Hysteretic modelling of x-braced shear walls^{*}

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Abstract

We present a differential model of the hysteretic behaviour of unsheathed x-braced frames. The model includes all the relevant aspects (perfect or hardening plasticity of diagonal straps under tension, buckling of diagonal straps under compression). The frame is treated as a single-degree-of-freedom system, and the hysteretic model is written as a small system of ordinary differential equations. In spite of this simplified modelling approach, the proposed model captures the key features of hysteretic cycles of x-braced frames (extreme pinching and slackness) and can be used to obtain the q -factor for simplified seismic design based on the reduction of elastic spectra. The model also allows to couple several x-braced frames to obtain the response of more complex structures such as complete façades. In fact, the agreement with a more sophisticated approach, such as a finite element model of the frame, is excellent, but the computational cost is considerably lower. This point is illustrated by various numerical examples.

Key words: hysteresis, shear walls, x-bracing, cold-formed steel structures, seismic design, q -factor.

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1 Introduction

The use of lightweight steel structures has increased remarkably in the last years [1]. With respect to other building technologies, the main advantages are reduction of costs, low weight, and quickness in construction.

Some recent earthquakes have sadly highlighted the importance of good seismic performance of residential buildings. Enough lateral stiffness is required to resist seismic loads without significant structural damage. In cold-formed steel structures this lateral stiffness is provided by means of shear walls.

Another important issue is ductility. Staying in the elastic range is an uneconomical way to design buildings against earthquakes. A good seismic design should provide energy dissipation mechanisms via plastic deformation.

The hysteretic behaviour of shear walls is thus a key factor in the seismic response of cold-formed steel structures. Cyclic tests of full-scale shear walls have been carried out in the context of the outgoing research project “Seismic Design of Light-Gauge Steel Framed Buildings”. Figure 1 shows a typical force-displacement curve. Note that the hysteretic cycles exhibit pinching, stiffness degradation (reduction of the elastic slope) and load deterioration (reduction of peak load) as the loading cycles go on. Qualitatively similar results have been reported by other researchers [2,3].

Figure 1. Experimental cyclic response of a full-scale shear wall

Two different types of hysteretic models for shear walls can be found in the literature. On the one hand, there are non-differential models which consist of force-displacement relations that reproduce the experimental behaviour of shear walls. White and Dolan [4] proposed a finite element model for timber shear walls based on a “sheathing-to-framing” element with a hysteretic force-displacement law. De Matteis and Landolfo proposed an analytical model with an envelope curve and hysteretic cycles, first for connections [5] and then for shear walls [2]. This force-displacement law was incorporated to a set of

fictitious diagonals in a finite element model of a frame [6]. Fülöp and Dubina [7] also incorporated an analytical force-displacement law to a set of fictitious diagonals in a finite element model. In all cases, the numerical treatment for these models is not described in the references.

On the other hand, there are a family of closed-form differential hysteresis models. Bouc (1967) proposed a model of hysteresis that depends on four parameters. This model was later modified by Wen (1980), Baber and Wen (1981) [8] and Baber and Noori (1985, 1986) to incorporate stiffness and strength deterioration and pinching (BWBN model). Foliente [9] modified the pinching function. In these models, hysteresis is modelled by an additional variable in the dynamic equation and an associated ordinary differential equation.

Another differential model is the one proposed by Mostaghel [10]. It is an analytical hysteretic model for bilinear and multilinear systems based on a single-degree-of-freedom (SDOF) mechanical system. Starting from an elastoplastic behaviour, the model is extended progressively to account for pinching, stiffness degradation and load deterioration.

In the case of differential models, numerical treatment is straightforward: hysteretic behaviour of shear walls can be obtained by solving a small system of ordinary differential equations (ODEs) with suitable integration schemes, such as a Runge-Kutta method. This can be done by means of standard scientific software such as Matlab, Maple or Mathematica. Differential models are, in our opinion, a more general and versatile approach.

From an structural point of view, there are two types of shear walls: (1) unsheathed shear walls, in which the structural contribution of cladding is neglected and steel x-braces are the only source of lateral stiffness, and (2) sheathed shear walls, where the diaphragm effect of cladding panels (of gypsum, plywood, steel or other materials) is accounted for and x-braces are not needed. In the case of x-braced frames, hysteretic curves are bilinear, with an extreme pinching caused by the global buckling of the x-braces. In the case of sheathed shear walls, hysteretic curves are smooth (see Figure 1) and present pinching, strength and stiffness degradation. We focus in this paper on unsheathed x-braced frames and leave sheathed shear walls for another contribution.

Our goal here is to simulate the hysteretic behaviour of x-braced frames by considering them as single-degree-of-freedom systems. We have modified the bilinear model proposed by Mostaghel [10]. This modified bilinear model accounts for two features of the hysteretic loops of x-braced frames not included in the original model, namely slackness and different sources of stiffness. Once this differential model is calibrated and validated, several analyses such as the prediction of response of complete façades or the evaluation of the q -factor

can be carried out.

This paper is organized as follows: two different approaches for predicting the hysteretic response of unsheathed shear walls are presented and compared in Section 2. The evaluation of the q -factor for x-braced frames is discussed in Section 3. Coupling of shear walls is developed in Section 4. Concluding remarks and future developments are presented in Section 5.

2 Modelling of x-braced frames

Cold-formed x-braced frames are made of studs, tracks and x-braces, see Figure 2(a). In our case, studs are 2.75 m long C-profiles, tracks are 1.2 m long U-profiles and x-braces are 3 m long steel flat straps. Joints between members are made with bolts or self-drilling screws.

Figure 2. Cyclic behaviour of x-braced frames: (a) geometry of the frame; (b) horizontal force-displacement cyclic law. Notice the extreme pinching in the cycle.

Studs and tracks are classified by the Eurocode 3 [11] as C4 cross sections under compression loads, which means that they are prone to local buckling. Thus, they are not ductile members. Consequently, energy dissipation of the x-braced frame is given by the plastic deformation of the diagonal straps which have to be designed to get into the plastic range before the studs buckle: this way, studs and tracks remain in the elastic range.

Two different approaches are discussed and compared here. On the one hand, the frame is modelled as a multi-degree-of-freedom (MDOF) system and discretized by the finite element method with proper constitutive laws. On the

other hand, we assume that it behaves as a SDOF system and present a modified bilinear system to predict its hysteretic response.

2.1 Treatment as a MDOF system

We can model the unsheathed shear wall as a MDOF system and discretise the x-braced frame by the finite element method. After spatial FE discretization, we obtain the nonlinear system of ODEs of dynamics

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}(\mathbf{x})\mathbf{x} = \mathbf{f}(t), \quad (1)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}(\mathbf{x})$ are the mass, damping and stiffness matrices, $\mathbf{x}$, $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ are nodal displacements, velocities and accelerations and $\mathbf{f}(t)$ is the external load vector. This system of ODEs can be solved by step-by-step time integration using the Newmark method [12].

The size of the system is three times the number of nodes in the FE mesh (two displacements and one rotation per node). Structural members are modelled with standard Euler-Bernoulli beam elements. Linear shape functions are used for axial displacements and cubic functions for transverse displacements [13].

Studs and tracks are regarded as elastic (Young modulus $E = 2.1 \times 10^5$ MPa). As discussed in Section 1, diagonal straps buckle out of the plane when compressed. Thus, straps can undergo plastic deformation under tension, but have no stiffness under compression. This unilateral behaviour of diagonal straps is modelled with a nonlinear constitutive law, see Figure 3.

Figure 3. Constitutive law for diagonal straps in the finite element model

A nonlinear finite element code with the nonlinear constitutive law for diagonal straps of Figure 3 is used to model the hysteretic response of the x-braced

frame of Figure 2. The resulting cyclic force-displacement response is shown in Figure 2(b). The fact that diagonal straps do not work in compression causes an extreme pinching in the hysteresis loops.

2.2 Treatment as a SDOF system

Let us now assume that the x-braced frame behaves as a SDOF system. By doing so, a hysteretic model can be written as a very small system of ODEs [9,10,14]. This system can be solved very efficiently and accurately by means of Runge-Kutta methods, which are readily available in standard scientific software such as Matlab, Maple or Mathematica. No advanced nonlinear FE code is required.

First of all, we review the bilinear model of Mostaghel [10]. After that, we propose a modified bilinear model which takes into account additional features of the hysteretic response of x-braced frames.

2.2.1 The bilinear model of Mostaghel

The bilinear model presented in reference [10] is based on the mechanical device of Figure 4. A mass m is connected to a spring of stiffness αk , a damper with damping coefficient c , and another spring of stiffness $(1 - \alpha)k$ connected to a slider. If x is the displacement of the system and z is the elongation of the spring connected to the slider, the equation of dynamics is

$$m\ddot{x} + c\dot{x} + \alpha kx + (1 - \alpha)kz = f(t), \quad (2)$$

where $f(t)$ is the applied external force. Note that the restoring force is given by $F(x, z) = \alpha kx + (1 - \alpha)kz$.

Figure 4. Mechanical device for the bilinear model of Mostaghel

When the spring connected to the slider reaches its maximum elongation δ , the slider starts sliding and the total stiffness decreases from k (two springs) to αk (only bottom spring). Note also that $-\delta \leq z \leq \delta$ and the maximum load at the upper spring is $(1 - \alpha)k\delta$. The resulting force-displacement relation turns out to be the one shown in Figure 5(a), which corresponds to the simplest case of hysteresis: plasticity.

Figure 5. Bilinear hysteretic cycles: (a) elastoplastic system; (b) pinched system

To model this behaviour, an evolution equation for elongation z is needed. For this purpose, it is useful to define the Heaviside function

$$N(\omega) = \begin{cases} 1 & \text{if } \omega \geq 0 \\ 0 & \text{if } \omega < 0 \end{cases} \quad (3)$$

and the related functions $M(\omega) = 1 - N(\omega)$, $\overline{N}(\omega) = M(-\omega)$ and $\overline{M} = N(-\omega)$. The differential equation for z is

$$\dot{z} = \dot{x} [\overline{N}(\dot{x}) \overline{M}(z - \delta) + M(\dot{x}) N(z + \delta)]. \quad (4)$$

Equation (4) simply states that $\dot{z} = 0$ or $\dot{z} = \dot{x}$, depending on whether there is sliding or not. Note that the cyclic plastic response of a SDOF system is governed by a system of two ODEs, Equations (2) and (4).

Pinching can be added into the system, see Figure 5(b), by reducing the maximum resistance of the spring from $(1 - \alpha)k\delta$ to $\lambda_p(1 - \alpha)k\delta$ when load changes sign ($0 \leq \lambda_p \leq 1$: pinching parameter). The differential equation (4) is modified into

$$\begin{aligned} \dot{z} = \dot{x} \bigg\{ & \overline{N}(\dot{x}) [\overline{M}(z - \lambda_p \delta) \overline{M}(x) + \overline{M}(z - \delta) \overline{N}(x)] \\ & + M(\dot{x}) [N(z + \lambda_p \delta) N(x) + N(z + \delta) M(x)] \bigg\}. \end{aligned} \quad (5)$$

Other phenomena such as stiffness-hardening pinching, stiffness degradation and load deterioration can also be modelled by modifying Equations (2) and (5), see reference [10]. No detail is given here because these phenomena are not exhibited by unsheathed x-braced frames.

2.2.2 The modified bilinear model

We have developed a modified bilinear model to describe the hysteretic behaviour of x-braced frames. Two basic features have been added to the bilinear model just described: (1) no compression capacity of diagonal straps and (2) two different sources of stiffness (frame and diagonal straps).

Diagonal strap without hardening

Diagonal straps are very slender so they have a very low compression capacity, which is neglected here. Under tension, on the other hand, they exhibit an elastoplastic response. Let us first consider the case of perfect plasticity (i.e. $\alpha = 0$ in Figure 4). We can model the unidirectional behaviour with the mechanical device of Figure 6(a) (for clarity, the damper is omitted in this and subsequent figures).

Figure 6. No-compression perfectly-plastic-tension system (diagonal strap without hardening): (a) mechanical device; (b) restoring force-displacement law

Table 1

Hysteretic model of a no-compression perfectly-plastic-tension system

Equation of dynamics	$m\ddot{x} + c\dot{x} + kz = f(t)$
Boundary displacement	$\dot{x}_b = \dot{x} \left\{ 1 - N(x_b) [1 - \overline{M}(z)M(\dot{x})] \right\}$
Elongation	$\dot{z} = \dot{x}N(x_b) \left\{ \overline{N}(z) [M(z - \delta) + N(z - \delta)M(\dot{x})] + \overline{M}(z)N(\dot{x}) \right\}$

Note the unidirectional connection of the mass-spring-slider system to the wall, which offers no resistance if the mass moves to the left. This means that negative elongations z are not possible in this device (i.e. no compression). Under tension (that is, for positive elongations z), this system has elastic stiffness k up to the maximum elongation $z = \delta$, and the restoring force is $F(z) = kz$.

The unidirectional connection is modelled by an additional variable, x_b (boundary displacement). Note that, with the sign criteria indicated in Figure 6(a), $x_b \leq 0$.

We need an evolution equation for x_b reflecting the following response:

- (1) If there is a gap in the boundary ($x_b < 0$), the system moves rigidly, either to the left or to the right ($\dot{x}_b = \dot{x}$);
- (2) If there is no gap ($x_b = 0$) and the spring is under tension ($z > 0$), then no gap will be created ($\dot{x}_b = 0$), because tension must be released first;
- (3) If there is no gap ($x_b = 0$) and no tension in the spring ($z = 0$), then (i) a gap is created if the system moves to the left ($\dot{x}_b = \dot{x}$ for $\dot{x} < 0$) or (ii) no gap is created if the system moves to the right ($\dot{x}_b = 0$ for $\dot{x} > 0$).

Such a response is provided by equation

$$\dot{x}_b = \dot{x} \left\{ M(x_b) + N(x_b) \overline{M}(z) M(\dot{x}) \right\} = \dot{x} \left\{ 1 - N(x_b) [1 - \overline{M}(z) M(\dot{x})] \right\}. \quad (6)$$

Regarding elongation z , we have modified the evolution equation (4) to take into account the rigid motion ($\dot{z} = 0$) of the system if there is a gap at the boundary ($x_b < 0$) or if there is no gap but it is about to be created ($x_b = 0$, $z = 0$ and $\dot{x} < 0$):

$$\dot{z} = \dot{x} N(x_b) \left\{ \overline{N}(z) [M(z - \delta) + N(z - \delta) M(\dot{x})] + \overline{M}(z) N(\dot{x}) \right\}. \quad (7)$$

The model for a no-compression perfectly-plastic-tension system is summarized in Table 1. Figure 6(b) shows an hysteretic cycle $ABCDEFGH$. Note the slackness caused by plastification and no compression (i.e. reloading with elastic stiffness occurs at $F \equiv D$, not at $F \equiv A$).

Diagonal strap with hardening

We can add hardening to the system by adding a second spring attached directly to the unilateral connection, see Figure 7(a). The restoring force is

Figure 7. No-compression hardening-plastic-tension system (diagonal strap with hardening): (a) mechanical device; (b) restoring force-displacement law

$F(y) = ky$, where $y = \alpha u + (1 - \alpha)z$ is the weighted average of the two spring elongations u and z .

The evolution equations for x_b and z are modified to take into account that now the weighted elongation y , rather than z , controls the sign of the restoring force. In addition, we need an evolution equation for u reflecting the no-compression response of the two-spring system. The complete model is summarized in Table 2. Figure 7(b) shows an hysteretic cycle $ABCDEFGH$. Due to hardening, branches BC and GH have slope αk .

Table 2

Hysteretic model of a no-compression hardening-plastic-tension system

Equation of dynamics $m\ddot{x} + c\dot{x} + ky = f(t)$ with $y = \alpha u + (1 - \alpha)z$

Boundary displacement $\dot{x}_b = \dot{x} \left\{ 1 - N(x_b) [1 - \overline{M}(y)M(\dot{x})] \right\}$

Elongations:

Spring with slider $\dot{z} = \dot{x}N(x_b) \left\{ \overline{N}(y) [M(z - \delta) + N(z - \delta)M(\dot{x})] + \overline{M}(y)N(\dot{x}) \right\}$

Spring without slider $\dot{u} = \dot{x}N(x_b) \left\{ 1 - \overline{M}(y)M(\dot{x}) \right\}$

X-braced frame

An x-braced frame can be idealized as the mechanical device of Figure 8. The mass is connected to two no-compression plastic-tension systems, see Figure 7, representing the two diagonal straps, and to a spring of stiffness βk which models the frame without diagonals. Parameter β represents the fraction of the total stiffness of the x-braced frame due to the frame itself, without the diagonals (i.e. $\beta = 0$ for a pinned frame).

Figure 8. Mechanical device representing an x-braced frame consisting of two no-compression plastic-tension systems (diagonal straps) and one elastic system (frame without diagonals)

The restoring force is given by

$$F(x, y_1, y_2) = \beta kx + \frac{(1-\beta)}{2}ky_1 - \frac{(1-\beta)}{2}ky_2, \quad (8)$$

where $y_i = \alpha u_i + (1-\alpha)z_i$ ($i = 1, 2$) is the averaged elongation of strap i . The hysteretic model for the x-braced frame is summarized in Table 3. The fact that $x_{b1} \leq 0$ but $x_{b2} \geq 0$, see the sign criteria in Figure 8, causes the differences in the models for strap 1 and strap 2.

Table 3

Hysteretic model of an x-braced frame

Equation of dynamics	$m\ddot{x} + c\dot{x} + k\left[\beta x + \frac{(1-\beta)}{2}(y_1 - y_2)\right] = f(t)$ <p>with $y_i = \alpha u_i + (1-\alpha)z_i$ ($i = 1, 2$)</p>
Diagonal strap 1	$\dot{x}_{b1} = \dot{x}\left\{1 - N(x_{b1})[1 - \overline{M}(y_1)M(\dot{x})]\right\}$ $\dot{z}_1 = \dot{x}N(x_{b1})\left\{\overline{N}(y_1)[M(z_1 - \delta) + N(z_1 - \delta)M(\dot{x})] + \overline{M}(y_1)N(\dot{x})\right\}$ $\dot{u}_1 = \dot{x}N(x_{b1})\left\{1 - \overline{M}(y_1)M(\dot{x})\right\}$
Diagonal strap 2	$\dot{x}_{b2} = \dot{x}\left\{1 - \overline{M}(x_{b2})[1 - \overline{M}(y_2)\overline{N}(\dot{x})]\right\}$ $\dot{z}_2 = -\dot{x}\overline{M}(x_{b2})\left\{\overline{N}(y_2)[M(z_2 - \delta) + N(z_2 - \delta)\overline{N}(\dot{x})] + \overline{M}(y_2)\overline{M}(\dot{x})\right\}$ $\dot{u}_2 = -\dot{x}\overline{M}(x_{b2})\left\{1 - \overline{M}(y_2)\overline{N}(\dot{x})\right\}$

Figure 9 shows an hysteretic cycle $ABCDEFGH$ of the x-braced frame. Three different values of the stiffness are clearly visible: $k[\beta + (1-\beta)/2]$ in branches AB , CD , EF and GH (one elastic strap + frame without straps); $k[\beta + \alpha(1-$

Figure 9. Hysteretic cycle for an x-braced frame

$\beta)/2]$ in branches BC and FG (one plastified strap + frame without straps); βk in branches DE and HA (frame without straps). The frame never exhibits its total stiffness k , because the two diagonal straps are never simultaneously under tension.

2.3 Comparison between SDOF and MDOF approaches

We compare here the seismic response of the x-braced frame of Figure 2 obtained with the two approaches discussed above: (1) by modelling it as a MDOF system (Section 2.1) and (2) by assuming it behaves as the SDOF system of Figure 8 (Section 2.2). The frame is 2.75 m high and 1.2 wide. Studs, tracks and straps are described in Table 4.

Table 4

Dimensions of studs, tracks and diagonal straps of the unsheathed shear wall

	<i>profile</i>	<i>web</i> (mm)	<i>flange</i> (mm)	<i>lip</i> (mm)
<i>Stud</i>	C-profile	150×1	50×1	15×1
<i>Track</i>	U-profile	154×1	50×1	—
<i>X-brace</i>	Strap	100×1	—	—

First of all, we present in Section 2.3.1 a simple validation example with analytical solution: all structural members remain elastic and buckling of the diagonals is the only source of nonlinearity. The second example, see Section 2.3.2, takes into account plastification of diagonal straps.

2.3.1 Elastic case

In this example, elastic behaviour (with a Young modulus $E = 2.1 \times 10^5$ MPa) is assumed for all the structural members, including straps. The only source of nonlinearity is the no-compression response of diagonal straps, represented by a bilinear stress-strain curve (see Figure 3).

The frame is pinned (zero bending stiffness at track-stud and strap-stud joints) and axial joints between straps and studs are semirigid, with a stiffness of 10 kN/mm. Mass at the top of the frame is $m = 300$ kg. It is submitted to a sinusoidal horizontal load in its left shoulder $f(t) = P_0 \sin(\bar{\omega}t)$, with amplitude $P_0 = 25$ kN and frequency $\bar{\omega} = 90$ rad/s. Damping is neglected.

We have computed the horizontal displacement of the frame with the two approaches previously discussed, see Figure 10(a).

When we model the x-braced frame as a MDOF system, we use the nonlinear finite element code CAST3M [15] to solve the system of ODEs (1). The frame is discretized with 28 finite elements and the Newmark method is used to integrate in time, with a time-step of $\Delta t = 0.1$ s (566 time-steps).

For the SDOF approach we need to solve the system of ODEs of Table 3. The corresponding parameters are summarized in Table 5. Note that $\alpha = 1$ and δ is not defined because the response is elastic, and that the stiffness ratio is $\beta = 0$ because the frame is pinned. The stiffness k is obtained from a simple linear static analysis (as the ratio between the applied horizontal load and the horizontal displacement).

Table 5

Parameters for the modified bilinear model in the elastic case

Symbol	Meaning	Value
m	Mass	300 kg
c	Damping	0
k	Stiffness	835 kN/m
α	Hardening ratio	1
δ	Yielding displacement	—
β	Stiffness ratio	0

To solve the system of seven ODEs, we use the Runge-Kutta-Felberg 45 method [16] with command `ode45` in Matlab. It is a fourth order method with adaptive time-step size based on error control. An error tolerance of 10^{-6} is prescribed. We can see in Figure 10(b) how the time-step indeed varies with time. Note that very small time-steps are needed whenever the displacement changes sign (i.e. when diagonal straps switch from elastic tension to no-

compression or vice versa). A total of 813 non-uniform time-steps are needed.

An analytical solution of the SDOF system can be obtained very simply upon noting that the stiffness is constant and equal to $k/2$ (i.e. stiffness of frame with *one* diagonal strap), so the SDOF system is linear: due to the no-compression elastic-tension response of straps, there is no slackness and there is always one of the two straps under tension. The analytical solution reads [17]

$$x(t) = \frac{P_0}{m\omega^2} \frac{1}{1 - (\bar{\omega}/\omega)^2} \left\{ \sin(\bar{\omega}t) - \frac{\bar{\omega}}{\omega} \sin(\omega t) \right\}, \quad (9)$$

where $\omega = \sqrt{(k/2)/m} = 37.3$ rad/s is the frequency of the linear SDOF system.

The two numerical solutions and the analytical solution are compared in Figure 10(a). We can see that there is an excellent agreement between the three curves, so we can conclude that the x-braced frame indeed behaves like a SDOF system.

Figure 10. Dynamic response of an elastic system: (a) displacement-time law for the MDOF approach, SDOF approach and analytical solution; (b) evolution of time-step in time for the SDOF approach

2.3.2 Plastic case

In this second example, plastic dissipation is taken into account. Hardening elastoplasticity (with a yield stress $\sigma_y = 250$ MPa and a plastic modulus $E_p = E/20$) is assumed for diagonal straps under tension. The rest of the members remain elastic. The second difference with respect to the previous example is that stud-track connections are rigid, so the x-braced frame has two sources of stiffness ($\beta \neq 0$): the frame itself and the x-brace.

With the MDOF approach, the same numerical parameters of the elastic case

are used (i.e. a mesh of 28 finite elements and a time-step $\Delta t = 0.1$ s). The parameters for the SDOF approach are summarized in Table 6. The hardening ratio α is computed as the ratio of plastic to elastic modulus, $\alpha = E_p/E = 0.05$. The stiffness ratio β and the yielding displacement δ can be computed with two linear static analyses. An error tolerance of 10^{-6} is prescribed in command `ode45` of Matlab. This results in 889 time-steps.

Table 6

Parameters for the modified bilinear model in the plastic case

Symbol	Meaning	Value
m	Mass	300 kg
c	Damping	0
k	Stiffness	945.5 kN/m
α	Hardening ratio	0.05
δ	Yielding displacement	22 mm
β	Stiffness ratio	1.85×10^{-2}

The two approaches are compared in Figure 11. There is a good agreement both in terms of the evolution of displacements with time, Figure 11(a), and in the force-displacement curve, Figure 11(b).

Figure 11. Comparison between MDOF and SDOF responses: (a) evolution of displacements in time and (b) force-displacement curves

The SDOF approach is computationally more efficient than the MDOF approach. Despite the higher order of time-integration (order 4 in the RKF 45 method vs. order 2 in the Newmark method), the significant reduction in the number of equations leads to a factor of about 10 in the computational cost in favour of the SDOF approach.

To sum up: the SDOF approach based on the modified bilinear model is a simple, accurate and computationally efficient way to compute the seismic response of x-braced frames. In the remainder of this paper we will illustrate the applicability of this approach with two cases of practical interest: the evaluation of the q -factor and the coupling of various frames.

3 Evaluation of the q -factor of x-braced frames

In common engineering practice, seismic analysis is carried out by means of response spectra [17]. This technique relies on modal analysis and only maximum responses are obtained. Since it is based on the principle of superposition, this approach is only valid for linear analysis. To take into account nonlinear phenomena such as plasticity or buckling, a nonlinear dynamic analysis would be needed.

However, these type of nonlinear analyses are not common in the practical design of structures. Instead, energy dissipation due to plasticity is taken into account by scaling the elastic response spectra by a reduction factor, the q -factor, which depends both on the structure and the loading. It is defined as the ratio

$$q = \frac{V_{\max}^{\text{elas}}}{V_{\max}^{\text{plas}}} \quad (10)$$

where $V_{\max}^{\text{plas}}$ is the maximum shear at the basis of the structure during the loading history and $V_{\max}^{\text{elas}}$ is the hypothetical maximum shear if the response was linear.

The two approaches discussed in Section 2 can be used to estimate the q -factor. If the x-braced frame is modelled as a MDOF system (finite element method), the procedure is:

- (1) Perform a nonlinear dynamic analysis of the frame, as described in Section 2.1, and obtain the maximum shear at basis $V_{\max}^{\text{plas}}$.
- (2) Perform a linear dynamic analysis of the frame, assuming elastic behaviour in all the structural members and including only one diagonal strap (so that the total stiffness of the frame is correctly reproduced), working both in tension in compression. Obtain the maximum shear at basis $V_{\max}^{\text{elas}}$.
- (3) Evaluate the q -factor as $q = V_{\max}^{\text{elas}}/V_{\max}^{\text{plas}}$.

If the x-braced frame is treated as a SDOF system, the procedure is very similar:

- (1) Obtain the nonlinear response of the frame by solving the system of ODEs

in Table 3. The maximum shear at basis $V_{\max}^{\text{plas}}$ is given by the maximum absolute value of

$$V^{\text{plas}} = k \left[\beta x + \frac{(1 - \beta)}{2} (y_1 - y_2) \right]. \quad (11)$$

- (2) To obtain the linear response of the system, solve the ordinary differential equation

$$m\ddot{x} + c\dot{x} + k'x = f(t), \quad (12)$$

where k' is the stiffness of the frame with only one diagonal strap (i.e. $k' = k(\beta + 1)/2$). The maximum shear at basis is computed as the maximum absolute value of

$$V^{\text{elas}} = k'x. \quad (13)$$

- (3) Again, the q -factor is evaluated as the ratio $q = V_{\max}^{\text{elas}}/V_{\max}^{\text{plas}}$.

Application example

The value of the q -factor of an x-braced frame has been evaluated with the two procedures just described. The dimensions of the members are those indicated in Table 4 (except for thickness of studs and tracks, which is equal to 1.5 mm instead of 1 mm). Besides, the frame has two sets of x-bracings. Bending connections are pinned and diagonal straps are elastic perfectly-plastic in tension and do not work in compression.

In order to simulate the behaviour of the frame if placed in a complete 13.2 m long façade, the mass at the top of the frame has been set equal to $m = 4400$ kg. The damping ratio has been assumed equal to zero. The frame has been submitted to the accelerogram of Figure 12, obtained from [18].

Figure 12. Accelerogram of the Gazli earthquake (1976) scaled by a factor 0.52

The parameters for the modified bilinear model (SDOF approach) are $\alpha = 0$ (perfect plasticity), $\beta = 0$ (pinned connections), yielding displacement $\delta = 12.4$ mm and stiffness $k = 3301$ kN/m.

The responses (displacement in time) obtained with the two approaches are compared in Figure 13, both for the linear case, Figure 13(a), and the non-linear case, Figure 13(b). The good agreement between the two curves in the two cases clearly illustrates that the SDOF approach is indeed a good approximation to predict the seismic response of the unsheathed shear wall.

Figure 13. Evaluation of the q -factor. Displacement-time responses for the SDOF and MDOF approaches: (a) linear case; (b) nonlinear case

Note that, for the SDOF system, the shear forces are simply the restoring forces, see Equations (11) and (13). These forces are plotted as a function of displacement in Figure 14. The ratio between the peak values results in a q -factor of 1.87.

With the MDOF approach, we have obtained a very similar value, $q = 1.73$, but the computational cost (especially to obtain the nonlinear response) has been quite higher. Since our goal is to estimate the q -factor for different x-braced walls and different accelerograms, the reduction in computational cost of the modified bilinear model (SDOF approach) is a relevant factor.

4 Coupling of shear walls

It has been shown in Section 2 how the dynamic response of a shear wall can be predicted by solving a small system of ordinary differential equations. Once the differential model is calibrated, experimental results can be easily

Figure 14. Evaluation of the q -factor. Force-displacement laws obtained with the SDOF approach.

extended to other loading cases.

Another case of interest is the extension of the experimental results to more complex structural units. Thus, we can think on predicting the seismic response of a whole façade with more than one x-braced frame.

4.1 Two shear walls

Consider the façade of Figure 15, consisting of two shear walls and submitted to an accelerogram at its basis, $a_g(t)$. The behaviour of each frame is governed by

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1\left[\beta_1x_1 + \frac{(1-\beta_1)}{2}(y_1^1 - y_1^2)\right] = -m_1a_g(t) \quad (14)$$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2\left[\beta_2x_2 + \frac{(1-\beta_2)}{2}(y_2^1 - y_2^2)\right] = -m_2a_g(t) \quad (15)$$

The hysteretic response is controlled by strap elongations. The averaged elongation of strap j in frame i ($i, j = 1, 2$) is defined as $y_i^j = \alpha_i u_i^j + (1 - \alpha_i)z_i^j$, see Table 3.

A compatibility condition is required to couple the two SDOF systems of Figure 15(b). If we assume that the horizontal displacement of the two frames

Figure 15. Façade with two shear walls submitted to an accelerogram: (a) geometry of the façade; (b) idealization of the structure

is the same, $x_1 = x_2 \equiv x$, Equations (14) and (15) can be added into

$$(m_1 + m_2)\ddot{x} + (c_1 + c_2)\dot{x} + (\beta_1 k_1 + \beta_2 k_2)x + \frac{1 - \beta_1}{2} k_1 (y_1^1 - y_1^2) + \frac{1 - \beta_2}{2} k_2 (y_2^1 - y_2^2) = -(m_1 + m_2)a_g(t). \quad (16)$$

The dynamic response of the two SDOF system is given by the equation of dynamics (16) complemented with equations

$$\dot{x}_{b_i}^j = \dot{x} \left\{ 1 - N(x_{b_i}^j) \left[1 - \overline{M}(y_i^j) M(\dot{x}) \right] \right\} \quad (17)$$

$$\dot{z}_i^j = \dot{x} N(x_{b_i}^j) \left\{ \overline{N}(y_i^j) \left[M(z_i^j - \delta_i) + N(z_i^j - \delta_i) M(\dot{x}) \right] + \overline{M}(y_i^j) N(\dot{x}) \right\} \quad (18)$$

$$\dot{u}_i^j = \dot{x} N(x_{b_i}^j) \left\{ 1 - \overline{M}(y_i^j) M(\dot{x}) \right\} \quad (19)$$

for $(i, j = 1, 2)$, where $\dot{x}_{b_i}^j$, $\dot{z}_i^j$ and $\dot{u}_i^j$ are the boundary displacement and spring elongations associated to strap j of frame i , see Table 3.

Application example

The façade of Figure 16 is submitted to a sinusoidal accelerogram $a_g(t) = A \sin(\bar{\omega}t)$, with frequency $\bar{\omega} = 75$ rad/s and amplitude $A = -30$ m/s². The total mass at the top of the façade is $m = 2100$ kg, so the maximum amplitude of the applied load is $P = 63$ kN.

The façade consists of seven pinned frames. The two frames at the edges are x-braced, with rigid axial strap-frame connections and elastic-perfectly plastic straps. All the studs are C150×1.5 profiles and the tracks are U154×1.5. Characteristics of the two x-braced frames are summarized in Table 7.

Since the horizontal response of the façade is governed by the x-braced frames,

Figure 16. Façade with two x-braced frames

Table 7

Properties of the x-braced frames

	<i>stud</i>	<i>track</i>	<i>strap width</i>	<i>strap thickness</i>
<i>frame 1</i>	C150×1.5	U154×1.5	65 mm	1.0 mm
<i>frame 2</i>	C150×1.5	U154×1.5	150 mm	1.5 mm

we can model it as a system of two coupled shear walls, see Figure 15, each of them with a mass $m_i = 1050$ kg.

The seismic response of the façade has been obtained in two different ways: a finite element analysis of the whole façade (MDOF approach) and the SDOF approach, Equations (16)-(19), with the parameters of Table 8.

Table 8

Parameters for the modified bilinear model of the coupled x-braced frames

Symbol	Meaning	Value frame 1	Value frame 2
m_i	Mass	1050 kg	1050 kg
c_i	Damping	0	0
k_i	Stiffness	1243.3 kN/m	3056.1 kN/m
α_i	Hardening ratio	0	0
δ_i	Yielding displacement	10.5 mm	15.0 mm
β_i	Stiffness ratio	0	0

The results are compared in Figure 17. Again, there is a good agreement between the two approaches. Note that the seismic response of the façade is indeed controlled by the two x-braced frames: a very similar behaviour is obtained with the SDOF approach, which only takes into account these two frames, and the finite element model, in which the whole structure is discretized.

Note the change in the slope of the force-displacement law of the façade in Figure 17(b). It is due to the fact that frame 1 gets into the plastic range before frame 2, so the initial stiffness of the façade is reduced after reaching a displacement of 10.5 mm. Besides, we can clearly see from Figures 17(c) and 17(d) the different load carrying capacity of both frames: frame 1, which has smaller diagonal straps, gets a smaller fraction of the total horizontal load than frame 2.

In this case, the computational efficiency of the SDOF approach over the FE model is a more relevant factor than in the modelling of one x-braced frame: since here the structure is more complex and more structural members are involved in the finite element modelling, the difference in computational cost between both approaches is much more significant.

4.2 Extension to n shear walls

The coupling strategy just discussed for two shear walls can be extended in a straightforward manner to the case of n shear walls, see Figure 18. If we assume that all the frames have the same horizontal displacement, $x_1 = x_2 = \dots = x_n \equiv x$ (i.e. axial deformation of the top track neglected), then the equations of dynamics for each frame can be summed up into

$$\left(\sum_{i=1}^n m_i\right) \ddot{x} + \left(\sum_{i=1}^n c_i\right) \dot{x} + \left(\sum_{i=1}^n \beta_i k_i\right) x + \sum_{i=1}^n \frac{1 - \beta_i}{2} k_i (y_i^1 - y_i^2) = - \sum_{i=1}^n m_i a_g(t). \quad (20)$$

The hysteretic responses of the frames are given by Equations (17)-(19), with $i = 1, \dots, n$ (n frames) and $j = 1, 2$ (two straps per frame).

5 Concluding remarks and future work

We have presented a differential model of the hysteretic behaviour of un-sheathed x-braced frames. Taking the bilinear model of Mostaghel [10] as our starting point, we have incorporated two key features of x-braced frames: slackness due to buckling of diagonal straps and two different sources of stiffness.

With this hysteretic model, the frame can be treated as a SDOF system, and its hysteretic response is modelled by a small system of ordinary differential equations (ODEs).

Figure 17. Seismic response of a whole façade obtained with the SDOF approach and the finite element method: (a) displacement-time of the complete façade; (b) force-displacement at the façade; (c) force-displacement for frame 1; (d) force-displacement for frame 2

We have compared this SDOF approach with a more sophisticated, MDOF approach, based on a finite element model of the x-braced frame. Our numerical results indicate a very good agreement between the two models, both at the qualitative level (e.g. extreme pinching and slackness in the hysteretic cycles caused by buckling of diagonal straps) and the quantitative level (e.g. estimation of the q -factor for simplified seismic analysis based on reduction of the elastic response spectra).

In comparison with finite element analysis, the proposed SDOF approach has a number of attractive computational features: (1) the computational cost is

Figure 18. Façade with n shear walls submitted to an earthquake: (a) geometry of the façade; (b) idealization of the structure

much lower, (2) the order of time-integration (and thus the time accuracy) is higher, and (3) the resulting small system of ODEs can be solved with any standard software like Matlab, Maple or Mathematica, with no need of a nonlinear finite element code.

With the seismic response of light-gauge steel structures in mind, we are already working in a number of related topics:

- The hysteretic modelling of sheathed shear walls by means of the BWN model [9];
- The analysis of complete façades consisting of various (sheathed or unsheathed) shear walls of different type (standard, x-braced, door- or window-frame, etc.)
- An automatic procedure for the calibration of the modified bilinear model (for unsheathed shear walls) and the BWN model (for sheathed shear walls) by means of an experimental testing campaign.

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