Abstract

In this work, a stabilized formulation to solve the inductionless magnetohydrodynamic (MHD) problem using the finite element method is presented. The important feature of this formulation resides in the design of the stabilization terms, which serve several purposes. First, convective dominated flows in the Navier-Stokes equations can be dealt with. Second, there is no need to use interpolation spaces subject to an inf-sup condition both for the pairs $u-p$ and $j-\phi$ and therefore linear interpolation spaces can be used. Finally, this formulation allows to deal with flows with high values of the Hartmann number, that is, flows where the electromagnetic forces are much higher than the viscous forces.

**Key words**: Inductionless MHD, primal-dual formulation, stabilized finite element, variational multiscale method, monolithic scheme, HCLL test blanket module.

Parallel: En aquest treball s’ha proposat una formulació estabilitzada del mètode dels elements finits per resoldre el problema de la magnetohidrodinàmica on el camp magnètic induït es considera negligiblement. L’aspecte important d’aquesta formulació consisteix en el disseny dels termes d’estabilització que tenen diferents objectius. Primer, fluxes dominats per la convecció a les equacions de Navier-Stokes poden ser resolts. Segon, s’elimina la necessitat de considerar espais d’interpolació que compleixin les condicions inf-sup, tant pel parell $u-p$ com pel parell $j-\phi$, i per tant es poden fer servir espais d’interpolació lineals. Finalment, aquesta formulació permet el tractament de fluxes amb valors del nombre de Hartmann alts, és a dir, fluxes on les forces electromagnètiques són més importants que les forces degudes a la viscositat del fluid.

**Paraules clau**: MHD sense inducció, formulació primal-dual, elements finits estabilitzats, mètode multi-escala variacional, esquema monolític, HCLL mòdul de test.
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1 Introduction

The objective of this work is to present a finite element method for the approximation of
the inductionless magnetohydrodynamic (MHD) problem which arises in physical phenomena
when the magnetic field induced by currents in the fluid is negligible compared to the external
magnetic field applied, $B$. The inductionless approximation to the MHD problem consists
of the equations of conservation for momentum, mass and charge together with Ohm’s law.
Therefore, the problem is written in terms of velocity $\mathbf{u}$, pressure $p$, current density $\mathbf{j}$ and
electric potential $\phi$. The structure of this system of partial differential equations corresponds
to the Navier-Stokes equations coupled with a Darcy’s problem.

The finite element approximation of the inductionless MHD faces several difficulties. First,
there is the classical and well-known problem of dealing with flows where the first order deriva-
tives, for instance the convective term in the Navier-Stokes equations, dominate the second
order derivatives. This situation eliminates the elliptic nature of the system of differential equa-
tions. This behavior may lead to oscillations and instabilities that can be overcome by means
of stabilized finite element methods. Second, there is the compatibility condition between the
approximation spaces for the velocity and the pressure, but also for the current density and
the electric potential. These conditions are expressed in a classical inf-sup form. Finally, the
coupling between the hydrodynamic and the electromagnetic problems may lead to numeri-
cal difficulties when solving the resulting discrete system of equations. In the Navier-Stokes
equations, the coupling comes from the Lorentz’s force, whereas in the magnetic problem the
coupling appears in Ohm’s law because the conducting fluid moves with velocity $u$. The goal
of this work is to design a stabilized finite element method able to deal with everyone of these
difficulties.

The stabilization technique presented in this work is developed in the variational multiscale
framework. It is based on a two-scale decomposition of the unknowns in the finite element
component and a subgrid scale or subscale which corresponds to the unknown component that
can not be captured by the finite element space. The idea followed in this paper was first
presented in [7]. In particular, the version for systems of equations was introduced in [3]. The
key point is the approximation of the subgrid scales. In this work it has been chosen the simplest
approach that consists of taking them proportional to a projection of the residual of the finite
element approximation multiplied by a matrix of stabilization parameters. Between the several
options to select the projection and the structure of the matrix of stabilization parameters,
the identity and a diagonal structure have been chosen, respectively. Up to this point, the only
missing issue to close the formulation is the design of the stabilization parameters. It has been
done based on the stability and convergence analysis of the method.

The literature about the numerical approximation of the inductionless MHD equations is
not very vast. There has been some research done in the finite difference and finite volume
community. For instance, a conservative scheme for incompressible MHD flows is proposed
in [12], [13]. The presented algorithm uncouples the computation of the unknowns through
the implementation of a coupling iterative scheme in order to converge to the global solution.
The finite volume method is also very much used to solve the inductionless MHD equations
in simulations of the HCLL test blanket module for nuclear fusion reactors, see [2], [9] for
examples in this field. The approach used to solve the problem is a Poisson equation for the
electric potential obtained by taking the divergence of the Ohm’s law and adding the Lorentz
force as a body force in the momentum equation. An iterative algorithm is used to converge
globally to the coupled solution. There exists several articles applying the finite element method
to solve the MHD equations in the general case of induced magnetic field non-negligible (see
for instance [4], [14], [15], [6]) but the authors have not found any paper regarding the solution of the inductionless MHD with the finite element method.

The paper is organized as follows. In Section 2 the problem to be solved is stated both in its continuous and its variational form. Issues regarding the time integration and the linearization of the non-linear term are discussed in Section 3, leading to a time discrete and linearized scheme. Next, the subgrid scale framework is explained in Section 4 and then it is applied to the inductionless MHD problem. After proposing the stabilization method, it is fully analyzed regarding its stability, accuracy and convergence properties. The final scheme proposed in this work is written in Section 5. Numerical experiments verifying the theoretical results are explained in Section 6 and finally some conclusions are drawn in Section 7.

2 Problem statement

2.1 Initial and boundary value problem

Let \( \Omega \subset \mathbb{R}^d \) (\( d=2 \) or \( 3 \)) be a domain where we want to solve the inductionless MHD problem during the time interval \([0, T]\). The unknowns of the problem are the fluid velocity \( u : \Omega \times (0, T) \rightarrow \mathbb{R}^d \), the pressure \( p : \Omega \times (0, T) \rightarrow \mathbb{R} \), the current density \( j : \Omega \times (0, T) \rightarrow \mathbb{R}^d \) and the electric potential \( \phi : \Omega \times (0, T) \rightarrow \mathbb{R} \), which are solution of the system of partial differential equations:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p - \frac{1}{\rho} (j \times B) &= f, \\
\nabla \cdot u &= 0, \\
j + \sigma \nabla \phi - \sigma (u \times B) &= 0, \\
\nabla \cdot j &= 0,
\end{align*}
\]

where \( \rho \) is the fluid density, \( B \) the external magnetic field, \( f \) the body forces of the flow motion and \( \sigma \) the electric conductivity. It is important to note that the pressure \( p \) we are working with here is the kinematic pressure (pressure divided by density).

Let us define two different partitions of the domain boundary \( \Gamma = \partial \Omega \). The first one, for imposing the boundary conditions of the hydrodynamic unknowns, is divided between the part of the boundary where imposing essential (Dirichlet) boundary conditions and the rest of the boundary where imposing natural (Neumann) boundary conditions. The other partition is used for imposing the boundary conditions of the magnetic problem. It consists of the part of the boundary that corresponds to perfectly conducting walls and the part that corresponds to perfectly insulated walls:

\[
\begin{align*}
\Gamma &= \Gamma_{E,u} \cap \Gamma_{N,u}, \\
\Gamma &= \Gamma_{C,j} \cap \Gamma_{I,j}.
\end{align*}
\]

The boundary conditions for the velocity at the walls are the non-slip wall conditions, that is,

\[
u = 0, \quad \text{on } \Gamma_{E,u}.
\]

On the other hand, the free boundary conditions for the velocity are zero traction conditions,

\[
-p n + \nu n \cdot \nabla u = 0, \quad \text{on } \Gamma_{N,u}.
\]
With respect to the boundary conditions for the magnetic equations, two different options have been considered. Firstly, perfectly insulating walls. In this situation, the electric currents cannot cross the wall surface which implies that the normal component of the density currents has to vanish, that is,

\[ j \cdot n = 0, \quad \text{on } \Gamma_{I,j} \]

The second option is perfectly conducting walls. In this case, the wall does not apply any resistance to the current and therefore, the electric currents cross the wall surface in an orthogonal way. This means that the tangential component of the density current has to vanish on the boundary, that is,

\[ j \times n = 0, \quad \text{on } \Gamma_{C,j} \]

Note that, because \( u = 0 \) on the wall boundary, the density current and the electric potential are related as \( j = -\sigma \nabla \phi \). Therefore, on a perfectly conducting wall it is verified that \( \nabla \phi \times n = 0 \). This means that \( \phi = \text{constant} \) on the boundary. It can be taken as \( \phi = 0 \) without loss of generality.

Finally, an initial condition for the velocity field has to be considered. For instance, \( u = u_0 \) in \( \Omega \) at instant \( t = 0 \).

### 2.2 Weak form

Let us introduce some notation. Let \( \langle f, g \rangle_\omega = \int_\omega fg \), where \( f \) and \( g \) are two generic functions defined on a region \( \omega \) such that the integral of their product is well defined. When \( f, g \in L^2(\Omega) \), we will write \( \langle f, g \rangle_\omega = (f, g)_\omega \). The norm in \( L^2(\Omega) \) will be denoted by \( \| f \| = (f, f)_\Omega^{1/2} \).

Let \( v, q, k \) and \( \psi \) be the test functions for \( u, p, j \) and \( \phi \) respectively. We consider them time-independent because time will be discretized using a finite difference scheme. To obtain the weak form of (1) – (4), the equations are multiplied by the corresponding test functions, integrated over the domain \( \Omega \) and the second order terms are integrated by parts, resulting in the variational form

\[
\begin{align*}
(\partial_t u, v) + (u \cdot \nabla u, v) + \nu (\nabla u, \nabla v) - (p, \nabla \cdot v) - \frac{1}{\rho} (j \times B, v) &= \langle f, v \rangle, \\
(q, \nabla \cdot u) &= 0, \\
(j, k) + \sigma (\nabla \phi, k) - \sigma (u \times B, k) &= 0, \\
- (\nabla \psi, j) &= - \langle \psi, j \cdot n \rangle_\Gamma,
\end{align*}
\]

which must hold for all test functions \( v, q, k \) and \( \psi \) in the functional spaces that will be defined next. Note that \( \sigma \) is assumed to be constant and that the boundary term appearing from integration by parts in (8) is zero both in the case of conducting walls and in the case of insulating walls.

The functional spaces considered in this work are

\[
\begin{align*}
V_u &= \{ v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma_{E,u} \}, \\
V_p &= \{ q \in L^2(\Omega) \mid \int_\Omega q = 0 \text{ if } \Gamma_{N,u} = \emptyset \}, \\
V_j &= \{ j \in L^2(\Omega)^d \mid j \cdot n = 0 \text{ on } \Gamma_{I,j} \}, \\
V_\phi &= \{ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_{C,j} \}.
\end{align*}
\]
Remark 1. It is important to note that the equations for $j \cdot \phi$ have the same structure of the Darcy’s problem. The formulation selected in this work corresponds to the primal version of the problem. However, there exists also the dual formulation which consists on considering a different functional setting of the problem: $j \in H^1(\Omega)$ and $\phi \in L^2(\Omega)$, see [1] for a complete definition and analysis of these two formulations of the Darcy’s problem. In this work, the dual approach is not suitable because imposing the boundary condition $\phi = 0$ is not well-defined for functions $\phi \in L^2(\Omega)$.

Remark 2. From (3), it follows that the trace of $j \cdot n$ is well defined if so is the trace of $n \cdot (u \times B) = (n \times B) \cdot u$. The former is well defined because $\phi \in H^1(\Omega)$ and the latter because $B \in H(\text{curl}; \Omega)$ (assuming $B$ to be a given datum solving the Maxwell equations), and thus $n \times B$ makes sense, and also $u \in H^1(\Omega)^d$ has trace on $\Gamma_{I,j}$ (for almost all $t$).

The multilinear forms appearing in the variational form of the problem are well defined and continuous for

$$u \in L^2(0,T; V_u), \quad v \in V_u,$$
$$p \in D'(0,T; V_p), \quad q \in V_p,$$
$$j \in D'(0,T; V_j), \quad k \in V_j,$$
$$\phi \in D'(0,T; V_\phi), \quad \psi \in V_\psi.$$

In these expressions, $L^2(0,T; X)$ denotes the set of mappings defined on $\Omega \times (0,T)$ such that their $X$-spatial norm is an $L^2(0,T)$ function. Similarly, $D'(0,T; X)$ denotes the set of mappings for which their $X$-spatial norm is a distribution in time.

The variational form of the problem (5) – (8) can be written as a single variational equation of the form

$$M(\partial_t U, V) + A(U, V) = L(V),$$

where

$$U = [u, p, j, \phi]^t, \quad V = [v, q, k, \psi]^t,$$
$$A(U, V) := \langle u \cdot (\nabla u, v) + \nu(\nabla u, \nabla v) - (p, \nabla \cdot v) + (q, \nabla \cdot u) - \frac{1}{\rho}(j \times B, v) + \alpha_j [(j, k) + \sigma (\nabla \phi, k) - \sigma (u \times B, k)] + \alpha_\phi [-\nabla \psi, j] \rangle,$$
$$L(V) := \langle f, v \rangle,$$
$$M(U, V) := \langle u, v \rangle.$$

The scaling coefficients $\alpha_j$ and $\alpha_\phi$ are introduced to make $A(U, U)$ dimensionally consistent. A possible choice of these coefficients is

$$\alpha_j = \frac{1}{\rho \sigma}, \quad \alpha_\phi = \frac{1}{\rho}.$$

3 Time integration and linearization

3.1 Time discretization

Consider the variational problem given by (9) and consider $\delta t$ the time step size of a uniform partition $[0,T]$. The method used in this work for the time integration is the generalized mid-point rule, which is a one-step time integration method. Its application to the problem (9) is
given by

\[ M(\delta t U^n, V) + A(U^{n+1}, V) = L(V), \tag{10} \]

where \( \delta t U^n = \delta t^{-1} (U^{n+1} - U^n) \). This time discretization corresponds to the Backward-Euler method, which is a first-order method in time. Other time integration schemes could also be applied to obtain the final discrete problem, but the time integration accuracy has not been an important factor in this work.

### 3.2 Linearization of the stationary inductionless MHD problem

The simplest way to linearize problem (9) is by a fixed point method, in this case Picard’s method. Let us assume there exist an estimate for the velocity at iteration \( k \), \( u^k \). Then, the approximation of \( A(U, V) \) at iteration \( k+1 \) using Picard’s method can be written as

\[
A^{k+1}(U^k, V) = \left( (u^k \cdot \nabla) u^{k+1}, v \right) + \nu(\nabla u^{k+1}, \nabla v) - (p^{k+1}, \nabla \cdot v) \\
+ (q, \nabla \cdot u^{k+1}) - \frac{1}{\rho} (j^{k+1} \times B, v) + \frac{1}{\rho \sigma} (j^{k+1}, k) + \frac{1}{\rho} (\nabla \phi^{k+1}, k) \\
- \frac{1}{\rho} (u^{k+1} \times B, k) - \frac{1}{\rho} (\nabla \psi, j^{k+1}).
\]

**Remark 3.** Note that in order to have a stable scheme for each iteration, the chosen linearization is the only possible one because it has to be guaranteed that \( A^{k+1}(U^{k+1}, U^{k+1}) \geq 0 \). This condition implies that the problem needs to be solved for \( u^{k+1}, p^{k+1}, j^{k+1} \) and \( \phi^{k+1} \) in a coupled way. Then, it is very convenient to have the nodal values of the four unknowns, which enforces the choice of a monolithic approach to solve the problem and also the choice of the mixed primal formulation of the Darcy-like problem for \( j \cdot \phi \).

Therefore, calling \( a \equiv u^k, u \equiv u^{k+1}, p \equiv p^{k+1}, j \equiv j^{k+1} \) and \( \phi \equiv \phi^{k+1} \), the linearization of the inductionless MHD scaled problem is

\[- \nu \Delta u + a \cdot \nabla u + \nabla p - \frac{1}{\rho} (j \times B) = f, \]
\[ \nabla \cdot u = 0, \]
\[ \frac{1}{\rho \sigma} j + \frac{1}{\rho} \nabla \phi - \frac{1}{\rho} (u \times B) = 0, \]
\[ \frac{1}{\rho} \nabla \cdot j = 0. \]

### 3.3 Time discrete and linearized scheme

The following step is to consider together the time discretization and the linearization schemes described previously. Thus, the problem to be solved is:

For \( n = 0, 1, 2, \ldots, T/\delta t \), given \( u^n, p^n, j^n \) and \( \phi^n \), find \( u^{n+1}, p^{n+1}, j^{n+1} \) and \( \phi^{n+1} \) as the
Converged solutions of the following iterative algorithm:

\[
\begin{align*}
\delta_t u^{n+1,k+1} + (u^{n+1,k} \cdot \nabla) u^{n+1,k} + \nu (\nabla u^{n+1,k}, \nabla v) - (p^{n+1,k}, \nabla \cdot v) \\
- \frac{1}{\rho} (j^{n+1,k+1} \times B, v) &= \langle f^{n+1}, v \rangle \tag{11}
\end{align*}
\]

\[
\begin{align*}
(q, \nabla \cdot u^{n+1,k+1}) = 0, \tag{12}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\rho \sigma} (j^{n+1,k+1} + 1)(\nabla \phi^{n+1,k+1}, k) - \frac{1}{\rho} (u^{n+1,k+1} \times B, k) = 0, \tag{13}
\end{align*}
\]

\[
\begin{align*}
- \frac{1}{\rho} (\nabla \psi, j^{n+1,k+1}) = 0. \tag{14}
\end{align*}
\]

Therefore, considering \( a \equiv u^{n+1,k}, \ u \equiv u^{n+1,k+1}, \ p \equiv p^{n+1,k+1}, \ j \equiv j^{n+1,k+1} \) and \( \phi \equiv \phi^{n+1,k+1} \), the differential equations associated to (11)-(14) are

\[
\begin{align*}
\delta_t u - \nu \Delta u + a \cdot \nabla u + \nabla p - \frac{1}{\rho} (j \times B) &= f, \\
\nabla \cdot u &= 0, \\
\frac{1}{\rho \sigma} j + \frac{1}{\rho} \nabla \phi - \frac{1}{\rho} (u \times B) &= 0, \\
\frac{1}{\rho} \nabla \cdot j &= 0.
\end{align*}
\]

This problem can be written as the vector differential equation

\[
M \delta_t U + \mathcal{L}(U) = F \quad \text{in } \Omega, \tag{15}
\]

where \( M = \text{diag}(I, 0, 0, 0) \), \( I \) being the \( d \times d \) identity, \( \delta_t U = (\delta t)^{-1}(U - U^n) \), \( F = [f, 0, 0, 0] \) a vector of \( n_{\text{unk}} = 2d + 2 \) components and the scaled operator \( \mathcal{L} \) is given by

\[
\mathcal{L}(U) = \begin{bmatrix}
- \nu \Delta u + a \cdot \nabla u + \nabla p - \frac{1}{\rho} (j \times B) \\
\nabla \cdot u \\
\frac{1}{\rho \sigma} j + \frac{1}{\rho} \nabla \phi - \frac{1}{\rho} (u \times B) \\
\frac{1}{\rho} \nabla \cdot j
\end{bmatrix}. \tag{16}
\]

The time discrete and linearized counterpart of (9) can be written as

\[
M(\delta_t U, V) + A^{\text{lin}}(U, V) = L(V),
\]

where

\[
A^{\text{lin}}(U, V) = \nu (\nabla u, \nabla v) + (a \cdot \nabla u, v) - (p, \nabla \cdot v) + (q, \nabla \cdot u) - \frac{1}{\rho} (j \times B, v)
\]

\[
+ \frac{1}{\rho \sigma} (j, k) + \frac{1}{\rho} (\nabla \phi, k) - \frac{1}{\rho} (u \times B, k) - \frac{1}{\rho} (\nabla \psi, j). \tag{17}
\]

**Remark 4.** We have assumed here that \( \nabla \cdot a = 0 \). This is not necessarily true at the discrete level, where \( a = u^{n+1,k} \). Technically speaking, we should work with the skew-symmetric expression of the convective term, \( (u \cdot \nabla) u + \frac{1}{2} (\nabla u) u \). However, the results obtained in the numerical analysis of the stabilized formulation would be the same. Therefore, we will keep working with the assumption \( \nabla \cdot a = 0 \) for simplicity.
4 Stabilized formulation for the stationary and linearized problem

4.1 Stability of the Galerkin approximation

Consider the linearized stationary problem. Its variational form is: Find \( U \in (V_u \times V_p \times V_j \times V_\phi) \) such that

\[
A^{lin}(U, V) = L(V) \quad \forall V \in (V_u \times V_p \times V_j \times V_\phi).
\] (18)

If we assume again that \( \nabla \cdot a = 0 \), \( A^{lin} \) satisfies the stability estimate

\[
A^{lin}(U, U) = \nu \|\nabla u\|^2 + \frac{1}{\rho \sigma} \|j\|^2.
\] (19)

In order to be able to guarantee that the linearized problem is well posed, the inf-sup conditions between \( V_u \) and \( V_p \) and between \( V_j \) and \( V_\phi \) have to be added to the stability estimate given by (19). These inf-sup conditions are

\[
\inf_{q \in V_p} \sup_{v \in V_u} (q, \nabla \cdot v) \|q\| \geq \beta^* > 0,
\] (20)

\[
\inf_{\psi \in V_\phi} \sup_{k \in V_j} (\nabla \psi, k) \|\nabla \psi\| \geq \gamma^* > 0,
\] (21)

where \( \beta^* \) and \( \gamma^* \) are constants. Therefore, for each iteration \( k \) and given \( u^k \) and \( j^k \), there exists a unique solution of the linearized problem (18) \( u^{k+1} \), \( p^{k+1} \), \( j^{k+1} \) and \( \phi^{k+1} \).

4.2 The subgrid scale framework for a general CDR system of equations

The basic idea of the stabilization method proposed in this work is based on the subgrid scale concept introduced in [7]. The next ideas are a summary of the approach described in [3].

The main idea is to split the continuous space as \( W = W_h \oplus \tilde{W} \), where \( W_h \) is the finite element space in which we will search the approximate solution. We call \( \tilde{W} \) the space of subscales or space of subgrid scales. It can be easily seen that the continuous problem can be written as the system of equations

\[
A^{lin}(U_h, V_h) + A^{lin}(\tilde{U}, V_h) = L(V_h) \quad \forall V_h \in W_h,
\] (22)

\[
A^{lin}(U_h, \tilde{V}) + A^{lin}(\tilde{U}, \tilde{V}) = L(\tilde{V}) \quad \forall \tilde{V} \in \tilde{W},
\] (23)

where \( U = U_h + \tilde{U} \) and \( U_h \in W_h, \tilde{U} \in \tilde{W} \).

We need to introduce some notation that will be used in the following

\[
\langle \cdot, \cdot \rangle_h := \sum_{e=1}^{n_{el}} \langle \cdot, \cdot \rangle_{\Omega^e} , \quad \langle \cdot, \cdot \rangle_{\partial h} := \sum_{e=1}^{n_{el}} \langle \cdot, \cdot \rangle_{\partial \Omega^e} ,
\]

where \( n_{el} \) is the number of elements of the finite element discretization, \( \Omega^e \) denotes the domain of element number \( e \) and \( \partial \Omega^e \) the boundary of \( \Omega^e \).
Integrating by parts all the terms in $A^{lin}(\tilde{U}, V_h)$ in (22) and the left-hand-side terms of (23) within each element domain, we have

$$A^{lin}(U_h, V_h) + \left\langle \tilde{U}, n_i (K_{ij} \partial_j V_h - (A_i^j)^t V_h) \right\rangle_{\partial h} + \left\langle \tilde{U}, L^*(V_h) \right\rangle_h = L(V_h),$$  \hspace{1cm} (24)

where $\tilde{V}, n_i \left[ K_{ij} \partial_j (U_h + \tilde{U}) - A_i^j (U_h + \tilde{U}) \right] \right\rangle_{\partial h} + \left\langle \tilde{V}, L(U) \right\rangle_h = \left\langle \tilde{V}, F - L(U_h) \right\rangle_h,$  \hspace{1cm} (25)

where $K_{ij}$ and $A_i$ are matrices of $n_{unk} \times n_{unk}$ components storing the diffusion coefficients and advection velocities, respectively. The indices $i, j$ are the number of unknowns. Furthermore, $n_i$ is the $i$-th component of the exterior normal to $\partial \Omega^e$ and $L^*$ is the adjoint operator of $L$ with homogeneous Dirichlet conditions, given by

$$L^*(U) := -\partial_t (A_i^j U) - \partial_t (K_{ij} \partial_j U).$$

Equation (25) is equivalent to

$$L(U) = F - L(U_h) + V_{h, ort} \quad \text{in } \Omega^e,$$  \hspace{1cm} (26)

$$\tilde{U} = \tilde{U}_{ske} \quad \text{on } \partial \Omega^e,$$  \hspace{1cm} (27)

where $V_{h, ort}$ is obtained from the condition that $\tilde{U}$ must belong to $\tilde{W}$ and $\tilde{U}_{ske}$ is a function defined on the element boundaries and such that

$$q_h := n_i \left[ K_{ij} \partial_j (U_h + \tilde{U}) - A_i^j (U_h + \tilde{U}) \right],$$

is continuous across interelement boundaries, and therefore the first term in the left-hand-side of (25) vanishes.

There exist several subgrid scale (SGS) stabilization methods depending on the way problem (26) – (27) is approximated. The purpose of this paper is to see how to apply a well established formulation to the inductionless MHD. This can be obtained by approximating the subscales by the algebraic expression

$$\tilde{U} \approx \tau \bar{P}[F - L(U_h)],$$  \hspace{1cm} (28)

where $\tau$ is a $n_{unk} \times n_{unk}$ matrix of stabilization parameters, the expression of which is discussed in the following subsection, and $\bar{P}$ is the projection onto the space of subscales. There are two main options for choosing $\bar{P}$ which determines the space of subscales. The most common one is to take $\bar{P} = I$, the identity, when applied to the finite element residual appearing in the right-hand-side of equation (28). The other main possibility is to take $\bar{P}$ as the $L^2$-projection onto the space orthogonal to the finite element space. In this paper we will restrict ourselves to the first option, $\bar{P} = I$. The design of the stabilization parameters is the same using both approaches.

To close the approximation, we neglect the interelement boundaries terms in (24), which can be understood as taking $U_{ske} = 0$ on the interelement boundaries. The final problem is : Find $U_h \in W_h$ such that

$$A^{lin}(U_h, V_h) + \left\langle \tilde{U}, L^*(V_h) \right\rangle_h = L(V_h) \quad \forall V_h \in W_h,$$

which, upon substitution of the subscales by (28) with $\bar{P} = I$, yields the following discrete problem : Find $U_h \in W_h$ such that

$$A^{lin}_{stab}(U_h, V_h) = L_{stab}(V_h) \quad \forall V_h \in W_h,$$  \hspace{1cm} (29)
where the bilinear form $A_{stab}^{lin}$ and the linear form $L_{stab}$ are given by

$$A_{stab}^{lin}(U_h, V_h) = A^{lin} - \langle \mathcal{L}^*(V_h), \tau \mathcal{L}(U_h) \rangle_h,$$  \hspace{1cm} (30)
$$L_{stab}(V_h) = L(V_h) - \langle \mathcal{L}^*(V_h), \tau F \rangle_h.$$  \hspace{1cm} (31)

### 4.3 Stabilized finite element approximation for the linearized problem

In this subsection we present a stabilized finite element formulation to approximate problem (18). The finite element approximation will be described in the setting of a system of linear convection-diffusion equations.

The objective is to apply the algebraic version of the SGS stabilization method to the inductionless linearized stationary MHD problem we are considering. The adjoint operator of this problem $\mathcal{L}^*(V_h)$ is given by

$$\mathcal{L}^*(V_h) = \begin{bmatrix} -\nu \Delta v_h - a \cdot \nabla v_h - \nabla q_h + \frac{1}{\rho} (k_h \times B) \\ -\nabla \cdot v_h \\ \frac{1}{\rho \sigma} k_h - \frac{1}{\rho} \nabla \psi_h + \frac{1}{\rho} (v_h \times B) \\ -\frac{1}{\rho} \nabla \cdot k_h \end{bmatrix}. \hspace{1cm} (32)$$

The next step is to define an expression for the matrix of stabilization parameters $\tau$. In the case we are considering, we will see in the following subsection that stability can be improved maintaining optimal accuracy by taking a diagonal expression for $\tau$, with one scalar component for each equation. In the 3D case we have

$$\tau = \text{diag}(\tau_1, \tau_1, \tau_2, \tau_3, \tau_3, \tau_3, \tau_4). \hspace{1cm} (33)$$

Using both expressions (32) and (33) in problem (29), the stabilized bilinear form is

$$A_{stab}^{lin}(U_h, V_h) = A^{lin}(U_h, V_h) - \langle \mathcal{L}^*(V_h), \tau \mathcal{L}(U_h) \rangle_h = A^{lin}(U_h, V_h) \hspace{1cm} (34)$$

where we have used the abbreviations

$$X_u(v_h, q_h, k_h) := a \cdot \nabla v_h + \nabla q_h - \frac{1}{\rho} (k_h \times B),$$
$$X_j(v_h, \psi_h) := \frac{1}{\rho} \nabla \psi_h - \frac{1}{\rho} (v_h \times B).$$

The right-hand-side of the stabilized problem is given by

$$L_{stab}(V_h) = L(V_h) - \langle \mathcal{L}^*(V_h), \tau F \rangle_h = L(V_h) + \langle X_u(v_h, q_h, k_h) + \nu \Delta v_h, \tau_1 f \rangle_h. \hspace{1cm} (35)$$
The definition of the stabilized finite element method only misses the expression of the stabilization parameters. The expressions proposed in this work are

\[ \alpha := c_1 \frac{a}{h} + c_2 \frac{\nu}{h^2}, \quad \beta := \frac{B}{\rho}, \quad \gamma := c_4 \frac{1}{\rho \sigma}, \]

\[ \tau_1 = \alpha^{-1} \left( 1 + \frac{1}{\sqrt{\alpha \gamma\beta}} \right)^{-1}, \quad \tau_2 = c_5 \frac{h^2}{\tau_1}, \]

\[ \tau_3 = \gamma^{-1} \left( 1 + \frac{1}{\sqrt{\alpha \gamma\beta}} \right)^{-1}, \quad \tau_4 = c_6 \frac{\rho^2 h^2}{\tau_3}. \]  

These expressions are evaluated element by element. Here, \( a \) is the maximum norm of the velocity field \( a \) computed in the element under consideration. Likewise, \( B \) is the maximum norm of the magnetic field \( B \) in this element, and \( h \) the element diameter.

### 4.4 Numerical analysis and design of the stabilization parameters

In this subsection we proceed to the numerical analysis of the formulation introduced before and to the justification of the stabilization parameters expression (36) – (37). For the sake of simplicity we assume that \( a \) is constant and that the finite element meshes are quasi-uniform. Thus, \( h \) can be taken the same for every element and therefore \( \tau_i, i = 1, 2, 3, 4 \) are also constant. Moreover, for quasi-uniform meshes the following inverse estimates hold

\[ \| \nabla v_h \| \leq \frac{C_{inv}}{h} \| v_h \|, \quad \| \nabla \nabla v_h \| \leq \frac{C_{inv}}{h} \| \nabla v_h \|, \]  

for any function \( v_h \) in the finite element space and for a certain constant \( C_{inv} \).

The stability and convergence analysis will be made using the mesh-dependent norm

\[ \| U_h \|^2 := \nu \| \nabla u_h \|^2 + \frac{1}{\rho \sigma} \| j_h \|^2 \]

\[ + \tau_1 \| a \cdot \nabla u_h + \nabla p_h - \frac{1}{\rho} (j_h \times B) \|^2 + \tau_2 \| \nabla \cdot u_h \|^2 \]

\[ + \tau_3 \| \frac{1}{\rho} \nabla \phi_h - \frac{1}{\rho} (u_h \times B) \|^2 + \tau_4 \frac{1}{\rho^2} \| \nabla \cdot j_h \|^2 \]

\[ = \nu \| \nabla u_h \|^2 + \frac{1}{\rho \sigma} \| j_h \|^2 \]

\[ + \tau_1 \| X_u(u_h, p_h, j_h) \|^2 + \tau_2 \| \nabla \cdot u_h \|^2 \]

\[ + \tau_3 \| X_j(u_h, \phi_h) \|^2 + \tau_4 \frac{1}{\rho^2} \| \nabla \cdot j_h \|^2. \]  

From now on, \( C \) will denote a positive constant independent of the mesh discretization and the physical parameters, not necessarily the same at different stages.
4.4.1 Coercivity

Let us start by proving stability in the form of coercivity of the bilinear form (34)

\[
A_{\text{lin}}^{\text{stab}}(U_h, U_h) = A_{\text{lin}}^{\text{stab}}(U_h, U_h) - \langle L^*(U_h), \tau L(U_h) \rangle_h \\
= \nu \|\nabla u_h\|^2 + \frac{1}{\rho \sigma} \|j_h\|^2 \\
+ \tau_1 \|X_u(u_h, p_h, j_h)\|^2 - \tau_1 \nu^2 \|\Delta u_h\|^2 + \tau_2 \|\nabla \cdot u_h\|^2 \\
+ \tau_3 \|X_j(u_h, \phi_h)\|^2 - \tau_3 \frac{1}{\rho^2 \sigma^2} \|j_h\|^2 + \tau_4 \frac{1}{\rho^2} \|\nabla \cdot j_h\|^2.
\]

Using the second inverse estimate in (38), a sufficient condition for \(A_{\text{lin}}^{\text{stab}}\) to be coercive is

\[
\nu - \tau_1 \nu^2 \frac{C^2_{\text{inv}}}{h^2} \geq \alpha \nu \iff \tau_1 \leq (1 - \alpha) \frac{1}{\nu} \frac{h^2}{C^2_{\text{inv}}}, \tag{40}
\]

\[
\frac{1}{\rho \sigma} - \tau_3 \frac{1}{\rho^2 \sigma^2} \geq \frac{\alpha}{\rho \sigma} \iff \tau_3 \leq (1 - \alpha) \rho \sigma, \tag{41}
\]

with \(0 < \alpha < 1\). Conditions (40) – (41) imply

\[
A_{\text{lin}}^{\text{stab}}(U_h, U_h) \geq C \|U_h\|^2, \tag{42}
\]

for a constant \(C\) independent of the discretization and of the physical parameters.

4.4.2 Optimal accuracy

The requirement that the stabilized formulation is optimally accurate will allow us to obtain new conditions on the stabilization parameters. These new conditions together with (40) – (41) from stability will lead to the final expression of the stabilization parameters.

For a function \(v\), let \(\pi_h(v)\) be its optimal finite element approximation. We assume that the following estimates hold

\[
\|v - \pi_h(v)\|_{H^i(\Omega)} \leq \varepsilon_i(v) := C h^{k+1-i} \|v\|_{H^{k+1}(\Omega)}, \quad i = 0, 1, \tag{43}
\]

where \(\|v\|_{H^q(\Omega)}\) is the \(H^q(\Omega)\)-norm of \(v\), that is, the sum of the \(L^2(\Omega)\)-norm of the derivatives of \(v\) up to degree \(q\), \(\|v\|_{H^q(\Omega)}\) the corresponding semi-norm, and \(k\) the degree of the finite element approximation.

We will prove next that the error function of the formulation is

\[
E(h) := \tau_1^{-1/2} \varepsilon_0(u) + \tau_2^{-1/2} \varepsilon_0(p) + \tau_3^{-1/2} \varepsilon_0(j) + \tau_4^{-1/2} \varepsilon_0(\phi). \tag{44}
\]

Let \(U\) be the solution of the continuous problem and \(\pi_h(U)\) its optimal finite element approximation. The accuracy estimate that will be needed to prove convergence is

\[
A_{\text{lin}}^{\text{stab}}(U - \pi_h(U), V_h) \leq CE(h) \|V_h\|, \tag{45}
\]

for any finite element function \(V_h\).
Let us prove this by showing that both the Galerkin and the stabilization terms in \( A_{\text{stab}}^{\text{lin}}(U - \pi_h(U), V_h) \) satisfy estimate (45) for sufficiently smooth functions solution of the continuous problem. Integrating by parts some terms in the Galerkin contribution we obtain

\[
A_{\text{lin}}^{\text{lin}}(U - \pi_h(U), V_h) = \nu(\nabla(u - \pi_h(u)), \nabla v_h) - (u - \pi_h(u), a \cdot \nabla v_h)
\]

\[
- (u - \pi_h(u), \nabla q_h) - (p - \pi_h(p), \nabla \cdot v_h) + \frac{1}{\rho}(u - \pi_h(u), k_h \times B)
\]

\[
+ \frac{1}{\rho}(j - \pi_h(j), v_h \times B) + \frac{1}{\rho\sigma}(j - \pi_h(j), k_h) - \frac{1}{\rho}(j - \pi_h(j), \nabla \psi_h)
\]

\[
- \frac{1}{\rho}(\phi - \pi_h(\phi), \nabla \cdot k_h)
\]

\[
\leq C \left( \varepsilon_0(u) \tau_1^{-1/2} \tau_1^{1/2} \| X_u(v_h, q_h, k_h) \| + \nu^{1/2} \varepsilon_1(u) \nu^{1/2} \| \nabla v_h \| \right)
\]

\[
+ \varepsilon_0(p) \tau_2^{-1/2} \tau_2^{1/2} \| \nabla \cdot v_h \|
\]

\[
+ \varepsilon_0(j) \tau_3^{-1/2} \tau_3^{1/2} \left[ \| X_j(v_h, \psi_h) \| + \frac{1}{\rho\sigma} \| k_h \| \right]
\]

\[
+ \varepsilon_0(\phi) \tau_4^{-1/2} \tau_4^{1/2} \frac{1}{\rho} \| \nabla \cdot k_h \| \right). \quad (46)
\]

Conditions (40) – (41) and the expression of the interpolation errors imply

\[
\nu^{1/2} \varepsilon_1(u) \leq C \varepsilon_0(u) \tau_1^{-1/2} \left( \frac{1}{\rho\sigma} \| k_h \| \right) \leq \tau_3^{-1/2} \| k_h \|
\]

and therefore from (46) it follows that the Galerkin contribution to \( A_{\text{stab}}^{\text{lin}}(U - \pi_h(U), V_h) \) can be bounded as indicated in (45). It remains to prove that also the stabilization terms can be bounded the same way

\[
- \langle L^*(V_h), \tau L(U - \pi_h(U)) \rangle_h
\]

\[
= (X_u(v_h, q_h, k_h) + \nu h \nabla v_h, \tau_1(X_u(v_h, q_h, k_h) + \nu h \nabla v_h))_h
\]

\[
+ (\nabla \cdot v_h, \tau_2 \nabla \cdot (u - \pi_h(u)))_h
\]

\[
+ \left( X_j(v_h, \psi_h) - \frac{1}{\rho\sigma} k_h, \tau_3(X_j(v_h, \psi_h) - \frac{1}{\rho\sigma} k_h) \right)_h
\]

\[
+ \frac{1}{\rho} \nabla \cdot k_h, \tau_4 \frac{1}{\rho} \nabla \cdot (j - \pi_h(j)) \right)_h
\]

\[
\leq C \left( \tau_1^{1/2} \| X_u(u - \pi_h(u), p - \pi_h(p), j - \pi_h(j)) \| + \tau_1^{1/2} \nu \| \Delta u - \pi_h(u) \| \right)
\]

\[
\times \left( \| V_h \| + \tau_1^{1/2} \nu \| \Delta v_h \| \right)
\]

\[
+ C \tau_2^{1/2} \varepsilon_1(u) \| V_h \|
\]

\[
+ C \left( \tau_3^{1/2} \| X_j(u - \pi_h(u), \phi - \pi_h(\phi)) \| + \tau_3^{1/2} \frac{1}{\rho\sigma} (j - \pi_h(j)) \right) \times \left( \| V_h \| + \tau_3^{1/2} \frac{1}{\rho\sigma} \| k_h \| \right)
\]

\[
+ C \tau_4^{1/2} \frac{1}{\rho} \varepsilon_1(j) \| V_h \|. \quad (47)
\]

Using again conditions (40) – (41) and the inverse estimates (38) we have

\[
\tau_1^{1/2} \nu \| \Delta v_h \| \leq C \tau_1^{1/2} \nu^{1/2} \frac{C_{\text{inv}}}{h} \| \nabla v_h \| \leq C \| V_h \|
\]

\[
\tau_3^{1/2} \frac{1}{\rho\sigma} \| k_h \| \leq C(\rho\sigma)^{1/2} \frac{1}{\rho\sigma} \| k_h \| \leq C \| V_h \|
\]

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Therefore, from (47) we have
\[- \langle L^*(V_h), \tau L(U - \pi_h(U)) \rangle_h \]
\[\leq C \| V_h \| \left[ \tau_1^{1/2} \left( \frac{\nu}{h^2} \delta_0(u) + \frac{a}{h} \delta_0(u) + \frac{1}{h} \delta_0(p) + \frac{B}{\rho} \delta_0(j) \right) \\
+ \tau_2^{1/2} \left( \frac{1}{h} \delta_0(u) \right) \\
+ \tau_3^{1/2} \left( \frac{1}{\rho} \delta_0(j) + \frac{1}{\rho h} \delta_0(\phi) + \frac{B}{\rho} \delta_0(u) \right) \\
+ \tau_4^{1/2} \left( \frac{1}{ph} \delta_0(j) \right) \right] \]
\[\leq C \| V_h \| \left[ \delta_0(u) \left[ \tau_1^{1/2} \left( \frac{\nu}{h^2} + \frac{a}{h} \right) + \tau_2^{1/2} \frac{1}{h} + \tau_3^{1/2} \frac{B}{\rho} \right] \\
+ \delta_0(p) \left[ \tau_1^{1/2} \frac{1}{h} \right] \\
+ \delta_0(j) \left[ \tau_1^{1/2} \frac{B}{\rho} + \tau_3^{1/2} \frac{1}{\rho} + \tau_4^{1/2} \frac{1}{ph} \right] \\
+ \delta_0(\phi) \left[ \tau_3^{1/2} \frac{1}{ph} \right] \right].

Using the definitions (36) – (37) of the stabilization parameters it is easily checked that these terms can also be bounded as indicated in (45).

**Remark 5.** The last step provides the crucial design condition for the stabilization parameters. Expressions (36) – (37) result from solving
\[\tau_1^{1/2} \left( \frac{\nu}{h^2} + \frac{a}{h} \right) + \tau_2^{1/2} \frac{1}{h} + \tau_3^{1/2} \frac{B}{\rho} \lesssim \tau_1^{-1/2}, \quad (48)\]
\[\tau_1^{1/2} \frac{1}{h} \lesssim \tau_2^{-1/2}, \quad (49)\]
\[\tau_1^{1/2} \frac{B}{\rho} + \tau_3^{1/2} \frac{1}{\rho} + \tau_4^{1/2} \frac{1}{ph} \lesssim \tau_3^{-1/2}, \quad (50)\]
\[\tau_3^{1/2} \frac{1}{ph} \lesssim \tau_4^{-1/2}, \quad (51)\]
where \(\sim\) stands for equality up to constants that do not depend on the physical variables nor the mesh discretization.

### 4.4.3 Convergence

The properties of stability and optimal accuracy, in the sense of (45) allow us to show that the method is optimally convergent. From the orthogonality property \(A_{\text{stab}}^{\text{lin}}(U - U_h, V_h) = 0\) for any finite element function \(V_h\), a consequence of the consistency of the method, we have that
\[C \| \pi_h(U) - U_h \|^2 \leq A_{\text{stab}}^{\text{lin}}(\pi_h(U) - U_h, \pi_h(U) - U_h) \]
\[\leq A_{\text{stab}}^{\text{lin}}(\pi_h(U) - U, \pi_h(U) - U_h) + A_{\text{stab}}^{\text{lin}}(U - U_h, \pi_h(U) - U_h) \]
\[\leq CE(h) \| \pi_h(U) - U_h \|, \]
so that
\[\| \pi_h(U) - U_h \| \leq CE(h).\]
If we apply the triangle inequality we get
\[ \| U - U_h \| \leq \| U - \pi_h(U) \| + \| \pi_h(U) - U_h \| \]
\[ \leq \| U - \pi_h(U) \| + CE(h). \]

It is trivial to check that \( \| U - \pi_h(U) \| \leq CE(h) \) using the expression of the norm (39), the interpolation estimates (43) and the stabilization parameters (36) – (37). Therefore,
\[ \| U - U_h \| \leq CE(h). \] (52)

The fact that this error estimate is exactly the same as the estimate for the interpolation error \( \| U - \pi_h(U) \| \leq CE(h) \) justifies why it can be considered optimal.

5 Final numerical scheme

The final numerical scheme proposed to solve the inductionless MHD problem results from applying the stabilized finite element approximation described in Section 4.2 to the time discrete and linearized problem (11)-(14). Therefore, the final algorithm reads: For \( n = 0, 1, 2, ..., T/\delta t \) and given \( u^n, p^n, j^n \) and \( \phi^n \), find \( u^{n+1}, p^{n+1}, j^{n+1} \) and \( \phi^{n+1} \) as the converged solutions of the following iterative algorithm:

\[
\begin{align*}
(\delta_t u_h^{n,k+1}, v_h) + & \left\langle (u_h^{n+1,k} \cdot \nabla)u_h^{n+1,k+1}, v_h \right\rangle + \nu(\nabla u_h^{n+1,k+1}, \nabla v_h) - (p_h^{n+1,k+1}, \nabla \cdot v_h) \\
& - \frac{1}{\rho} (j_h^{n+1,k+1} \times B, v_h) \\
& + \langle u_h^{n+1,k} \cdot \nabla v_h + \nu \Delta v_h, \tau_1^{n+1,k} R_h^{n+1,k+1} \rangle_h \\
& + \langle \nabla \cdot v_h, \tau_2^{n+1,k} R_h^{n+1,k+1} \rangle_h \\
& - \left\langle \frac{1}{\rho} (v_h \times B), \tau_3^{n+1,k} R_h^{n+1,k+1} \right\rangle_h = \langle f^{n+1}, v_h \rangle \\
(q_h, \nabla \cdot u_h^{n+1,k+1}) + & \left\langle \nabla q_h, \tau_1^{n+1,k} R_h^{n+1,k+1} \right\rangle_h = 0, \\
\frac{1}{\rho \sigma} (j_h^{n+1,k+1}, k_h) + & \frac{1}{\rho} (\nabla \phi_h^{n+1,k+1}, k_h) - \frac{1}{\rho} (u_h^{n+1,k+1} \times B, k_h) \\
& - \left\langle \frac{1}{\rho} (k_h \times B), \tau_1^{n+1,k} R_h^{n+1,k+1} \right\rangle_h - \left\langle \frac{1}{\rho \sigma} k_h, \tau_3^{n+1,k} R_h^{n+1,k+1} \right\rangle_h \\
& + \left\langle \frac{1}{\rho} \nabla \cdot k_h, \tau_4^{n+1,k} R_{h,\phi}^{n+1,k+1} \right\rangle_h = 0, \\
& - \left\langle \frac{1}{\rho} (\nabla \phi_h, j_h^{n+1,k+1}) + \left\langle \frac{1}{\rho} \nabla \phi_h, \tau_3^{n+1,k} R_{h,j}^{n+1,k+1} \right\rangle_h = 0. \\
\end{align*}
\]

where the expression of the residuals is,
\[
R_{h,u} := \delta_t u_h + a \cdot \nabla u_h - \nu \Delta u_h + \nabla p_h - \frac{1}{\rho} (j_h \times B) - f,
\]
\[
R_{h,p} := \nabla \cdot u_h,
\]
\[
R_{h,j} := \frac{1}{\rho \sigma} j_h + \frac{1}{\rho} \nabla \phi_h - \frac{1}{\rho} (u_h \times B),
\]
\[
R_{h,\phi} := \frac{1}{\rho} \nabla \cdot j_h.
\]
6 Numerical experimentation

6.1 Shercliff’s case

The first numerical experimentation that have been carried out is the simulation of the Shercliff’s case. It corresponds to a fully developed flow in a channel with square section where both the Hartmann walls, which are the walls orthogonal to the external magnetic field direction, and the side walls, which are the walls parallel to the external magnetic field, electrically insulating. The fluid flows with unidirectional velocity in the \( z \)-direction driven by a constant pressure gradient. The channel is exposed to an external magnetic field applied in the \( y \)-direction. This problem has an analytical solution in form of a Fourier series that was developed by J.A. Shercliff [16]. A more appropriate version of this solution for the implementation in a computer can be found in [13], although there are some typographical errors in two of the formulae that have to be corrected. The formulae used in this work to compute the analytical solution and compare with the numerical approximation are explained in the following.

Let the side walls be of length \( 2a \), the Hartmann of length \( 2b \) and \( l = b/a \). The Hartmann walls are considered to have arbitrary conductivity with \( d_B = (t_w \sigma_w)/(a \sigma) \), where \( \sigma_w \) is the conductivity of the wall, \( t_w \) its thickness and \( \sigma \) the conductivity of the fluid. The analytical
solution was given by Hunt [8] as a Fourier series in $\xi = x/a \in [-l, l]$ and $\eta = y/a \in [-1, 1]$:

$$V = \sum_{k=0}^{\infty} \frac{2(-1)^k \cos(\alpha_k \xi)}{l\alpha_k^3} (1 - V2 - V3)$$  \hspace{1cm} (57)

$$V2 = \left( d_B r_{2k} + \frac{1 - \exp(-2r_{2k})}{1 + \exp(-2r_{2k})} \right) \frac{\exp(-r_{1k}(1-\eta)) + \exp(-r_{1k}(1+\eta))}{2}$$
$$+ \frac{1 - \exp(-2(r_{1k} + r_{2k}))}{1 + \exp(-2r_{2k})} \frac{d_B N}{2}$$ \hspace{1cm} (58)

$$V3 = \left( d_B r_{1k} + \frac{1 - \exp(-2r_{1k})}{1 + \exp(-2r_{1k})} \right) \frac{\exp(-r_{2k}(1-\eta)) + \exp(-r_{2k}(1+\eta))}{2}$$
$$+ \frac{1 - \exp(-2(r_{1k} + r_{2k}))}{1 + \exp(-2r_{2k})} \frac{d_B N}{2}$$ \hspace{1cm} (59)

$$H = \sum_{k=0}^{\infty} \frac{2(-1)^k \cos(\alpha_k \xi)}{l\alpha_k^3} (H2 - H3)$$ \hspace{1cm} (60)

$$H2 = \left( d_B r_{2k} + \frac{1 - \exp(-2r_{2k})}{1 + \exp(-2r_{2k})} \right) \frac{\exp(-r_{1k}(1-\eta)) - \exp(-r_{1k}(1+\eta))}{2}$$
$$+ \frac{1 - \exp(-2(r_{1k} + r_{2k}))}{1 + \exp(-2r_{2k})} \frac{d_B N}{2}$$ \hspace{1cm} (61)

$$H3 = \left( d_B r_{1k} + \frac{1 - \exp(-2r_{1k})}{1 + \exp(-2r_{1k})} \right) \frac{\exp(-r_{2k}(1-\eta)) - \exp(-r_{2k}(1+\eta))}{2}$$
$$+ \frac{1 - \exp(-2(r_{1k} + r_{2k}))}{1 + \exp(-2r_{2k})} \frac{d_B N}{2}$$ \hspace{1cm} (62)

where, $N = (Ha^2 + 4\alpha_k^2)^{1/2}$ \hspace{1cm} (63)

$$r_{1k}, r_{2k} = \frac{1}{2} \left( \pm Ha + (Ha^2 + 4\alpha_k^2)^{1/2} \right)$$ \hspace{1cm} (64)

$$\alpha_k = \left( k + \frac{1}{2} \right) \frac{\pi}{l}$$ \hspace{1cm} (65)

and then, $V_z = \frac{V}{\mu} \left( -\frac{\partial p}{\partial z} \right) a^2$ \hspace{1cm} (66)

$$H_z = \frac{H}{\mu^{1/2}} \left( -\frac{\partial p}{\partial z} \right) a^2 \sigma^{1/2}$$ \hspace{1cm} (67)

$$j_x = \frac{\partial H_z}{\partial y}, \quad j_y = -\frac{\partial H_z}{\partial x}$$ \hspace{1cm} (68)

$V_z, j_x$, and $j_y$ are precisely the analytical solution of the problem. Note that in the Shercliff’s case the Hartmann walls are perfectly insulating, and therefore $d_B = 0$ in the above formulae. Note also that the formulae in [8] have been written in terms of exponential functions to allow its computation in a computer. The original formulae in terms of hyperbolic functions is not suitable for computing at high values of the Hartmann number.

This problem has a 2D behavior that has been simulated setting as the computational domain a very thin section of the channel of width 1/100 times the section sides. The boundary conditions at the inflow and outflow sections have been set as periodic conditions to enforce the condition of fully developed flow. Therefore, the constant pressure gradient that drives the
flow has to be set as an external body force. Its value can be computed as (see [11] for details)

\[
\frac{dp}{dz} = \frac{KL^3}{\rho \nu^2 \text{Re}}
\]

where, 
\[
K = \frac{\text{Ha}}{1 - 0.825 \text{Ha}^{-1/2} - \text{Ha}^{-1}},
\]

\[
\text{Ha} = BL \sqrt{\frac{\sigma}{\rho \nu}},
\]

\[
\text{Re} = \frac{UL}{\nu}.
\]

In the preceding formulae, \(L\) is a characteristic length, \(U\) a characteristic velocity of the fluid and \(B\) the norm of the externally applied magnetic field. Every physical property of the problem, that is, density, viscosity and electrical conductivity has been set equal to one. In this way, the Hartmann number \(\text{Ha}\) is equal to the norm of the external magnetic field. There have been several meshes used to perform the computations. The coarser one consists of 2028 nodes and 7500 tetrahedral elements whereas the finer one consists of 121203 nodes and 480000 tetrahedral elements. Furthermore, there have been considered two different configurations of meshes, a uniformly structured one and a structured one but concentrating the elements near the boundaries. Figure 1 shows the two different configurations in section for a mesh of 30000 elements.

![Uniformly structured mesh](a) Uniformly structured mesh

![Structured mesh but concentrating elements near the boundaries](b) Structured mesh but concentrating elements near the boundaries

**Figure 1:** Mesh configurations.

The first simulation done is a fluid with \(\text{Ha}=10\) and \(\text{Re}=10\). Figure 2 shows the velocity field and the current paths obtained using a mesh of 30603 nodes and 120000 tetrahedral elements.

The second simulation is a fluid with \(\text{Ha}=100\) and \(\text{Re}=10\). In this case, the uniformly structured meshes do not lead to a proper solution because the Hartmann layer is much thinner than the mesh size \(h\). Therefore, this case has been solved with the meshes with element concentration near the boundaries. The results for a mesh of 30603 nodes and 120000 tetrahedral elements are shown in Figure 3.

Figure 4 shows the convergence study of both \(\text{Ha}=10\) and \(\text{Ha}=100\) cases depending on the mesh size \(h\) in a logarithmic scale. Note that the mesh size for the meshes with element concentration is not constant. Therefore, the results have been plotted related to an equivalent
mesh size $h^*$ which corresponds to the same number of degrees of freedom than a uniformly structured mesh. The values shown in this study correspond to the $L^2$-norm of the error in the velocity $||e_u||$, the velocity gradient $||\nabla e_u||$, the current density $||e_j||$ and the divergence of the current density $||\nabla \cdot e_j||$. It can be clearly seen that in both cases, $Ha=10$ and $Ha=100$, the convergence rates are very good for every computed error. Actually, the convergence rates for the errors in the velocity gradient and in the divergence of the current density are higher than the theoretical value, which can be found in [1]. The results of this study show a superconvergence rate, which it also happened in the convergence study done in the same article [1] for the Darcy’s problem.

![Velocity field](image1.png) ![Current paths](image2.png)

(a) Velocity field  
(b) Current paths

Figure 2: Shercliff’s case : $Ha = 10$, $Re = 10$.

![Velocity field](image3.png) ![Current paths](image4.png)

(a) Velocity field  
(b) Current paths

Figure 3: Shercliff’s case : $Ha = 100$, $Re = 10$.

6.2 Hunt’s case

The next simulation done is the Hunt’s case. It corresponds to a fully developed flow in a channel with square section where the Hartmann walls are perfectly conducting and the side

![Velocity field](image5.png) ![Current paths](image6.png)

(a) Velocity field  
(b) Current paths

Figure 2: Shercliff’s case : $Ha = 10$, $Re = 10$.
walls are electrically insulated. Similarly to the Shercliff’s case, this problem has an analytical solution in the form of a Fourier series that can be found in an article from J.C.R. Hunt [8]. The analytical solution is computed using the formulae (57)-(68). In this case, the Hartmann walls are perfectly conducting and therefore $d_B \to \infty$. Therefore, the modifications in the formulae (57)-(68) for Hunt’s case consist of taking the limit $d_B \to \infty$ in the Fourier series:

\[
V_2 = \frac{r_{2k}}{N} \cdot \frac{\exp(-r_{1k}(1 - \eta)) + \exp(-r_{1k}(1 + \eta))}{1 + \exp(-2r_{1k})}
\]  

(69)

\[
V_3 = \frac{r_{1k}}{N} \cdot \frac{\exp(-r_{2k}(1 - \eta)) + \exp(-r_{2k}(1 + \eta))}{1 + \exp(-2r_{2k})}
\]  

(70)

\[
H_2 = \frac{r_{2k}}{N} \cdot \frac{\exp(-r_{1k}(1 - \eta)) - \exp(-r_{1k}(1 + \eta))}{1 + \exp(-2r_{1k})}
\]  

(71)

\[
H_3 = \frac{r_{1k}}{N} \cdot \frac{\exp(-r_{2k}(1 - \eta)) - \exp(-r_{2k}(1 + \eta))}{1 + \exp(-2r_{2k})}
\]  

(72)

This problem has a similar 2D behavior than Shercliff’s case that has been simulated setting the same computational domain, that is, a very thin section of the channel of width 1/100 times the section sides. The boundary conditions at the inflow and outflow sections have been set as periodic conditions to enforce the condition of fully developed flow. Therefore, the constant pressure gradient that drives the flow has to be set as an external body force. Its value can be computed with a slightly different formula from Shercliff’s case as (see [11] for details)

\[
\frac{dp}{dz} = \frac{KL^3}{\rho \nu^2 \text{Re}}
\]

where, $K = \frac{\text{Ha}}{1 - 0.95598\text{Ha}^{-1/2} - \text{Ha}^{-1}}$

Every physical property involved in the calculation has been set equal to one. Therefore, the Hartmann number is computed directly as the norm of the external magnetic field. The meshes used to solve this problem and obtain the convergence rates are the same meshes that were used in the previous Shercliff’s case.
The same two simulations than in the Shercliff’s case have been done here. The first one is a fluid with $Ha=10$ and $Re=10$. Figure 5 shows the velocity field and the current paths solution of this problem when using a structured mesh of 30603 nodes and 120000 tetrahedral elements.

The second simulation corresponds to a fluid flowing with $Ha=100$ and $Re=10$. Figure 6 shows the velocity distribution and the current paths obtained with a mesh of 30603 nodes and 120000 tetrahedral elements but concentrating the elements near the boundaries to improve the capturing of the Hartmann layers.

Figure 7 shows the convergence rates obtained for both $Ha=10$ and $Ha=100$ cases in a logarithmic scale. It is important to note that the mesh size for the meshes with element concentration is not constant. Therefore, the representation variable selected to plot the results has been an equivalent mesh size $h^*$ which corresponds to the same number of degrees of freedom than a uniformly structured mesh. The values shown in this study are the same computed in the Shercliff’s case: the $L^2$-norm of the error in the velocity $||e_u||$, the velocity gradient $||\nabla e_u||$, the current density $||e_j||$ and the divergence of the current density $||\nabla \cdot e_j||$. Similarly than in the Shercliff’s case, the results show that in both cases the convergence rates are very good. Furthermore, the errors in the velocity gradient and the divergence of the current density also present a superconvergent behavior in relation to the theoretical expected value, see again [1] for details.

6.3 HCLL test blanket

The helium cooled lead lithium (HCLL) blanket is a liquid metal blanket concept developed in the framework of the European breeding blanket programme for a DEMO reactor to be tested in ITER. Figure 8(a) shows the geometry considered as computational domain, see [9], [10] for details. It consists of a U-shaped channel which measures 360 mm in its longitudinal direction ($x$-axis). The total height is 390 mm ($z$-axis) divided into two subchannels of 190 mm and a transition zone of 10 mm. The section width ($y$-axis) is 206.5 mm. In every one of the subchannels, there are 3 cooling plates whose dimensions are $280 \times 206.5 \times 12$ mm.

Figure 8(b) shows the mesh generated to perform the calculations. It consists of 266,072 nodes and 1,417,435 linear tetrahedral elements. This mesh leads to 2,128,576 degrees of freedom.
The physical properties of the eutectic Pb-17Li fluid have been considered to be constant. The adopted values in this work are: fluid density $\rho = 9.2 \times 10^3 \text{ kg/m}^3$, fluid viscosity $\nu = 1.4 \times 10^{-7} \text{ m}^2/\text{s}$ and fluid electrical conductivity $\sigma = 7.4 \times 10^3 \text{ 1/\Omega m}$ (see [2] and [5] for more details). The external magnetic field applied to the fluid has a value of 10 T and has a direction in the $y$-axis, $\mathbf{B} = 10\hat{y} \text{T}$. Considering that the characteristic magnetic length is half the length of the side walls, $L = 0.103 \text{ m}$, the Hartmann number associated to this flow is $Ha=2470$.

The hydrodynamic boundary conditions have been set as $\mathbf{u} = 0$ at the walls, both the external walls and the cooling plates, $\mathbf{u} = 0.001 \hat{x} \text{ m/s}$ at the inlet, which corresponds to the bottom subchannel, and free condition at the outlet, the top subchannel.

On the other hand, the magnetic boundary conditions have been set as perfectly insulating material in the exterior walls, that is $\mathbf{j} \cdot \mathbf{n} = 0$, and perfectly conducting material in the cooling plates which corresponds to $\mathbf{j} \times \mathbf{n} = 0$.

The solution to this problem converges to a stationary solution. In Figure 9 there have been plotted the solutions in the plane $y = 0.103 \text{ m}$. Those graphics show the longitudinal
behavior of the flow in the $x$-direction. The velocity field shows clearly that the distribution of the cooling plates in the top subchannel is not optimal because almost the entire flow takes place in the top part of the subchannel whereas in the bottom part the fluid has velocity equal to zero. Furthermore, the high values of the velocity near the top part of the top subchannel results on higher values of the current density in the same zone, instead of the distribution that could be expected, similar to the Shercliff’s case solution, which actually is the solution in the bottom subchannel.

Figure 10 shows the obtained solution in a section orthogonal to the flow, the plane $x = 0.150$ m. In this case, it is clearly seen that the current density distribution differs from the theoretical solution of the Shercliff’s case in the top subchannel because the velocity field is not uniform in the four subchannels generated by the presence of the cooling plates.

Finally, Figure 11 shows the streamlines of the velocity field. It is clearly seen how the fluid entering the blanket from the inlet surface goes to the outlet through only the top 2 subchannels, leaving the bottom 2 subchannels of the superior module with almost zero velocity. On the other hand, Figure 12 displays the current density streamlines in section $x = 0.150$m. The streamlines in the inferior module reproduce almost perfectly the streamlines of the Shercliff’s case where both the Hartmann and side walls are perfectly insulating. However, the top module behavior is different. The velocity field concentrating in the top 2 subchannels produces a different distribution of the current density field.
Figure 9: Results in section $y = 0.103 \text{ m}$. 

(a) Velocity norm

(b) Velocity field

(c) Current norm

(d) Current field

(e) Pressure

(f) Electric potential
Figure 10: Results in section $x = 0.150$ m.
Figure 11: Velocity field streamlines

Figure 12: Current density field streamlines at section $x = 0.150$ m
7 Conclusions

In this paper, a numerical formulation to solve the inductionless MHD equations that consists on a stabilized finite element method has been presented. Its design is based on the variational multiscale framework which is derived from a splitting of the unknown into two parts, a finite element component and a subscale that corresponds to the part of the unknown that can not be captured by the discretization. The crucial point in this approach resides in the subscale approximation.

The most important aspects of this formulation are that it allows to use equal interpolation for all the unknowns without having to satisfy the inf-sup conditions. Furthermore, it is stable and optimally convergent in a norm that is meaningful for every value of the physical parameters of the fluid.

Another key point of this formulation is the monolithic approach to solving the problem instead of the possibility of uncoupling the global problem by solving a Laplacian equation for the electric potential. This latter option needs a block iteration algorithm to converge to the coupled solution but there exists no guarantee that it will converge to the solution nor the number of iterations needed in case it converges.

The approximation of the subscales leads to the introduction of some stabilization parameters that need to be proposed. An interesting point of this work is that these parameters have been designed based on the stability and convergence analysis of the method.

The time integration and linearization of the problem considered here is the simplest possible which leads to a method easy to implement but without losing any robustness and convergence properties. The numerical experimentation presented in this article validate these statements and the theoretical development of the method.

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