Application of a Kalman Filter with Augmented Measurement Model in Non-Invasive Cardiac Imaging

Master-Thesis
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1 Introduction

1.1 Motivation and Nature of the Problem

The aim of solving the inverse problem of electrocardiography is to reconstruct the evolution of the action potentials without any direct testing on the heart, just measuring the ECG of a patient.

When a patient with a potential heart disease is submitted to an invasive cardiac intervention to find out what kind of disease he is suffering from, the patient is subject to the potential risk of the intervention and stress, problems that may be avoided in the future with the help of non-invasive cardiac imaging techniques.

The problem of obtaining transmembrane voltages (TMV) from an ECG is ill-posed which means that for a certain ECG there are infinite TMV distributions which lead to the same ECG. It can only be solved by imposing regularization on the reconstructed sources which are the solution to the problem: a-priori knowledge needs to be introduced to the formulation of the problem in order to achieve a practically sensible solution. To this end, there is a need to write this problem in a way which permits a real diagnosis of the disease the patient is suffering from, a need for a solution with physical meaning and a small admissible error that makes the solution correspond to the measured ECG.

Even with regularization techniques in place, the cardiac source signals obtained with common reconstruction methods are highly unstable. The aim at IBT is to increase the performance of established methods by using filters that improve the robustness of the solutions.

Recently, IBT has achieved more stable and precise results in the imaging of TMVs in the heart with a Kalman filter [Schulze et al., 2009]. The filter combines the reconstructed signals with a prediction that results from a model of their evolution over time.

This work will focus on improving the Kalman filter by use of an extended measurement model [Kaipio et Somersalo, 1999] which introduces spatial regularization terms into the filter. This model has been applied in solvers of the inverse problem of electrical impedance tomography [Hiltunen et al., 2010], though this tomography inverse problem is non-linear, it is mathematically very similar to the imaging of electric sources in the heart.
1.2 Main Focus

The main points of this work will be to introduce the Augmented Measurement Model (AMM) into the known Kalman filter algorithm [Kaipio et Somersalo, 1999], introducing spatial regularization into the filter, which will impose physical spatial constraints on the solution, trying to make results more robust and stable.

In this work it is shortly explained how the Kalman filter works. After that the AMM is introduced. Then it is shown how the space model for the Kalman filter is obtained. After that the tuning of parameters to make the Kalman filter with AMM work is discussed, along with strategies to make the solution converge.

Finally, with the objective of obtaining an optimal reconstruction of the sources, the advantages and/or disadvantages of this algorithm are discussed, attempting to find an optimal reconstruction of the sources.
1.3 The Heart

The focus of this work is on solving the inverse problem of ECG. To give an overview of the biomedical context, this section introduces the main anatomical features of the heart and the electrophysiological behavior of its cells.

1.3.1 Anatomy and Contraction

The heart is a muscle that takes care of pumping blood to every cell in the human body. This chapter describes the medical physiology of the heart. It is based on [Guyton et Hall, 1972; Netter et al., 1997] and also [Institute, 2010] which draws information from a very intuitive introduction about the pumping blood process.

The heart is divided into two parts, the right side and the left side. The heart has a pair of atria and a pair of ventricles, on the right side one atrium and one ventricle, and on the left side another atrium and ventricle, altogether four chambers. The heart also has four valves that regulate the flow of the blood between these chambers and the arteries that conduct the blood through the body [see Fig. 1.1].

![Heart physiology, heart contraction and blood flow](image)

**Fig. 1.1.** Heart physiology, heart contraction and blood flow, modified from [Institute, 2010].

The right side of the heart supplies blood to the lungs through the pulmonary artery, while the left side of the heart provides blood to the rest of the body through the aorta.

The pumping process starts when the blood with few oxygen coming from the superior and inferior vena cava fills the right atrium. Once the right atrium is full, the tricuspid valve is opened and the blood flows from the right atrium to the right ventricle. This part of the process is called atrial systole [see Fig. 1.2].
Fig. 1.2. Atrial systole, heart right side in the foreground, modified from [Institute, 2010].

As the right ventricle is full, the tricuspid valve is then closed. Thus the blood cannot flow back to the right atrium.

At this point the right ventricle contracts and the pulmonary valve is opened to let the blood flow to the pulmonary artery and the lungs. This part of the process is called ventricular systole. Promptly, the pulmonary valve is then closed to prevent the blood from running back [see Fig. 1.3].
The blood that was once sent to the lungs is now rich in oxygen when it comes back from the pulmonary veins to fill the left atrium of the heart. Then at the same time the contraction is taking place in the heart’s right atrium, the left atrium is contracted to push the blood through the mitral valve [see Fig. 1.4].

Like on the right side of the heart, as the left ventricle is completely full of blood, the mitral valve is closed, preventing once again the blood from flowing backwards to the left atrium.
While the left ventricle is being contracted the aortic valve is opened to bring fresh blood with oxygen to the aorta and consequently to the rest of the body. This mechanism happens at the same time as the contraction of the right ventricle and finishes with the closing at the aortic valve, which prevents the blood from flowing back into the left ventricle [see Fig. 1.5].

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**Fig. 1.4.** Atrial systole, heart left side in the foreground, modified from [Institute, 2010]

**Fig. 1.5.** Ventricular systole, heart left side in the foreground, modified from [Institute, 2010]
1.3. The Heart

The process of contracting the atria and ventricles is continuously repeated.

The heart contractions are coordinated by the electrical changes that happen in cardiac cells. These electrical changes (see section 1.3.2) are transmitted from one cell to another and make them contract or relax.

1.3.2 Cardiac Action Potentials, Transmembrane Voltages (TMV)

This section is based on explanations of the electrical activity of the heart by [Netter et al., 1997; Sherwood, 2008, Chapter 9 p. 309].

To make blood circulate through the body, the heart beats with a certain rhythm as described in section 1.3.1, due to the propagation of the action potentials. To allow for this process to happen, there are two types of muscle cells in the heart:

- Contractile cells. These are the most common muscle cells in the heart (approx. 99% of the heart is made of muscle cells). They just contract and relax when the signal comes to them. Under normal conditions these cells are not supposed to start their own action potential.
- Auto-rhythmic cells. These cells are much less common and they do not contract but they initiate the action potentials to make the contractile cells work. Therefore, they make the heart beat, and are established as its pacemaker.
- Signal conducting cells (Purkinje fibers). These cells conduct the electrical stimulus that allows the heart to contract in a coordinated way.

The contraction and posterior relaxation of the heart explained in section 1.3.1 is controlled by the generation and spreading of the action potentials. Action potentials spread like a wave to all cells that are capable of contraction (see Fig. 4.2). In the figure the action potentials start to spread at the area where the auto-rhythmic cells are located. In this case the cells are self-activated by their own potential and distribute the signal to the contractile cells. After the wave is started to spread, the contractile cells realize the mechanical contraction, which results in the overall contraction of the heart and therefore, in the flow of blood.

![Action potential of a cell versus contractile response](Sherwood, 2008)
2 Mathematical Methods

2.1 The Forward Problem of Electrocardiography

The forward problem is to obtain the ECG that is generated by a certain TMV distribution [Shahidi et Savard, 1994; Skipa, 2004; Schulze, 2009].

In the simulation studies of this work, the aim is to obtain models that can describe biophysical processes that determine the heart electrical activity and body surface potentials $z_{BSPM}$, to analyze and diagnose the value of these potential distributions and then establish a relation between the volume and conduction of the body, results that can be used to solve the inverse problem of electrocardiography [Shahidi et Savard, 1994].

![Fig. 2.1. The Forward Problem of Electrocardiography](image)

2.2 The Inverse Problem of Electrocardiography

The inverse problem is to solve the forward problem retrospectively, i.e., trying to discover the TMV distribution that has generated a certain ECG [Schulze, 2009; Skipa, 2004].
This problem is ill-posed due to attenuation, spatial smoothing and discretization effects. As a result the inverse problem and the solution are automatically ill-conditioned because small perturbations in the measured data or the forward solution produce great errors in the inverse solutions. The produced solution can therefore not be considered as a valid solution [Brooks et Gee, 1999].

Taking a look at table 2.1 the ill-posed nature of the problem becomes evident, where one must reconstruct the 2018 nodes defined on the heart with just 63 nodes of the ECG measurements, besides the noise of the measurements.

Traditionally, the inverse problem is solved with regularization methods such as Tikhonov [Schulze, 2009]. In this work, the regularization will be integrated into a Kalman filter. The respective regularization terms and methods are introduced in section 2.4.1 and 2.4.3.

2.3 Kalman Filter

The fundamental mathematics of the Kalman filter, which are introduced in this section, are provided in a useful compact book by [Grewal et al., 1988], which also gives a practical point of view on the Kalman filter. [Welch et Bishop] provides useful explanatory resources and other links related to the sources in this field.

The Kalman filter for a state vector of a linear dynamic system perturbed by Gaussian noise \( w \), either white or not, the Kalman filter can be derived as the best linear state estimator [Shimkin, 2009]. The Kalman filter provides an algorithm to derive missing information from indirect noisy measurements, but it can also be used to predict the future states of a system [Grwal et Andrews, 2001].
This filter uses two types of information to produce the best estimate it can provide, the measurements and the system model which describes the dynamics in the relation between the measurements and the sources. To produce the estimate it weights both sources of information, and it bases the weights depending on the noise covariance provided to the filter.

2.3.1 State Space Model

The state space model defines the system evolution which relates the next state to the current. The measurement model relates the sources to the measurements on the body surface.

2.3.1.1 System evolution

In our case the state vector $x_k$ would be the TMVs that are distributed in the volume of the heart wall. The transition matrix $D_k$ defines how the system is going to evolve over time. This matrix relates the next state vector to the current. In our case $D_k$ is chosen not to change over time. The transition matrix is obtained through a Multidimensional Least-Squares Estimator as described in [Schulze, 2009, p. 20].

Finally the noise of the process $w_k$ is defined, with covariance $Q$ (see section 2.6.1). The noise of the process is everything we can not take into account in the model, such as for example aleatory perturbations.

\[
x_{k+1} = Dx_k + w_k \tag{2.1}
\]

\[
x_k = \begin{bmatrix} x_1^k \\ \vdots \\ x_S^k \end{bmatrix}, w_k = \begin{bmatrix} w_1^k \\ \vdots \\ w_S^k \end{bmatrix} \tag{2.2}
\]

In equation (2.2) the subscript $k$ indicates the time step and the superscript indicates the random variable (in our case it is the node on the heart, where we want to find the TMV), where $S$ is the dimension of the state vector.

\[
D = \begin{bmatrix} d_{11} & \ldots & d_{1S} \\ \vdots & \ddots & \vdots \\ d_{S1} & \ldots & d_{SS} \end{bmatrix} \tag{2.3}
\]

2.3.1.2 Measurement

The measurement of the body surface potential map (BSPM) $z_{k}^{BSPM}$ is related to the state vector $x_k$ through a transfer matrix $A$.

\[
z_k = Ax_k + v_k \tag{2.4}
\]

In equation (2.5) the subscript $k$ indicates the time step and the superscript indicates the random variable (node where the $BSPM$ is measured), where $M$ is the dimension of the measurement vector. In other words, $M$ is the number of points where the $BSPM$ is measured. Measurements are considered to be corrupted by noise $v_k$ with covariance matrix $R$ (see section 2.6.2).
\[ z_k^{BSPM} = \begin{bmatrix} z_1^k \\ \vdots \\ z_M^k \end{bmatrix}, \quad v_k = \begin{bmatrix} v_1^k \\ \vdots \\ v_M^k \end{bmatrix} \] (2.5)

The transfer matrix \( A \) defines the relation between the sources and the measurements. For a certain distribution of the TMV \( x_k \) a unique \( z_k^{BSPM} \) can be obtained, but not backwards. Due to the ill-posedness of the problem as commented in 2.2 the transfer matrix is not square, it has many more columns than rows, which means that one has many more source points than points of measurement, so it is not possible to calculate it’s inverse.

\[ A = \begin{bmatrix} a_{11} & \cdots & a_{1S} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MS} \end{bmatrix} \] (2.6)

### 2.3.2 Kalman Filter Equations

The Kalman filter equations are separated in two subgroups, the prediction equations and the correction equations. Both equations have a recursive character and are executed in a loop and the initial values for the algorithm are discussed in section 2.3.2.3.

#### 2.3.2.1 Prediction

The prediction equation makes the algorithm evolve over time. An a-priori state estimate is obtained using the system evolution model:

\[ \tilde{x}_k^- = \mathbf{D}\tilde{x}_{k-1} \] (2.7)

An a-priori estimate error covariance matrix is calculated, projecting the error covariance to the next step in time:

\[ P_k^- = \mathbf{D}P_{k-1}\mathbf{D}^T + Q \] (2.8)

#### 2.3.2.2 Correction

A second set of equations is called correction equations, because a better estimate is produced by modification of the prediction 2.7.

The correction equation is weighted with the Kalman gain:

\[ K_k = P_k^- A^T(A P_k^- A^T + R)^{-1} \] (2.9)

The corrected estimate is produced by updating the estimate with the measurement, the optimal estimate estimation of the current step will be used in the next step of the loop:

\[ \hat{x}_k = \tilde{x}_k^- + K_k(z_k - A\tilde{x}_k^-) \] (2.10)

In the end, we update the a-posteriori estimate error covariance matrix, which goes into the next step of the loop:

\[ P_k = (\mathbf{I} - K_k A) P_k^- \] (2.11)
2.3. Kalman Filter

2.3.2.3 Initial values for the Kalman filter algorithm

In case of having a-priori knowledge of the initial values of the sources $\hat{x}_{k=0}$, it would be possible to provide a realistic initial guess, in our case the initial $\hat{x}_{k=0}$ is set to zero because such a-priori-knowledge is supposed to be unavailable.

As explained by G. Welch and G. Bishop [Welch et Bishop, 2006] the a-posteriori estimate error covariance matrix $P_{k=0}$ must be nonzero in order to have a proper initiation of the algorithm. In case $P_{k=0}$ is chosen to be zero and $\hat{x}_{k=0} = 0$ the algorithm remains caught at $\hat{x}_k = 0$. For proper convergence $P_{k=0}$ is chosen in our case to be the identity matrix $P_{k=0} = I$, though any matrix different from zero could be chosen $P_{k=0} \neq 0$.

2.3.2.4 Array Dimensions

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Dimension $n_{\text{rows}} \times m_{\text{columns}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Transfer matrix</td>
<td>$M \times S$</td>
</tr>
<tr>
<td>D</td>
<td>Transition matrix</td>
<td>$S \times S$</td>
</tr>
<tr>
<td>$K_k$</td>
<td>Kalman filter gain</td>
<td>$S \times M$</td>
</tr>
<tr>
<td>I</td>
<td>Identity matrix</td>
<td>$S \times S$</td>
</tr>
<tr>
<td>$P_{-k}$</td>
<td>A-priori error covariance matrix</td>
<td>$S \times S$</td>
</tr>
<tr>
<td>$P_k$</td>
<td>A Posteriori error covariance matrix</td>
<td>$S \times S$</td>
</tr>
<tr>
<td>Q</td>
<td>Process noise covariance matrix</td>
<td>$S \times S$</td>
</tr>
<tr>
<td>R</td>
<td>Measurement noise covariance matrix</td>
<td>$S \times M$</td>
</tr>
<tr>
<td>$v_k$</td>
<td>Measurement noise</td>
<td>$M \times 1$</td>
</tr>
<tr>
<td>$w_k$</td>
<td>Process noise vector</td>
<td>$S \times 1$</td>
</tr>
<tr>
<td>$x_k$</td>
<td>State vector</td>
<td>$S \times 1$</td>
</tr>
<tr>
<td>$\hat{x}_{-k}$</td>
<td>A-priori estimate state vector</td>
<td>$S \times 1$</td>
</tr>
<tr>
<td>$\hat{x}_k$</td>
<td>Corrected estimate state vector</td>
<td>$S \times 1$</td>
</tr>
<tr>
<td>$z_k$</td>
<td>Measurement vector</td>
<td>$M \times 1$</td>
</tr>
</tbody>
</table>

Table 2.1. Array Dimensions of the Kalman filter

\(^1\) $M = 63$ nodes on the body surface where the ECG is measured

\(^2\) $S = 2018$ nodes on the heart where the TMV is calculated
2.4 Augmented Measurement Model (AMM)

The AMM is presented by [Kaipio et Somersalo, 1999] and used by [Hiltunen et al., 2010] to introduce a state space regularization in the non-stationary inverse problem for diffuse optical tomography.

2.4.1 Minimization Function

Taking a look to the measurement model where the measurements are corrupted by noise, one would like to take out the noise from the measurements to find the optimal estimate of the state vector, in our case the TMVs.

\[ z_{BSPM} = Ax_k + v_k \] (2.12)

Therefore, the residual to minimize is the difference between the ECG produced by the transfer matrix and the measurements:

\[ F(x) = \|Ax - z_{BSPM}\|^2 \] (2.13)

But minimizing this function cannot lead to valid solutions because of the ill-posedness of the inverse problem, see chapter 2.2. There are a lot of TMVs distributions which lead to a certain BSPM. When the operator \( A \) is ill-conditioned, it is necessary to regularize the solution, and one needs to rewrite the problem in such way that a solution with physical meaning is obtained.

For this reason additional information is added to the minimization function. According to Tikhonov [Schulze et al., 2009] the additional cost term is the weighted \( L^2 \)-norm of \( L(x - \bar{x}) \), where \( L \) is a linear regularization operator (see section 2.4.3) and \( \bar{x} \) is a desired target value for \( x \).

\[ F(x) = \|Ax - z_{BSPM}\|^2 + \alpha^2 \|L(x - \bar{x})\|^2 \] (2.14)

\[ \Omega = \alpha^2 \|L(x - \bar{x})\|^2 \] (2.15)

This function can be rewritten as [Kaipio et Somersalo, 1999, p. 8]:

\[ F(x) = \left\| \begin{pmatrix} Ax \\ \alpha Lx \end{pmatrix} - \begin{pmatrix} z_{BSPM} \\ \alpha L\bar{x} \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} A \\ \alpha L \end{pmatrix} x - \begin{pmatrix} z_{BSPM} \\ \alpha L\bar{x} \end{pmatrix} \right\|^2 \] (2.16)

Having the function (2.12) rewritten this way we can perceive the regularization in a different way. Comparing (2.13) and (2.16), we could see the regularization as if it augmented the measurement (in our case the measurement of the \( BSPM \)) by a noisy measurement \( \alpha Lx \).

2.4.2 Derivation of the Augmented Measurement Model

From equation (2.16) we can derive a new state model that will later be incorporated in the Kalman filter equations, to integrate the spatial regularization into the Kalman filter.

The measurement model is augmented by the target value of \( x \) projected by the weighted regularization matrix.
2.4. Augmented Measurement Model (AMM)

\[ z_{AMM} = \begin{pmatrix} z_{BSPM} \\ \alpha L x \end{pmatrix} \]  \hspace{1cm} (2.17)

The transfer matrix is also augmented by the regularization matrix multiplied by alpha

\[ A_{AMM} = \begin{pmatrix} A \\ \alpha L \end{pmatrix} \]  \hspace{1cm} (2.18)

Since the measurement state model (2.4) is augmented (2.17) the noise is also augmented by a new fictitious noise. From equation (2.16) we can obtain the new augmented noise \( v_{AMM} \) that disturbs the augmented measurement \( z_{AMM} \):

\[ v = z_{BSPM} - Ax \]  \hspace{1cm} (2.19)

\[ u = \alpha L x - \alpha L x \]  \hspace{1cm} (2.20)

\[ v_{AMM} = \begin{pmatrix} v \\ u \end{pmatrix} \]  \hspace{1cm} (2.21)

Then the new state space model extracted from the new interpretation of the minimization function can be represented by the next scheme.

2.4.3 Spatial Regularization Matrix, \( L \)

The second term of the equation (2.15) adds spatial regularization to the minimization of \( \|Ax - z_{BSPM}\|^2 \).

The regularization matrix imposes a constraint in the space of \( x_{TMV} \), which means that solutions breaking this constraint will be penalized more and less probable to be chosen as best estimate, while the solutions which do not break or are closer not to break the constraint will have more chances to be chosen as the optimal estimate.
For example we could impose \( L \) as an identity matrix and the target value \( \bar{x} \) as \( \bar{0} \). Therefore, when the term \( \Omega \) is forced to 0 every term of the vector \( x \) is forced to zero, and the solution is forced to have small values around zero.

Therefore, when the term \( \Omega \) is forced to be zero under the conditions of the example above \( L = I \) and \( \bar{x} = 0 \):

\[
\|L(x - \bar{x})\|^2 = 0 \\
\|Lx - L\bar{x}\|^2 = 0 \\
\|I(x - \bar{0})\|^2 = 0 \Rightarrow x = 0
\]

(2.22)

Common regularization matrices according to [Skipa, 2004, p. 67] are:

- The unit matrix.
- The discrete first order derivative operator.
- The discrete second order derivative operator.

In [Schulze et al., 2009] the Laplace operator is used, which constraints the solution and forces it to be smoother than the solution obtained with no constraints.

\( L \) plays the role of a spatial derivative, which is a measures of how much the signal is changing in space. In the case of this work, it reflects how much are changing the TMVs. Therefore, it imposes how different the TMV can be between different points defined in the heart in a certain time step.

Instead of the Laplace operator, in this work the relation between different nodes defined in the heart is established. This forces the TMVs to be similar between nodes in the neighborhood, because it is known that when a cell is activated this also activates its neighbors.

### 2.4.4 Regularization Parameter, \( \alpha \)

Parameter \( \alpha \) controls the strength of the regularization term \( \Omega \) (2.15) with respect to the residual norm \( \|Ax - z_{BSPM}\|^2 \).

On one hand if \( \alpha \) has a strong weight, then the regularization term \( \alpha^2 \|L(x - \bar{x})\|^2 \) becomes very important, therefore it’s error will be highly minimized. On the other hand if its value is close to 0, then it is not that important to minimize \( \Omega \), and reducing the norm of the first term \( \|Ax - z_{BSPM}\|^2 \) of the equation (2.12) acquires more importance.

A compromise between the minimization of the first and the second term of the equation (2.12) has to be found, because both terms at the same time are described to be as small as possible, and because there is no sense in minimizing strongly one of both terms if the residual error of the other one is becoming extremely big.
2.5 Kalman Filter with Augmented Measurement Model

When the Augmented Measurement Model is integrated into the Kalman filter (see section section 2.4) some matrix dimensions change and the new augmented measurement size is $M' = S + M$:

<table>
<thead>
<tr>
<th>Variable $\quad$</th>
<th>Name</th>
<th>Dimension $n_{rows} \times m_{columns}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{AMM}$</td>
<td>Transfer matrix (2.18)</td>
<td>$M' \times S$</td>
</tr>
<tr>
<td>$D$</td>
<td>Transition matrix</td>
<td>$S \times S$</td>
</tr>
<tr>
<td>$I$</td>
<td>Identity matrix</td>
<td>$S \times S$</td>
</tr>
<tr>
<td>$K_k$</td>
<td>Kalman filter gain</td>
<td>$S \times M'$</td>
</tr>
<tr>
<td>$P_k$</td>
<td>A-priori error covariance matrix</td>
<td>$S \times S$</td>
</tr>
<tr>
<td>$P_{AMM}$</td>
<td>A posteriori error covariance matrix</td>
<td>$S \times S$</td>
</tr>
<tr>
<td>$Q$</td>
<td>Process noise covariance matrix</td>
<td>$S \times S$</td>
</tr>
<tr>
<td>$R^{AMM}$</td>
<td>Augmented measurement noise covariance matrix</td>
<td>$S \times M'$</td>
</tr>
<tr>
<td>$u_k$</td>
<td>Measurement noise increaser</td>
<td>$S \times 1$</td>
</tr>
<tr>
<td>$v_k$</td>
<td>Measurement noise</td>
<td>$M \times 1$</td>
</tr>
<tr>
<td>$v_{AMM}^k$</td>
<td>Augmented measurement noise (2.21)</td>
<td>$M' \times 1$</td>
</tr>
<tr>
<td>$w_k$</td>
<td>Process noise vector</td>
<td>$S \times 1$</td>
</tr>
<tr>
<td>$x_k$</td>
<td>State vector</td>
<td>$S \times 1$</td>
</tr>
<tr>
<td>$\hat{x}_k$</td>
<td>A-priori estimate state vector</td>
<td>$S \times 1$</td>
</tr>
<tr>
<td>$\hat{x}_k$</td>
<td>Corrected estimate state vector</td>
<td>$S \times 1$</td>
</tr>
<tr>
<td>$x$</td>
<td>Target value of x</td>
<td>$S \times 1$</td>
</tr>
<tr>
<td>$z_k$</td>
<td>Measurement vector</td>
<td>$M \times 1$</td>
</tr>
<tr>
<td>$z_{AMM}^k$</td>
<td>Augmented Measurement vector (2.17)</td>
<td>$M' \times 1$</td>
</tr>
</tbody>
</table>

Table 2.2. Array dimensions of the Kalman filter when using Augmented Measurement Model

For the variables of the Augmented Measurement Model superscript AMM is used.

![Fig. 2.5. Kalman filter with Augmented Measurement Model diagram](image)
Chapter 2. Mathematical Methods

2.6 Noise Covariance Matrices

The covariance matrices play a very important role in getting a good estimation with the Kalman filter or not. In case these matrices are not properly estimated the filtering cannot result in a good reconstruction of the sources.

The main inconvenience of the Kalman filter is that it cannot estimate the noise covariance matrices. They are supposed to be known and must be given to the filter in order to make it work.

2.6.1 Process Noise Covariance Matrix, Q

The process noise covariance matrix gives us information about the second order moment of the noise in every point defined in the heart. The diagonal of this matrix informs about the power of the noise in each node while the rest of the values outside the diagonal give information about how strong the correlation between the two respective nodes is.

To isolate the process noise from equation (2.1), an estimated transition matrix \( \hat{D} \) is used which estimates the real system dynamics \( D_k \). This way the noise can be estimated from the simulated data, including information about the process noise and the error about the estimated transition matrix:

\[
w_k = x_{k+1} - \hat{D}x_k
\]  

(2.23)

The covariance operator in maths, with \( E[\cdot] \) being the expectancy operator is usually defined as:

\[
Q = \text{cov}(w) = E[(w - \mu_w)(w - \mu_w)'] =
\begin{bmatrix}
E[(w_1 - \mu_w)(w_1 - \mu_w)] & \cdots & E[(w_1 - \mu_w)(w_S - \mu_w)] \\
E[(w_1 - \mu_w)(w_1 - \mu_w)] & \ddots & \vdots \\
E[(w_S - \mu_w)(w_1 - \mu_w)] & \cdots & E[(w_S - \mu_w)(w_S - \mu_w)]
\end{bmatrix}
\]  

(2.24)

In the Kalman filter, however, the covariance operator is rewritten as follows. At the diagonal one can see the terms term \( E[w_iw_i] \) which provides information about the power of \( w_i \) (noise in the node \( i \)):

\[
Q = \begin{bmatrix}
E[w_1w_1] & \cdots & E[w_1w_M] \\
\vdots & \ddots & \vdots \\
E[w_Mw_1] & \cdots & E[w_Mw_M]
\end{bmatrix}
\]  

(2.25)

In case the noise is white Gaussian noise with no mean the mathematical covariance \( \text{cov}(w_k) \) and \( E[(w_k)(w_k)'] \) are the same due to \( \mu = 0 \):

\[
Q = \text{cov}(w_k) = E[(w_k)(w_k)']
\]  

(2.26)

2.6.2 Measurement Noise Covariance Matrix, R

The measurement noise covariance matrix gives us information about the second order moment of the noise in every point defined on the body surface. The diagonal of this matrix informs about the power of the measurement noise in each node while the rest of the values outside the diagonal denote how strong the correlation between any two nodes is.
2.6. Noise Covariance Matrices

The augmented measurements, the respective noise and the augmented noise covariance matrix are defined as:

\[ z^{\text{AMM}} \equiv \psi \]  
\[ v^{\text{AMM}} = \begin{pmatrix} v \\ u \end{pmatrix} \equiv \nu \]  
\[ R^{\text{AMM}} \equiv \mathbf{r}_\nu \]

Isolating measurement noise from (2.4) and omitting superscript AMM for clarity in (2.30) yields:

\[ R^{\text{AMM}} = E[\nu \nu'] = 
\begin{bmatrix}
E[\nu_1 \nu_1] & \cdots & E[\nu_1 \nu_M] \\
\vdots & \ddots & \vdots \\
E[\nu_M \nu_1] & \cdots & E[\nu_M \nu_M]
\end{bmatrix} \]  

With the covariance matrix **R**, exactly the same is true as for equation (2.26): the second order moment coincides with the Kalman filter covariance in the case of null mean.

\[ \text{cov}(\nu_k) = E[(\nu_k^{\text{AMM}} - \mu_{\nu_k})(\nu_k^{\text{AMM}} - \mu_{\nu_k})'] = 
\begin{bmatrix}
E[(\nu^1 - \mu_{\nu_1})(\nu^1 - \mu_{\nu_1})] & \cdots & E[(\nu^1 - \mu_{\nu_1})(\nu^M - \mu_{\nu_M})] \\
\vdots & \ddots & \vdots \\
E[(\nu^M - \mu_{\nu_M})(\nu^1 - \mu_{\nu_1})] & \cdots & E[(\nu^M - \mu_{\nu_M})(\nu^M - \mu_{\nu_M})]
\end{bmatrix} \]  

In case the noise is white Gaussian noise with no mean \((\mu = 0)\):

\[ \text{cov}(v_k) = E[\nu \nu'] = R \]

### 2.6.3 Obtaining Noise Covariance Matrices from Simulations

In the simulation case where a group of samples are available, to calculate the noise covariance matrices **Q** and **R** in (2.25) and (2.30) respectively, the expectancy operator \(E[\cdot]\) of a random variable is replaced by an unbiased estimator, where the superscripts \(n\) and \(m\) denote the nodes (random variable) from which we are trying to find its correlation.

The covariance between a nodes \(n\) and \(m\) is approximated by:

\[ \text{cov}(x^n, x^m) \approx \frac{1}{N-1} \sum_{i=1}^{N} (x^n_i - \mu_{x^n})(x^m_i - \mu_{x^m}) \]

The expectancy operator between nodes \(n\) and \(m\) is approximated by:
\[
E[x^n x^m] \approx \frac{1}{N-1} \sum_{i=1}^{N} x^n_i x^m_i
\]  
(2.34)

In the equations above the subscript \(i\) denotes index of time, and the mean is extracted from a set of \(N\) samples.

### 2.6.4 Obtaining Noise Covariance Matrices without A-Priori TMV Knowledge

Obtaining a good reconstruction of the sources without access to the simulated data \(x\) is a tough issue to overcome. The approach here is to get a huge set of simulations and calculate their covariance matrices which are then a mean of covariance matrices \(\bar{Q}\) and \(\bar{R}\) respectively. From the definition of covariance function in (2.25) and (2.30) for \(Q\) and \(R\) respectively, one can develop the expressions (2.37) and (2.46) for \(Q\) and \(R\).

#### 2.6.4.1 Process Noise Covariance Matrix \(Q\) development

Inserting (2.23) into (2.25) we can develop the next expressions:

Defining the following notation:

\[
r_{x(r,s)} \equiv E[x_{k+r} x'_{k+s}]
\]  
(2.35)

\[
r_x \equiv r_{x(r,r)} \equiv E[x_{k+r} x'_r]
\]  
(2.36)

One can develop \(Q\) as:

\[
Q = E[w_k w'_k] = E[(x_{k+1} - D x_k)(x'_{k+1} - x'_k D')] =
\]

\[
r_x - r_{x(1,0)} D' - D r_{x(0,1)} + D r_x D'
\]  
(2.37)

As seen in equation (2.37) \(Q\) does not depend on the AMM parameters, so when changing the Augmented Model parameters this covariance matrix does not change at all. In the next section the same aspect is studied for \(R\).

#### 2.6.4.2 Measurement Noise Covariance Matrix \(R\) development

In the following, the measurement noise covariance matrix is developed. Since the measurement was augmented by the AMM (see section 2.4), the measurement noise was also augmented by a fictitious noise and the new covariance matrix will therefore depend on the AMM parameters.

Using definitions described in section 2.6.2, equation (2.30) can be written (avoiding for clarity the subscript \(k\)) as:

\[
R^{AMM} = E[\nu \nu'] = E \left[ \begin{pmatrix} \nu \\ u \end{pmatrix} \begin{pmatrix} \nu' \\ u' \end{pmatrix} \right] =
\]

\[
E \begin{pmatrix} \nu \\ u \end{pmatrix} \begin{pmatrix} \nu' \\ u' \end{pmatrix} = \begin{pmatrix} r_{vv} & r_{vu} \\ r_{uv} & r_{uu} \end{pmatrix}
\]  
(2.38)
In the following formulation, the subscripts define the variables whose expectation is taken:

\[ r_v \equiv r_{vv} = E[vv'] \]  \hspace{1cm} (2.39)

\[ r_{vu} = E[vu'] = \begin{bmatrix} E[v_1u_1] & \cdots & E[v_1u_M] \\ \vdots & \ddots & \vdots \\ E[v Mu_1] & \cdots & E[v Mu_M] \end{bmatrix} \]  \hspace{1cm} (2.40)

Remembering the augmented noise formulation in (2.19), (2.20) and (2.21) we can take out the dependency from \( \alpha \), taking care that \( A \) was also augmented, and is also related to \( \alpha \), see (2.18). This clearly shows the dependency of \( R \) on the regularization parameter:

\[
\begin{align*}
\mathbf{v} &= \mathbf{z}_{BSPM} - \mathbf{A} \mathbf{x} \\
\mathbf{u} &= \alpha \mathbf{L} \mathbf{x} - \alpha \mathbf{L} \mathbf{x} = \alpha \mathbf{L} (\mathbf{x} - \mathbf{x})
\end{align*}
\]

With the two equations above we can obtain the next relations. For the first term in equation (2.38):

\[
r_v = E[(\mathbf{z}_{BSPM} - \mathbf{A} \mathbf{x})(\mathbf{z}_{BSPM} - \mathbf{A} \mathbf{x})'] = \\
r_x - r_{xx} A' - A r_{xx} + A r_{x} A' 
\]  \hspace{1cm} (2.41)

For the second term:

\[
r_{vu} = E[(\mathbf{z}_{BSPM} - \mathbf{A} \mathbf{x})(\mathbf{x} - \mathbf{x})' L'] = \\
\alpha (r_{xx} - r_{xx} A' - r_{xx} + r_{x} A') 
\]  \hspace{1cm} (2.42)

Third term:

\[
r_{uv} = E[(\alpha \mathbf{L} (\mathbf{x} - \mathbf{x}))(\mathbf{z}_{BSPM} - \mathbf{A} \mathbf{x})'] = \\
\alpha \mathbf{L} (r_{xx} - r_{xx} A' - r_{xx} + r_{x} A') 
\]  \hspace{1cm} (2.43)

Fourth term:

\[
r_u = E[(\alpha \mathbf{L} (\mathbf{x} - \mathbf{x}))(\mathbf{x} - \mathbf{x})' L'] = \\
\alpha^2 \mathbf{L} (r_{xx} - r_{xx} A' - r_{xx} + r_{x} A') 
\]  \hspace{1cm} (2.44)

Note that in these four equations above \( \mathbf{A} \) and \( \mathbf{z} \) are not augmented.

One can finally obtain the covariance matrix, putting all intermediate results together:

\[
\mathbf{R}^{AMM} = \begin{bmatrix} r_v & r_{vu} \\ r_{uv} & r_u \end{bmatrix} 
\]  \hspace{1cm} (2.45)

\[
\mathbf{R}^{AMM} = \begin{bmatrix} [r_z - r_{xx} A' - A r_{xx} + A r_{x} A'] & [\alpha (r_{xx} - r_{xx} - A r_{xx} + A r_{x}) L'] \\ [\alpha \mathbf{L} (r_{xx} - r_{xx} A' - r_{xx} + r_{x} A')] & [\alpha^2 \mathbf{L} (r_{xx} - r_{xx} - r_{xx} + r_{x}) L'] \end{bmatrix} 
\]  \hspace{1cm} (2.46)

From the expression above one can see that the part of the matrix called \( r_v \) does not depend on the AMM parameters at all, while the rest of the matrix does. The separation of \( \alpha \) allows for a quick recalculation of \( \mathbf{R}^{AMM} \) for different regularization parameters.

In case no target value is used (\( \mathbf{x} = 0 \)) and \( \mathbf{x} \) being constant while the estimation is done \( \mathbf{R}^{AMM} \) can be particularized by the expression:
Another way to calculate the measurement noise covariance matrix would be supposing that the augmented measurement is independent from the state vector:

\[ \mathbf{r}_\psi = E[\psi\psi'] = E[(A\mathbf{x} + \nu)(A\mathbf{x} + \nu)'] \]  

(2.48)

The generic expression would be:

\[ \mathbf{r}_\psi = A\mathbf{r}_x^{A'} + A\mathbf{r}_x\nu + r_\nu A^{A'} + r_\nu \]  

(2.49)

Using the hypothesis of independency:

\[ \mathbf{r}_\psi = A\mathbf{r}_x^{A'} + r_\nu \]  

(2.50)

The augmented noise covariance matrix could be obtained as:

\[ \mathbf{R}^{AMM} = \mathbf{r}_\nu = \mathbf{r}_\psi - A\mathbf{r}_x^{A'} \]  

(2.51)

2.6.4.3 Covariance matrices from stochastic basis

With a stochastic database for \( \mathbf{x}_{TMV} \) and \( \mathbf{z}_{BSPM} \), it is possible to obtain covariance matrices of generic form. With equation (2.46) one can calculate the mean covariance matrix of these datasets as:

\[ \bar{\mathbf{R}} = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} \mathbf{R}_i \]  

(2.52)

The same relation holds true for calculating \( \mathbf{Q} \) using (2.37):

\[ \bar{\mathbf{Q}} = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} \mathbf{Q}_i \]  

(2.53)

Once these matrices are calculated from the stochastic basis, they can be used in the Kalman filter to get a reconstruction of the TMV without any a-priori knowledge of the specific simulation that is being reconstructed.

2.7 Implementation of the Kalman Filter with AMM

When implementing the Kalman filter, to solve the inverse problem of electrocardiography one problem to overcome is the inversion of matrices that are nearly singular. When calculating the Kalman filter gain (2.9) it is necessary to invert the term:

\[ \lambda = \mathbf{A}\mathbf{P}_k^{+} \mathbf{A}^T + \mathbf{R} \]  

(2.54)

In reference to [Berry et al., Apr; Skipa, 2004], the SDV and the pseudo-inverse of \( \lambda \) can be introduced as follows:

\[ \text{SV}
\]D(\lambda) = \mathbf{U}\Sigma
\]V'  

(2.55)

\footnote{Number of simulations performed.}
\[
\lambda^l = U^T \Sigma^{-1} V
\]  
(2.56)

Where:

- The matrix \( U \) contains the left singular vectors. Its determinant is \(|U| = 1\) and its inverse \( U^{-1} = U'\).
- The matrix \( \Sigma \) is a nonnegative diagonal matrix, containing in its diagonal the singular values associated with \( \lambda \). Its inverse is obtained by the inversion of the diagonal elements.
- The matrix \( V \) contains the right singular vectors. Its determinant is \(|V| = 1\) and its inverse \( V^{-1} = V'\).

Due to the nature of \( A \) the resulting matrix to invert is nearly singular (\(|\lambda| \simeq 0\)). When the determinant of a matrix is zero, it cannot be inverted, and if the determinant is very close to zero, then the inversion becomes very inaccurate.

2.7.1 Inversion of \( \lambda \) Term and the Truncated SVD

To overcome the issue of inverting \((AP_kA^T + R)\) which is an almost singular matrix, the inversion is done using the truncated singular value decomposition (truncated SVD), which computes the SVD but treats as zero any singular values that are smaller than a certain value (tolerance).

This method is not an exact method of decomposition, but it approximates the pseudo-inverse in a more stable way.

Truncated SVD is needed because the singular values of \( \lambda \) have very big numbers and numbers close to zero. In the first loop of the simulation of section 4.2 a maximum singular value of \(2.1974 \times 10^7\) and a minimum singular value of \(7.3020 \times 10^{-11}\) very close to zero were obtained. Due to machine precision issues the determinant of \( \lambda \) is equal to zero. That illustrates why the SVD can not be used and why the truncated SVD has to be used instead.

In this work the function pinv of Matlab is used to calculate all pseudo-inverses. This function is an implementation of the truncated SDV where the tolerance value used for truncation is:

\[
tolerance = \max(size(M)) \cdot \|M\| \cdot \text{eps}
\]  
(2.57)

Where the first term is the number of rows or columns depending on which is the biggest one, the second term is the norm of the matrix and the third term is the floating-point relative accuracy of the machine.

2.8 RTS Two-Pass Kalman Smoother

Instead of the Kalman filter, which is a causal filter, the RTS two-pass Kalman smoother is a non-causal filter that takes into account also the future measurements at any sample in time.

The smoother was defined by H. E. Rauch, F. Tung and C. T. Striebel in [Rauch et al., 1965]. Other references to this kind of smoother are [Welch et Bishop; Grwal et Andrews, 2001; Crassidis et Junkins, 2004]. A summary is provided in [Terejannu, 2009] which is an article about different smoothing algorithms.
The RTS smoother is initialized with:

\[ \hat{x}_{[s]N} = \hat{x}_N \]  

(2.58)

Then the smoothed estimates are estimated recursively:

\[ \hat{x}_{[s]k} = \hat{x}_k + B_k \left( \hat{x}_{[s]k+1} - \hat{x}_{k+1} \right) \]  

(2.59)

\[ B_k = \begin{bmatrix} P_k D_k \end{bmatrix}^T (P_{k+1}^{-1})^{-1} \]  

(2.60)

In these equations the subscript \([s]\) designates smoother and all other matrices and vectors are defined in the Kalman filter.

When running this type of smoother, one needs to run the Kalman filter (see section 2.3) first, and intermediate results \(\hat{x}_k, \hat{x}_{k-1}, P_k, P_{k-1}\) are saved. Once the Kalman filter was run the smoother runs back in time from step \(N\) and computes the smoothed estimates denoted as \(\hat{x}_{[s]k}\).

When implementing this algorithm with its inverse of \(P_{k+1}^{-1}\), the problem of inverting a nearly singular matrix comes once again into play. The problem is solved again using the truncated SVD (see section 2.7.1).
Covariance Matrices Analysis

The covariance matrices as defined in section 2.6 reflect the second order moment of the respective random variables. As it will be shown, assuming certain noise properties or not assuming them can make the difference between getting a useful reconstruction or obtaining a solution that is not similar to reality at all. When having access to the simulated data one can obtain the noise that is added to the signals by using the linear models. Let us highlight that this cannot be achieved in a real scenario, where one does not have access to the ground truth which in this situation would be the real TMV of the patient.

3.1 Matrix Analysis

In this section, the covariance matrices will be analyzed as obtained from the simulation in section 4.2 and the corresponding forward calculation.

The shape of the covariance matrices will be explained by looking at the representation of the values of the matrices, the range of these values for covariance matrices 2.25 and 2.30, as well as the auto correlation of the noise in different nodes and the cross-correlation between different nodes.

3.1.1 Process Noise

A close look on the process noise is needed to understand the main reason for the shape of matrix $Q$ (see fig. 3.2). Even though the process noise seems to be Gaussian, see e.g. node 24, there are some nodes that turn out to have non-Gaussian noise, see nodes 1130 and 1080 in Gaussian fig. 3.1.
Fig. 3.1. Process noise autocorrelation, cross-correlation and noise itself for nodes 24, 1130 and 1080. The mean of the signal has been removed.

### 3.1.1.1 Analysis of Matrix Q

The process covariance matrix is not changed at all by the augmented measurement model. Due to this, the dimension of this matrix (2018 rows & 2018 columns) corresponds to the number of nodes defined on the state vector.
The range of values covered by $Q$ is between $-1.4003e+03$ and $7.3128e+03$. Even though most of the values are concentrated around 36, there are terms in the matrix which are less common but have a big impact (see fig. 3.3).

![Fig. 3.3. Zoom fig. 3.2 of matrix $Q$](image)

### 3.1.2 Measurement Noise

The measurement noise as shown in the next figures 3.4 and 3.6 is not Gaussian at all, because the process noise in the nodes is highly autocorrelated, as one can see in the dispersion diagrams and the autocorrelation representations. Besides this, the noise is highly cross-correlated between different nodes as seen in the cross-correlation between nodes 6 and 7.

Comparing figures 3.4 and 3.6 there is little difference between the cross-correlation when the mean is taken out and when not, so statistics are a little bit different when calculated from a random variable with or without mean. The mean introduces a bias into the auto/cross-correlation calculation. For example, the autocorrelation of node 35 is notably increased when the mean is considered. Therefore when covariance matrices of section 2.6 are calculated it makes a big difference to consider the mean or not to consider it.
Fig. 3.4. Measurement noise autocorrelation, cross-correlation and noise itself for nodes 6, 7 and 35. The mean of the signal has been removed.

Fig. 3.5. Measurement noise autocorrelation, cross-correlation and noise itself for nodes 6, 7 and 35. The mean of the signal has not been removed.
3.1. Matrix Analysis

Fig. 3.6. Measurement noise autocorrelation for nodes with augmented measurement noise, cross-correlation and noise itself for nodes 6, 7 and 35. The mean of the signal has not been removed.

The augmented measurement noise is very similar in every node. This explains the shape of matrix \( R_{AMM} \) in fig. 3.8 where \( r_u \) commonly has very high values compared to the values of other parts of the matrix (\( r_v \), \( r_{uv} \) and \( r_{vu} \)), both in its diagonal terms and cross terms.

3.1.2.1 Analysis of Matrix R

In this paragraph, the measurement matrix as explained in section 2.6.2 is analyzed. Since the measurement vector was augmented, the covariance matrix of its noise is also augmented by fictitious noise. Therefore, the dimension of this covariance matrix is 2081 rows and 2081 columns, while the real measurement of BSPM is just only 63 nodes, the augmented measurement is this 63 nodes plus 2018 nodes from the state vector.
The range of values for the different parts of the matrix as defined in (2.45) are:

- between -0.396 and 0.984 for $r_v$,
- between -28.75 and 28.43 for $r_{vu}$.
between -28.75 and 28.43 for $r_{uv}$, as it is the transpose of $r_{vu}$.

- between 9.6404e+03 and 1.1280e+04 for $r_u$.

Therefore matrix $R_{AMM}$ covers a range of values between -28.75 and 1.1280e+04, but the range of values is not distributed randomly. Every sub-matrix has its own range of values with boundaries that are relatively close to each other.

Values on the diagonal of the matrix are not more important than values in the cross-terms of the matrix, which means that all terms are required to get a good reconstruction with the Kalman filter with AMM.

3.2 General Consideration of the Kalman Filter

**Gaussian noise**

The Kalman filter is the best estimator for linear models in case of independent Gaussian noise, therefore:

$$E[w_i w_j] = q\delta_{ij}$$

$$E[v_i v_j] = r\delta_{ij}$$

$$E[w_i v_j] = 0$$

The Kalman filter is then the estimator that minimizes the minimum mean square error (MMSE) of the estimation error covariance and under these circumstances provides the optimal estimation.

**Non-Gaussian noise**

For linear models with correlated noise or non-Gaussian noise where the covariance matrices are not diagonal, the Kalman filter is derived as the best linear estimator that minimizes the minimum mean square error (LMMSE) [Shimkin, 2009], therefore the filter can allow for non-Gaussian noise, but it does not mean that the Kalman filter is the best estimator for this situation, it gives the best estimation that a linear estimator can give.

3.2.1 White Gaussian Noise Assumption

Normally when using Kalman filtering the process noise and the measurement noise are supposed to be Gaussian and independent between different sources. In this case the covariance matrices (see section 2.6) are diagonal.

Under the assumption of optimal (opt.) linear transition and transfer models ($D_k^{opt.}$, $A_k^{opt.}$) where no model error exists, one could suppose the noises affecting them are Gaussian and, inclusively, uncorrelated between different nodes. This last assumption is very likely to be violated, because of spatial proximity of the nodes and imperfections of the models:

- The transition model (see eq. 2.1) could be rewritten as:

$$x_{k+1} = D_k^{opt.} x_k + w_k^{model} + w_k$$
The measurement model (see eq. 2.4) could be rewritten as:

$$x_k = A_k^{opt}.x_k + v_k^{model} + v_k$$  \hspace{1cm} (3.5)

When the model is almost perfect then the error committed by the model is so close to zero that the error of the model can be ignored. In this case one could achieve a good reconstruction using diagonal covariance models.

$$Q = \begin{bmatrix}
E[w_1^tw_1] & \cdots & 0 \\
0 & \cdots & E[w_S^tw_S^S]
\end{bmatrix} \hspace{1cm} (3.6)$$

$$R = \begin{bmatrix}
E[v_1^tv_1] & \cdots & 0 \\
0 & \cdots & E[v_M^tv_M^M]
\end{bmatrix} \hspace{1cm} (3.7)$$

More specifically, when the noise is equally powerful at all nodes, these matrices are the identity multiplied by a scalar factor.

$$R = rI$$ \hspace{1cm} (3.8)

$$Q = qI$$ \hspace{1cm} (3.9)

The results obtained under these conditions are unreliable and they are not close at all to the simulated ones, loosing physical meaning. This proofs that the models are not close to the optimal models and the simple error model cannot reflect this.

**Process Noise**

The process noise cannot be considered to be Gaussian, because it considers the noise of the process which is supposed to be Gaussian indeed, but also allows for modeling errors that are not necessarily Gaussian.

Since the transition matrix is chosen to be constant during the whole reconstruction. This assumption is not completely true, because the "real" transition matrix $D_k$ is changing with time $D_k$. Therefore, a modeling error is added which needs to be modeled.

**Measurement Noise**

The measurement noise, cannot be assumed to be Gaussian because it is a combination of the error of the estimated transfer matrix and the artificially added noise (additive Gaussian noise). Methods that take into account the nature of noise could lead to better estimation results [Schulze et al., 2009].

When working with simulated data, one can have access to the noise and both noises can be estimated considering all the contributions that was commented in the paragraphs above. The noise can be calculated as seen in equations (2.30) and (2.25) using the estimator described in equation (2.34).
Reconstruction of Transmembrane Voltages, TMV

As commented in the introduction due to the inverse problem being strongly ill-posed, there is a need to incorporate a-priori knowledge into the algorithms that solve the inverse problem (see section 2.2).

4.1 A-Priori Knowledge

A-priori knowledge can be included into the algorithm in different ways, forcing the solution into realistic subspaces.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Type of a-priori-knowledge</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D )</td>
<td>Transition matrix</td>
<td>Includes information about the TMV transitions.</td>
</tr>
<tr>
<td>( P_0^{-} )</td>
<td>A-priori error covariance matrix</td>
<td>The first a-priori error covariance matrix is needed to start the algorithm, it can be any matrix except all zeros matrix.</td>
</tr>
<tr>
<td>( Q )</td>
<td>Process noise covariance matrix</td>
<td>Includes prior information about the nature of the process noise, here comes everything we can not model inside the transition matrix.</td>
</tr>
<tr>
<td>( R^{AMM} )</td>
<td>Augmented measurement noise covariance matrix</td>
<td>Includes prior information about the nature of the augmented measurement noise, a-priori knowledge about the real measurement noise is also included inside the first part of the matrix ( r_v ) (see section 3.8).</td>
</tr>
<tr>
<td>( \bar{x} )</td>
<td>Target value of ( x )</td>
<td>Here one can incorporate any desired target value to which the solution is desired to converge.</td>
</tr>
</tbody>
</table>

In a simulated extrasystole The transition matrix \( D \) is not time-invariant, and with this matrix changing over time, it becomes complicated to solve the problem. To approximate this matrix two options are considered in this work, it is approximated by a identity matrix which is a very poor model or we can extract the mean of the different shapes that the transition matrix \( (D_k) \) adopts.
The covariance noise matrices $Q$ and $R^{AMM}$, can be supposed to be diagonal in case white Gaussian noise is assumed. In such cases the prior information included is the hypothesis that noises are Gaussian and moreover if we suppose all nodes to have the same noise power, then each matrix can be tuned by only one parameter controlling the assumed power of the noise. For these matrices one has mainly two options, diagonal matrices (see section 3.2.1) or full matrices (see section 2.6).

The target value of the sources $\bar{x}$ can incorporate information about the value of the solution which forces the reconstruction to converge to that value. Of course this value can be changed at every iteration of the algorithm. An alternative is to set it to the mean of values of the expected solution.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>How a-priori-knowledge can be included into the Kalman filter with $AMM$</th>
</tr>
</thead>
</table>
| D        | Transition matrix                                         | • Identity matrix  
|          |                                                           | • RLMS                                                                                                                                   |
| $P_0$    | A-priori error covariance matrix                          | • Any matrix except a null matrix, changed by the Kalman filter as needed to make the residual error converge to zero.                                                                                                                                 |
| Q        | Process noise covariance matrix                            | • Identity matrix  
|          |                                                           | • from simulation (see eq. 2.25)  
|          |                                                           | • from database (see eq. 2.53)                                                                                                                |
| $R^{AMM}$| Augmented measurement noise covariance matrix              | • Identity matrix  
|          |                                                           | • from simulation (see eq. 2.30)  
|          |                                                           | • from database (see eq. 2.52)                                                                                                                |
| $\bar{x}$| Target value of $x$                                       | • All zeros  
|          |                                                           | • Mean value of the expected solution  
|          |                                                           | • The output of the Kalman filter in the previous step $\hat{x}_{k-1}$  
|          |                                                           | • Prior state estimation $D\hat{x}_{k-1}$, 2.7                                                                                               |

Table 4.2. Terms that can include a-priori knowledge to the Kalman filter and their options
4.2 Simulated TMV

In fig. 4.1 the 2018 points defined on the ventricle are presented. For each node the evolution of the voltage over time is shown, indicating which of the cells are in activated state or inactive state, showing an extrasystole.

The action potential behavior of the transmembrane voltages with the overall length of the simulation being 200 milliseconds is produced by the consecutive activation of cells that activate their neighbor cells, as seen in fig. 4.2, which leads to a change of their potential from a non-activated state to an activated state. The dynamic range of the signals is between -90 mV and 26 mV. The problem has been discretized at a sampling rate of 2018 samples in space per millisecond in time.
4.3 Reconstruction with A-Priori Knowledge of the Covariance Matrices

To reconstruct the TMVs from BSPM, the only information available is the one provided by the 63 sensors that are on the body surface.

In this reconstruction the simulation is used to calculate the process noise covariance matrix and the measurement noise covariance matrix, which means that a-priori knowledge about the solutions is incorporated in the covariance matrices. Once estimated at the beginning of the algorithm and incorporated in the Kalman filter, these noise covariance matrices remain constant throughout the simulation.

4.3.1 Reconstruction with Transition Matrix Estimated from RLMS, Taking Out the Mean for the Covariance Calculation

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Initial value</th>
<th>Constant during Reconstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>Transition matrix</td>
<td>System model estimated with RLMS algorithm</td>
<td>✓</td>
</tr>
<tr>
<td>$z_{k}^{AMM}$</td>
<td>Augmented Measurement vector (2.17)</td>
<td>$z_{1}^{AMM} = (z_{1}^{BSPM} - \alpha L\bar{x})$</td>
<td>... 1</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Regularization parameter</td>
<td>0.001</td>
<td>✓</td>
</tr>
<tr>
<td>$L$</td>
<td>Regularization matrix</td>
<td>Laplacian operator</td>
<td>✓</td>
</tr>
<tr>
<td>$A^{AMM}$</td>
<td>Transfer matrix (2.18)</td>
<td>System model</td>
<td>✓</td>
</tr>
<tr>
<td>$\hat{x}_k$</td>
<td>Corrected estimate state vector</td>
<td>$\hat{x}_0 = \bar{0}$</td>
<td>...</td>
</tr>
<tr>
<td>$\bar{x}$</td>
<td>Target value of $x$</td>
<td>$\bar{x} = \bar{0}$</td>
<td>✓</td>
</tr>
<tr>
<td>$P_k$</td>
<td>Posterior error covariance matrix</td>
<td>$P_0 = I$</td>
<td>...</td>
</tr>
<tr>
<td>$Q$</td>
<td>Process noise covariance matrix</td>
<td>Estimated from simulation 2.37 2</td>
<td>✓</td>
</tr>
<tr>
<td>$R^{AMM}$</td>
<td>Augmented measurement noise covariance matrix</td>
<td>Estimated from simulation 2.38 3</td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 4.3. Reconstruction initial conditions

1. Not applicable: The measurements are provided by the measured BSPM.
2. The covariance is calculated taking out the mean (2.24).
3. The covariance is calculated taking out the mean (2.31).
The RTS smoother is initialized with the following values:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Initial value</th>
<th>Constant during Smoothing</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{x}_{[s]}^{k} )</td>
<td>Corrected smoothed estimation ( \hat{x}^{N} )</td>
<td>( \hat{x}_{[s]}^{N} = \hat{x}^{N} )</td>
<td>No (^4)</td>
</tr>
<tr>
<td>( \hat{x}^{k} )</td>
<td>Corrected Kalman filter estimation ( \hat{x}^{N} )</td>
<td>( \hat{x}^{N} )</td>
<td>No (^5)</td>
</tr>
<tr>
<td>( B_{k} )</td>
<td>RTS smoother gain ( B_{N} = I )</td>
<td>( B_{N} = I )</td>
<td>No (^4)</td>
</tr>
<tr>
<td>( P_{k} )</td>
<td>Posteriori error covariance matrix ( P_{N} )</td>
<td>( P_{N} )</td>
<td>No (^5)</td>
</tr>
<tr>
<td>( P_{k}^{-} )</td>
<td>A-priori error covariance matrix ( P_{N}^{-} )</td>
<td>( P_{N}^{-} )</td>
<td>No (^5)</td>
</tr>
</tbody>
</table>

Table 4.4. Smoothed reconstruction initial conditions

Though the shape of the reconstruction is pretty similar to the form of the simulation, the range of values of the reconstruction is totally erroneous, because it contains values between 0 mV and 120 mV, while the simulation is approximately between -90 mV and 20 m (see fig. 4.3). This erroneous range of values is due how the covariance matrices are calculated: these matrices were estimated taking out the mean (see table 4.3).

![Fig. 4.3. Reconstructed TMV 3D time evolution with the Kalman filter](image)

The smoothed solution stabilized the error to a constant value (see fig. 4.4) but it also does not solve the range problem with the reconstructed data.

\(^4\) Parameter changed by the smoother.

\(^5\) Parameter coming from the outputs of the Kalman filter.
Fig. 4.4. Reconstruction’s absolute error $\|\hat{x}_k - x_k\|$ (top), reconstruction’s relative error $\frac{\|\hat{x}_k - x_k\|}{\|x_k\|}$ (bottom)

Though the residual norm is very small compared to the norm of the state vector, the relative error is bigger than 100%. The obtained reconstruction although good in shape is therefore totally out of range with a dynamic range between approximately 0 mV and 120 mV.

Fig. 4.5. Residual norm $\|A\hat{x}_k - z_k^{RSPM}\|$
4.3. Reconstruction with A-Priori Knowledge of the Covariance Matrices

4.3.2 Reconstruction with Transition Matrix estimated from RLMS

Table 4.5 shows to what initial conditions the filter was set and the parameters changed during the reconstruction of TMV.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Initial value</th>
<th>Constant during Reconstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>Transition matrix</td>
<td>System model estimated with RLMS algorithm</td>
<td>✓</td>
</tr>
<tr>
<td>$z_{k}^{AMM}$</td>
<td>Augmented Measurement vector (2.17)</td>
<td>$z_{1}^{AMM} = \left( \begin{array}{c} z_{1}^{BSPM} \ \alpha L \bar{x} \end{array} \right)$</td>
<td>...</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Regularization parameter</td>
<td>0.001</td>
<td>✓</td>
</tr>
<tr>
<td>$L$</td>
<td>Regularization matrix</td>
<td>Laplacian operator</td>
<td>✓</td>
</tr>
<tr>
<td>$A_{k}^{AMM}$</td>
<td>Transfer matrix (2.18)</td>
<td>System model</td>
<td>✓</td>
</tr>
<tr>
<td>$\hat{x}_k$</td>
<td>Corrected estimate state vector</td>
<td>$\bar{x} = 0$</td>
<td>...</td>
</tr>
<tr>
<td>$\pi$</td>
<td>Target value of $x$</td>
<td>$\bar{x}$</td>
<td>✓</td>
</tr>
<tr>
<td>$P_k$</td>
<td>Posterior error covariance matrix</td>
<td>$P_0 = I$</td>
<td>...</td>
</tr>
<tr>
<td>$R_{k}^{AMM}$</td>
<td>Augmented measurement noise covariance matrix</td>
<td>Estimated 2.37</td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 4.5. Reconstruction initial conditions

The smoother is not itself a reconstruction because it uses the results from a reconstruction and tries to smooth them. The RTS smoother of section 2.8 was initialized with the following values (see table 4.6).

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Initial value</th>
<th>Constant during Smoothing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{x}_{[s]k}$</td>
<td>Corrected smoothed estimation</td>
<td>$\hat{x}_{[s]N} = \hat{x}_N$</td>
<td>No 2</td>
</tr>
<tr>
<td>$\hat{x}_k$</td>
<td>Corrected Kalman filter estimation</td>
<td>$\hat{x}_N$</td>
<td>No 3</td>
</tr>
<tr>
<td>$B_k$</td>
<td>RTS smoother gain</td>
<td>$B_N = I$</td>
<td>No 2</td>
</tr>
<tr>
<td>$P_k$</td>
<td>Posterior error covariance matrix</td>
<td>$P_N$</td>
<td>No 3</td>
</tr>
<tr>
<td>$P^-_k$</td>
<td>A-priori error covariance matrix</td>
<td>$P^-_N$</td>
<td>No 3</td>
</tr>
</tbody>
</table>

Table 4.6. Smoothed reconstruction initial conditions

1 Not applicable: The measurements are provided by the measured BSPM.
2 Parameter changed by the smoother.
3 Parameter coming from the outputs of the Kalman filter.
The Kalman filter is initialized to zero, that is why at the beginning of the reconstruction the sources are close to zero and the Kalman filter needs around 20 steps to reach a stable solution. The smoother does not always provide better results as can be seen in fig. 4.8. In step 131 one can recognize a difference in TMV represented by an orange color instead of a red one, which is the peak of the error situated at 130ms approximately.
4.3. Reconstruction with A-Priori Knowledge of the Covariance Matrices

The error of the smoothed solution in comparison to the not smoothed one is shown in fig. 4.8, the RTS algorithm improves the solution giving better performance at the beginning, but it leads to slightly worse estimations in some of the further steps.

Comparing 4.8 and 4.9 one can see that having the minimum residual norm does not imply necessarily to obtain the correct solution to the problem, as one can see at the first 20ms, where the residual norm is close to zero, but the relative error is close to one, which means a 100% of error with respect to the simulation, something that can not be admissible a correct solution.
4.3.3 Reconstruction with Transition Matrix as Identity Matrix

This reconstruction is the same as section 4.3 but with the difference that no a-priori knowledge is incorporated in the transition matrix. Therefore an identity matrix is chosen, which is a very poor model of the dynamics of the system.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Initial value</th>
<th>Constant during Reconstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>Transition matrix</td>
<td>( D = I )</td>
<td>✓</td>
</tr>
<tr>
<td>( z_{k}^{AMM} )</td>
<td>Augmented Measurement vector (2.17)</td>
<td>( z_{1}^{AMM} = (z_{1}^{BSPM} \alpha L \bar{x}) )</td>
<td>... 1</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Regularization parameter</td>
<td>0,001</td>
<td>✓</td>
</tr>
<tr>
<td>( L )</td>
<td>Regularization matrix</td>
<td>Laplacian operator</td>
<td>✓</td>
</tr>
<tr>
<td>( A_{AMM} )</td>
<td>Transfer matrix (2.18)</td>
<td>System model</td>
<td>✓</td>
</tr>
<tr>
<td>( \hat{x}_{k} )</td>
<td>Corrected estimate state vector</td>
<td>( \hat{x}_{0} = \bar{0} )</td>
<td>...</td>
</tr>
<tr>
<td>( \bar{x} )</td>
<td>Target value of ( x )</td>
<td>( \bar{x} = \bar{0} )</td>
<td>✓</td>
</tr>
<tr>
<td>( P_{k} )</td>
<td>Posteriori error covariance matrix</td>
<td>( P_{0} = I )</td>
<td>...</td>
</tr>
<tr>
<td>( R_{AMM} )</td>
<td>Augmented measurement noise covariance matrix</td>
<td>Estimated 2.38</td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 4.7. Reconstruction initial conditions

The RTS smoother is started with the next initial conditions:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Initial value</th>
<th>Constant during Smoothing</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{x}_{[s]k} )</td>
<td>Corrected smoothed estimation</td>
<td>( \hat{x}<em>{[s]N} = \hat{x}</em>{N} )</td>
<td>No 2</td>
</tr>
<tr>
<td>( \hat{x}_{k} )</td>
<td>Corrected Kalman filter estimation</td>
<td>( \hat{x}_{N} )</td>
<td>No 3</td>
</tr>
<tr>
<td>( B_{k} )</td>
<td>RTS smoother gain</td>
<td>( B_{N} = I )</td>
<td>No 2</td>
</tr>
<tr>
<td>( P_{k} )</td>
<td>Posteriori error covariance matrix</td>
<td>( P_{N} )</td>
<td>No 3</td>
</tr>
<tr>
<td>( P_{k}^{-} )</td>
<td>A-priori error covariance matrix</td>
<td>( P_{N}^{-} )</td>
<td>No 3</td>
</tr>
</tbody>
</table>

Table 4.8. Smoothed reconstruction initial conditions

The first 40 steps is where the biggest error takes place because of the bad model and because of the initialization of the Kalman filter that needs some steps to stabilize the solution.

1 Not applicable: The measurements are provided by the measured BSPM.
2 Parameter changed by the smoother.
3 Parameter coming from the outputs of the Kalman filter.
The impact of a bad model of the system, in case the transition matrix is modeled as an identity matrix, is great at the first 50 iterations of the reconstruction in which the error is clearly bigger than the error achieved in reconstructions according to 4.3 where the transition matrix is approximated with a RLMS algorithm.

Fig. 4.10. Reconstructed TMV 3D time evolution with the Kalman filter

Fig. 4.11. Reconstruction’s absolute error $\| \hat{x}_k - x_k \|$ (top), reconstruction’s relative error $\frac{\| \hat{x}_k - x_k \|}{\| x_k \|}$ (bottom)
Fig. 4.12. Residual norm $\| A\hat{x}_k - z_k^{BSPM} \|$
4.3.4 Reconstruction with Dynamical Target Value

In this reconstruction the goal is to improve the results achieved in section 4.3.2 by including dynamic information as target value $\bar{x}$.

The action potentials spread like a wave front, similar to the wave produced by a little rock thrown into a lake. Therefore we can say that if we make a snapshot of the action potentials, between two time points near to each other there is little difference between them. The same is done in this reconstruction. Our solution in the actual step should be similar to the solution that was estimated in the previous iteration, but since the target value is changed, the augmented measurement vector needs to be changed and the augmented measurement noise covariance matrix needs to be updated for every step as well.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Initial value</th>
<th>Constant during Reconstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$</td>
<td>Transition matrix</td>
<td>System model estimated with RLMS algorithm</td>
<td>✓</td>
</tr>
<tr>
<td>$z_{k}^{AMM}$</td>
<td>Augmented Measurement vector (2.17)</td>
<td>$z_{k}^{AMM} = \left(\begin{array}{c} z_{k}^{BSPM} \ \alpha \mathbf{L} \bar{x} \end{array}\right)$</td>
<td>... 1</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Regularization parameter</td>
<td>0.001</td>
<td>✓</td>
</tr>
<tr>
<td>$L_{AMM}$</td>
<td>Regularization matrix</td>
<td>Laplacian operator</td>
<td>✓</td>
</tr>
<tr>
<td>$A_{AMM}$</td>
<td>Transfer matrix (2.18)</td>
<td>System model</td>
<td>✓</td>
</tr>
<tr>
<td>$\hat{x}_{k}$</td>
<td>Corrected estimate state vector</td>
<td>$\hat{x}_{0} = 0$</td>
<td>...</td>
</tr>
<tr>
<td>$\bar{x}$</td>
<td>Target value of $x$</td>
<td>$\bar{x}_{0} = 0$</td>
<td>Modified at every step $\bar{x}<em>{k} = \hat{x}</em>{k-1}$</td>
</tr>
<tr>
<td>$P_{k}$</td>
<td>Posteriori error covariance matrix</td>
<td>$P_{0} = \mathbf{I}$</td>
<td>...</td>
</tr>
<tr>
<td>$Q$</td>
<td>Process noise covariance matrix</td>
<td>Estimated from simulation 2.37</td>
<td>✓</td>
</tr>
<tr>
<td>$R_{AMM}$</td>
<td>Augmented measurement noise covariance matrix</td>
<td>Estimated from simulation 2.38 with $\bar{x}_{0} = 0$</td>
<td>Depends on $\bar{x}_{k}$ (see section 2.46)</td>
</tr>
</tbody>
</table>

Table 4.9. Reconstruction initial conditions

Under these conditions the algorithm needs about 20 steps to stabilize the solution. The smoothed solution looks better with less irregularities (see the error in fig. 4.14).

1 Not applicable: The measurements are provided by the measured BSPM.
The performance of this setup really improves the results obtained in the solution of section 4.3.2, and it needs just only 12 steps to stabilize the solution compared to the 20 steps needed with no dynamic target value.

Fig. 4.13. Reconstructed TMV 3D time evolution with the Kalman filter

Fig. 4.14. Reconstruction’s absolute error $\|\hat{x}_k - x_k\|$ (top), reconstruction’s relative error $\frac{\|\hat{x}_k - x_k\|}{\|x_k\|}$ (bottom)
Fig. 4.15. Residual norm $\| A\hat{x}_k - z_k^{BSPM} \|$
4.3.5 Reconstruction with Mean Target Value

Table 4.10 shows what initial conditions the filter was set to and the parameters changed during the reconstruction of TMV.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Initial value</th>
<th>Constant during Reconstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>Transition matrix</td>
<td>System model estimated with RLMS algorithm</td>
<td>✓</td>
</tr>
<tr>
<td>$z_{k}^{AMM}$</td>
<td>Augmented Measurement vector (2.17)</td>
<td>$z_{l}^{AMM} = \left( z_{l}^{BSPM} \alpha \mathbf{L} \mathbf{x} \right)$</td>
<td>... 1</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Regularization parameter</td>
<td>0.001</td>
<td>✓</td>
</tr>
<tr>
<td>$L$</td>
<td>Regularization matrix</td>
<td>Laplacian operator</td>
<td>✓</td>
</tr>
<tr>
<td>$A_{AMM}$</td>
<td>Transfer matrix (2.18)</td>
<td>System model</td>
<td>✓</td>
</tr>
<tr>
<td>$\hat{x}_{k}$</td>
<td>Corrected estimate state vector</td>
<td>$\hat{x}_{0} = \bar{0}$</td>
<td>...</td>
</tr>
<tr>
<td>$\mathbf{x}$</td>
<td>Target value of $\mathbf{x}$</td>
<td>$\mathbf{x} = E[\mathbf{x}_{TMV}]$</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathbf{P}_{k}$</td>
<td>Posteriori error covariance matrix</td>
<td>$\mathbf{P}_{0} = \mathbf{I}$</td>
<td>...</td>
</tr>
<tr>
<td>$Q$</td>
<td>Process noise covariance matrix</td>
<td>Estimated form simulation</td>
<td>✓</td>
</tr>
<tr>
<td>$R_{AMM}$</td>
<td>Augmented measurement noise covariance matrix</td>
<td>Estimated from simulation 2.38</td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 4.10. Reconstruction initial conditions

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Initial value</th>
<th>Constant during Smoothing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{x}_{[s]k}$</td>
<td>Corrected smoothed estimation</td>
<td>$\hat{x}<em>{[s]N} = \hat{x}</em>{N}$</td>
<td>No 2</td>
</tr>
<tr>
<td>$\hat{x}_{k}$</td>
<td>Corrected Kalman filter estimation</td>
<td>$\hat{x}_{N}$</td>
<td>No 3</td>
</tr>
<tr>
<td>$B_{k}$</td>
<td>RTS smoother gain</td>
<td>$B_{N} = \mathbf{I}$</td>
<td>No 2</td>
</tr>
<tr>
<td>$\mathbf{P}_{k}$</td>
<td>Posteriori error covariance matrix</td>
<td>$\mathbf{P}_{N}$</td>
<td>No 3</td>
</tr>
<tr>
<td>$\mathbf{P}_{-k}$</td>
<td>A-priori error covariance matrix</td>
<td>$\mathbf{P}_{-N}$</td>
<td>No 3</td>
</tr>
</tbody>
</table>

Table 4.11. Smoothed reconstruction initial conditions

Though the Kalman filter is initialized with zero, the reconstruction starts closer to the real value of the simulation due to the value used inside the target value which contains the mean of the total simulation including a-priori knowledge of the simulation. This way the algorithm can quickly follow the evolution of the signal, and starts with more realistic values.

1 Not applicable: The measurements are provided by the measured BSPM.  
2 Parameter changed by the smoother.  
3 Parameter coming from the outputs of the Kalman filter.
4.3. Reconstruction with A-Priori Knowledge of the Covariance Matrices

In this particular case the reconstruction starts closer to the simulated values due to the a-priori knowledge included in the target value.

The error of the reconstruction is from the early beginning closer to zero, around 2%, and the RTS smoothed reconstruction can even reduce the error.
Fig. 4.18. Residual norm $\|A\hat{x}_k - z_k^{BSPM}\|$
4.4 Reconstruction without A-Priori Knowledge of the Covariance Matrices

In the next reconstructions since the Kalman filter does not provide the noise covariance matrices, they must be given to the filter assuming we have some kind of a-priori knowledge about the process noise and the measurement noise. This a-priori knowledge is obtained from simulations of myocardial ischemia as published in [Loewe et al., 2011], where 238 ischemia regions were simulated.

4.4.1 Reconstruction

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
<th>Initial value</th>
<th>Constant during Reconstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$</td>
<td>Transition matrix</td>
<td>System model estimated with RLMS algorithm</td>
<td>✓</td>
</tr>
<tr>
<td>$z_{k}^{AMM}$</td>
<td>Augmented Measurement vector (2.17)</td>
<td>$z_{1}^{BSPM}$</td>
<td>$\alpha L \bar{x}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Regularization parameter</td>
<td>0.001</td>
<td>✓</td>
</tr>
<tr>
<td>$L$</td>
<td>Regularization matrix</td>
<td>Laplacian operator</td>
<td>✓</td>
</tr>
<tr>
<td>$A_{k}^{AMM}$</td>
<td>Transfer matrix (2.18)</td>
<td>System model</td>
<td>✓</td>
</tr>
<tr>
<td>$\bar{x}_{k}$</td>
<td>Corrected estimate state vector</td>
<td>$\bar{x} = E[x_{TMV}]$</td>
<td>✓</td>
</tr>
<tr>
<td>$P_{k}$</td>
<td>Posteriori error covariance matrix</td>
<td>$P_{0} = I$</td>
<td>✓</td>
</tr>
<tr>
<td>$Q$</td>
<td>Process noise covariance matrix</td>
<td>Estimated from database 2.53</td>
<td>✓</td>
</tr>
<tr>
<td>$R_{k}^{AMM}$</td>
<td>Augmented measurement noise covariance matrix</td>
<td>Estimated from database 2.52 with $\bar{x}_{0} = 0$</td>
<td>Depends on $\bar{x}_{k}$ (see section 2.46)</td>
</tr>
</tbody>
</table>

Table 4.12. Reconstruction initial conditions

In the simulation in fig. 4.19 an ischemia was placed, visible from step 31 to step 171, when a reconstruction is performed using a-priori knowledge from the simulation like in section 4.3 good reconstructions are achieved.

Under conditions of table 4.12 no good result was achieved, because the error introduced by the models is different for every simulation of the database, making it impossible for the filter to compensate the error introduced by the transition and the transfer model.

1 Not applicable: The measurements are provided by the measured BSPM.
The covariance matrices are both estimated from a database including no specific a-priori knowledge from the simulation. In fig. 4.20 it can be observed that the evolutions of the action potentials are physically not coherent and the ischemia cannot be seen.
5

Discussion and Outlook

5.1 Discussion

Including a-priori knowledge in the algorithm that tries to estimate $x_{TMV}$ is a way to choose between the infinite solutions that minimize $\|Ax - z_{BSPM}\|^2$. When finding a solution with very little residual norm $\|A\hat{x} - z_{BSPM}\|^2 \simeq 0$, the solution is mathematically correct but can have lack of physical sense.

To select the solution with medical sense among all the possible solutions, additional information has to be introduced in the solution. This information is represented in the cost terms as an additional regularization term (see section 2.4.1).

5.1.1 Results

An analysis of the results exposes the advantages and/or disadvantages of the different reconstruction methods.

Obtaining a good reconstruction depends highly on the a-priori knowledge that is incorporated in the algorithm.

The reconstruction in section 4.3 gives the best results since it incorporates prior knowledge from the simulation in the noise covariance matrices. On the one hand, with this type of reconstruction one can obtain a satisfactory reconstruction, the best one obtained in this work. On the other hand there is a big drawback when reconstructing the sources this way, since it cannot be applied to real scenarios, because the estimation of the noise covariance matrices depends on the sources, to which there is no access when dealing with real patients.

The smoothed reconstruction which is done offline once the Kalman filter is run through and after all required intermediate results from the Kalman filter are saved (see section 2.8), is supposed to work better. This is true for most of the reconstructions, but sometimes it gives worse results than the reconstruction using just the Kalman filter (see fig.
5.1). Therefore the smoothed solution is not necessarily better than the one obtained with the Kalman filter.

Taking a look at both figures 5.1 and 5.2, it can be seen that lower residual norms do not necessarily imply less erroneous solutions. In the first iterations one can see how the RTS smoother increases the residual norm with respect to the Kalman filter reconstruction, but it clearly improves the solution in the first steps of the algorithm. Sometimes however, the RTS smoother increases the relative error although it does not change the residual norm, see points next to 140ms and 180ms.

When using the identity matrix as transition matrix, and therefore taking a very simple system model of the dynamics of the system, the Kalman filter needs around 50 steps to stabilize the solution (see fig. 5.3), instead of the 20 steps needed when the dynamics of the system are approximated by a RLMS matrix (see fig. 5.1).
In this case one can see that the error of the smoothed solution is always below the error produced by the Kalman filter, and the reconstruction is always better.

The parameter $\alpha$ is also very important. It quantifies how strong we want the regularization to be and it must be carefully tuned for that reason. If not, the error of the solution can be too big to consider it a valid reconstruction. The least reconstruction error was achieved with alpha values of $\alpha \approx 0.001$. 

![Fig. 5.3. Absolute and relative residual error comparison](image1)

![Fig. 5.4. Absolute and relative residual error comparison](image2)
Chapter 5. Discussion and Outlook

For the regularization parameter, the residual norm of different reconstructions can be again very similar but its error can be totally different (see fig. 5.5 and 5.6) achieving a good reconstruction with good relative error (in the case of $\alpha = 0.001$) or a bad reconstruction with intolerable relative error (case of $\alpha = 0.1$).

As an additional factor, the choice of the target value can improve the error with respect to the simulation for the same reconstruction conditions.
5.1. Discussion

In this work, instead of using parameter optimization methods such as the L-curve [Hansen, 1992], the regularization parameter was chosen based on patient specific studies.

The best target values achieved in this work, which give the least reconstruction error, are the mean value of the simulation and the dynamic target value. The mean value of the simulation incorporates a-priori knowledge, but the dynamic target value has the advantage that it does not need a-priori knowledge from the simulated data. It just uses the output of the previously estimated data of the Kalman filter. On the other hand, it needs more steps to achieve a stable solution.

5.1.2 Comparison to other Methods

With the Kalman filter with Augmented Measurement Model one can obtain reconstructions with smaller error than with the Kalman filter with Integrated Tikhonov-Regularization [Schulze et al., 2009] (see fig. 5.5 where $\alpha = 0$ represents the reconstruction with no regularization).

In the work of [Brooks et Gee, 1999] the L-Curve method is used which minimizes both the residual norm and the regularization term, but as seen in fig. 5.6 similar residual norms or even norms close to zero can give results with bigger reconstructions errors, see fig. 5.5 and 5.6.

In comparison to the work of [Jiang et al., 2007] where the MAP-Method is used, Gaussian noise is assumed, while in this work the modeling error is also taken into account and introduced into the covariance matrices, the modeling error as seen in section 3 is non-Gaussian and identity noise covariance matrices leaded to bad results, under non-Gaussian noise the Kalman filter gives the best linear estimation [Shimkin, 2009], therefore good reconstructions could be obtained with the Kalman filter with AMM.
5.2 Outlook

5.2.1 Check Model Linearity

The better the model, the better the reconstruction one will achieve. This can be observed in simulations of section 4.3.1 and 4.3.3.

To deal with the non-linearity of the models, the Unscented Kalman filter (UKF) could be used instead of the Kalman filter which uses linear transition and transfer models. With UKF instead of using linear model one could use more complicated transition and transfer functions that reduce the model error and then apply again the augmented measurement model to the UKF [Wan et Van Der Merwe, 2000].

To implement it the idea would be similar to what has been done in this work. First the UKF has to be designed and then the AMM has to be applied to the state space model to regularize the solution.

5.2.2 Improving the Noise Covariance Matrices Estimation

To obtain better noise covariance more realizations of both processes are needed, both action potentials distributions and their associated body surface potential maps. Then it will be necessary to study the nature of the noise and their associated covariance matrices to see if useful covariance matrices can be obtained.

These matrices should be calculated for multiple cycles of the action potentials for better estimation. Since the processes are assumed to be stationary, having more samples of the process one will achieve a better estimation of the second moment order of the process.

One should take a close look if the noise covariance matrices change in case of an ischemia, because in case of scar tissue no currents propagate through this part of the heart.

5.2.3 Studying Noise Covariance Matrices in Different Case Scenarios

Once a database with realizations of measurements and sources is created, a study should be done taking a look at the variance of the covariance matrices. In case the variance of different covariance matrices is very big, it could mean that it is necessary to have different covariance matrices for different types of scenarios to obtain a good reconstruction of the sources, because wrong covariance matrices will not lead to a valid reconstruction of the action potentials of the patient’s heart.

5.2.4 Online Noise Covariance Matrices Estimation

*Auto-Covariance Least-Squares (ALS)*

Another idea would be to create some kind of adaptive Kalman filter that is able to estimate both noise covariance matrices from the output of the Kalman filter. One could try to use auto-covariance least-squares (ALS), a method that uses output data to estimate the covariances [Abdel-Hafez, 2010].
5.2. Outlook

Dual Kalman filter

Another way to try to solve this problem would be to use a Dual Kalman filter estimation [Paul et Wan, 2005] or Dual UKF estimation, where \( x_k \) and the model parameters (indicated by the letter \( w_k \) not to be confused with noise \( w_k \) used in previous sections) are estimated from the measurements [Wan et van der Menve, 2002; Haykin, 2001].

![Fig. 5.8. Dual Kalman filter scheme, modified from [Paul et Wan, 2005]](image1)

\[
\hat{x}_k = \begin{bmatrix} x_k \\ w_k \end{bmatrix}
\]

(5.1)

The Dual Kalman filter could also be modified to produce an estimation of both noises \( \hat{w} \) and \( \hat{v} \), the process noise and the measurement noise respectively.

Joint Kalman filter

Another possibility is to use the joint Kalman filter [Haykin et al., 2001]. Here the reasoning is similar to what was done in the Augmented Measurement Model (see section 2.4.1), the unknown system variables are concatenated to the state vector.

![Fig. 5.9. Dual Kalman filter scheme. One filter is used to estimate the process state and another one to estimate the process noise, modified from [Haykin et al., 2001]](image2)
If that works out, one could try to augment the state with even more unknown parameters, like the measurement noise.

\[ \tilde{x}_k = \begin{bmatrix} x_k \\ w_{1k} \\ w_{2k} \\ w_{3k} \end{bmatrix} \] (5.2)

When this state augmentation is done the system model needs to be changed properly to make things work, because since the state vector is augmented the transition matrix and the process noise are also augmented.
Acknowledgements

The writing of this work comes to its end, so in this section I will thank the people who somehow contributed to make this words possible and of course i will try to make a slight reference to them.

Per la meva família:

Deixaré aquí unes paraules d’amor en la meva llengua paterna per als meus pares, agraint-vos el vostre amor i paciència, i tot i que ja sabeu que us estimo amb tot el meu cor, queda així escrit per la posteritat, amb amor el vostre fill Francesc, un petó!!.

Dejaré también escrito por supuesto unas palabras de amor para mi abuela que me quiere, dificilmente haya otra persona que sienta más orgullo por mí que tú, un beso yaya!!.

Diplomandenraum:

Also gut, ich würde gerne ein paar Worte zur Kaffeepause und zum Kickerspielen sagen. Ich liebe diese Zeit des Tages, es klingt immer gut, wenn irgendjemand sagt “Kaffeepause, geh ma hoch!”.

Dipl.-Ing. Walther Schulze:

Es sieht so aus, dass die Endphase meiner Arbeit gekommen ist und ich will dir für deine Hilfe danken, die zahlreiche Probleme gelöst hat, und natürlich für die Korrekturen, die unerlässlich für diese Arbeit waren. Am Ende ist alles gut gelaufen, danke!.

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Ich danke Ihnen für die Möglichkeit in diesem schönen Bereich zu arbeiten. Ich wünsche viel Erfolg und Glück mit den Ergebnissen aller Projekte, die man hier durchführt.

Also Leute, anstatt “ciao ciao” hoffe ich, dass ich “Bis bald” sagen kann!! 😊
References


[Welch et Bishop 2006] Welch, Greg; Bishop, Gary: An Introduction to the Kalman Filter. In: University of North Carolina at Chapel Hill, Chapel Hill, NC 7 (2006), Nr. 1