Boundary Element Method Based on Preliminary Discretization

J. Poblet-Puig, V. Yu. Valyaev, and A. V. Shanin

Jordi Girona 1, Campus Nord, B1, E-08034 Barcelona, Spain

Department of Physics, Moscow State University, Moscow, 119899 Russia

e-mail: a.v.shanin@gmail.com

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Abstract—A new numerical method for solving wave diffraction problems is given. The method is based on the concept of boundary elements; i.e., the unknown values are the field values on the surface of the scatterer. An analog of a boundary element method rather than a numerical approximation of the initial (continuous) problem is constructed for an approximate statement of the problem on the discrete lattice. Although it reduces the accuracy of the method, it helps to simplify the implementation significantly since the Green functions of the problem are no longer singular. In order to ensure the solution to the diffraction problem is unique (i.e., to suppress fictitious resonances), a new method is constructed similarly to the CFIE approach developed for the classical boundary element method.

Keywords: wave diffraction, Helmholtz equation, boundary element method, boundary algebraic equations

INTRODUCTION

We consider an exterior wave diffraction problem on a sufficiently smooth barrier. For the sake of definiteness, we consider a stationary acoustic problem with the rigid boundary; i.e., the inhomogeneous Helmholtz equation holds in the infinite domain $\tilde{\Omega}'$, while the Neumann boundary condition holds at the boundary of the domain

$$\Delta u(\mathbf{x}) + \mathcal{H}^2 u(\mathbf{x}) = f(\mathbf{x}) \text{ in } \tilde{\Omega}', \quad (1)$$

$$\frac{\partial u(\mathbf{x})}{\partial n} = 0 \text{ on } \partial\tilde{\Omega}', \quad (2)$$

where $\partial\tilde{\Omega}'$ is the scatterer's boundary and $\mathcal{H}$ is the wave number. The dependence of all unknown variables on time has the form $e^{i\omega t}$. The sources $f(\mathbf{x})$ are assumed to be nonzero in the finite domain of the space. The problem statement is added with the radiation condition at infinity, for instance, in the form of the limit-absorption principle.

One can use various methods to solve such problems in practice. These methods can be sorted into space discretization methods and boundary discretization methods. Methods based on space discretization (such as the finite element method (FEM) and the finite difference method (FD)) require the domain of the space to be bounded and the boundary conditions to be stated. To do this, one can apply, for instance, the DtN map method [1] or the method of perfectly matched boundary layers (PML) [2].

The boundary element method is a boundary discretization method. Although it does not entail any difficulties in meeting the conditions at infinity, it entails difficulties of another kind. First, the matrix of the system of linear equations suppress this text depends on the frequency (the frequency is included in the Green function). Thus, for each frequency, we need to run a resource intensive procedure to calculate the matrix. Second, the dimension of the matrix for the boundary element method is significantly smaller than, for instance, for the finite element method; however, the matrix is sparse for the finite element method and dense for the boundary element method. This does not allow us to always choose one method rather than the other [3]. Another issue associated with the boundary element method is that its implementation requires accurately calculating singular surface integrals. The order of the integrals depends on the specific integral formulation used and the boundary smoothness. Although a number of examples are developed for calculating such integrals, all these procedures are sufficiently resource intensive. Angular points and edges of the boundary are also a challenge in terms of implementation of the boundary element method. Finally, when it comes to applying the boundary element method, the biggest challenge is the

existence of fictitious resonances. This phenomenon is caused by the ill-condition of the exterior Neumann problem matrix near the frequencies that correspond to the eigenvalues of the interior Dirichlet problem with the same form of the boundary [4]. This is a purely mathematical artifact since the solution to the exterior problem generally has no resonances. There are several ways to deal with it. The Combined Field Integral Equation (CFIE) proposed in [5] has been the most common approach so far. It implies that the original integral equation is added with some of its derivatives multiplied by an arbitrary complex constant. The resulting equation has the matrix with a smaller condition number. However, applying this method requires calculating integrals with stronger singularity than that of the original equation.

In this work, we create a method that is in a sense free from the challenges mentioned above. Its structure is as follows. We consider the diffraction problem on a discrete lattice rather than in a continuous space. The scatterer form is approximated in some way (here we assume that the scatterer consists of elementary coordinate cubes). One can use any of the existing methods, for instance, FEM or FD, to obtain the discrete approximation of Eq. (1). The lattice is originally assumed to be infinite (we use this lattice to state the diffraction problem rather than perform calculations). For such discretized diffraction problem, we construct an analog of the boundary integral equation; i.e., the set of unknown variables are the values of the field variable on the surface of the scatterer at the lattice nodes. Obviously, their number is finite. For these unknowns, we state the analog of the boundary integral equation using the discrete analog of Green's formula with the lattice Green function. Since the statement of the integral equation is conceptually close to the CFIE method, the solution of the obtained system is unique (we prove this as a theorem). The new method combines some advantages of the space discretization and boundary discretization methods.

The advantages of the proposed method are as follows. Since, in the discrete statement, integration is replaced by summation over nodes from the very beginning, we have no difficulties with singular integrals. With the CFIE statement, we can be sure that the method does not result in fictitious resonances. Finally, the new method imposes no requirements on the boundary's smoothness. The challenges associated with the new method are connected with the fact that the discrete Green function is more difficult to calculate than the Green function of a continuous problem. Moreover, the approximate boundary of the scatterer is jagged, which means the accuracy of the method cannot be high.

The idea of such a method has been proposed before. The analog of the boundary element method for the discrete problem was first constructed in [6, 7], where the method was called the method of boundary algebraic equations (BAE). In [8], the method of difference potentials was evolved. This method is based on discrete projections. It was used for active noise suppression [9]. The closest ideas to this work were published in [10]. The principal difference of the proposed method from BAE is that it uses a technique similar to CFIE to ensure that the solution is unique and to suppress fictitious resonances.

2. PROBLEM STATEMENT ON A DISCRETE LATTICE

Suppose a homogeneous cubic lattice is given in three-dimensional space (or a quadratic lattice is given in two-dimensional space). Suppose a part of the nodes that occupy the domain representing the approximation of the scatterer's form is deleted from the lattice. We use Ω to designate the full lattice and Ω' to designate the lattice with some nodes deleted from it. The deleted nodes form the set $\Omega \setminus \Omega'$. For auxiliary theoretical purposes, we will need the lattice Ω'' that is a sufficiently big yet finite fragment of the lattice Ω' (for instance, bounded by a big cube or square with side $2R$). Figure 1 shows the lattices involved.

We assume that the scatterer is bounded in space. Although we do not impose any formal restrictions on the mutual positions of the points of the scatterer $\Omega \setminus \Omega'$, we take approximations of convex bodies as examples for illustrations. For the sake of definiteness, we consider the coordinate origin to belong to the set $\Omega \setminus \Omega'$.

We call two nodes *neighboring* if they belong to the same elementary cube of the full lattice Ω in the three-dimensional case or the same elementary square in the two-dimensional case.

We define the boundary of the scatterer $\partial\Omega'$ as the set of all neighbors of the nodes $\Omega \setminus \Omega'$ in the lattice Ω' . We call the nodes belonging to the set $\Omega' \setminus \partial\Omega'$ internal nodes of the lattice Ω' . We also define the external boundary $\partial\Omega''$ as a set of neighbors $\Omega' \setminus \Omega''$ in $\partial\Omega''$.

We give the linear operators L , L' , and L'' on the lattices Ω , Ω' , and Ω'' , respectively, as follows

$$L[u]_j \equiv \sum_{q \in \Omega} \beta_{j,q} u_q, \quad j \in \Omega, \quad (3)$$

$$L'[u]_j \equiv \sum_{q \in \Omega'} \beta'_{j,q} u_q, \quad j \in \Omega', \quad (4)$$

Fig. 1. (a) The non-bounded space without a scatterer Ω ; (b) nonbounded space with nodes deleted (with a scatterer) Ω' ; (c) the final fragment Ω' .

$$L'[u]_j \equiv \sum_{q \in \Omega''} \beta''_{j,q} u_q, \quad j \in \Omega'', \quad (5)$$

where the variables u_q are the node values of the field coefficients and the matrix elements $\beta_{j,q}$, $\beta'_{j,q}$, and $\beta''_{j,q}$ have the following properties:

- (1) The elements $\beta_{j,q}$, $\beta'_{j,q}$, and $\beta''_{j,q}$ are zero if the nodes with the subscripts q and j are not neighbors;
- (2) The matrices are symmetric, i.e.,

$$\beta_{j,q} = \beta_{q,j}, \quad \beta'_{j,q} = \beta'_{q,j}, \quad \beta''_{j,q} = \beta''_{q,j} \quad (6)$$

for all admissible j and q ;

- (3) The matrix of the operator L' is the submatrix of the operator L and the matrix of the operator L'' is the submatrix of the operator L' in the following sense:

$$\beta'_{j,q} = \beta_{j,q} \text{ if } j, q \in \Omega', \quad j \notin \partial\Omega', \quad q \notin \partial\Omega'. \quad (7)$$

The latter condition means that j or q is the internal node of the lattice Ω' . Similarly,

$$\beta''_{j,q} = \beta'_{j,q} \text{ if } j, q \in \Omega'', \quad j \notin \partial\Omega'', \quad q \notin \partial\Omega''. \quad (8)$$

- (4) The matrix β is supposed to be homogeneous with respect to translations along the coordinate axes. This property allows us to apply the Fourier transform to calculate the lattice Green function of the operator L .

By the first property, the sums in (3)–(5) are finite for any j .

The operators introduced above should be difference approximations of the operator of Helmholtz's equation (2) in the respective domains. We also assume that the operator L' approximates the Neumann boundary conditions at the boundary of the domain, i.e., the heterogeneous equation

$$L'[u]_j = f_j \quad \forall j \in \Omega' \quad (9)$$

approximates Eq. (1) with the Neumann conditions at the scatterer's boundary. The behavior of the operator L'' at the boundary $\partial\Omega''$ does not matter for us.

We give an example of the operator L in the two-dimensional case

$$\beta_{j,q} = \begin{cases} h^{-2}, & j \neq q, \text{ } j \text{ is adjacent with respect to the edge with } q, \\ 4h^{-2} + \mathcal{K}^2, & j = q, \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

Here, h is the lattice step.

3. DISCRETE STATEMENT OF BOUNDARY INTEGRAL EQUATIONS

We use G_j^m to denote the lattice Green function on Ω ; i.e., we suppose

$$L[G_j^m] = \delta_{j,m} \quad (11)$$

where $\delta_{j,m}$ is the Kronecker symbol. We assume that the Green function satisfies the radiation principle with respect to subscript j .

We consider the operator L'' on the finite lattice Ω'' . It follows from (6) that the equality

$$\sum_{j \in \Omega''} L''[u]_j w_j = \sum_{j \in \Omega''} L''[w]_j u_j. \quad (12)$$

holds for any lattice functions u_j and w_j given on Ω' . Note that here the operator L'' is applied to the functions given on Ω' ; i.e., we consider the restrictions of these functions on the lattice Ω'' .

Suppose $L[u]_j$ and $L[w]_j$ are nonzero only in a finite domain and suppose the functions u and w satisfy the radiation condition stated above. The operator L'' differs from L' on Ω'' only at the nodes of the boundary $\partial\Omega''$. By the radiation conditions, this difference decreases as $R \rightarrow \infty$. Thus, we can calculate this limit and move from Ω'' to Ω'

$$\sum_{j \in \Omega'} L[u]_j w_j = \sum_{j \in \Omega'} L[w]_j u_j. \quad (13)$$

Choosing Ω as Ω' , G_j^m as u_j , and G_j^n as w_j in (13), one can easily show that

$$G_m^n = G_n^m. \quad (14)$$

We consider the restriction of the function G_j^m on the lattice Ω' . Although formally this restriction is another function, we will use the same symbol G_j^m for it since this will not be misleading.

We apply the operator L' to G_j^m . Obviously, relation (11) does not hold for the operator L' since the operators L' and L differ at the boundary $\partial\Omega'$. For $m \in \Omega'$, we give the variables b_j^m as

$$L'[G^m]_j = \delta_{j,m} + b_j^m. \quad (15)$$

These variables act as the residuals of the operator L' applied to Green's function. The variables b_j^m are nonzero only for $j \in \partial\Omega'$.

Choosing $w_j = G_j^m$ in (13), we obtain the following boundary integral representation of the field

$$u_m = \sum_{j \in \Omega'} (f_j G_j^m - u_j b_j^m) - \sum_{j \in \Omega'} f_j G_j^m - \sum_{j \in \partial\Omega'} u_j b_j^m, \quad m \in \Omega'. \quad (16)$$

The case $m \in \partial\Omega'$ is of special interest since then the boundary representation of the field immediately leads to the simplest analog of the boundary integral equations with unknowns u_j , $j \in \partial\Omega'$.

Equation (16) with $m \in \partial\Omega'$ act as the boundary integral equation in the direct Kirchhoff statement. Although it is ready to be used to find the variables u at the nodes of the boundary, in practice such an approach results in fictitious resonances. Our next step is to modify Eq. (16) in the CFIE style to suppress such resonances.

We apply the operator L' to both parts of Eq. (16) and take into account that $L'[u]_m = f_m$

$$f_m = \sum_{j,n \in \Omega'} f_j G_j^n \beta'_{m,n} - \sum_{j \in \partial\Omega'} u_j \sum_{n \in \Omega'} b_j^n \beta'_{m,n}. \quad (17)$$

This equation holds for any $m \in \Omega'$ but we put $m \in \partial\Omega'$ as we construct the boundary integral equation. According to the general scheme of the CFIE method (see [5]), we form the linear combination of (16) and (17)

$$u_m + v f_m = \sum_{j,n \in \Omega'} f_j (\delta_{m,n} + v \beta'_{m,n}) G_j^n - \sum_{j \in \partial\Omega'} u_j \sum_{n \in \Omega'} (\delta_{m,n} + v \beta'_{m,n}) b_j^n. \quad (18)$$

The parameter v is an arbitrary complex number with a nonzero imaginary part. Equation (18) taken for $m \in \partial\Omega'$ is a linear system for finding the vector composed of the variables u_m . Note that formally summation over n in the second term is done over all nodes of the lattice Ω' but only a finite number of terms that correspond to the nodes neighboring to the nodes of the boundary $\partial\Omega'$ are nonzero.

Equation (18) corresponds to radiation problem (9) stated at the beginning of the work. In many practically important cases, one needs to solve the diffraction problem, i.e., the incident wave u^{in} (for instance, a plane wave coming from infinity) rather than sources of the field in the environment is given. Suppose $L[u^{\text{in}}]_j = 0$ on the lattice Ω . Find the scattered field u^{sc} such that the complete field $u = u^{\text{in}} + u^{\text{sc}}$ satisfies

the homogeneous equation $L[u]_j = 0$ everywhere on Ω' . To solve such diffraction problem, one can also use Eq. (18) and choose

$$f_j = L[u^{\text{in}}]_j = \sum_{q \in \Omega'} \beta'_{j,q} u_q^{\text{in}}. \quad (19)$$

We can rewrite Eq. (18) in a more convenient form

$$\sum_{j \in \partial\Omega'} M_{m,j} u_j = \sum_{j \in \Omega'} F_{m,j} f_j, \quad m \in \partial\Omega', \quad (20)$$

where

$$M_{j,q} = v \sum_{n,q \in \Omega'} \beta'_{m,n} G_q^n \beta'_{q,j} + \sum_{q \in \Omega'} G_q^m \beta'_{q,j} = v \beta'_{m,j}, \quad j, q \in \partial\Omega', \quad (21)$$

$$F_{m,j} = v \sum_{q \in \Omega'} \beta'_{m,q} G_j^q + G_j^m = v \delta_{m,j}, \quad m \in \partial\Omega', \quad j \in \Omega'. \quad (22)$$

Equation (20) is solved with respect to the vector of variables u_j . In terms of the acoustic problem, this means that we search for pressures on the surface of the scatterer. After the values on the surface are found, we can use representation (16) to find the field in any other point of the lattice Ω' .

4. PROOF OF UNIQUENESS OF SOLUTION OF (20)

We show that when used to construct the discrete analog of the boundary integral equation, the CFIE idea results in the unique solution to the problem for all frequencies; i.e., there are no fictitious resonances.

We introduce a number of new designations. We define the operator $\tilde{L}'$ using

$$\tilde{\beta}'_{m,n} \equiv \beta'_{m,n} - \beta'_{n,m}. \quad (23)$$

In this formula, we assume that the coefficients β' are given on the entire lattice Ω and are zero if m or n belong to $\Omega' \setminus \partial\Omega'$. Obviously,

$$\tilde{\beta}'_{m,n} = -\tilde{\beta}'_{n,m}. \quad (24)$$

Suppose some function u_j is given only at the boundary nodes $\partial\Omega'$. We give the *internal potential of the double layer* as follows:

$$M_i[u]_m = \sum_{n \in \Omega'} \sum_{j \in \partial\Omega'} G_{m,n} \beta'_{n,j} u_j, \quad m \in \Omega. \quad (25)$$

Similarly, we give the external potential of the double layer

$$M_e[u]_m = \sum_{n \in \Omega} \sum_{j \in \partial\Omega'} G_{m,n} \tilde{\beta}'_{n,j} u_j. \quad (26)$$

We give the operator of calculating the *external normal derivative* as

$$D_e[v]_m = \sum_{n \in \Omega'} \beta'_{m,n} \nabla_n, \quad m \in \partial\Omega'. \quad (27)$$

Similarly, we give the operator of calculating the *internal normal derivative*

$$D_i[v]_m = \sum_{n \in \Omega} \tilde{\beta}'_{m,n} \nabla_n. \quad (28)$$

We give the exterior Neumann problem as follows. We need to find the function u_n , $n \in \Omega'$ such that

$$L[u]_m = \beta'_{m,n} u_n = 0, \quad m \in \Omega' \setminus \partial\Omega', \quad (29)$$

$$D_e[u]_m = f_m, \quad m \in \partial\Omega', \quad (30)$$

given also that u_n satisfies the radiation condition at infinity.

We prove a number of basic propositions stated similarly to the common potential theory.

Proposition 1.

$$D_i[M_i[u_j]] = D_e[M_e[u_j]]. \quad (31)$$

Proof. For $m \in \partial\Omega'$,

$$\beta'_{m,n} G_{n,k} \tilde{\beta}'_{k,j} u_j = \beta'_{m,n} G_{n,k} (\beta'_{k,j} - \beta'_{m,n}) u_j = \beta'_{m,n} G_{n,k} \beta'_{k,j} u_j + \beta'_{m,n} \delta_{n,j} u_j = \beta'_{m,n} G_{n,k} \beta'_{k,j} u_j + \beta'_{m,j} u_j, \quad (32)$$

$$\tilde{\beta}'_{m,n} G_{n,k} \beta'_{k,j} u_j = (\beta'_{m,n} - \beta'_{m,n}) G_{n,k} \beta'_{k,j} u_j = \beta'_{m,n} G_{n,k} \beta'_{k,j} u_j + \delta_{m,n} \beta'_{n,j} u_j = \beta'_{m,n} G_{n,k} \beta'_{k,j} u_j + \beta'_{m,j} u_j. \quad (33)$$

Here and in what follows, for the sake of brevity, summation is over the repeated indices. The summation range is clear from the definitions given above.

Proposition 2.

$$M_e[u_j]_m + M_i[u_j]_m = u_m, \quad m \in \partial\Omega'. \quad (34)$$

Proof.

$$G_{m,n} \tilde{\beta}'_{n,j} u_j + G_{m,n} \beta'_{n,j} u_j = G_{m,n} \beta'_{n,j} u_j + \delta_{m,n} u_j = u_m. \quad (35)$$

Proposition 3.

$$\tilde{L}[M_i[u_j]]_m = 0, \quad m \in \tilde{\Omega}' \setminus \partial\Omega'. \quad (36)$$

Proof. For $m \in \tilde{\Omega}' \setminus \partial\Omega'$

$$\tilde{\beta}'_{m,n} G_{n,k} \beta'_{k,j} u_j = \beta'_{m,n} G_{n,k} \beta'_{k,j} u_j + \delta_{m,n} \beta'_{n,j} u_j = 0. \quad (37)$$

Proposition 4.

$$L[M_e[u_j]]_m = 0, \quad m \in \Omega' \setminus \partial\Omega'. \quad (38)$$

The **proof** is similar to that of Proposition 3.

We can rewrite Eq. (18) in the new designations as

$$M_i[u_j]_m = \nu D_i[M_i[u_j]]_m \quad r.h.s., \quad m \in \partial\Omega'. \quad (39)$$

For the constructed discrete analog of the CFIE method, uniqueness means that the homogeneous equation

$$M_i[u_j]_m = \nu D_i[M_i[u_j]]_m = 0, \quad m \in \partial\Omega' \quad (40)$$

has a trivial solution only.

Theorem. Equation (40) for $\text{Im}[\nu] \neq 0$ possesses only a trivial solution if the exterior Neumann problem $L[v] = 0$ possesses only a trivial solution.

Proof. Suppose there exists a nontrivial solution u_j of Eq. (40). We construct the function $v_m = M_i[u_j]$. It follows from (40) that

$$v_m = \nu D_i[v]_m, \quad m \in \partial\Omega'. \quad (41)$$

We use the overline to designate complex conjugation. Note that

$$0 = \sum_{n,k \in \Omega} [\bar{v}_n \tilde{\beta}'_{n,k} \bar{v}_k - v_n \tilde{\beta}'_{n,k} v_k] = \sum_{j \in \partial\Omega'} [\bar{v}_j D_i[v]_j - v_j D_i[\bar{v}]_j] = 2\text{Im}[\nu] \sum_{j \in \partial\Omega'} |D_i[v]_j|^2 \quad (42)$$

and hence

$$D_i[v]_j = v_j = 0, \quad j \in \partial\Omega'. \quad (43)$$

Further, we consider the function $w = M_e[u_j]$. By (31) and (43), $D_e[w]_j = 0$ for $j \in \partial\Omega'$. By (38), the function w_m satisfies the homogeneous exterior Neumann problem. Since, by the theorem hypothesis, such a problem possesses a trivial solution only

$$w_j = 0, \quad j \in \partial\Omega'. \quad (44)$$

It follows from (43) and (44), $M_e[u_j]_m + M_i[u_j]_m = 0$. Finally, applying (34), we have $u_m = 0$.

5. NUMERICAL IMPLEMENTATION

In this work, we describe the implementation of the proposed method in the two-dimensional case.

Calculating Discrete Green Function of Unbounded Domain

We assume that the matrix $\beta_{j,q}$ is given by (10). Green's function G_j^m is constructed using the Fourier transform [11]. Suppose the source m coincides with the coordinate's origin. The variables j_1 and j_2 are the integer Cartesian coordinates of the node—the observation points of j . We represent Green's function as

$$G_j^0 = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} A(\xi_1, \xi_2) \exp\{i(\xi_1 j_1 + \xi_2 j_2)\} d\xi_1 d\xi_2. \quad (45)$$

The Fourier transform yields

$$A(\xi_1, \xi_2) = \frac{1}{4\pi^2} \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} G_j^0 \exp\{i(\xi_1 j_1 + \xi_2 j_2)\}. \quad (46)$$

We apply the operator L to (45)

$$\delta_{j_1,0} \delta_{j_2,0} = \frac{2}{h^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[\cos \xi_1 + \cos \xi_2 - 2 + \frac{\mathcal{H}^2 h^2}{2} \right] A(\xi_1, \xi_2) \exp\{i(\xi_1 j_1 + \xi_2 j_2)\} d\xi_1 d\xi_2, \quad (47)$$

hence

$$A(\xi_1, \xi_2) = \frac{h^2}{8\pi^2 \cos \xi_1 + \cos \xi_2} \frac{1}{2 + h^2 \mathcal{H}^2 / 2}. \quad (48)$$

We substitute (48) to (45) and calculate the integral with respect to ξ_2

$$G_j^0 = \frac{h^2}{4\pi i} \int_{-\pi}^{\pi} \frac{\exp\{i(\xi_1 j_1 + \Xi(\xi_1) |j_2|)\}}{\sin \Xi(\xi_1)} d\xi_1. \quad (49)$$

Here,

$$\Xi(\xi_1) = \arccos(2 \cos \xi_1 - h^2 \mathcal{H}^2 / 2). \quad (50)$$

Integration in (49) is performed in the complex plane ξ_1 over the contour passing below the segment $(0, \pi)$ and above the segment $(-\pi, 0)$.

Prior to the calculation of the matrices included in the principal equation, Green's function is tabulated numerically for j_1 and j_2 smaller than particular values. For higher j_1 and j_2 , it is replaced by its asymptotics.

Constructing the Matrix $\beta'_{m,n}$ Using the Matrix $\beta_{m,n}$

The choice of the matrix $\beta'_{m,n}$ is somewhat arbitrary because the coefficients associated with the points of $\partial\Omega'$ can be chosen arbitrarily. This choice certainly influences implementation of boundary conditions and the method's efficiency. In the numerical examples given below, this arbitrariness is eliminated as follows. To construct the matrix β' , we use the idea of it being assembled of elementary matrices, which underlies the finite element method. To be more exact, we link each elementary coordinate square (suppose its index is α) to the elementary matrix $\beta_{m,n}^\alpha$ such that the nonzero values $\beta_{m,n}^\alpha$ correspond only to the nodes m and n included in this elementary square and the property

$$\beta_{m,n} = \sum_{\alpha} \beta_{m,n}^\alpha,$$

holds; i.e., the entire matrix β is split between the elementary matrices. Obviously, there is only one way to do this—if we require the matrices β^α to allow translations along the lattice, as well as all geometric symmetries.

Suppose the lattice Ω' is some (infinite) subset of elementary coordinate squares that belong to Ω (note that this assumption does not hold always). Then, we assume that

$$\beta'_{m,n} = \sum_{\alpha \in \Omega'} \beta_{m,n}^\alpha.$$

Fig. 2. Scheme of the test problem.

We can assert that such a choice of the matrix β' ensures all restrictions imposed above are met. In particular, the matrix β' approximates the Neumann conditions at the boundary.

Calculating Field Directivity Pattern

The usual requirement on the solution to the diffraction problem is to find the directivity pattern of the scattered field. To be more exact, for the original (continuous) problem, we can write the field at a large distance H from the coordinate's origin in the two-dimensional case as

$$u^{\text{sc}}(x_1, x_2) \sim \sqrt{\frac{2}{\pi KH}} \exp\{iKH - i\pi/4\} D(\theta) + O((KH)^{3/2}), \quad (51)$$

where we use the formulas

$$x_1 = H \cos \theta, \quad x_2 = H \sin \theta.$$

to move to the cylindrical coordinates. The function D is the sought directivity pattern.

We construct the analog of the directivity pattern for the numeric solution found by the method described above. Suppose G_j^m is Green's function of the problem. Suppose the source (the node m) corresponds to the remote point with the geometric coordinates $(H \cos \theta, H \sin \theta)$. We apply formula (16) to the field u^{sc} given (19). As a result, we have the expression

$$u_m^{\text{sc}} = \sum_{q \in \partial'\Omega'} \sum_{j \in \partial\Omega'} (u_q^{\text{in}} \beta'_{q,j} G_j^m + G_q^m \beta'_{q,j} u_j^{\text{sc}}). \quad (52)$$

The set $\partial'\Omega'$ is the union of the boundary $\partial\Omega'$ and its neighbors in Ω' , u^{in} is the incident field (the plane wave that satisfies the homogeneous discrete Helmholtz equation on Ω), and u_j^{sc} are the values of the scattered field at the boundary $\partial\Omega'$ found by the method described above.

We approximate Green's function near the coordinate origin. If the index j_1 corresponds to the coordinate x_1 and the index j_2 corresponds to the coordinate x_2 , this approximation has the form

$$G_j^m \approx A(\theta) \sqrt{\frac{2}{\pi KH}} \exp\{iKH - i\pi/4\} w_j^\theta, \quad w_j^\theta = \exp\{i(\xi_1(\theta)j_1 + \xi_2(\theta)j_2)\}. \quad (53)$$

Approximation of this type can be constructed using the asymptotic analysis of integral representations for the discrete Green function (see [11]). Below, we will use approximation formulas that hold only for small h

$$A \approx \frac{ih^2}{4}, \quad \xi_1 \approx Kh \cos \theta, \quad \xi_2 \approx Kh \sin \theta. \quad (54)$$

Obviously, the directivity pattern can be calculated by the formula

$$D(\theta) = A(\theta) \sum_{q \in \partial'\Omega'} \sum_{j \in \partial\Omega'} (u_q^{\text{in}} \beta'_{q,j} w_j^\theta + w_q^\theta \beta'_{q,j} u_j^{\text{sc}}). \quad (55)$$

Fig. 3. The field u^{sc} at the scatterer's boundary for $K = 1$.

Fig. 4. The field u^{sc} at the scatterer's boundary for $K = 6$.

6. NUMERICAL RESULTS

We use the proposed method to solve the test, two-dimensional problem for diffraction on the circle under the Neumann boundary conditions. We chose this problem since it has an exact solution. Figure 2 gives the scheme of the problem. The incident plane wave propagates along the axis x_1 . The circle of the radius $a = 1$ is approximated by the set of coordinate squares with the side h . We study the directivity pattern as a function of the angle θ .

Field at the Boundary

We construct the field at the scatterer's boundary. For the fixed radius of the circle, we choose three values of the wave number $K = 1, 6, 12$ and three values of the lattice step $h = 0.2, 0.05, 0.02$. Figures 3–5 give the obtained results. The values of the field at the node points are given. For each of the node points, we calculate the value $\theta = \arctg(x_2/x_1)$ and put the respective dot on the graph. We compare the obtained values to the theoretical ones obtained using the Fourier series (the solid line). We give the values of the real part of the field. One can see from the graphs that the acceptable values of the error (several percents) are attained for $Kh < 0.1$.

Fig. 5. The field u^{sc} at the scatterer's boundary for $K = 12$.

Fig. 6. Directivity pattern for $K = 1$.

Fig. 7. Directivity pattern for $K = 6$.

Fig. 8. Directivity pattern for $K = 12$.

Fig. 9. Relative error for the directivity pattern as the function of the parameter Kh .

Directivity Pattern

For the same values of a , K , and h , we calculated the directivity patterns for the scattered field. Figures 6–8 give the real parts of the diagrams. Figure 9 gives the relative error for the directivity patterns for $a = 1$ and $h = 0.05$. We calculated the relative error by the formula

$$\left(\frac{\int_0^{2\pi} |D(\theta) - D_{exact}(\theta)|^2 d\theta}{\int_0^{2\pi} |D_{exact}(\theta)|^2 d\theta} \right)^{1/2},$$

where D_{exact} is the exact directivity pattern. The graph shows that the relative error is close to the variable Kh .

7. CONCLUSIONS

We gave the modification of the method of boundary algebraic equations constructed based on the CFIE approach to suppress fictitious resonances. We used numerical examples to show the method's efficiency. The method requires a sufficiently dense lattice (at least 20 points per wave length) and is not accurate. Nevertheless, it is not resource intensive and is easy-to-implement. We expect the method to be

applied in practical problems associated with the propagation and diffraction of acoustic and electromagnetic waves.

One of the ways to improve the method is to extend it to include heterogeneous (curvilinear) lattices to improve the boundary approximation accuracy. Although applying such lattices does not change our reasoning here, it requires a different approach to calculating and tabulating Green's function, since a heterogeneous lattice does not allow translations along the coordinate axes. A possible way is to calculate Green's function on the basic homogeneous lattice and then interpolate it to the nodes of the heterogeneous lattice.

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