l} \) and by \( \pi_j \circ \phi_{j,pq} \) of each \( S_{j,pq} \).

**Corollary 5.12 (Desingularization theorem).** Let \( X \) be a closed subanalytic subset of \( M \). There exists a desingularization of \( X \).

**Proof.** Use proposition \([4.59]\) and then the Uniformization theorem. \( \square \)
Example 5.13. Consider $X$ the zero set of the function $p(x, y) = x^2 - y^3$. From basic algebraic geometry [Sha94], an irreducible algebraic variety $X$ is singular at a point $a$ if and only if the dimension of the variety is smaller than the dimension of the tangent space at $a$. In this case $X$ has dimension 1 and the tangent space is defined by the kernel of the differential

$$dp = (2x, -3y),$$

which has dimension 2 at the point $0 = (0, 0)$. The blow-up of $X$ at 0 is the set

$$V = \{(x, y; u, v) \in \mathbb{R}^2 \times \mathbb{P}^1 \mid x^2 = y^3, xv = yu, u + v = 0\}.$$

Since the differential

$$\begin{pmatrix} 2x & -3y & 0 & 0 \\ v & u & y & x \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

has constant rank 2 on $V$, the dimension of the tangent space is 1 and $V$ is non-singular. $V$ with the induced projection is a desingularization of $X$.

5.3 Subanalytic curves

**Lemma 5.14.** Let $I = [t_0, t_1]$ be an interval and $x : I \to M$ be a continuous subanalytic curve.

1. If its graph is a relatively compact subset of $\mathbb{R} \times M$, then there exist a uniquely defined continuous subanalytic prolongation $\tilde{x} : I \to M$.

2. If its graph is not a relatively compact subset then it is closed.

*Proof.*

1. Let $I = [t_0, t_1)$. First, we prove that there exists the limit

$$x_1 = \lim_{t \to t_1^-} x(t).$$

We know that the closure of a subanalytic set is subanalytic. Thus, $\overline{\Gamma}_x$ is subanalytic and compact and $\overline{\Gamma}_x \cap (\{t_1\} \times M)$ is also subanalytic, compact and non-empty. Let $(t_1, x_1) \in \overline{\Gamma}_x \cap (\{t_1\} \times M)$. Then (using an appropriate coordinate chart), for any small $\varepsilon > 0$, the set

$$\Gamma^\varepsilon_x = \Gamma_x \cap ((t_0, t_1) \times B_\varepsilon(x_1))$$

is nonempty and relatively compact. Thus, it has a finite number of connected components, which we will denote by $C_i$, $i = 1, \ldots, n_x$. Then, $C_i = \text{graph}(x|_{(\alpha_i, \beta_i)})$, with $\beta_i \leq \alpha_{i+1}$, $i = 1, \ldots, n_x - 1$ (reordering if necessary). Since $x_1$ is a limit point of $\Gamma^\varepsilon_x$, then $\beta_{n_x} = t_1$ which implies that $x(t) \in B_\varepsilon(x_1)$ for all $t \in (\alpha_{n_x}, \beta_{n_x})$ and, therefore, the extension is continuous.

The fact that it is subanalytic comes again from the fact that the closure of a subanalytic set is subanalytic.

2. Now suppose that $\Gamma_x$ is not relatively compact. Then $\overline{\Gamma}_x \cap (\{t_1\} \times M)$ must be empty (if not, proceeding as in first part, we would prove that $\Gamma_x$ was relatively compact). Since all limit points of $\Gamma_x$ are in $\overline{\Gamma}_x \cap (\{t_1\} \times M)$, then $\Gamma_x$ is closed. 

\[\square\]
Lemma 5.15. Let $M$ be a real analytic manifold.

1. Let $A \subseteq M$ be a semianalytic subset of dimension 1. Let $a \in \overline{A}$. Assume $A - \{a\}$ is locally connected at $a$. Then there exists $\varepsilon > 0$ and a real analytic mapping $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = a$ and $\gamma([0, \varepsilon))$ is a neighborhood of $a$ in $A$.

2. Conversely, let $\gamma : I \to M$ be a real analytic mapping, where $I$ is an interval containing $0 \in \mathbb{R}$. If $\gamma$ is not constant, then there exists $\varepsilon > 0$ such that $\gamma((0, \varepsilon))$ is a smooth semianalytic subset of $M$.

Proof. 1. Immediate from the desingularization theorem.

2. For any $\varepsilon > 0$,
$$
\Gamma_{\gamma|(0,\varepsilon)} = \{(t, x) \in \mathbb{R} \times M \mid 0 < t < \varepsilon, x = \gamma(t)\}.
$$

\square

Theorem 5.16. Let $M$ be a real analytic manifold and let $X$ be a subanalytic subset of $M$. Then:

1. If $\dim(X) \leq 1$, $X$ is semianalytic.

2. If $\dim(M) \leq 2$, $X$ is semianalytic.

Proof. 1. Let $N$ be a real analytic manifold and let $\pi : M \times N \to M$ be the projection. Using the fiber-cutting lemma (lemma 4.48), it is enough to prove that, if $X$ is a relatively compact semianalytic subset of $M \times N$ and $\dim(X) = 1$, then $\pi(X)$ is semianalytic. By lemma 5.15,1, $X$ is locally a union of finitely many sets of the form $A = \gamma((0, \varepsilon))$, where $\gamma : (-\varepsilon, \varepsilon) \to M \times N$ is a nonconstant analytic mapping, perhaps together with a point. Each $\pi(A) = (\pi \circ \gamma)((0, \varepsilon))$ is a semianalytic, by lemma 5.15.

2. $\overline{X} - \text{int}(X)$ and $\overline{X} - X$ are subanalytic of dimension $\leq 1$. Using (1), they are semianalytic. But $\overline{X}$ is the union of $\overline{X} - \text{int}(X)$ and some connected components of its complement, hence semianalytic. Therefore, $X = \overline{X} - (\overline{X} - X)$ is semianalytic.

\square
6 Solutions

Let \((M, F, \pi, A, \sigma)\) be an analytic LSS. Let \(X\) be a closed subanalytic subset of \(M\). Using the desingularization theorem, there exists a desingularization \(\psi : M' \to M\) of \(X\). Now, we can use lemma 3.8 to construct another LSS which we will denote \((M', F', \pi', A', \sigma')\).

We already proved that every solution of the LSS on \(M'\) can be projected onto a solution of the original system in \(M\) (Proposition 3.10). In this section, we deal with the converse question: can all solutions of the original system be lifted to this new system?

Unfortunately, the answer is “No”. The so-called “as-solutions” (analytic solution with semianalytic graph) will be partially lifted. The main result of this section states that, if \(X\) is an appropriate set and \(\xi : I \to M\) is an as-solution whose image is a subset of \(X\) then \(\xi\) is “piecewise” lifted. This may seem a weak result but, using the desingularization process, \(X\) is desingularized into a manifold with (usually) many connected components. Thus, there are solutions that cannot be lifted as a continuous solution. This is done by generalizing some results that can be found in [CE06].

Theorem 3.16 states that a subanalytic curve is semianalytic. Therefore, from now on we will not differentiate between these two kinds of sets.

Definition 6.1. Let \(I \subset \mathbb{R}\) be an interval. An analytic-semianalytic curve (or as-curve) \(\gamma : I \to M\) is a continuous map such that

- \(\gamma|_I\) is analytic.
- \(\gamma\) is semianalytic. (i.e., its graph \(\Gamma_\gamma \subset \mathbb{R} \times M\) is semianalytic.)

Lemma 6.2. 1. Let \(\gamma : I \to M\) be an analytic map, where \(I = (t_0, t_1)\). Then any map \(\gamma|_{(\tilde{t}_0, \tilde{t}_1)} : (\tilde{t}_0, \tilde{t}_1) \to M\), where \(t_0 < \tilde{t}_0 < \tilde{t}_1 < t_1\), is an as-curve in \(M\). This result extends to non-open subintervals.

2. Let \(\gamma : I \to M\) be an as-curve and assume that its graph \(\Gamma_\gamma\) is a relatively compact subset of \(\mathbb{R} \times M\). Then, there is a unique continuous extension \(\tilde{\gamma} : \tilde{I} \to M\). Moreover, this extension is an as-curve.

Proof. 1. Just notice that \(\Gamma_{\gamma|_{(\tilde{t}_0, \tilde{t}_1)}}\) is a the subset of \(\mathbb{R} \times M\) defined as

\[
\{(t, x) \in I \times M \mid \tilde{t}_0 < t < \tilde{t}_1, x = \gamma(t)\},
\]

so it is a semianalytic subset.


Example 6.3. Let \(f(x) = x \sin(1/x)\) and \(X = f(\mathbb{R}^+)\). This set is not a semianalytic subset of \(\mathbb{R}^2\), but the restriction \(f|_{(\varepsilon, \infty)}\) is an as-curve for any \(\varepsilon > 0\).

Lemma 6.4. 1. Let \(\gamma : I \to M\) be an as-curve and assume that there is a continuous extension \(\tilde{\gamma} : \tilde{I} \to M\). Then \(\tilde{\gamma}\) is an as-curve.

2. Let \(\gamma : I \to M\) an as-curve. Then, for any \(\tilde{I} \subset I\), \(\gamma : \tilde{I} \to M\) is an as-curve.
Proof. 1. By corollary 4.32, the closure of a semianalytic set is semianalytic. 2. It is clear by the local description of semianalytic sets.

Lemma 6.5. Let $X$ be the graph of a function $f : N \to M$. Then $X$ is a $C^k$-submanifold if and only if $f \in C^k(N, M)$.

Lemma 6.6. Let $\gamma : I \to M$ be a continuous semianalytic map. Then $\gamma$ is piecewise an as-curve.

Proof. Since the restriction of a semianalytic map to a compact interval $I' \subset I$ is semianalytic, we can assume that $I$ is compact. Moreover, assume there exists a local chart $(U, x)$ such that $\gamma(I) \subset U$. Since $\gamma$ is a continuous map and $\Gamma_\gamma$ is compact, using the fiber-cutting lemma, $\Gamma_\gamma$ is the union of finitely many pure analytic manifolds $A_i$ such that

$$\pi : \mathbb{R} \times U \to \mathbb{R}$$

is an immersion restricted on each one. By lemma 6.4, $\gamma|_{\pi^{-1}(\mathbb{R})}$ is an as-curve.

Lemma 6.7. Let $f : M \to N$ be an analytic map. Let $\gamma : I \to M$ be an as-curve. Then $(f \circ \gamma)|_{I'}$ is an as-curve for any compact interval $I' \subset I$. Moreover, if the graph $\Gamma_\gamma$ is relatively compact, then $(f \circ \gamma)$ itself is an as-curve and $\Gamma_{f \circ \gamma}$ is relatively compact.

Proof. Note first that $f \circ \gamma : I \to N$ is an analytic map. We may assume that $I$ is open or compact. If $I$ is open, then $I' \subset I$, then (lemma 6.2) $(f \circ \gamma)|_{I'}$ is semianalytic, thus an as-curve. If $I$ is compact, then

$$\Gamma = \Gamma_{\gamma|_{I'}}$$

is a semianalytic compact subset. Therefore, $\Gamma_{f \circ \gamma|_{I'}} = \Gamma_{f|_{I'}}$ is semianalytic compact (lemma 6.2). Therefore, $(f \circ \gamma)$ is an as-curve.

If $\Gamma_\gamma$ is relatively compact, using lemma 5.14 there exists a continuous extension $\overline{\gamma}$ of $\gamma$. Apply this lemma to $\overline{\gamma} : \overline{I} \to M$ and restrict into the original interval.

Lemma 6.8. Let $U$ be a finite dimensional vector space. Let $\gamma : I \to U$ an analytic immersion such that there exists a common complement $V$ of all $\gamma'(t)$, $t \in I$. Then, $\gamma$ is an embedding.

Proof. Claim: each plain parallel to $V$ intersects (at most) one time with $\gamma(I)$. Denote $V_x$ the parallel plane to $V$ which passes through $x$. Consider the canonical inclusion $V_x \subset T_x U$. Then, $V = \ker(\alpha)$ for some $\alpha \in U^*$. Consider the map

$$\alpha \circ \gamma = \langle \alpha, \gamma(\cdot) \rangle : I \to \mathbb{R}.$$ 

Suppose there exist two points $t_0, t_1 \in I$ such that $\gamma(t_0), \gamma(t_1) \in V_x$. Then $\alpha(\gamma(t_0)) = \alpha(\gamma(t_1))$. Therefore, there exists a $t_2 \in (t_0, t_1)$ such that

$$0 = D(\alpha \circ \gamma)(t_2) = \alpha \cdot \gamma'(t_2)$$

and $\gamma'(t_2) \in V_{\gamma(t_2)}$, which contradicts the hypothesis.
To prove that $\gamma$ is an embedding, we only need to check that it is an open map. Let $(t_0, t_1) \subset I$ an open interval. Let $z_i = \alpha(\gamma(t_i))$. Consider $W \subset U$ the open set

$$W = \{ u \in U \mid z_1 < \alpha(u) < z_2 \}$$

$U$ is clearly an open set and, due to the injectivity of $\alpha$ over $\gamma(I)$,

$$\gamma((t_0, t_1)) = W \cap \gamma(I),$$

thus $\gamma$ is open.

**Proposition 6.9.** Let $I$ be a compact interval and $M$ be a given manifold. Let $\gamma : I \to M$ be an as-curve which is not a constant. Then, there exists a finite family of open intervals $I_i \subset I$ such that

- $\gamma|_{I_i}$ is a diffeomorphism with its image.
- $\gamma|_{I_i}$ is a homeomorphism with its image.
- $\bigcup_i \gamma(I_i) = \gamma(I)$.

**Proof.** Assume $\gamma(I) \subset U$, where $(U, x)$ is a coordinate chart. Notice that $I$ is homeomorphic to $\Gamma_\gamma$ and $\bar{I}$ is diffeomorphic to $\Gamma_{\gamma|_{\bar{I}}}$. Since $I$ is compact, by lemma 6.7, $\Gamma_{\gamma(I)}$ is a semianalytic set. Since it is also compact, use the fiber-cutting lemma for the second projection

$$\pi_2 : \mathbb{R} \times U \to U$$

to get a family of open subsets $I_i \subset I$ such that

- For each $i$, $\gamma|_{I_i}$ is a local diffeomorphism.
- $\bigcup_i \gamma(I_i) = \gamma(I)$.

By lemma 4.51, the number of points in the fibers of $\gamma|_{I_i}$ is bounded, so it is also bounded on $\gamma|_{\bar{I}}$. This lets us (using compactness) to refine this partition in such a way that $\gamma|_{I_i}$ is injective. Let $I_i$ one of these new sets. Apply lemma 6.8 to get the result.

Before continuing with our lemmas, we need introduce some notions that will be studied with detail in the next section. Consider a LSS $(M, F, \pi, A, \sigma)$, where $M$ is a $n$-dimensional analytic manifold. Define

$$S_i(M) := \{ x \in M \mid \text{rank}(A(x)) \leq i \} \quad 0 \leq i \leq n$$

$$L_i(M) := \{ x \in S_i(M) \mid \text{dim}(\langle \sigma(x), \text{Im}(A(x)) \rangle) \leq i \} \quad 0 \leq i \leq n$$

**Lemma 6.10.** $S_i(M)$ and $L_i(M)$ are closed subanalytic subsets of $M$.

**Proof.** Locally, these sets are given by determinant conditions on the entries of $A(x), \sigma(x)$. 

44
Let
\[ S_{k_1}(M) \subset S_{k_2}(M) \subset \cdots \subset S_{k_r}(M) = M \]
where \( S_i(M) \) are supposed to be nonempty and distinct. Moreover, consider the corresponding chain
\[ L_{k_1}(M) \subseteq L_{k_2}(M) \subseteq \cdots \subseteq L_{k_r}(M). \]
Depending on these sets, use this chart to define the sets \( M_0, M_1 \) and \( M_2 \):

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<td>( M_0 )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( M_1 )</td>
<td>( M )</td>
</tr>
</tbody>
</table>

In the next section we will discuss these sets with more detail. But in a first approach (studying these sets on a local chart), the LSS has the form \( A(x) \dot{x} = \sigma(x) \) and, fixed \( x \in M \), can be considered as a linear system on \( \dot{x} \). Moreover, \( M_2 \) are the points where the system has maximal rank (therefore, there are solutions), \( M_1 \) are points where the linear system has no solution, and \( M_0 \) are the remaining points, yet to be analyzed.

**Proposition 6.11.** If \( M_0 \) is nonempty, there exists a desingularization of \( M_0 \).

**Proof.** Clearly, \( M_0 \) is always a subanalytic set.

A definition just to clarify notation:

**Definition 6.12.** A map \( \gamma : I \rightarrow M \) is an as-solution of a LSS \( (M,F,\pi,A,\sigma) \) if it is an as-curve such that \( \gamma|_I \) is a solution of the LSS.

**Lemma 6.13.** Consider the LSS \( (M,F,\pi,A,\sigma) \). Let \( \varphi : M^1 \rightarrow M \) a desingularization of \( M_0 \). Consider the induced system \( (M^1,\varphi^*(F),\varphi^*(\pi),\varphi^*(A),\varphi^*(\sigma)) \). Let \( \gamma : I \rightarrow M^1 \) be an as-solution of \( (M^1,\varphi^*(F),\varphi^*(\pi),\varphi^*(A),\varphi^*(\sigma)) \). Then, for each \( I' \subset I \) open subinterval strictly included on \( I \) (on both sides), \( \gamma|_{I'} \) is projected via \( \pi \) into an as-solution \( \gamma' \) of the LSS \( (M,F,\pi,A,\sigma) \), \( \gamma' = (\varphi \circ \gamma)|_{I'} \).

**Proof.** Use lemmas 6.2 and 3.10.

From last lemma, it may seem that when we substitute our original system for the lifted one, we add points to the solutions on \( M_1 \) which do not project into original solutions. This is not true. As we saw in lemma 3.10 the projection of a solution is a solution, but the projection of an as-solution may not be semi-analytic. But it will still be a solution in the classical sense. Next theorem will help us to study the converse of the lemma: as-solutions are partially lifted.

**Theorem 6.14.** Let \( \gamma' : I \rightarrow M \) be an as-solution of \( (M,F,\pi,A,\sigma) \) where \( I \) is a compact interval such that \( \gamma'(I) \subset M_0 \). Let \( \varphi : M^1 \rightarrow M \) a desingularization of \( M_0 \). Then there exists a family of intervals \( J_j \subset I \) such that

- \( \bigcup_j \gamma'(J_j) = \gamma'(I) \)
- For each \( j \), there exists a lifted as-solution \( \gamma_j : J_j \rightarrow M^1 \) of \( \gamma'|_{J_j} \) of the system on \( M^1 \), so \( \gamma'|_{J_j} = (\varphi \circ \gamma_j) \).
Proof. Using lemma 6.9 assume (shrinking I if necessary) that $\gamma'$ is a diffeomorphism between $\hat{I}$ and $\Gamma_{\gamma}|_{\hat{I}}$ and an homeomorphism over I. Moreover, assume that $\gamma'(I) \subset U$, where $(U, x)$ is a given coordinate chart.

Since $\gamma'(I)$ is compact and desingularizations are proper maps, the set $K$ defined as

$$K = \varphi^{-1}(\gamma'(I))$$

is compact and admits a finite cover using coordinate charts $(V_i, y_i)$. For simplicity in notation, assume that the original desingularization was $\varphi : V \to U$ and $\gamma'(I) \subset \varphi(V)$.

The map $\varphi$ can be described as the restriction $p|_{\Gamma_{\varphi}}$ to the graph $\Gamma_{\varphi} \subset V \times U$ of the projection $p : V \times U \to U$. Use the fiber-cutting lemma to decompose $\Gamma_{\varphi}(K)$ into finitely many analytic manifolds $A$ such that on each one $p$ is an immersion. Let $A$ be one of these manifolds such that $\dim(A) = 1$. As in lemma 6.8 we can assume that $p|_A$ is an embedding.

Denote $J = \gamma'^{-1}(p(A))$ and define the as-curve

$$z = (p|_A^{-1} \circ \gamma') : J \to V \times U.$$

Let $p_1 : V \times U \to V$ be the first projection and define

$$\gamma = p_1 \circ z.$$

Clearly, $p_1$ is a diffeomorphism when restricted to $\Gamma_{\varphi}$ so $\gamma$ is an as-curve. Moreover, $\gamma' = \varphi \circ \gamma$. Using lemma 3.11 $\gamma$ is an as-solution of the lifted system.

Moreover, since $\gamma(J)$ is relatively compact, by lemma 6.2 it admits a continuous extension to $J$. □

Corollary 6.15. Let $\gamma : I \to M_0$ be an as-solution. Then, $\gamma$ is lifted to a piecewise as-solution of the lifted system.

Proof. Last theorem proves that there exists a family of subsets $I_i \subset I$ such that, on each $I_i$, the solution is lifted. Suppose that $\bigcup I_i$ is not dense on $I$. Let $J \subset I$ be an open interval such that $J \cap I_i = \emptyset$ for all $i$ and $\gamma|_J$ is a diffeomorphism. Then, there exists a subfamily $\{I_{rk}\}_k$ such that $\bigcup_{rk} \gamma(I_{rk}) = \gamma(J)$. Then, $\gamma|_J$ is also lifted using a finite number of parts. □

As we said, this is a really strong result. We cannot expect a solution $\gamma : I \to M$ to be lifted as a continuous path. Corollary 6.15 shows (up to some points) $\gamma$ is lifted to a solution on $M^1$. 46
7 Algorithm

In this section, we present (simplifying the work in [CE06]) the algorithm itself. Consider a LSS \((M, F, \pi, A, \sigma)\) where \(M\) is a real analytic manifold of dimension \(n\). Our goal is to find the subset of \(M\) where solutions live and decompose it as union of analytic manifolds. That is to say, to find a family of linearly singular systems \((M_i, F_i, \pi_i, A_i, \sigma_i)\) such that each \(A_i\) has (locally) constant rank and

\[
\text{Im}(\sigma_i) \subset \text{Im}(A_i).
\]

This algorithm is recursive and the number of steps is bounded by the dimension of the manifold. First we will explain the recursive step and bellow we will discuss the correctness and finiteness of the algorithm. Later on, we will discuss about the implementation and limitations of the algorithm and show some examples.

7.1 Recursive step

Let \((M, F, \pi, A, \sigma)\) be a linearly singular system where \(M\) is a real analytic manifold of dimension \(n\). Assume that \(M\) is connected (if not, repeat the algorithm on each connected component).

As we saw in Section 6, we are going to decompose our manifold \(M\) in three parts, namely \(M_0\), \(M_1\) and \(M_2\).

First of all, define

\[
S_i(M) := \{ x \in M \mid \text{rank}(A(x)) \leq i \}, \quad 0 \leq i \leq n
\]

\[
L_i(M) := \{ x \in S_i(M) \mid \dim(\langle \sigma(x), \text{Im}(A(x)) \rangle) \leq i \}, \quad 0 \leq i \leq n
\]

If we fix a chart \(U\), points of \(S_i(U)\) are those points \(x \in U\) such that \(\text{rank}(\tilde{A}(x)) \leq i\) and points of \(L_i(U)\) are those such that \(\text{rank}[\tilde{A}(x) \tilde{\sigma}(x)] \leq i\). Remember that, locally, a linearly singular system is an equation of the form

\[
\tilde{A}(x) \dot{x} = \tilde{\sigma}(x).
\]

**Lemma 7.1.** Consider the couple of equations

\[
\tilde{A}(x) \dot{x} = \tilde{\sigma}(x) \quad \text{for } x : I \to M \tag{9}
\]

and

\[
\tilde{A}(x)u = \tilde{\sigma}(x) \quad (x, u) \in M \times \mathbb{R}^m \tag{10}
\]

defined on an open set \(U\). Consider the sets \(S_i(U)\) and \(L_i(U)\). Suppose that \(x \in S_j(U) \backslash S_{j-1}(U)\).

Then

1. If \(\xi\) is a solution of (9) \((\xi(t), \dot{\xi}(t))\) is a solution of (10) for all \(t \in I\).
2. If \(j = n\) and \(x \notin L_{n+1}(U)\), then the system (9) has locally constant rank at \(x\) and the system (10) has a solution \((x, u)\).
3. If \(x \in L_{j+1}(U) \backslash L_j(U)\) neither the system (9) nor (10) have solutions at \(x\).
4. If \(x \in L_j(U)\), the system (10) has a solution \((x, u)\).

Then, in a first approach, we do not need to work with all of \(M\) but only with those points in which the system (10) has solution (described by lemma above). Next lemma will help us working with these sets.
Lemma 7.2. $S_i(M)$ and $L_i(M)$ are closed subanalytic subsets of $M$.

Proof. Locally, $S_i(M)$ and $L_i(M)$ are just inequalities defined by determinants. Note that these sets form two chains

$$S_0(M) \subset S_1(M) \subset \cdots \subset S_n(M) = M$$

and

$$L_0(M) \subset L_1(M) \subset \cdots \subset L_n(M) = M.$$ 

For simplicity, we will form another chain,

$$S_{k_1}(M) \subset S_{k_2}(M) \subset \cdots \subset S_{k_r}(M) = M,$$

where $S_i(M)$ are supposed to be nonempty and distinct. Moreover, consider the corresponding chain:

$$L_{k_1}(M) \subseteq L_{k_2}(M) \subseteq \cdots \subseteq L_{k_r}(M).$$

We must separate three cases:

1. If $L_{k_r}(M) = \emptyset$, define $M_0 = M_2 = \emptyset$ and $M_1 = M$.
2. If $L_{k_r}(M) \neq \emptyset$ and $\dim(L_{k_r}(M)) < n$, define $M_0 = L_{k_r}(M)$, $M_1 = M - M_0$ and $M_2 = \emptyset$.
3. If $L_{k_r}(M) \neq \emptyset$ and $\dim(L_{k_r}(M)) = n$, define $M_0 = S_{k_{r-1}}(M)$, $M_1 = \emptyset$ and $M_2 = M - M_0$.

<table>
<thead>
<tr>
<th>$L_{k_r}(M)$</th>
<th>$L_{k_r}(M) \neq \emptyset$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_0$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$M_1$</td>
<td>$M$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\dim(L_{k_r}(M)) &lt; n$</td>
<td>$S_{k_{r-1}}(M)$</td>
</tr>
<tr>
<td>$\dim(L_{k_r}(M)) = n$</td>
<td>$M - M_0$</td>
</tr>
</tbody>
</table>

As showed earlier, $S_{k_{r-1}}(M) \subset M$ and $L_{k_{r-1}}(M) \subset M$ are closed subanalytic subsets of $M$. Thus, $M_1$ and $M_2$ are open submanifolds of $M$. With this, we have proved that

Lemma 7.3. 1. The equation of motion has no solution passing through any point $x \in M_1$.

2. For any $x \in M_2$, there exists a solution of the equation of motion passing through $x$.

Proof. It follows from last lemma.

On the other hand, $M_0$ is not a manifold. In fact, it is a closed set defined by analytic equations. Thus, $\dim M_0 < n$.

Proposition 7.4. There exists a desingularization of $M_0$.

Proof. $M_0$ is a subanalytic set by lemma 7.2. Then apply desingularization theorem to get a desingularization $\pi_0 : M^1 \to M$ of $M_0$.

Using lemma 3.8, we can construct a new linearly singular system

$$(M^1, \pi_0^*(F), \pi_0^*(\pi), \pi_0^*(A), \pi_0^*(\sigma)).$$

Moreover, $\dim M^1 < \dim M$. Repeat the process with this new linearly singular system.
7.2 Discussion

- Every time we use above method on a linearly singular system, we get a new linearly singular system whose dimension is strictly smaller than the original. Therefore, this process ends and we get a finite sequence

\[ M^q \xrightarrow{\pi_{q-1}} M^{q-1} \xrightarrow{\pi_{q-2}} \cdots \xrightarrow{\pi_1} M^1 \xrightarrow{\pi_0} M^0 = M. \]

Therefore we can define a “global” linearly singular system

\[(\tilde{M}_2, \tilde{F}, \tilde{\pi}, \tilde{A}, \tilde{\sigma})\]

were \(\tilde{M}_2 = \bigsqcup_{k=0}^{q} M^k_2\) and the functions are defined in the natural way. Moreover, each subsystem

\[(M^k_2, F^k, \pi^k, A^k, \sigma^k)\]

is such that \(\text{Im}(\sigma^k) \subset \text{Im}(A^k)\). Therefore, there exists a solution (maybe not unique) passing through any point \(x \in \tilde{M}_2\). To find these solutions, we only need to find the vector fields \(X \in \mathcal{X}(M^k_2)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
T M^k_2 & \xrightarrow{F^k} & A^k \\
\downarrow X & & \downarrow \sigma^k \\
M^k_2 & \xrightarrow{\pi^k} & \tilde{M}_2
\end{array}
\]

Locally, this is equivalent to solve the solutions of a system of the form

\[A^k(x)\dot{x} = \sigma(x)\]

where \(A^k(x)\) has constant rank.

- Proposition 3.10 shows that any solution \(\xi\) of the system \((\tilde{M}_2, \tilde{F}, \tilde{\pi}, \tilde{A}, \tilde{\sigma})\) (analytic or not) can be projected into a solution of the original system. Therefore, the algorithm is correct in the sense that it does not introduce false solutions.

- As is stated at corollary 6.15, any as-solution of \((M, F, \pi, A, \sigma)\) defined on a relatively compact interval is lifted to a piecewise as-solution of \((\tilde{M}_2, \tilde{F}, \tilde{\pi}, \tilde{A}, \tilde{\sigma})\).

7.3 Implementation notes

Since the goal of this work is to give an algorithm to solve non-regular differential equations, we must do some remarks about the implementation of the algorithm.

- The sets \(M_0, M_1\) and \(M_2\) are defined with rank conditions. Therefore, an algorithmic function/procedure which separates a manifold onto these tree sets is easily implemented.

- Desingularization process is just a sequence of blow-ups so we must define a “projective space” class. The center of the blow-ups are also easily identified with rank conditions, so once it is well defined, the equations defining the blow-up are easily implemented. We must be careful that the blow-up of a subset \(X\) of \(\mathbb{R}^n\) is not the restriction of the blow-up of \(\mathbb{R}^n\) to the inverse image of \(X\).
• Probably the most annoying point about implementation is to know when a set defined by some analytic equations is empty. We implemented a simplified version of the algorithm, which assumes that $M_0$ is a manifold. We could not check the emptiness of the sets defined by rank conditions and, therefore, we couldn’t check the regularity of $M_0$. It is possible to do an algorithm which assumes that every set of equations defines a nonempty set. This process would be finite, but it could take a factorial computational time and we would end with multiple empty systems (in general, there would be exponentially more empty systems than nonempty systems). Our implementation can be seen in Appendix C.
8 Examples

In this section we present two examples of application of the algorithm. They are simple examples, but long enough so that any reader can understand the behavior of the algorithm.

Example 1

First, we will show [GP92] an example of a system whose solutions can be easily found:

\[ A(x)\dot{x} = \sigma(x) \]

where \[ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

and \[ \sigma(x) = \begin{pmatrix} \frac{1}{2}x_2^2 \\ x_3 \\ x_1x_2 \end{pmatrix}. \]

This system can be written as:

\[ \begin{cases} \dot{x} = 1/2y^2 \\ \dot{y} = z \\ 0 = xy \end{cases} \]

It is clear that its solutions are the constant paths \( \xi(t) = (x_0, 0, 0) \). Now we analyze this system using the desingularization algorithm. As stated on Section 7, define the sets

\[ S_0 = \emptyset \quad S_1 = \emptyset \quad S_2 = \mathbb{R}^3 \quad S_3 = \mathbb{R}^3 \]

\[ L_0 = \emptyset \quad L_1 = \emptyset \quad L_2 = \{x \mid x_1x_2 = 0\} \quad L_3 = \mathbb{R}^3. \]

Then, eliminate the repeated ones:

\[ S_0 \quad S_2 \]

and so

\[ L_0 \quad L_2. \]

Then

\[ M_0 = \{x \mid x_1x_2 = 0\} \]

\[ M_1 = \mathbb{R}^3 \setminus \{x \mid x_1x_2 = 0\} \]

\[ M_2 = \emptyset. \]

We already know that there are no solutions on \( M_1 \). Now, we want to desingularize the set \( M_0 = \{x \mid x_1x_2 = 0\} \). Clearly, a desingularization of it is the disjoint union of two planes \( \pi_1 : P^1 \rightarrow \{x \mid x_1 = 0\} \) and \( \pi_2 : P^2 \rightarrow \{x \mid x_2 = 0\} \).

First, we will work in \( P^1 = \{(x_2, x_3) \in \mathbb{R}^2\} \). Then, the projection is

\[ \pi_1(x_2, x_3) = (0, x_2, x_3). \]
Take the lifted system

\[ \pi_1^*(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \]

\[ \pi_1^*(\sigma) = \begin{pmatrix} x_2^2 \\ x_3 \\ 0 \end{pmatrix}. \]

Proceeding as before,

\[ S_0 = \emptyset \quad S_1 = \mathbb{R}^2 \quad S_2 = \mathbb{R}^2 \]

\[ L_0 = \emptyset \quad L_1 = \{ x \mid x_2 = 0 \} \quad L_2 = \mathbb{R}^2. \]

Therefore

\[ P_0^1 = \{ x \mid x_2 = 0 \} \]
\[ P_1^1 = P^1 \setminus \{ x \mid x_2 = 0 \} \]
\[ P_2^1 = \emptyset \]

and there are no solutions in \( P_1^1 \). A proper algorithm would continue working in the (already desingularized) line \{ \( x \mid x_2 = 0 \) \}. However, it can be thought as a subset of \( P^2 \). Therefore, we will continue with the study of \( P^2 = \{ (x_1, x_3) \in \mathbb{R}^2 \} \) and its projection

\[ \pi_2(x_1, x_3) = (x_1, 0, x_3). \]

The lifted system is

\[ \pi_2^*(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \]

\[ \pi_1^*(\sigma) = \begin{pmatrix} 0 \\ x_3 \\ 0 \end{pmatrix}. \]

One more iteration of the algorithm (we omit details) give us that

\[ P_0^2 = \{ x \mid x_3 = 0 \} \]
\[ P_1^2 = \mathbb{R}^2 \setminus \{ x \mid x_3 = 0 \} \]
\[ P_2^2 = \emptyset. \]

Using the trivial desingularization

\[ \pi_L : \quad L \rightarrow P^2 \]

\[ x_1 \rightarrow (x_1, 0) \]
we have a final system

\[ \pi_L^*(\pi_2^*(A)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]

\[ \pi_L^*(\pi_2^*(\sigma)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

It is easy to show that the decomposition of \( L \) is

\[ L_0 = \emptyset \]
\[ L_1 = \emptyset \]
\[ L_2 = L. \]

Hence, \( L \) is the final manifold and there exists a solution passing through any point \( x \in L \). In fact, the only vector field \( \dot{x} \frac{\partial}{\partial x} \) compatible with the equations is the zero vector field, whose integral curves are \( \xi(t) \equiv x_0, \ x_0 \in \mathbb{R} \), which project via \( \pi_1 \circ \pi_L \) to \( \xi(t) \equiv (x_0, 0, 0) \), as stated.

**Example 2**

Consider the singular Lagrangian [Der10, p. 238]

\[ L = q_1^2 q_2^2 + 2 q_1 q_2 q_1 q_2 + q_2^2 q_1^2 + q_1^2 + q_2^2. \]

To simplify notation, we will denote (as in algebraic geometry):

\[ V(f) := \{ x \mid f(x) = 0 \}. \]

The Euler-Lagrange equation is:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
2 v_2 q_2 & 4 v_1 q_2 + 2 v_2 q_1 & 2 q_2^2 & 2 q_1 q_2 \\
2 v_1 q_2 + 4 v_2 q_1 & 2 v_1 q_2 & 2 q_1 q_2 & 2 q_1^2
\end{pmatrix}
\begin{pmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{v}_1 \\
\dot{v}_2
\end{pmatrix}
= \begin{pmatrix}
\dot{v}_1 \\
\dot{v}_2 \\
2 v_1 v_2 q_2 + 2 v_2^2 q_1 + 2 q_1 \\
2 v_1^2 q_2 + 2 v_1 v_2 q_1 + 2 q_2
\end{pmatrix}
\]

Identify \( \mathbb{R}^2 \cong \mathbb{R}^4 \). Using the definitions,

\[ S_0 = \emptyset \quad S_1 = \emptyset \quad S_2 = V(q_1, q_2) \quad S_3 = \mathbb{R}^4 \quad S_4 = \mathbb{R}^4 \]

\[ L_0 = \emptyset \quad L_1 = \emptyset \quad L_2 = V(q_1, q_2) \quad L_3 = V(q_1^2 - q_2^2) \quad L_4 = \mathbb{R}^4. \]

Consider the sub-chain

\[ S_0 \subset S_2 \subset S_3. \]

Since \( L_3(M) \neq \emptyset \) and \( \dim(L_3(M)) = 3 < 4 \), then

\[ M_0 = V(q_1^2 - q_2^2) \]
\[ M_1 = \mathbb{R}^4 \setminus V(q_1^2 - q_2^2) \]
\[ M_2 = \emptyset. \]
We know that there are no solution passing through points of $M_1$. A desingularization of $M_0$ is the set of two disjoint hyperplanes

\[ \pi_+: H_+ \to V(q_1 - q_2) \subset \mathbb{R}^4 \]

and

\[ \pi_-: H_- \to V(q_1 + q_2) \subset \mathbb{R}^4. \]

First let’s solve the system on $H_+$:

\[ \pi_+: H_+ \to V(q_1 - q_2) \subset \mathbb{R}^4 \]

\[ (x, v_1, v_2) \to (x, x, v_1, v_2) \]

Consider the lifted system

\[ \pi^*_+(A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ \pi^*_+ (\sigma) = \begin{pmatrix} v_1 \\ v_2 \\ v_1 v_2 + v_2^2 + 2x \\ 2v_1 v_2 + 2v_2 + 2x \end{pmatrix} . \]

Now that we have lifted a system, we apply again the algorithm to this new system:

\[ S_0 = \emptyset \quad S_1 = V(x) \quad S_2 = \mathbb{R}^3 \quad S_3 = \mathbb{R}^3 \]

\[ L_0 = \emptyset \quad L_1 = V(x, v_1 - v_2) \quad L_2 = V(x(v_1 - v_2)) \quad L_3 = \mathbb{R}^3 \]

Therefore,

\[ H_{+,0} = V(x(v_1 - v_2)) \]

\[ H_{+,1} = \mathbb{R}^3 \setminus H_{+,0} \]

\[ H_{+,2} = \emptyset. \]

Again, there are no solution passing through points of $H_{+,1}$. A desingularization of $H_{+,0}$ is again the disjoint union of two planes

\[ \phi_0: P^0 \to V(x) \subset \mathbb{R}^3 \]

and

\[ \phi: P \to V(v_1 - v_2) \subset \mathbb{R}^3. \]
We are not going to proceed on $P^0$: since $\dot{q}_i = v_i$ and on $P^0$ the $q_i$ are constant, the only possible solution is such that $v_i = 0$ (and in fact, it is a solution). Consider the projection

$$\phi : \quad P \longrightarrow V(v_1 - v_2) \subset \mathbb{R}^3$$

$$(x, v) \longrightarrow (x, v, v)$$

The lifted system is

$$\phi^*(\pi^*_+(A)) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 8vx & 4x^2 \\ 8vx & 4x^2 \end{pmatrix}$$

$$\phi^*(\pi^*_+(\sigma)) = \begin{pmatrix} v \\ v \\ 4v^2x + 2x \\ 4v^2x + 2x \end{pmatrix}.$$ 

Again,

$$S_0 = \emptyset \quad S_1 = V(x) \quad S_2 = \mathbb{R}^2$$

$$L_0 = \emptyset \quad L_1 = V(x) \quad L_2 = \mathbb{R}^2.$$ 

This time $\text{dim}(L_2) = 2$ and

$$P_0 = V(x)$$

$$P_1 = \emptyset$$

$$P_2 = \mathbb{R}^2 \setminus V(x).$$

We already discussed $P_0$. For points on $P_2$ we can invert the system and get the vector field

$$v \frac{\partial}{\partial x} - \left( \frac{2v^2}{x} + \frac{2v^2x + v}{2x^2} \right) \frac{\partial}{\partial v}.$$ 

Now, we are solving the problem on

$$\pi_- : \quad H_- \longrightarrow V(q_1 + q_2) \subset \mathbb{R}^4.$$ 

$$(x, v_1, v_2) \longrightarrow (x, -x, v_1, v_2)$$

Then,
\[ \pi^+_{x}(A) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 4v_1 x - 4v_2 x & 2x^2 & -2x^2 \\ -4v_1 x + 4v_2 x & -2x^2 & 2x^2 \end{pmatrix} \]

\[ \pi^+_{x}(\sigma) = \begin{pmatrix} v_1 \\ v_2 \\ -2v_1 v_2 x + 2v_3^2 x + 2x \\ 2v_1 v_2 x - 2v_3^2 x - 2x \end{pmatrix} \]

In this new system,

\[
S_0 = \emptyset \quad S_1 = V(x) \quad S_2 = \mathbb{R}^3 \quad S_3 = \mathbb{R}^3
\]

\[
L_0 = \emptyset \quad L_1 = V(x, v_1 + v_2) \quad L_2 = V(x(v_1 + v_2)) \quad L_3 = \mathbb{R}^3.
\]

Therefore,

\[
H_{-0} = V(x(v_1 + v_2))
\]

\[
H_{-1} = \mathbb{R}^3 \setminus H_{-0}
\]

\[
H_{-2} = \emptyset.
\]

There are no solutions passing through points of \(H_{-1}\). A desingularization of \(H_{-0}\) is, like in the case of \(H_{+0}\), the disjoint union of two planes

\[
\phi_0 : P^0 \rightarrow V(x) \subset \mathbb{R}^3
\]

and

\[
\psi : Q \rightarrow V(v_1 + v_2) \subset \mathbb{R}^3.
\]

Again, we are not doing the plane \(V(x)\) for mechanical reasons. For \(Q\), we have

\[
\psi : \quad Q \rightarrow V(v_1 + v_2) \subset \mathbb{R}^3
\]

\[
(x, v) \rightarrow (x, v, -v)
\]

and

\[
\psi^*(\pi^+_{x}(A)) = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 8vx & 4x^2 \\ -8vx & -4x^2 \end{pmatrix}
\]

\[
\psi^*(\pi^+_{x}(\sigma)) = \begin{pmatrix} v \\ -v \\ 4v^2 x + 2x \\ -4v^2 x - 2x \end{pmatrix}.
\]
A final use of the algorithm give us

\[ S_0 = \emptyset \quad S_1 = V(x) \quad S_2 = \mathbb{R}^2 \]

\[ L_0 = \emptyset \quad L_1 = V(x) \quad L_2 = \mathbb{R}^2. \]

This time \( \text{dim}(L_2) = 2 \) and

\[ Q_0 = V(x) \]
\[ Q_1 = \emptyset \]
\[ Q_2 = \mathbb{R}^2 \setminus V(x). \]

We already discussed \( Q_0 \). For points on \( Q_2 \) we can invert the system and a vector field with the same local expression as before

\[
v \frac{\partial}{\partial x} - \left( \frac{2v^2}{x} + \frac{2v^2x + v}{2x^2} \right) \frac{\partial}{\partial v}.
\]

Note that even if the vector field is the same, the projections are different. In fact, the integral curves of one of these last vector fields are symmetric, with respect to \( q_2 \), to the ones of the other vector field.
A  Tarski–Seidenberg theorem

In this Appendix we explain the proof of the Tarski–Seidenberg theorem, as can be found in [Cos02]. This theorem is used in Section 4 to prove that the projection of a semialgebraic set is semialgebraic.

Theorem A.1 (Tarski–Seidenberg theorem – first form). There exists an algorithm which, given a system of polynomial equations and inequalities with coefficients in \( R \)

\[
S(X,Y) \begin{cases} 
    P_1(X,Y) \sigma_1 0 \\
    \ldots \\
    P_l(X,Y) \sigma_l 0
\end{cases}
\]

(where \( \sigma_i \in \{<,>,=\} \) and \( X = (X_1, \cdots, X_n) \)) produces a finite list \( C_j(X) \) of systems of polynomial equations and inequalities in \( X \) with coefficients in \( R \) such that, for every \( x \in R^n \), the system \( S(x,Y) \) has a real solution if and only if one of \( C_j(x) \) is satisfied.

All polynomials appearing on this section will be supposed to have real coefficients.

A.1  Sturm theorem

Let \( P \) and \( Q \) be non zero polynomials, \( P \) non constant. Denote:

- \( P_0 = P \)
- \( P_1 = Q \)
- \( P_{i+1} = P_i Q_i - P_{i-1} \) the negative of the remainder of the euclidean division of \( P_{i-1} \) by \( P_i \) stopping just before we get a \( P_{k+1} = 0 \).

Definition A.2. The Sturm sequence of \( P \) and \( Q \) is \( P_0, \cdots, P_k \).

Theorem A.3 (Sturm’s theorem). Let \( P \) be a non constant polynomial. Let \( a < b \) in \( R \), neither \( a \) nor \( b \) being a root of \( P \). Then, the number of different roots of \( P \) in the interval \( (a,b) \) is equal to \( v_P(a) - v_P(b) \), where \( v_P(a) \) denotes the number of sign changes in the sequence \( P_0(a), \cdots, P_k(a) \), where \( P_1, \cdots, P_k \) denotes the Sturm sequence of \( P \) and \( P' \).

Proof. First consider \( P \) without multiple roots. Consider the signs of how \( v_P(x) \) change when \( x \) passes through a root \( c \) of a polynomial of the Sturm sequence of \( P \) and \( P' \).

- If \( c \) is a root of \( P \), the signs of \( P_0 \) and \( P_1 \) behave as follows.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_0 )</td>
<td>-</td>
</tr>
<tr>
<td>( P_1 )</td>
<td>+</td>
</tr>
</tbody>
</table>

Thus, the contribution to \( v_P(x) \) decreases by 1.
• Suppose that $c$ is also a root of $P_i$, $0 < i < k$. By construction, $P_{i-1}(c) = -P_{i+1}(c)$ and different from 0 (since $P$ has no multiple roots). Thus, the contribution to $v_P(x)$ remains unchanged (equal to 1). For instance, the sign structure may be:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P_{i-1}$</th>
<th>$P_i$</th>
<th>$P_{i+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$- - -$</td>
<td>$- 0 +$</td>
<td>$+ + +$</td>
</tr>
</tbody>
</table>

Now let $P$ be an arbitrary non constant polynomial. The first part of the proof relies on the following properties of the Sturm sequence:

1. $P$ and $P_0$ have the same zeros and $P_k$ is a nonzero constant.
2. If $c$ is a root of $P_0$, the product $P_0P_1$ is negative on some interval $(c - \varepsilon, c)$ and positive on some interval $(c, c + \varepsilon)$.
3. If $c$ is a root of $P_i$, $0 < i < k$, then $P_i - 1(c)P_i + 1(c) < 0$.

In fact, the algorithm used to calculate the Sturm sequence of $P$ and $P'$ is (up to signs) the Euclidean algorithm. Therefore, $P_k = \pm \gcd(P, P')$. Consider the polynomials $P_0/P_k, \ldots, P_k/P_k$. This sequence satisfies the properties above, so its changes of signs also determine the number of real roots of $P$. Moreover, $P_0/P_k, \ldots, P_k/P_k$ and $v_P$ have the same number of sign changes for any non root $c$.

Example A.4. Let $P(x) = x^3 - x$, whose roots are $\{-1, 0, 1\}$. Then

- $P_0 = P = x^3 - x$
- $P_1 = P' = 3x^2 - 1$
- $P_2 = \frac{3}{2}x$
- $P_3 = 1$

Let $(a, b) = (-1/2, 2)$. Then $v_P(-1/2) - v_P(2) = 2 - 0 = 2$

Proposition A.5. Let $P = a_0X^d + \cdots + a_d$, $a_0 \neq 0$. Define

$$M = \max_{i=1, \ldots, d} \left( d \left| \frac{a_i}{a_0} \right| \right)^{1/i}.$$

If $c \in \mathbb{C}$ is a root of $P$, then

$$|c| \leq M$$

Proof. Suppose $|z| > M$. Then $|a_i| < |a_0||z|^i/d$ and

$$|a_1z^{d-1} + \cdots + a_d| \leq |a_1||z|^{d-1} + \cdots + |a_d| < |a_0z^d|.$$

Thus, $P(z) \neq 0$
Then, $v_p(x)$ is constant in $(-\infty, M)$ (resp. $(M, +\infty)$). Using limits, $v_p(-\infty)$ (resp. $v_p(+\infty)$) is the number of sign changes of the leading coefficients of $P_0(-X), \ldots, P_k(-X)$ (resp. $P_0(X), \ldots, P_k(X)$).

**Proposition A.6.** The total number of different real roots of $P$ is

$$v_p(-\infty) - v_p(\infty).$$

### A.2 Real roots satisfying inequalities

Now we are going to discuss the number of solutions of a polynomial which belongs to some subset defined by polynomial inequalities.

#### Systems with one inequality

Let $P, Q \in \mathbb{R}[X]$. We want to count the number of real roots $c$ of $P$ such that $Q(c) > 0$. Consider the Sturm sequence of $P$ and $P'Q$ and, for any $a \in \mathbb{R}$ not a root of $P$, denote $v_{P,Q}(a)$ the number of sign changes in its Sturm sequence evaluated at $X = a$.

**Theorem A.7.** Let $a < b$ be real numbers which are not roots of $P$. Then, $v_{P,Q}(a) - v_{P,Q}(b)$ is equal to the number of different roots $c$ of $P$ in $(a, b)$ such that $Q(c) > 0$ minus the number of those such that $Q(c) < 0$.

**Proof.** Suppose that $P$ and $P'Q$ are relatively prime (if, not, divide by $P_k$ as before). This means that $P$ has no multiple roots and no common roots with $Q$. Replace the second condition from Sturm’s theorem by

2’ If $c$ is a root of $P_0$, the product $P_0P_1Q$ is negative on some interval $(c - \varepsilon, c)$ and positive on some interval $(c, c + \varepsilon)$.

**Corollary A.8.** The number of different roots $c$ of $P$ in $(a, b)$ such that $Q(c) > 0$ is equal to

$$\frac{1}{2} \left( v_{P,Q^2}(a) + v_{P,Q}(a) - v_{P,Q^2}(b) - v_{P,Q}(b) \right).$$

**Proof.** Use that $v_{P,Q^2}(a) - v_{P,Q^2}(b)$ counts the number of distinct roots of $P$ in $(a, b)$ which are not real roots of $Q$.

#### Systems with several inequalities

Now we want to count the number of real roots of a system $P = 0, Q_1 > 0, \ldots, Q_n > 0$. To do this, we will generalize the formula of corollary A.8 using matrices. Assume that $P$ is relatively prime with all $Q_i$. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in \{0, 1\}^k$ and $Q^\varepsilon = Q_1^{\varepsilon_1} \cdots Q_k^{\varepsilon_k}$. Using theorem A.7, $s_\varepsilon = v_{P,Q^\varepsilon}(a) - v_{P,Q^\varepsilon}(b)$ is equal to the number of distinct real roots $c$ of $P$ such that $Q^\varepsilon(c) > 0$ minus the number of roots such that $Q^\varepsilon(c) < 0$. If $\varphi = (\varphi_1, \ldots, \varphi_l) \in \{0, 1\}^l$, denote $e_\varphi$ the number of distinct real roots of $P$ such that $\text{sign}(Q_i(c)) = (-1)^{\varphi_i}$. Let $s$ (resp. $e$) be the vectors whose coordinates are all the $s_\varepsilon$ (resp. $e_\varphi$). Then, $S$ and $e$ are related by:

$$s = A_l \cdot e$$

where
• $A_0 = 1$

• $A_l = \begin{pmatrix} A_{l-1} & A_{l-1} \\ A_{l-1} & -A_{l-1} \end{pmatrix}$

These matrices are invertible: $A_l^{-1} = \frac{1}{2} \begin{pmatrix} A_{l-1}^{-1} & A_{l-1}^{-1} \\ A_{l-1}^{-1} & -A_{l-1}^{-1} \end{pmatrix}$ Then, $c = A_l^{-1}$ and recover the number of solutions of $P = 0$, $Q_1 > 0$, $\cdots$, $Q_l > 0$. If $P$ and $Q_i$ are not coprime, we do the same trick as before.

Systems with several equalities and inequalities

In a more general case, we want to count the number of real roots of a system $P_1 = 0$, $\cdots$, $P_s = 0$, $Q_1 > 0$, $\cdots$, $Q_l > 0$. Define $P = P_1^2 + \cdots + P_s^2$ and apply last case. If there is an equation $Q \geq 0$, do the disjoint union for the cases $Q = 0$ and $Q > 0$.

A.3 Tarsi–Seidenberg theorem

Lemma A.9. Let $(P, Q_1, \cdots, Q_l)$ be given polynomials on $X = (X_1, \cdots, X_n)$ and $Y$. There exists an algorithm which produces a finite list $R_1, \cdots, R_k \in \mathbb{R}[X]$ and a function $c : \{-1, 0, 1\}^k \to \mathbb{N}$ such that, for every $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_k) \in \{-1, 0, 1\}^k$ and $x \in \mathbb{R}^n$ which satisfies

\[ l_{CY}(P)(x) \neq 0 \quad l_{CY}(Q_i)(x) \neq 0 \quad \text{sign}(R_j)(x) = \varepsilon_j \quad i = 1, \cdots, l \quad j = 1, \cdots, k \]

the system

$P(x, Y) = 0 \quad Q_i(x, Y) > 0 \quad i = 1, \cdots, l$

has exactly $c(\varepsilon)$ solutions.

Proof. Compute all the Sturm sequences as polynomials on $Y$ and, for every new polynomial obtained, test whether its leading coefficient is zero or nonzero. In the case where the leading coefficient is zero, replace the polynomial with its truncation (we do not test the leading coefficients of the original polynomials because we assume they are all nonzero).

In this way, we obtain a tree of computation of Sturm sequences. Every branch gives us a system of polynomial equations and inequations in $X$ and the Sturm sequence corresponding to all parameters $x$ satisfying this system. The signs of the leading coefficient of the polynomials in this Sturm sequence determine the difference $v(-\infty) - v(+\infty)$ between the numbers of sign changes.

The leading coefficients are rational fractions $A(X)/B(X)$, where $B$ is assumed to be nonzero in the branch. Note that the sign of $A(x)/B(x)$ is the same as the sign of $A(x)B(x)$. Take $R_1, \cdots, R_k$ all $A(x)B(x)$ appearing in all branches of trees of computation of Sturm sequences. If we fix the sign of each $R_i(x)$, and assume $D(x)$ holds, then last section will give us the number of real solutions of the system

$P(x, Y) = 0, \quad Q_i(x, Y) > 0, \quad i = 1, \cdots, l.$
**Theorem A.10** (Tarski–Seidenberg theorem - first form). *There exists an algorithm which, given a system of polynomial equations and inequalities with coefficients in \( \mathbb{R} \)

\[
S(X,Y) \begin{cases} 
P_1(X,Y) \sigma_1 \geq 0 \\ ... \\ P_l(X,Y) \sigma_l \geq 0 
\end{cases}
\]

(wheret \( \sigma_i \in \{\geq, >, =, \neq\} \) and \( X = (X_1, \cdots, X_n) \)) produces a finite list of systems of polynomial equations and inequalities in \( X \) with coefficients in \( \mathbb{R} \) such that, for every \( x \in \mathbb{R}^n \), the system \( S(t, X) \) has a real solution if and only if one \( C_j(t) \) is satisfied.

**Proof.**

First of all (separating cases if necessary), write the system as

\[
S(X,Y) \begin{cases} 
P_i(X,Y) = 0 \\ Q_j(X,Y) > 0 
\end{cases}
\]

and write \( P = P_1^2 + \cdots + P_l^2 \). And apply lemma above. In the case that there is no equation of positive degree with respect to the variable \( Y \), looking at the signs of the leading coefficients of the \( Q_j \), it can be decided whether the system is satisfied on an unbounded interval. The existence of an interval, whose end points are roots of \( Q = \prod_{j=1}^{l} Q_j \), can be decided using by discussing the system obtained by the adjuntion of the equation \( Q' = 0 \). Then use lemma above.
B  Weierstrass’ preparation theorem

In this Appendix, some results about analytic functions are given. In particular, Weierstrass’ preparation theorem is proved following \[Lew11\]. This theorem is used many times along this thesis.

B.1 Analytic functions: definitions and some results

The purpose of this section is to explain some results we use on our work. Some notation we use is:

- $B^n(x_0, r) = \{ x \in \mathbb{R}^n \mid \| x - x_0 \|_2 < r \}.$
- $D^n(x_0, r) = \{ x \in \mathbb{R}^n \mid \| x - x_0 \|_\infty < r \}.$
- If $I \in \mathbb{N}^n$, denote $|I| = i_1 + \cdots + i_n$.
- If $I \in \mathbb{N}^n$, denote $I! = i_1! \cdots i_n!$.
- If $I,J \in \mathbb{N}^n$ and $x \in \mathbb{R}^n$, denote $x^I = x_1^{i_1} \cdots x_n^{i_n}$.
- If $I,J \in \mathbb{N}^n$, then $I \leq J \iff i_k \leq j_k, k = 1, \cdots, n$.

Here we list some definitions and basic results which any reader with a little background on real analysis should already know.

**Definition B.1.** Let $U \subset \mathbb{R}^n$ be an open set. A map $f : U \to \mathbb{R}$ is real analytic or of class $C^\omega$ on $U$ if, for every $x_0$, there exists $r > 0$ such that $f$ coincides with its Taylor series on $B^n(r, x_0)$:

$$f(x) = \sum_{I \in \mathbb{N}^n} \alpha_{x_0, I}(x - x_0)^I$$

**Definition B.2.** A map $f : U \to \mathbb{R}^m$ is real analytic on $U$ if its components $f_1, \cdots, f_m : U \to \mathbb{R}$ are real analytic.

**Theorem B.3** (Identity theorem). Let $U \subset \mathbb{R}^n$ be a connected open set and let $V \subset U$ be a nonempty open set. Suppose that $f,g : U \to \mathbb{R}$ are real analytic such that $f \mid_V = g \mid_V$. Then $f = g$.

In real analysis of one variable, there is a well defined “radius of convergence” $r$. A series is absolutely convergent for any $|x| < r$ and divergent for any $|x| > r$. The following definitions and theorems give us an equivalent notion for multivariable real analysis.

**Definition B.4.** Let $\xi = (\xi_1, \cdots, \xi_n)$. Let $\alpha$ denote a formal series

$$\alpha = \sum_{I \in \mathbb{N}^n} \alpha_I \xi^I.$$

We say that $\alpha$ converges absolutely at $x \in \mathbb{R}^n$ if

$$\sum_{I \in \mathbb{N}^n} |\alpha_I||x^I| < \infty.$$
Definition B.5. We say that $\alpha$ is convergent at $x \in \mathbb{R}^n$ if there exists a bijection $\phi : \mathbb{Z}^+ \to \mathbb{N}^n$ such that

$$\sum_{j=1}^{\infty} \alpha_{\phi(j)} x_{\phi(j)}$$

is convergent.

We denote

$$R_{\text{conv}}(\alpha) := \{x \in \mathbb{R}^n \mid \alpha \text{ converges at } x\}.$$

Trivially, for any $\alpha \in \mathbb{R}[[\xi]]$, $0 \in R_{\text{conv}}(\alpha)$.

Let also denote

$$\hat{\mathbb{R}}[[\xi]] := \{\alpha \in \mathbb{R}[[\xi]] \mid R_{\text{conv}}(\alpha) \neq \{0\}\}$$

and

$$C(x) = \{(c_1 x_1, \cdots, c_n x_n) \in \mathbb{R}^n \mid (c_1, \cdots, c_n) \in D^n(0,1)\}.$$

We could define a third sense of convergence: something would be “commutatively-convergent” if, for any bijection $\phi : \mathbb{Z}^+ \to \mathbb{N}^n$, then

$$\sum_{j=1}^{\infty} \alpha_{\phi(j)} x_{\phi(j)}$$

is convergent. But this notion of convergence is equivalent to absolute convergence ([Rud76], theorem 3.55).

Theorem B.6. Let $\alpha \in \mathbb{R}[[\xi]]$ be a formal power series and suppose that $\alpha$ converges at a point $x_0 \in \mathbb{R}^n$. Then $\alpha$ converges uniformly and absolutely on every compact subset of $C(x_0)$

Proof. Let $K \subset C(x_0)$ be compact. Proposition is trivial if $K = \{0\}$, so we suppose this is not the case. Let $\lambda \in (0,1)$ be such that $|x_j| \leq \lambda |x_0,j|$ for $x \in K$, $j \in \{1, \cdots, n\}$. Let $\phi : \mathbb{Z}^+ \to \mathbb{N}^n$ be a bijection such that

$$\sum_{j=1}^{\infty} \alpha_{\phi(j)} x_{\phi(j)}^0 < \infty$$

converges. This implies, in particular, that the sequence $(\alpha_{\phi(j)} x_0^\phi(j))_{j \in \mathbb{Z}^+}$ is bounded. Thus, there exists a certain $M > 0$ such that $|\alpha_I|^I x_0^I \leq M$ for every $I \in \mathbb{N}^n$. Then $|\alpha_I|^I x^I \leq M\lambda^{|I|}$ for every $x \in K$. An easy computation shows that, for any $x \in (-1,1)$,

$$\sum_{j=0}^{\infty} \frac{(m+j)!}{j!} x^j = \frac{d^m}{dx^m} \left( \frac{x^m}{1-x} \right).$$

Therefore, for any $x \in K$ and $m \in \mathbb{N}$ we have

$$\sum_{I \in \mathbb{N}^n \mid |I| \leq m} |\alpha_I| x^I \leq \sum_{I \in \mathbb{N}^n \mid |I| \leq m} |\alpha_I|x^I|\lambda^{|I|} \leq \sum_{I \in \mathbb{N}^n \mid |I| \leq m} M\lambda^{|I|} \leq M \sum_{j=0}^{\infty} \binom{n+j-1}{n-1} \lambda^j$$

$$< M \sum_{j=0}^{\infty} \frac{(n+j-1)!}{(n-1)!} \lambda^j = M d^{n-1} \frac{d^n}{d\lambda^{n-1}} \left( \frac{\lambda^{n-1}}{1-\lambda} \right).$$

Thus, it converges absolutely on $K$ and uniformly in $K$, since this bound is independent of $x$. □
This theorem implies that, if we have convergence for a formal power series at some nonzero point \( x_0 \in \mathbb{R}^n \), then we have absolute convergence in some neighborhood of the origin. Define

\[
R_{\text{abs}}(\alpha) = \{ x \in \mathbb{R}^n \mid \exists r > 0 \text{ st } \sum_{I \in \mathbb{N}^n} |\alpha_I y^I| < \infty \text{ for all } y \in B^n(x, r) \},
\]

which we call the region of absolute convergence.

**Proposition B.7.** Let \( \alpha \) be a formal series. Then \( \text{Int}(R_{\text{conv}}(\alpha)) = R_{\text{abs}}(\alpha) \)

**Proof.** Let \( x \in \text{int}(R_{\text{conv}}(\alpha)) \). Then, there exists \( \lambda > 1 \) such that \( \lambda x \in R_{\text{conv}}(\alpha) \). For that \( \lambda \), \( x \in C(\lambda x) \) and \( r > 0 \) be such that \( B^n(r, x) \subset K \). By theorem B.6 it follows that

\[
\sum_{I \in \mathbb{N}^n} |\alpha_I y^I| < \infty.
\]

Conversely, if \( x \in R_{\text{abs}}(\alpha) \), then there exists \( r > 0 \) such that

\[
\sum_{I \in \mathbb{N}^n} |\alpha_I y^I| < \infty
\]

for \( y \in B^n(r, x) \). In particular, \( \alpha \) converges at every \( y \in B^n(r, x) \) and so, \( x \in \text{int}(R_{\text{conv}}(\alpha)) \). \( \square \)

**Corollary B.8.** Let \( \alpha \in \hat{\mathbb{R}}[[\xi]] \) and \( x \in R_{\text{abs}}(\alpha) \) then there exists \( C, \varepsilon > 0 \) such that

\[
|\alpha_I| \leq \frac{C}{(|x_1| + \varepsilon)^{i_1} \cdots (|x_n| + \varepsilon)^{i_n}}
\]

for every \( I \in \mathbb{N}^n \).

**Proof.** Note that, if \( x \in R_{\text{abs}}(\alpha) \), then

\[
(|x_1|, \ldots, |x_n|) \in R_{\text{abs}}(\alpha)
\]

by definition of the region of absolute convergence and theorem B.6. Now, by proposition B.7 there exists \( \varepsilon > 0 \) such that

\[
(|x_1| + \varepsilon, \ldots, |x_n| + \varepsilon) \in R_{\text{abs}}(\alpha).
\]

Thus, there exists a bijection \( \phi : \mathbb{Z}^+ \to \mathbb{N}^n \) such that

\[
\sum_{j=1}^{\infty} \alpha_{\phi(j)}(|x_1| + \varepsilon)^{\phi(j)_1} \cdots (|x_n| + \varepsilon)^{\phi(j)_n}
\]

converges. Therefore, the terms in this series are bounded, Thus, there exists \( C > 0 \) such that

\[
|\alpha_I|(|x_1| + \varepsilon)^{i_1} \cdots (|x_n| + \varepsilon)^{i_n} \leq C
\]

for every \( I \in \mathbb{N}^n \). \( \square \)

**Corollary B.9.** Let \( \alpha \in \hat{\mathbb{R}}[[\xi]] \). Then the series

\[
\alpha = \sum_{I \in \mathbb{N}^n} \alpha_I x^I
\]

converges in \( R_{\text{abs}} \) to an infinitely differentiable function whose derivatives are obtained by differentiating the series term-by-term.
Proof. By induction, it suffices to show that any partial derivative of \( f \) is defined on \( \mathbb{R}_{abs} \) by convergent power series. Consider a term \( \alpha_I x^I \) in the power series, \( I \in \mathbb{N}^n \). For \( j \in \mathbb{Z}^+ \) we have

\[
\frac{\partial}{\partial x_j} = \begin{cases} 
0 & i_j = 0 \\
\alpha_I x^{I-e_j} & i_j \geq 1 
\end{cases}
\]

Thus, when differentiating the terms in the power series with respect to \( x_j \), the only nonzero contribution will come from terms corresponding to multi-indices of the form \( I + e_j \). Therefore, the power series whose terms are partial derivatives for the given power series with respect to \( x_j \) is

\[
\sum_{I \in \mathbb{N}^n} (i_j + 1)\alpha_{I+e_j}x^I.
\]

Now let \( x \in \mathbb{R}_{abs} \) and, according to corollary B.8, let \( C, \varepsilon > 0 \) be such that

\[
|\alpha_I| \leq \frac{C}{(|x_1| + \varepsilon)^i_1 \cdots (|x_n| + \varepsilon)^i_n}, \quad I \in \mathbb{N}^n.
\]

Let \( y \in \mathbb{R}_{abs} \) be such that \( y \in D^n(\varepsilon/2, x) \). Note that

\[
|y_j| \leq |x_j| + |y_j - x_j| < |x_j| + \frac{\varepsilon}{2}.
\]

Also let

\[
\lambda = \max \left\{ \frac{|x_1| + \frac{\varepsilon}{2}}{|x_1| + \varepsilon}, \cdots, \frac{|x_n| + \frac{\varepsilon}{2}}{|x_n| + \varepsilon} \right\} \in (0, 1).
\]

Then, we compute

\[
\sum_{I \in \mathbb{N}^n} |i_j + 1||\alpha_{I+e_j}||y^I| \leq \sum_{I \in \mathbb{N}^n} C|i_j + 1| \left( \frac{|x_1| + \frac{\varepsilon}{2}}{|x_1| + \varepsilon} \right)^{i_1} \cdots \left( \frac{|x_n| + \frac{\varepsilon}{2}}{|x_n| + \varepsilon} \right)^{i_n}
\]

\[
\leq \sum_{m=0}^{\infty} \sum_{|I|=m} C|m+1| \lambda^m \leq \sum_{m=0}^{\infty} C(m+1) \left( \frac{n-m-1}{n-1} \right) \lambda^m.
\]

The ratio test shows that this last series converges. Thus the power series whose terms are the partial derivatives of those for the given power series with respect to \( x_j \) converges uniformly and absolutely in a neighborhood of \( x \). Thus

\[
\frac{\partial}{\partial x_j} \left( \sum_{I \in \mathbb{N}^n} \alpha_I x^I \right) = \sum_{I \in \mathbb{N}^n} (i_j + 1)\alpha_I x^I
\]

which (after induction) gives the corollary.

The next theorem gives us an equivalent definition of real analytic function.

Lemma B.10. Let \( J \in \mathbb{N}^n \) and let \( x \in \mathbb{R}^n \) such that \( |x_k| < 1, \ k \in \{1, \cdots, n\} \). Then

\[
\sum_{j=0}^{\infty} J! \begin{pmatrix} i_1 + j_1 \\ j_1 
\end{pmatrix} \cdots \begin{pmatrix} i_n + j_n \\ j_n \end{pmatrix} |x^J| = \frac{\partial^{|J|}}{\partial x^J} \left( \prod_{k=1}^{n} \frac{x_k^j}{1-x_k} \right).
\]
Proof. First, note that
\[ \sum_{j_k=0}^{\infty} (x_k x_k + j_k)^k k! x_k^k = \sum_{j_k=0}^{\infty} \frac{(x_k^k + j_k)^k k!}{k!} x_k^k = \frac{d^{j_k}}{dx_k^{j_k}} \left( \frac{x_k^{j_k}}{1-x_k} \right), \quad k \in \{1, \ldots, n\}. \]
Therefore
\[ \sum_{I \in \mathbb{N}^n} J! \left( \begin{array}{c} i_1 + j_1 \cr j_1 \end{array} \right) \cdots \left( \begin{array}{c} i_n + j_n \cr j_n \end{array} \right) x^I = \sum_{I \in \mathbb{N}^n} J! \left( \begin{array}{c} i_1 + j_1 \cr j_1 \end{array} \right) x_1^i \cdots \left( \begin{array}{c} i_n + j_n \cr j_n \end{array} \right) x_n^i \]
\[ = \left( \sum_{j_1=0}^{\infty} j_1! \left( \begin{array}{c} i_1 + j_1 \cr j_1 \end{array} \right) x_1^i \right) \cdots \left( \sum_{j_n=0}^{\infty} j_n! \left( \begin{array}{c} i_n + j_n \cr j_n \end{array} \right) x_n^i \right) \]
\[ = \left( \frac{d^{j_1}}{dx_1^{j_1}} \left( \frac{x_1^{j_1}}{1-x_1} \right) \right) \cdots \left( \frac{d^{j_n}}{dx_n^{j_n}} \left( \frac{x_n^{j_n}}{1-x_n} \right) \right) \]
\[ = \frac{\partial^{j_1}}{\partial x_1^{j_1}} \left( \frac{\prod_{k=1}^{n} x_k^{j_k}}{1-x_k} \right). \]

Lemma B.11. For each \( r \in (0, 1) \) there exist \( A, \lambda > 0 \) such that, for each \( m \in \mathbb{N} \)
\[ \sup \left\{ \frac{d^m}{dx^m} \left( \frac{x^m}{1-x^m} \right) \mid x \in [-r, r] \right\} \leq A m! \lambda^{-m}. \]
Proof. Remember that
\[ \sum_{j=0}^{\infty} \left( \begin{array}{c} m+j \cr j \end{array} \right) x^j = \frac{1}{(1-x)^{m+1}} \]
and the convergence is uniform and absolute on \([-r, r]\), for \( r \in (0, 1) \). We have
\[ \frac{d^m}{dx^m} \left( \frac{x^m}{1-x^m} \right) = \sum_{j=0}^{\infty} \frac{(m+j)!}{j!} x^j \]
for \( x \in (-1, 1) \). If \( x \in [-r, r] \) then
\[ \left( \frac{d^m}{dx^m} \left( \frac{x^m}{1-x^m} \right) \right) \frac{(1-r)^m}{m!} = (1-r)^m \sum_{j=0}^{\infty} \frac{(m+j)!}{m! j!} x^j = (1-r)^m \sum_{j=0}^{\infty} \left( \begin{array}{c} m+j \cr j \end{array} \right) x^j \]
\[ = \left( \frac{1-r)^m}{(1-x)^{m+1}} \right) \frac{1}{1-x} \leq \frac{1}{1-r}. \]
Thus
\[ \frac{d^m}{dx^m} \left( \frac{x^m}{1-x^m} \right) \leq \frac{1}{1-r} m! (1-r)^{-m} \]
and so the lemma follows taking \( A = \frac{1}{1-r} \) and \( \lambda = 1-r \). □

Theorem B.12. Let \( U \subset \mathbb{R}^n \) be an open set. Let \( f : U \to \mathbb{R} \) be infinitely differentiable. Then, the following statements are equivalent:

1. \( f \) is real analytic.

67
2. for each \( x_0 \in U \), there exists a neighborhood \( V \subset U \) of \( x_0 \) and \( C, r > 0 \) such that
\[
|D^I f(x)| \leq CI r^{-|I|}
\]
for all \( x \in V \) and \( I \in \mathbb{N}^n \).

**Proof.**

- \((i) \Rightarrow (ii)\): Suppose that \( f \) is real analytic. Take \( x_0 \) and a neighborhood \( V \) of \( x_0 \) such that its Taylor series converges absolutely on \( V \). Then, for any \( x \in V \) we have
\[
f(x) = \sum_{I \in \mathbb{N}^n} \frac{1}{I!} D^I f(x_0)(x - x_0)^I.
\]
Denote \( \alpha_I = \frac{1}{I!} D^I f(x_0). \) By corollary \([B.8]\) there exists \( C', \sigma > 0 \) such that
\[
|\alpha_I| \leq C' \sigma^{-|I|}, \quad I \in \mathbb{Z}_{\geq 0}^n.
\]
By corollary \([B.9]\) we can write (formally)
\[
\frac{1}{J!} D^J f(x) = \sum_{I \in \mathbb{Z}_{\geq 0}^n} \left( \frac{i_1 + j_1}{j_1} \right) \cdots \left( \frac{i_n + j_n}{j_n} \right) \alpha_{I+J}(x - x_0)^I (11)
\]
for \( J \in \mathbb{N}^n \) and \( x \) in a neighborhood of \( x_0 \). Therefore, there exists \( \rho \in (0, \sigma) \) sufficiently small such that, if \( x \in \mathbb{R}^n \) satisfies \( |x_j - x_{0,j}| < \rho, \ j \in \{1, \cdots, n\} \), equation (11) holds. Let \( C_J \) denote
\[
C_J = \left. \frac{\partial^{|J|}}{\partial x^J} \left( n \left( \prod_{k=1}^{n} \frac{x_j^{j_k}}{1 - x_j^{j_k}} \right) \right) \right|_{x = (\xi, \cdots, \xi)}.
\]
Let \( r \in (0,1) \) satisfy \( r > \frac{\xi}{\sigma}. \) By lemma \([B.11]\) there exist \( A, \lambda > 0 \) such that, for each \( k \in \{1, \cdots, n\} \) and each \( x_k \in [-r, r] \), we have
\[
\frac{d^{j_k}}{dx^{j_k}} \left( \frac{x_j^{j_k}}{1 - x_j^{j_k}} \right) \leq A j_k! \lambda^{-j_k}.
\]
Therefore,
\[
\left. \frac{\partial^{|J|}}{\partial x^J} \left( \prod_{k=1}^{n} \frac{x_j^{j_k}}{1 - x_j^{j_k}} \right) \right|_{x = (\xi, \cdots, \xi)} \leq A^n J! \lambda^{-|J|}
\]
whenever \( x = (x_1, \cdots, x_n) \) satisfies \( |x_j| < r \), for \( j \in \{1, \cdots, n\} \). In particular,
\[
C_J \leq A^n J! \lambda^{-|J|}.
\]
Then, for any \( x \) such that \( |x_j - x_{0,j}| < \rho, \ j \in \{1, \cdots, n\} \) we have
\[
|D^I f(x)| \leq \sum_{I \in \mathbb{Z}_{\geq 0}^n} J! \left( \frac{i_1 + j_1}{j_1} \right) \cdots \left( \frac{i_n + j_n}{j_n} \right) |\alpha_{I+J}| (x - x_0)^I \|I| \\
\leq \sum_{I \in \mathbb{Z}_{\geq 0}^n} J! \left( \frac{i_1 + j_1}{j_1} \right) \cdots \left( \frac{i_n + j_n}{j_n} \right) C' \sigma^{-|J|} \left( \frac{\rho}{\sigma} \right)^{|J|} \\
\leq C' \sigma^{-|J|} C_J \leq C' A^n J!(\lambda + \sigma)^{-|J|},
\]
using lemmas \([B.10] \) and \([B.11]\). Thus the second condition holds with \( C = C'A^n \) and \( r = \lambda + \sigma. \)
\begin{itemize}
  \item (ii) ⇒ (i): Let \( x_0 \in U \) and a neighborhood \( V \subset U \) of \( x_0 \) and \( C, r > 0 \) such that
  \[ |D^I f(x)| \leq CI!r^{-|I|} \]
  for all \( x \in V \) and \( I \in \mathbb{N}^n \). We must show that the Taylor series of \( f \) converges absolutely in a neighborhood of \( x_0 \). Let \( k \in \mathbb{Z}^+, x \in B^n(\rho, x_0) \). Recall that there exists
  \[ z \in \{(1-t)x_0 + tx | t \in [0, 1]\} \]
  such that
  \[ f(x) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \frac{1}{I!} D^I f(x_0)(x-x_0)^I + \sum_{I \in \mathbb{Z}^n_{\geq 0}} \frac{1}{I!} D^I f(z)(x-x_0)^I. \]

  Thus
  \[
  \left| f(x) - \sum_{I \in \mathbb{Z}^n_{\geq 0}} \frac{1}{I!} D^I f(x_0)(x-x_0)^I \right| \leq \sum_{I \in \mathbb{Z}^n_{\geq 0}} \frac{1}{I!} |D^I f(z)||I| \leq \sum_{I \in \mathbb{Z}^n_{\geq 0}} \frac{1}{I!} (\frac{\rho}{r})^{|I|} = \left( \frac{n-k}{k} \right) \left( \frac{\rho}{r} \right)^{k+1}.
  \]

  Using ratio test,
  \[
  \sum_{k=0}^{\infty} \left( \frac{n-k-1}{k-1} \right) \left( \frac{\rho}{r} \right)^k
  \]
  converges. Therefore
  \[
  \lim_{n \to \infty} \left( \frac{n-k}{k} \right) \left( \frac{\rho}{r} \right)^{k+1} = 0
  \]
  and
  \[
  \lim_{n \to \infty} \left| f(x) - \sum_{I \in \mathbb{Z}^n_{\geq 0}} \frac{1}{I!} D^I f(x_0)(x-x_0)^I \right| = 0.
  \]

  Thus, \( f \) is equal to its Taylor series on \( V \).
\end{itemize}

\textbf{B.2 Weierstrass’ preparation theorem}

Let \( U \subset \mathbb{R}^n \) be a neighborhood of \( 0 \in \mathbb{R}^n \) and \( V \subset \mathbb{R} \) be a neighborhood of \( 0 \in \mathbb{R} \). If a real analytic function \( f : U \times V \to \mathbb{R} \) admits a power series expansion valid on all \( U \times V \), then we will write this power series expansion as
\[
  f(x, y) = \sum_{I \in \mathbb{N}^n} \sum_{j=0}^{\infty} f_{I,j}x^I y^j
\]
where \( x \in U \) and \( y \in V \).
Definition B.13. Let \( U \times V \subset \mathbb{R}^n \times \mathbb{R} \) be a neighborhood of 0. A real analytic function \( W : U \times V \to \mathbb{R} \) is a Weierstrass polynomial of degree \( k \) if there exists real analytic functions \( w_0, w_1, \cdots, w_{k-1} : U \to \mathbb{R} \) such that

1. \( w_j(0) = 0, j = 0, \cdots, k \)
2. \( W(x,y) = y^k + w_{k-1}y^{k-1} + \cdots + w_1(x)y + w_0(x) \) for all \( (x,y) \in U \times V \)

Lemma B.14. Let \( 0 < b < a \) and let \( I \in \mathbb{N}^n \) be such that \( i_k = 0 \) for some \( k \in \{1, \cdots, n\} \). Then

1. \[
\sum_{\substack{I \leq J \\text{j} \in I_k \lessdot j_k \leq j_k}} \left( \frac{a}{b} \right)^{|J|} \leq \frac{ba^{n-1}}{(a-n)^n} \left( \frac{a}{b} \right)^{|I|}
\]
2. \[
\sum_{\substack{I \leq J \\text{|} J \leq |I|}} \left( \frac{a}{b} \right)^{|J|} \leq \frac{nba^{n-1}}{(a-n)^n} \left( \frac{a}{b} \right)^{|I|}
\]

Proof. 1. Note that, for \( \alpha \neq 1 \) and \( r \in \mathbb{Z}^+ \) we have

\[
(\alpha - 1) \sum_{s=0}^r \alpha^s = \alpha^{r+1} - 1 \Rightarrow \sum_{s=0}^r \alpha^s = \frac{\alpha^{r+1} - 1}{\alpha - 1}.
\]

Using this fact, we compute

\[
\sum_{\substack{I \leq J \\text{j} \in I_k \lessdot j_k \leq j_k}} \left( \frac{a}{b} \right)^{|J|} = \left( \sum_{j_k = 0}^{i_k - 1} \frac{a}{b} \right) \left( \prod_{l \neq k} \sum_{j_l = 0}^{i_l} \frac{a}{b} \right) = \frac{(a/b)^{i_k} - 1}{(a/b) - 1} \prod_{l \neq k} \left( \frac{(a/b)^{i_l+1} - 1}{(a/b) - 1} \right)
\]

\[
= \frac{b|J| + (a/b)^{i_k} - 1}{b|J| + (a/b) - 1} \prod_{l \neq k} \left( \frac{(a/b)^{i_l+1} - 1}{(a/b) - 1} \right)
\]

\[
= \frac{b}{b|J|} \left( \frac{a^{i_k} - b^{i_k}}{a-b} \prod_{l \neq k} \frac{a^{i_l+1} - b^{i_l+1}}{a-b} \right)
\]

\[
\leq \frac{b}{b|J|} \left( \frac{a^{i_k}}{a-b} \prod_{l \neq k} \frac{a^{i_l+1}}{a-b} \right) = \frac{b}{b|J|} \left( \frac{a^{i_k + 1}}{(a-b)^n} \right)
\]

\[
= \frac{ba^{n-1}}{(a-n)^n} \left( \frac{a}{b} \right)^{|I|}.
\]

2. Simply use that

\[
\sum_{\substack{I \leq J \\text{|} J \leq |I|}} \left( \frac{a}{b} \right)^{|J|} \leq \sum_{k=1}^m \sum_{\substack{I \leq J \\text{j} \in I_k \lessdot j_k \leq j_k}} \left( \frac{a}{b} \right)^{|J|}.
\]
Proposition B.15. Let $U_A \times V_A \subset \mathbb{R}^n \times \mathbb{R}$ be a neighborhood of $(0,0)$ and suppose that the real analytic function $A : U_A \times V_A \to \mathbb{R}$ is given by

$$A(x,y) = \sum_{I \in \mathbb{N}^n} \sum_{j=0}^{\infty} A_{I,j} x^I y^j$$

on $U_A \times V_A$, where $A_{0,0} = \cdots = A_{0,k-1} = 0$ and $A_{0,k} = 1$ for some $k \in \mathbb{Z}^+$. Let $B : U_B \times V_B \to F$ be a real analytic function such that, in $U_B \times V_B$, it has a convergent power series

$$B(x,y) = \sum_{I \in \mathbb{N}^n} \sum_{j=0}^{\infty} B_{I,j} x^I y^j.$$

Then, there exist unique real analytic functions $Q : U_Q \times V_Q \to \mathbb{R}$ and $R : U_R \times V_R \to \mathbb{R}$ such that:

1. $Q$ and $R$ are represented by power series

$$Q(x,y) = \sum_{I \in \mathbb{N}^n} \sum_{j=0}^{\infty} Q_{I,j} x^I y^j$$

$$R(x,y) = \sum_{I \in \mathbb{N}^n} \sum_{j=0}^{\infty} R_{I,j} x^I y^j$$

where $R_{I,j} = 0$ for all $I \in \mathbb{N}^n$ and $j \geq k$

2. $B(x,y) = Q(x,y) A(x,y) + R(x,y)$ for all $x \in U \subset U_A \cap U_B \cap U_C \cap U_D$ and $y \in V \subset V_A \cap V_B \cap V_C \cap V_D$.

Proof. Note that, in fact, every real analytic function is (in a small neighborhood of each point) of this form, up to a multiplicative constant. If necessary, do a linear change of variable.

Let us first show that, at the level of formal power series, there exist unique formal power series in $\xi = (\xi_1, \cdots, \xi_n)$ and $\eta$

$$\sum_{I \in \mathbb{N}^n} \sum_{j=0}^{\infty} Q_{I,j} \xi^I \eta^j$$

$$\sum_{I \in \mathbb{N}^n} \sum_{j=0}^{\infty} R_{I,j} \xi^I \eta^j$$

with $R_{I,j} = 0$ for all $I \in \mathbb{N}^n$ and $j \geq k$ and such that, at the level of formal power series, $B = QA + R$. Note that this formula reads

$$B_{I,j} = \sum_{J \leq I} \sum_{m=0}^{j} Q_{J-m} A_{J-j-m} + R_{I,j}.$$  \hspace{1cm} (12)

In particular, for $j = k$, using the fact that $R_{I,k} = 0$ for all $I \in \mathbb{N}^n$:

$$B_{I,k} = \sum_{J \leq I} \sum_{m=0}^{j} Q_{J-m} A_{J-j-m} = Q_{I,0} + \sum_{|J|<|I|} \sum_{m=0}^{j} Q_{J-m} A_{J-j-m}.$$
Thus
\[ Q_{I,0} = B_{I,k} - \sum_{j \leq I} \sum_{m=0}^{j} Q_{J,m} A_{I-J,j-m}. \] (13)

From this, we infer that \( Q_{I,0} \) is determined uniquely from \( A \) and \( B \), and the set of \( Q_{J,m} \) such that \( J \leq I, |J| < |I| \) and \( m \in \{0, \ldots , I\} \). In particular
\[ Q_{0,0} = B_{0,k} \] (14)

and we can recursively and uniquely define \( Q_{I,0} \) for all \( I \in \mathbb{N}^n \).

Now take \( j = k + l \), for \( l \in \mathbb{Z}^+ \). Then (using that \( R_{I,j} = 0 \) for \( j \geq k \)),
\[
B_{I,k+l} = \sum_{j \leq I} \sum_{m=0}^{j} Q_{J,m} A_{I-J,k+l-m} = \sum_{m=0}^{k+1} Q_{I,m} A_{0,k+l-m} + \sum_{j < I} \sum_{m=0}^{k+1} Q_{J,m} A_{I-J,k+l-m}
\]
\[
= Q_{I,l} + \sum_{m=0}^{l-1} Q_{I,m} A_{0,k+l-m} + \sum_{j < I} \sum_{m=0}^{k+1} Q_{J,m} A_{I-J,k+l-m}
\]
using the fact that \( A_{0,j} = 0 \), for \( j = 0, \ldots , k \) and \( A_{0,k} = 1 \). Thus, we have
\[ Q_{I,l} = B_{I,k+1} - \sum_{m=0}^{l-1} Q_{I,m} A_{0,k+l-m} - \sum_{j < I} \sum_{m=0}^{k+1} Q_{J,m} A_{I-J,k+l-m} \] (15)

for \( I \in \mathbb{N}^n \) and \( l \in \mathbb{Z}^+ \). From this, we infer that we can solve uniquely for \( Q_{I,l} \), \( I \in \mathbb{N}^n \), \( l \in \mathbb{Z}^+ \), in terms of \( A \) and \( B \), and the set of \( Q_{J,m} \) with \( J \geq I, |J| < |I| \), and \( m \in \{0, \ldots , k\} \). In particular, when \( I = 0 \) last formula reads
\[ Q_{0,l} = B_{0,k+l} - \sum_{m=0}^{l-1} Q_{0,m} A_{0,k+l-m}, \]
showing that we can recursively define \( Q_{0,l} \), for \( l \in \mathbb{Z}^+ \) and then, apply recursively for all \( Q_{I,l} \), \( I \in \mathbb{N}^n \), \( l \in \mathbb{Z}^+ \).

Finally, for \( I \in \mathbb{N}^n \) and \( j \in \{0, \ldots , k-1\} \), formula (12) gives us that
\[
R_{I,j} = B_{I,j} - \sum_{J \leq I} \sum_{m=0}^{j} Q_{J,m} A_{I-J,j-m}
\]
\[
= B_{I,j} - \sum_{m=0}^{j} Q_{I,m} A_{0,j-m} - \sum_{j < I} \sum_{m=0}^{j} Q_{J,m} A_{I-J,j-m}
\] (16)
\[
= B_{I,j} - \sum_{J \leq I} \sum_{m=0}^{j} Q_{J,m} A_{I-J,j-m}
\]
using that $A_{0,j} = 0$, for $j = 0, \cdots, k - 1$ and $A_{0,k} = 1$. Thus $R_{I,j}$ is uniquely determined from those $A$ and $B$, and from the set of $Q_{J,m}$ with $J \geq I$, $|J| < |I|$, and $m = 0, \cdots, k - 1$.

These computations show that the equality holds as formal series. It only remains to show that the power series for $Q$ and $R$ converge. By theorem B.12, there exists $b, c > 0$ such that

$$\max\{|A_{I,j}|, |B_{I,j}|\} \leq bc|I|^{1+j}, \quad I \in \mathbb{N}^n, j \in \mathbb{Z}^+.$$ 

Let $\alpha, \beta, \gamma > 0$ be chosen so that $\alpha > b$, $\beta, \gamma > c$ and

$$\frac{bc^k}{\alpha} < \frac{1}{3}, \quad \frac{bc^{k+1}}{\beta - c} < \frac{1}{3}, \quad \frac{nb\beta^{k+1}}{c^{k-1}(\beta - c)^n (\gamma - c)^n} < \frac{1}{3}.$$ 

We claim that

$$|Q_{I,j}| \leq \alpha \beta^j |I|^{1+j}, \quad I \in \mathbb{N}^n, j \in \mathbb{N}. \quad (17)$$

We will prove this by induction on $|I| + j$. By (14), we have

$$|Q_{0,0}| = |B_{0,k}| \leq b < \alpha.$$ 

We can think this as (14) for $|I| + j = 0$. Now assume that it holds for $I \in \mathbb{N}^n$ and $j \in \mathbb{N}$ such that $|I| + j = r - 1$. Then let $I \in \mathbb{N}^n$ and $j \in \mathbb{N}$ such that $|I| + j = r$. By (15), we have

$$|Q_{I,j}| \leq bc|I|^{1+k+j} + \sum_{m=0}^{l-1} \alpha \beta^m \gamma |I| bc^{j+m}$$

$$+ \sum_{|J| < |I|} \sum_{m=0}^{j+k} \alpha \beta^m \gamma |I| bc^{j-|J|+k+j-m}$$

$$= \alpha \beta^j |I| \left( \frac{bc^k}{\alpha} \left( \frac{c}{\beta} \right)^j \left( \frac{c}{\gamma} \right)^{|I|} + \left( \frac{c}{\beta} \right)^j bc^k \sum_{m=0}^{j-1} \left( \frac{\beta}{c} \right)^m \right)$$

$$+ \left( \frac{c}{\beta} \right)^j \left( \frac{\beta}{c} \right)^{|I|} b \left( \sum_{|J| < |I|} \left( \frac{\beta}{c} \right)^{m} \left( \sum_{J \leq I, |J| < |I|} \left( \frac{\gamma}{c} \right)^{|J|} \right) \right).$$

By definition of $\alpha$, and since $\beta, \gamma > c$

$$\frac{bc^k}{\alpha} \left( \frac{c}{\beta} \right)^j \left( \frac{c}{\gamma} \right)^{|I|} < \frac{1}{3}$$

then

$$\left( \frac{c}{\beta} \right)^j bc^k \sum_{m=0}^{j-1} \left( \frac{\beta}{c} \right)^m = \left( \frac{c}{\gamma} \right)^j bc^k (\beta/c)^j - 1 \frac{1}{(\beta/c)^j - 1} = bc^{k+1} \frac{1}{\beta^j - c}$$

$$\leq bc^{k+1} \frac{\beta^j}{\beta - c} = \frac{bc^{k+1}}{\beta - c} < \frac{1}{3}.$$
Using lemma B.14

\[
\left(\frac{c}{\beta}\right)^j \left(\frac{c}{\gamma}\right)^{|I|} b \left(\sum_{m=0}^{k+j} \left(\frac{\beta}{c}\right)^m\right) \left(\sum_{|J|<|I|} \left(\frac{\gamma}{c}\right)^{|J|}\right)
\]

\[
\leq \left(\frac{c}{\beta}\right)^j \left(\frac{c}{\gamma}\right)^{|I|} b \left(\frac{(\beta/c)^{k+j+1} - 1}{\beta/c - 1} \frac{nce^{n-1}}{\gamma/c}\right)^{|I|}
\]

\[
= \frac{\beta^{k+1}}{e^k (\beta - c)} \frac{nce^{n-1}}{(\gamma - c)^n} < \frac{1}{3}.
\]

Combining the previous three estimates we obtain

\[|Q_{I,j}| \leq \alpha \beta^j \gamma^{|I|} \]

Now that we have proved (17), we claim that \(Q_{I,j}, I \in \mathbb{N}^n, j \in \mathbb{N}\) defines a convergent power series. To see this, let \(\lambda \in (0, 1)\) and let \(r, \rho > 0\) such that \(r \beta = \rho \gamma = \lambda\). If \((x, y) \in \mathbb{R}^n \times \mathbb{R}\) satisfy \(|x_j| < r, j = 1, \cdots, n\) and \(|y| < \rho\), then

\[
\sum_{I \in \mathbb{N}^n} \sum_{j=0}^{\infty} |Q_{I,j}| |x|^{|I|} |y|^{j} \leq \sum_{m=0}^{\infty} \sum_{I \in \mathbb{N}^n} \sum_{|J|<|I|} \alpha(r \beta)^j (\rho \gamma)^m
\]

\[
= \sum_{m=0}^{\infty} \sum_{J \in \mathbb{N}^{n+1}} \alpha \lambda^m = \sum_{m=0}^{\infty} \binom{n+m}{n} \alpha \lambda^m,
\]

which converges. Therefore, the series

\[
\sum_{I \in \mathbb{N}^n} \sum_{j=0}^{\infty} Q_{I,j} x^I, y^j
\]

is absolutely convergent for \((x, y)\) satisfying \(|x_j| < r, j = 1, \cdots, n\) and \(|y| < \rho\). Now that we know that \(Q\) is real analytic and its series is absolutely convergent, using that \(R = B - QA, R\) is also real analytic and its power series also converges absolutely.

**Lemma B.16.** Let \(U \times V \subset \mathbb{R}^n \times \mathbb{R}\) be a neighborhood of \((0, 0)\) and suppose that \(f, g, W \in C^\omega(U \times V)\). The following statements hold:

1. If \(f\) is a polynomial in \(y\)

\[f(x, y) = f_k(x)y^k + \cdots + f_0(x)\]  

for some real analytic functions \(f_j : U \to \mathbb{R}, j = 0, \cdots, k\), if \(W\) is a Weierstrass polynomial, and if \(f = gW\), then \(g\) is a polynomial in \(y\):

\[g(x, y) = g_m(x)y^m + \cdots + g_0(x)\]  

for real analytic functions \(g_j : U \to \mathbb{R}, j = 0, \cdots, m\)
2. If $W$ is a Weierstrass polynomial, if $f$ and $g$ are polynomials in $y$ as in \cite{18} and \cite{19}, respectively, and if $W = fg$, then there exist real analytic functions $E, F : U \to \mathbb{R}$ such that $E(0) \neq 0$, $F(0) \neq 0$ and $Ef$ and $Fg$ are Weierstrass polynomials.

Proof.

1. Since the coefficient of the highest degree term of $y$ in $W$ is 1 (a unit in $C^\infty$), and since $f$ is a polynomial in $y$, we can perform a polynomial division in $y$ to write $f = QW + R$ where the degree of $R$ as a polynomial in $y$ is less than that of $W$ and where $Q$ is a polynomial in $y$. Uniqueness of division comes from the proof of proposition \cite{B.15}. Since $f = gW$, then $f = Q$ and $R = 0$. In particular, $g$ is a polynomial in $y$.

2. Let $k$ and $m$ be the degrees of $f$ and $g$ (i.e., $f_k$ and $g_m$ are nonzero). Let $r$ be the degree of $W$. Then

$$y^r = W(0, y) = f(0, y)g(0, y) = f_k(0)g_m(0)y^{k+m},$$

implying that $f_k(0)$ and $g_m(0)$ are nonzero, and so $f_k$ and $g_m$ are invertible in a neighborhood of 0. Thus, the result follows by taking $E(x) = f_k(x)^{-1}$ and $F(x) = g_m(x)^{-1}$.

Theorem B.17 (Weierstrass’ preparation theorem). Let $U_A \times V_A \subset \mathbb{R}^n \times \mathbb{R}$ be a neighborhood of $(0,0)$ and suppose that the real analytic function $A : U_A \times V_A \to \mathbb{R}$ is given by

$$A(x, y) = \sum_{I \in \mathbb{N}^n} \sum_{j=0}^{\infty} A_{I,j} x^I y^j$$
on $U_A \times V_A$, where $A_{0,0} = \cdots = A_{0,k-1} = 0$ and $A_{0,k} = 1$ for some $k \in \mathbb{Z}^+$. Then, there exist unique real analytic functions $W : U_W \times V_W \to \mathbb{R}$ and $E : U_E \times V_E \to \mathbb{R}$ defined on neighborhoods $U_W \times V_W$ and $U_E \times V_E$, respectively, of 0, such that

1. $W(x, y)$ is a Weierstrass polynomial in $y$ of degree $k$.

2. $E(0, y) \neq 0$.

3. $E(x, y)A(x, y) = W(x, y)$ for all $x \in U \subset U_A \cap U_B \cap U_W \cap U_E$ and $y \in V \subset V_A \cap V_B \cap V_W \cap V_E$.

Proof. Define $B(x, y) = y^k$ and apply the proposition. Then, there exists two real analytic functions $Q$ and $R$ such that $B = QA + R$ in a neighborhood of 0. Define $W = B - R$ and $E = Q$. Clearly $W = EA$. Since $B_{0,k} = 1$, like in \cite{14} we have $Q(0, 0) = Q_{0,0} = 1 \neq 0$. If we apply \cite{16} with $I = 0$ we have $R_{0,j} = B_{0,j}$ for $j = 1, \cdots, n$, giving $W_{0,j} = 0$ for $j = 1, \cdots, n$. Therefore noting that $R_{I,j} = 0$ for $j \geq k$, we have that

$$W(x, y) = \sum_{j=0}^{k} \sum_{I \in \mathbb{N}^n_{|I|>0}} (B_{I,j} - R_{I,j}) x^I y^j,$$

which shows that $W$ is a Weierstrass polynomial. □
C Implementation of the algorithm

In this last section, we present a (simplified) implementation of the algorithm. This version uses the constraint algorithm (which can be found at [GP92]) and it is implemented using SAGE. It was done to fully understand the algorithm and to have more examples of the applied algorithm.

```python
def df(f,carta):  #Returns a vector with the differential of a function "f"
    eq=[];
    for i in range(len(carta)):
        eq.append(f.diff(carta[i]));
    return vector(eq);

def matext(M,i,j):  #Given a matrix "M" and a list of rows "i" and columns "j",
    returns the minor of M consisting of the elements given by the lists
    RRR=x*matrix(len(i),len(j));
    ii=vector(i);
    jj=vector(j);
    for r in range(len(i)):
        for s in range(len(j)):
            RRR[r,s]=M[ii[r],jj[s]];
    return RRR;

def RankM(M,llig1,llig2,carta):  #Just returns the rank of M. It's a function
    return rank(M);

def simplify_ultra(temp,lligams,carta):  #Combine some simplify functions to
    get a very simplified function (this is supposed to be done by simplify_full,
    but at time of writing this algorithm, it has some problems)
    temp2=temp;
    temp2=temp2.expand()
    temp2=temp2.simplify()
    temp2=temp2.simplify_exp()
    temp2=temp2.simplify_factorial()
    temp2=temp2.simplify_full()
    temp2=temp2.simplify_radical()
    temp2=temp2.simplify_rational()
    temp2=temp2.simplify_log()
    temp2=temp2.simplify_trig()
    temp2=temp2.simplify_full()
    if temp2==0:
        return 0;
    return temp

def pasdeGauss(A,b,i,j,fil,col):
    for a in range(i+1,fil):
        b[a]=b[a]-A[a,j]/A[i,j]*b[i]
    A=A.with_added_multiple_of_row(a,i,-A[a,j]/A[i,j])
    return A,b

def Gauss(C,d,lligams,carta):  #Does Gauss elimination to the system "Cx=d"
    A=1*C; b=1*d;  #Copy of the matrix so we don't modify the originals.
    col = A.ncols()
    fil = A.nrows()
```

76
i=j=0;
while((i<col) and (j<fil)):
    k=j
    coef=A[k,i]
    coef=simplify_ultra(coef,lligams,carta);
    while((k<fil) and (coef==0)): #Search a non-zero coefficient
        k=k+1
        if k<fil:
            coef=A[k,i]
            coef=simplify_ultra(coef,lligams,carta)
    if k<fil:
        A.swap_rows(j,k)
        swapval=b[j]
        b[j]=b[k]
        b[k]=swapval
        A,b=pasdeGauss(A,b,j,i,fil,col)
        i=i+1
        j=j+1
    if k>=fil:
        i=i+1
return A,b

def eliminazeros(C,d,i): #Erase one row of the matrix "C" and the element of "d" corresponding to this row
    C=block_matrix([[C[:i],C[i+1:]]],ncols=1,subdivide=false);
    r=[];
    for j in range(len(d)):
        if i!=j:
            r.append(d[j]);
    if len(r)==0:
        return C,d,1;
    d=vector(r);
    return C,d,0;

def Lligams(A,b): #Searches for constraints
    C=A;
    d=b;
    c=[0*x]
    no_lligams=0;
    for s in range(A.nrows()):
        i=A.nrows()-s-1;
        for j in range(A.ncols()):
            if bool(A[i,j]!=0):
                break
            tempbol= bool((j==(A.ncols()-1)))
            if tempbol:
                c.append(b[i]);
                C,d,no_lligams=eliminazeros(C,d,i);
    return c,C,d,no_lligams;

def Dcarta(carta): #Make new variables with "D" before the name
    temp=[];
    for i in range(len(carta)):
        temp.append(var("D"+str(carta[i])));
    return vector(temp);
def ALI(L,carta): #Given a list of constraints, calculates the
differential matrix
    temp=0*x*Matrix(len(L),len(carta));
    for j in range(len(L)):
        for i in range(len(carta)):
            temp[j,i]=L[j].diff(carta[i]);
    return L,temp;

def afegeixlligam(llig,dllig,l,carta,ltemp,L): #Given a list of
    constraints and another one, decides if it’s independent
    temp=0*x*Matrix(1,len(carta));
    for i in range(len(carta)):
        temp[0,i]=l.diff(carta[i]);
    temp2=block_matrix([dllig,temp],ncols=1,subdivide=false);
    if RankM(temp2,llig,L,carta)==len(llig)+1:
        llig.append(l);
        dllig=vector(dllig);
        ltemp.append(l);
    return llig, dllig, ltemp;

def afegeixlligams(llig,dllig,L,carta): #From a list of constraints,
    selects an independant maximal set
    temp=[0*x];
    for i in range(len(L)):
        llig,dllig,temp=afegeixlligam(llig,dllig,L[i],carta,temp,L);
    return llig, dllig, vector(temp);

def Soluciona(A,b,Sol,Incog): #Returns the solution of a triangular system
    SolT=Incog;
    temp=[];
    tempsol=[];
    for s in range(A.nrows()):
        i=A.nrows()-s-1;
        for j in range(A.ncols()):
            if A[i,j]!=0:
                temp.append(j);
                eq=[A[i]*SolT==b[i],A[i]*SolT==b[i]];
                s=solve(eq,Incog[j],solution_dict=True);
                Sol=Sol.subs(s[0]);
                SolT=SolT.subs(s[0]);
                tempsol.append(s[0]);
                break;
    return tempsol;

def Ajusta(A,b,NLLig,carta,Sol,Incog,OLLig): #Once we have done a
    iteration, it reajust the data to do another iteration.
    lligam=[];
    while(len(NLLig)>0):
        pl=NLLig.pop();
        a=df(pl,carta);
        equa=a*Sol;
        lligam.append(equ);
        OLLig.append(pl);
        if len(lligam)==0: #If it’s solved, it stops
            78
def Fi(A,b,carta,Sol,OLLig):
    return A,b,OLLig,1;
C=0*x*Matrix(len(lligam),len(Incog));
equ=vector(lligam);
d=[];
for i in range(C.nrows()):
    for j in range(C.ncols()):
        C[i,j]=equ[i].diff(Incog[j]);
    d.append(-equ[i]+C[i]*vector(Incog));
return C,vector(d),OLLig,0;
def Fi(C_red,d_red,carta,Sol,Llig): #Writes the solution
    print "Final constraints:";
    print Llig;
    print "Solution (defined only in the constraint manifold):";
    print Sol;
def Itera(A,b,carta,Olligams,lligam,Sol,Incog): #Iterative part
    At=0*x*A;A=At+A;bt=0*x*b;b=bt+b;
    C,d=Gauss(A,b,Olligams,carta);
    L,C_red,d_red,no_matrix=Lligams(C,d);
    if len(L)==1:
        if no_matrix==0:
            Solt=Soluciona(C_red,d_red,Sol,Incog);
            for i in range(len(Solt)):
                Sol=Sol.subs(Solt[i]);
            Fi(C_red,d_red,carta,Sol,Olligams);
            return 1;
        L.remove(0);
        templl=Olligams[:];
        llig,dllig,L=afegeixlligams(vector(templl),dlligam,L,carta);
        if no_matrix==0:
            Solt=Soluciona(C_red,d_red,Sol,Incog);
            for i in range(len(Solt)):
                Sol=Sol.subs(Solt[i]);
            tempincog=set(Matrix(Sol).variables()).intersection(set(Dcarta(carta)));
            if tempincog==set():
                Fi(C_red,d_red,carta,Sol,OLLigam);
                return 1;
            Incog=vector(tempincog);
            L=L.list();
            L.remove(0);
            C_red,d_red,lligams,acab=Ajusta(C_red,d_red,L,carta,Sol,Olligam);
            if acab:
                return 1;
            Itera(C_red,d_red,carta,OLLigam,dlligam,Sol,Incog);
def noind(pL,lligam,carta): #Finds a first independent constraint
    temp=0*x*Matrix(1,len(carta));
    for i in range(len(carta)):
        temp[0,i]=pL.diff(carta[i]);
    if RankM(temp,[pL],lligam,carta)==0:
        print "The constraint ",pL," is zero over the other constraints.
        Therefore, it is dependant with them";
        return true;
    return false;
def SingularSolve(A,b,chart):  # Main function. It receives the matrix A and b of the linear system Ax=b, and the list of variables in the vector "carta"
carta=chart;
At=0*x*A;A=At+A;bt=0*x*b;b=bt+b;  # Converts a constant matrix to functional matrix
Sol=Dcarta(carta);
Incog=Dcarta(carta);
C,d=Gauss(A,b,[0,0],carta);
L,C_red,d_red,no_matrix=Lligams(C,d);
if len(L)==1:
    print "There are no constraints";
    Solt=Soluciona(C_red,d_red,Sol,Incog);
    for i in range(len(Solt)):
        Sol=Sol.subs(Solt[i]);
        Fi(C_red,d_red,carta,Sol,vector([0,0]));
    return 1;
L.remove(0);
pL=L.pop();
while(noind(pL,L,carta)):  # Finds a first constraint
    if len(L)==0:
        print "There are no constraints";
        if no_matrix==0:
            Solt=Soluciona(C_red,d_red,Sol,Incog);
            for i in range(len(Solt)):
                Sol=Sol.subs(Solt[i]);
                Fi(C_red,d_red,carta,Sol,vector([0,0]));
            return 1;
            pl=L.pop();
        llig,dllig=ALI(vector([pL]),carta);
temp2=pL;
        llig,dllig,L=afegeixlligams(llig,dllig,L,carta);
temp=L.list();temp.append(temp2);L=vector(temp);
        if no_matrix==0:
            Solt=Soluciona(C_red,d_red,Sol,Incog);
            for i in range(len(Solt)):
                Sol=Sol.subs(Solt[i]);
        if no_matrix:
            print "WARNING: No system to solve";
            Incog=vector(set(Matrix(Sol).variables()).intersection(set(Dcarta(carta))));  # Decides which variables are not found yet
            L=L.list();
            L.remove(0);
            C_red,d_red,ligams,acab=Ajusta(C_red,d_red,L,carta,Sol,Incog,[]);
            Itera(C_red,d_red,carta,ligams,dllig,Sol,Incog);

To use the algorithm on a system $A(x)\dot{x} = b(x)$, use the following command

SingularSolve(A,b,chart);

where $A$ is a matrix, "b" a vector and "chart" is a vector, whose components are the variables of the system.
References


