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BOSE-HUBBARD MODEL OF A BOSE-EINSTEIN CONDENSATE IN A DOUBLE-WELL

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Bose-Hubbard model of a Bose-Einstein condensate in a double-well

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Abstract. We study the static and dynamic behavior of a Bose-Einstein condensate in a double-well potential in the framework of the Bose-Hubbard model. Particular attention is devoted to analyze the structure of the ground state and the appearance of cat-like states which signal the presence of quantum correlations and the fragmentation of the condensate.

1. Theoretical framework

The physics of Bose-Einstein condensates (BEC) in a symmetric double-well potential has attracted a lot of attention since the theoretical prediction of Josephson-like oscillations of the atom’s population and the existence of self-trapped states [1, 2]. In addition, the recent experimental realization of such a system by the Heidelberg group using $^{87}$Rb atoms [3], has opened the possibility of practical applications and extensions to other physical scenarios [4, 5].

In a wide range of physical conditions the behavior of the system is well described by a two-state model. In this particular case, the two states approximately correspond to the ground state built by the system around the vicinity of each minimum of the double-well potential [2]. These two spatially localized modes are labeled by $L$ ($R$) for the left (right) well of the external potential.

A good description of a system with $N$ particles that populate two weakly coupled states ($L$ and $R$ in this case), and with an interaction between the atoms that occupy the same mode, is obtained by means of the Bose-Hubbard model [2, 6]:

$$H = -\varepsilon (a_L^\dagger a_L - a_R^\dagger a_R) - J (a_L^\dagger a_R + a_L a_R^\dagger) + \frac{U}{2} \left( a_L^\dagger a_L a_L^\dagger + a_R^\dagger a_R a_R^\dagger a_R^\dagger \right),$$  \hspace{1cm} (1)

where the term with $J$ takes into account the coupling between the two modes (in our case it is directly related to the tunneling rate between the two sides of the double-well). $U$ characterizes the interaction between the particles and it is taken to be the same for both wells, where $U > 0$ ($U < 0$) describes a repulsive (attractive) interaction. The $\varepsilon$ term introduces a bias between the two states, such that $\varepsilon > 0$ promotes the $L$ state.
A natural basis to study the system is the Fock basis, which is characterized by the number of particles at each side of the barrier, $|N_L, N_R\rangle$. The dimension of the basis is $N + 1$, where $N = N_L + N_R$. Both the static and dynamical properties of the system will be analyzed in terms of the parameter $\Lambda = NU/J$. In all our study, we fix $J = 1$, which is equivalent to measure the energy in units of $J$, and we vary the number of particles $N$ and the strength of the interaction, which will be always considered attractive, $U < 0$. The bias term, that can be related to a possible small asymmetry of the external potential, will be taken very small: $\epsilon/J = 10^{-8}$.

To make contact with the experiment and determine the explicit values of the parameters $J$ and $U$, one should rely on the Gross-Pitaevskii equation [7, 8], which is a mean-field approximation that provides a good description of dilute Bose-systems with a sufficiently large number of particles:

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) + gN|\Psi(\vec{r}, t)|^2 \right] \Psi(\vec{r}, t), \quad (2)$$

where $m$ is the mass of the particles, $V(\vec{r})$ is the external potential (the double-well in our case) and $g = 4\pi\hbar^2a_s/m$ takes into account the effective interaction between the atoms, with $a_s$ the $s$-wave scattering length.

When both sides of the double-well potential are weakly linked, the wave function $\Psi(\vec{r}, t)$ can be approximately written as a superposition of two time-independent spatial wave functions $\Phi_{L(R)}(\vec{r})$ mostly localized at the left (right) side of the trap [1, 9, 10]:

$$\Psi(\vec{r}, t) = \Psi_L(t)\Phi_L(\vec{r}) + \Psi_R(t)\Phi_R(\vec{r}). \quad (3)$$

The left and right modes can be expressed as linear combinations of the ground, $\Phi_+(\vec{r})$, and the first excited, $\Phi_-(\vec{r})$, states of the double-well potential including the interaction term as $\Phi_L(\vec{r}) = (\Phi_+(\vec{r}) + \Phi_-(\vec{r}))/\sqrt{2}$ and $\Phi_R(\vec{r}) = (\Phi_+(\vec{r}) - \Phi_-(\vec{r}))/\sqrt{2}$. Assuming that the wave function has a well-defined quantum phase, $\phi_{L(R)}(t)$, at each side of the trap, which is independent of the position but changes during the time evolution, one can write $\Psi_i(t) = \sqrt{N_i(t)}e^{i\phi_i(t)}$, where $N_{L(R)}(t)$ corresponds to the number of atoms on the left (right) well. Inserting the two-mode ansatz, Eq. (3), in the Gross-Pitaevskii equation (2) and neglecting terms involving mixed products of $\Phi_L(\vec{r})$ and $\Phi_R(\vec{r})$ of order larger than one, one arrives to a system of equations which quantization gives rise to the Bose-Hubbard Hamiltonian. This procedure allows to express the tunneling rate, $J$, and the interaction, $U$, in terms of the wave functions of the two modes:

$$J = -\int d\vec{r} \left[ \frac{\hbar^2}{2m} \nabla \Phi_L(\vec{r}) \cdot \nabla \Phi_R(\vec{r}) + \Phi_L(\vec{r})V(\vec{r})\Phi_R(\vec{r}) \right]$$

$$U = g \int d\vec{r} \Phi^4_{L(R)}(\vec{r}). \quad (4)$$

Moreover, it is useful to introduce the Rabi frequency $\omega_R = 2J/\hbar$, which is the frequency that governs the dynamics in the non-interacting case, $U = 0$. 


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2. Static properties

By diagonalizing the $N + 1$ dimensional Hamiltonian (1), we have found the eigenvalues and the corresponding eigenstates of the system. In Fig. 1 we plot the energy of the first three excited states measured with respect to the ground-state energy as a function of $\Lambda$ for two different number of particles: $N = 50$ (left panel) and $N = 500$ (right panel). For zero interactions, $\Lambda = 0$, the energy gap between consecutive states is equal (except for the bias), and the gap is independent of the number of particles. As $|\Lambda|$ increases, the eigenvalues start to merge in pairs (the ground with the first excited, the second with the third, etc) but due to both $\varepsilon$ and $J$, they do not reach complete degeneracy. Moreover, the convergence of the merging process depends on the number of particles: for higher $N$ it occurs for smaller values of $|\Lambda|$.

In Fig. 2 we plot the spectral decomposition of the ground (solid black) and the first excited states (dotted red) of the Hamiltonian (1) in the Fock space for different values of $\Lambda$, and for $N = 50$. It is worth to recall that a state $|\phi\rangle$ in the Fock basis is written as $|\phi\rangle = \sum_{k=0}^{N} c_k |k, N - k\rangle$. The plotted values $|c_k|^2$ give the probability that the state $|\phi\rangle$ has $k$ particles in the left well and $N - k$ particles in the right one. Instead of plotting the discretized values $|c_k|^2$ it is more practical to report a smooth curve by considering the parameter $k$ continuous. Notice that if the spectral decomposition of the state is peaked at high values of $k$, means that for this state most of the atoms are located on the left side of the double-well.

As it can be observed in the figure, for weak interactions, $|\Lambda| < 2.6$, the spectral decomposition of the ground and the first excited states are clearly different, (as were also their energies in Fig. 1, left panel). For stronger interactions, $-3.2 \leq \Lambda \leq -2.6$, these two states become very close in energy (Fig. 1, left panel), and their spectral decompositions $|c_k|^2$ are very similar. However, these states are not equal, as the first excited one is asymmetric ($c_k = -c_{-k}$), while the ground state is symmetric ($c_k = c_{-k}$). Finally, for $|\Lambda| > 3.2$, the two states become again clearly different: the ground state is peaked at a high value of $k$, with a large amount of atoms in the left well, while the first excited state has its peak at the opposite well. Notice that both of them have a very
similar energy without reaching complete degeneracy. In this regime the role of the bias is crucial to determine the ground state.

An efficient way to characterize the states is by means of the imbalance, \( z = (N_L - N_R)/N \). In the upper panel of Fig. 3, we plot the imbalance of the ground state (solid black) as a function of \( \Lambda \), together with its dispersion \( \sigma_z = \sqrt{\langle z^2 \rangle - \langle z \rangle^2} \) (dotted red). For small interactions, \( |\Lambda| < 2 \), the imbalance is zero and the dispersion is small. Note that \( \sigma_z \) is non-zero even when \( \Lambda = 0 \), as it corresponds to the binomial shape of the spectral decomposition of the state. In the region of \(-3.2 < \Lambda < -2\), the dispersion \( \sigma_z \) starts to grow while the imbalance stays at \( z = 0 \). The increase of \( \sigma_z \) gives a measure of the quantum effects [11, 12]. If we go back to Fig. 2, we can see that the ground state (as well as the first excited state) becomes cat-like, where a maximally entangled cat-state in this context is defined as an equal superposition of all the particles being at \( |L\rangle \) and \( |R\rangle \) simultaneously, \( |\phi\rangle = (|N, 0\rangle + |0, N\rangle)/\sqrt{2} \), [13].

Finally, for \( |\Lambda| > 3.2 \), due to the bias term, the ground state localizes on the left side of the double-well potential. The imbalance dispersion decreases substantially at the same time that the imbalance starts to grow. On the other hand, the first excited state, which is almost degenerate with the ground state, localizes on the other side of the double-well.

The use of the small bias is determinant to break the degeneracy between the ground and the first excited state. If we had considered a perfect symmetric double-well potential (no bias), the symmetry breaking induced by the term \( J \) would be of the same order of the numerical accuracy and the ground state would not be well determined.

To characterize the degree of condensation of the system, we will make use of the one-body density matrix operator:

\[
\hat{\rho} = \begin{pmatrix}
a_L^\dagger a_L & a_R^\dagger a_L \\
 a_L^\dagger a_R & a_R^\dagger a_R
\end{pmatrix}.
\] (5)
In particular, the diagonalization of the expectation value of \( \hat{\rho} \) in the ground state of the system, \( \rho_{\text{GS}} = \langle \Psi_{\text{GS}} | \hat{\rho} | \Psi_{\text{GS}} \rangle \), provides the condensed fractions \( n_1 \) and \( n_2 \), with \( n_1 + n_2 = 1 \). These condensed fractions measure the macroscopic occupations of the single-particle states \(|n_1\rangle\) and \(|n_2\rangle\), eigenfunctions of \( \rho_{\text{GS}} \). In the lower panel of Fig. 3 we plot \( n_1 \) (solid blue) and \( n_2 \) (dashed green), as a function of \( \Lambda \) for \( N = 50 \). The region where these values are not simply 1 and 0, signals the occurrence of fragmentation of the ground state, \(-2.5 < \Lambda < -3.5\), which is around the same interval where the cat-like structure takes place.

Moreover, in the regions where the eigenvalues of the density matrix are strictly \( n_1 = 1 \) and \( n_2 = 0 \), it is possible to express the ground state as a mean-field wave function constructed as \( |\phi_N\rangle = |n_1\rangle \otimes \ldots \otimes |n_1\rangle = |n_1\rangle^{\otimes N} \). In our case, the eigenvector \(|n_1\rangle\) tends to \(|n_1\rangle = (|L\rangle + |R\rangle)/\sqrt{2}\) when \( \Lambda \to 0 \), and to \(|n_1\rangle = |L\rangle\) when \(|\Lambda| \to \infty\).

3. A variational ansatz for the ground state

In this section, we propose a mean-field N-particle state with some free parameters, \( \alpha \) and \( \beta \), that minimize the expected value of the energy for this state. We start from a general single particle state \(|\phi\rangle_1 = \alpha |L\rangle + \beta |R\rangle\), with \(|\alpha|^2 + |\beta|^2 = 1\), and construct a \( N \)-particle state, with all the particles in the same single-particle state, \(|\phi\rangle_1\):

\[
|\phi\rangle_N = \frac{1}{\sqrt{N!}} [\alpha a_L^\dagger + \beta a_R^\dagger]^N |0\rangle. \tag{6}
\]

The expectation value of the Hamiltonian (1) is [14]:

\[
E(\alpha, \alpha^*, \beta, \beta^*) = \langle \phi_N | H | \phi_N \rangle = -\varepsilon N (\alpha\alpha^* - \beta\beta^*) - JN(\alpha^*\beta + \alpha\beta^*) + \frac{U}{2} N(N - 1) (|\alpha|^4 + |\beta|^4). \tag{7}
\]
The minimization of the energy with respect to $\alpha$, $\alpha^*$, $\beta$, and $\beta^*$, together with the normalization condition $|\alpha|^2 + |\beta|^2 = 1$, yields to the following equation:

$$2\varepsilon N - JN\left(\frac{\alpha^2 - \beta^2}{\alpha\beta}\right) + UN(N-1)(|\beta|^2 - |\alpha|^2) = 0.$$  \hspace{1cm} (8)

As $(\alpha^2 - \beta^2)/\alpha\beta$ has to be a real number, the possible solutions will be of the type $(\alpha, \pm \beta)$ with both $\alpha$ and $\beta$ positive real numbers. Explicit simple analytic solutions to the previous equation can be obtained by neglecting the bias term. Therefore, taking $\varepsilon = 0$ and introducing $\tilde{\Lambda} = \Lambda(N - 1)/N$, one gets the following set of solutions:

$$\alpha_0 = \beta_0 = \frac{1}{\sqrt{2}} ; \quad \alpha_\pm = \beta_\pm = \sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{2}^2 - \frac{1}{\tilde{\Lambda}^2}}} ,$$ \hspace{1cm} (9)

giving rise to the multi-particle states:

$$|\phi_i^\pm\rangle_N = \frac{1}{\sqrt{N!}}\left[\alpha_i\alpha_L^\dagger \pm \beta_i\beta_L^\dagger\right]^N |0\rangle ,$$ \hspace{1cm} (10)

with $i = 0, +, -$. Note that the solutions $\alpha_\pm$ and $\beta_\pm$ only exist when $|\tilde{\Lambda}| > 2$. The expectation value of the energy in these states is:

$$E^+_0 = \frac{U}{4} N(N-1) \mp JN \hspace{1cm} (11)$$

$$E^+_\pm = E^\pm_0 = \frac{U}{2} N(N-1) - \frac{NJ}{\tilde{\Lambda}} \left[1 \mp 2\right] .$$ \hspace{1cm} (12)

Therefore, the states $|\phi_i^m\rangle_N$ and $|\phi_i^m\rangle_N$ have the same average energy. Finally, to be consistent with the variational principle, we must identify the lowest energy among the possible solutions: in our case, as $U < 0$ and $J = 1$, the states $|\phi_i^+\rangle_N$ have a lower energy than the $|\phi_i^-\rangle_N$ for $i = 0, +$ or $-$, and therefore we study the energy difference between the states $|\phi_i^+\rangle_N$:

$$E^+_\pm - E^+_0 = NJ \left[\frac{1}{4}\tilde{\Lambda} + \left(\frac{1}{\tilde{\Lambda}} + 1\right)\right] .$$ \hspace{1cm} (13)

The equation $E^+_\pm - E^+_0 = 0$ defines for which value of $\tilde{\Lambda}$ both solutions have the same expected energy. Actually, by solving this equation one can see that for interactions $|\tilde{\Lambda}| < 2$ the lowest energy state is $|\phi_0^+\rangle_N$, while for $\tilde{\Lambda} < -2$, both functions $|\phi_+^\pm\rangle_N$ and $|\phi^-\rangle_N$ have the same minimum mean value energy.

For the case where $E^+_\pm$ are the smallest mean-field energies, $|\tilde{\Lambda}| > 2$, one can propose an alternative ansatz for the multi-particle wave function [14], that goes beyond the mean-field framework as:

$$|\phi_{cat}\rangle_N = \frac{1}{\sqrt{2}}\left(|\phi_0^+\rangle_N + |\phi^+\rangle_N\right) .$$ \hspace{1cm} (14)

The expectation value $\langle \phi_{cat} | H | \phi_{cat}\rangle_N$ is smaller than $E^+_\pm$, and tries to incorporate the cat-like structure. The existence of this cat-like state shows up as a bifurcation in the mean-field approach with two different mean-field wave functions having the same energy expectation value.
Figure 4. Fock space decomposition of the ground state of the system computed by exact diagonalization (solid black), and the wave functions $|\phi^+_0\rangle_N$ for $|\tilde{\Lambda}| < 2$ (dashed red) and both $|\phi^+_+\rangle_N$ (dashed blue) and $|\phi_{\text{cat}}\rangle_N$ (solid green) for $|\tilde{\Lambda}| > 2$, for different values of $\Lambda$, and with $N = 50$.

In Fig. 4 we plot the coefficients $c_k$ of the ground state of the system (solid black) computed by exact diagonalization of the many-body Hamiltonian \((1)\), together with the wave functions $|\phi^+_0\rangle_N$ for $|\tilde{\Lambda}| < 2$ (dashed-red), and both $|\phi^+_+\rangle_N$ (dashed blue) and $|\phi_{\text{cat}}\rangle_N$ (solid green) for $|\tilde{\Lambda}| > 2$. Notice that in the case of $N = 50$, $\Lambda = N\tilde{\Lambda}/(N - 1) = 1.02\tilde{\Lambda}$, and therefore the critical value of $\Lambda$ where the mean-field many-body wave functions $|\phi^+_+\rangle_N$ appear is $\Lambda \sim \tilde{\Lambda} = -2$.

The coefficients $c_k$ of the exact ground state and the different approximations are in good agreement for a broad range of the parameter $\Lambda$. For $|\tilde{\Lambda}| < 2$, the mean-field best representation of the ground state is $|\phi^+_0\rangle_N$, which is exactly the same we found with the density matrix, $|\Phi_\rho\rangle_N$ for $\Lambda \to 0$. However, in this region the exact ground state is slightly broader. The energy difference $E^+_0 - E_{\text{GS}}$ between these two states relative to the exact ground state energy is shown in the left panel of Fig. 5 (red line), while their overlap $\langle \phi_{\text{GS}} | \phi^+_0 \rangle_N$ is shown in the right panel of the same figure (red line).

In the region $-3.2 < \tilde{\Lambda} \leq -2$, the cat-like state $|\phi_{\text{cat}}\rangle_N$ is the one that better describes the Fock-space structure of the ground state. In addition, it has the closest energy to $E_{\text{GS}}$, and the largest overlap (see Fig. 5, green line in both panels).

For stronger interactions, $|\tilde{\Lambda}| > 3.2$, the bias becomes important and the ground state localizes in the left well. The energy of $|\phi_{\text{cat}}\rangle_N$ is still very close to the ground state energy, but the overlap decreases considerably. However, in this region the expected energies of $|\phi^+_+\rangle_N$ and $|\phi^+_+\rangle_N$ are very similar to $\langle \phi_{\text{cat}} | H | \phi_{\text{cat}}\rangle_N$, as the overlap $\langle \phi^+_+ | \phi^+_+ \rangle_N$ becomes smaller. Furthermore, the overlap of $|\phi^+_+\rangle$ with the ground state becomes very large. In the right panel of Fig. 5 there are plotted the overlaps $\langle \phi_{\text{GS}} | \phi^+_+ \rangle_N$ (blue line) and $\langle \phi_{\text{GS}} | \phi^+_+ \rangle_N$ (magenta line) as a function of $\Lambda$. In conclusion, when the bias is
**Figure 5.** Left panel: Energy difference between $E_0^+$ (red), or $E_{\pm}$ (blue), or $E_{cat}$ (green), and the exact ground state energy $E_{GS}$, relative to $E_{GS}$, for different values of $\Lambda$. Right panel: overlap between the GS of the system (eigenfunction of the Bose-Hubbard Hamiltonian) and the lower energy states $|\phi_0\rangle$ in red, $|\psi_+\rangle$ in blue, $|\psi_-\rangle$ in magenta and $|\psi_{cat}\rangle$ in green, as a function of $\Lambda$. In both panels $N = 50$.

dominant, the mean-field state $|\phi^+_N\rangle$ is the best mean-field state that represents the system and asymptotically tends to the ground state.

4. Time evolution of the population imbalance

The equation of motion for the imbalance operator $\hat{z} = (a_L^\dagger a_L - a_R^\dagger a_R)/N$ in the Heisenberg representation reads:

$$i \frac{d\hat{z}}{dt} = \frac{1}{\hbar} [H, \hat{z}] = \frac{2J}{\hbar N} (a_L^\dagger a_R - a_R^\dagger a_L).$$

(15)

Besides, solving this equation we have also calculated the evolution of the condensed fractions $n_1$ and $n_2$ by diagonalizing the one-body density matrix of the evolving state at each time step. In Fig. 6 we plot $z(t)$, $n_1(t)$ and $n_2(t)$ for an initial state with maximal imbalance $|N, 0\rangle$, in units of $t_{\text{rabi}} = 2\pi/\omega_R = \pi\hbar/J$, for different values of $\Lambda$ and $N$. Note that the runs with the same $\Lambda$ correspond to weaker interactions for higher $N$, as $\Lambda = NU/J$. Besides, the initial state of $z = 1$ starts with $n_1 = 1$ and $n_2 = 0$, because this initial state can be expressed with a mean-field wave function as $|N, 0\rangle = |L\rangle^\otimes N$.

In the left panels, as the interaction is weak, $\Lambda = -1$, the evolution of the state is characterized by oscillations with a frequency close to $\omega_R$. These oscillations, are more quickly dumped for smaller number of particles. The condensed fractions, on the other hand, seem to evolve asymptotically to a fixed value close to $1/2$, almost independent of the number of particles.

For $\Lambda = -3$ the dynamics is more complicated than in the previous cases, specially for lower number of particles. The condensed fractions also evolve around a constant value of $\sim 0.8$ and $\sim 0.2$, but not in a smooth way as for $\Lambda = -1$.

Finally, in the right panels, $\Lambda = -5$, we are in a self-trapped situation. The initial state is close to the ground state, which has become very asymmetric and therefore
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Figure 6. Evolution of the initial state $|N,0\rangle$ as a function of time (in units of $t_{\text{rabi}} = 2\pi/\omega R$) for different values of $\Lambda$ and $N$. In all panels we plot the imbalance (solid black), and the condensed fractions $n_1$ (solid red) and $n_2$ (solid green). In the lower panels we also plot the semi-classical evolution of $z$ (dashed blue).

provides a $<z>_{t\neq 0}$. Besides, as the initial state is of mean-field type and almost stationary, the condensed fractions stay rather constant and very close to 1 and 0.

5. Semi-classical limit

It is also helpful to explore the semi-classical limit of the Bose-Hubbard model, for $N \to \infty$. An operative way to implement this limit is to substitute the operators $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$ by c-numbers:

$$\hat{a}_\alpha \rightarrow \sqrt{N_\alpha}e^{i\phi_\alpha} \; ; \; \hat{a}_\alpha^\dagger \rightarrow \sqrt{N_\alpha}e^{-i\phi_\alpha}.$$ (16)

Using the Hamiltonian (1) and neglecting the bias term, the energy per particle in units of $J$ becomes:

$$\tilde{H} = \frac{H}{NJ} = -\frac{U}{2J} + \frac{\Lambda}{4} [1 + z^2] - \sqrt{1 - z^2} \cos (\phi_R - \phi_L).$$ (17)

The equations of motion for the imbalance and for the phase difference between the two wells, $\delta \phi = \phi_R - \phi_L$, can be obtained as:

$$\dot{z} = -\frac{\partial \tilde{H}}{\partial \delta \phi} = -\sqrt{1 - z^2} \sin \delta \phi$$

$$\delta \phi = \frac{\partial \tilde{H}}{\partial z} = \frac{\Lambda}{2} z + \frac{z}{\sqrt{1 - z^2}} \cos \delta \phi.$$ (18)

These equations are the two-mode equations, that one also obtains using the two-mode ansatz Eq. (3) into the Gross-Pitaevskii equation (2), [1]. The evolution of the imbalance
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in this semi-classical limit is plotted in the lower panels of Fig. 6 (dashed-blue), and it is independent of the number of particles. One can see that the agreement with the Bose-Hubbard description is better as \( N \) increases.

6. Summary and conclusions

In this work we have presented the Bose-Hubbard model for a Bose-Einstein condensate trapped in a symmetric double-well potential. We have studied both static and dynamic properties, and we have searched for a simpler many-body wave function that better describes the exact ground state of the system.

The evolution of the ground state as a function of \( \Lambda \) can be described in three different regions. The first one corresponds to weak interactions, \( |\Lambda| < 2 \), and is well represented by a mean-field solution that places all the atoms in the single particle state \((|L\rangle + |R\rangle)/\sqrt{2}\). The dynamics for the state \(|N,0\rangle\) in this region is mostly governed by Rabi oscillations, quickly dumped for small \( N \), and approaching the non-dumped semi-classical regime for large \( N \).

For \( 2 < |\Lambda| < 3.5 \), we have used a variational wave function built with two mean-field wave functions able to reproduce the cat-like structure of the ground state, and that has a better expected energy and overlap with the exact ground state than any mean-field wave function.

And finally, in the third region, \( |\Lambda| > 3.5 \), the ground state localizes in the left well. In this case, a mean-field wave function again represents correctly the structure of the exact ground state. The dynamics however, are governed by a self-trapping regime, where the studied initial state \(|N,0\rangle\), evolves with \( <z> \neq 0 \), and closer to 1 as \(|\Lambda|\) increases.

References