where $\hat{\eta} \in \mathbb{R}^m$; $\mathbf{Q} \in \mathbb{R}^{m \times m}$ is non-Hurwitz; $\theta(\cdot) \in \mathbb{R}^m$ is a causal forcing term described by the *unknown* LTI system of differential (12).

The vector $g(t) \in \mathbb{R}^m$ is constrained by

$$\begin{aligned} \hat{\eta}^{(k)} + p_{k-1}\hat{\eta}^{(k-1)} + \dots + p_1\dot{\hat{\eta}} + p_0\hat{\eta} \\ = \mathbf{F}_{k-1}g(t)^{(k-1)} + \dots + \mathbf{F}_1\dot{g}(t) + \mathbf{F}_0g(t) \end{aligned} \quad (36)$$

where the numbers $p_0, p_1, \dots, p_{k-1}$ are the coefficients of the characteristic polynomial (17), and the matrices $\mathbf{F}_{k-1}, \dots, \mathbf{F}_1, \mathbf{F}_0 \in \mathbb{R}^{m \times m}$ are to be selected.

One can uncouple (35) and (36) with respect to $g(t)$ as

$$\begin{aligned} (\mathbf{I} - \mathbf{F}_{k-1})g(t)^{(k)} + (p_{k-1}\mathbf{I} + \mathbf{Q}\mathbf{F}_{k-1} - \mathbf{F}_{k-2})g(t)^{(k-1)} \\ + \dots + (p_1\mathbf{I} + \mathbf{Q}\mathbf{F}_1 - \mathbf{F}_0)\dot{g}(t) + (p_0\mathbf{I} + \mathbf{Q}\mathbf{F}_0)g(t) \\ = -\left(\theta^{(k)} + p_{k-1}\theta^{(k-1)} + \dots + p_1\dot{\theta} + p_0\theta\right) \end{aligned} \quad (37)$$

and

$$\begin{aligned} (\mathbf{I} - \mathbf{F}_{k-1})\hat{\eta}^{(k)} + (p_{k-1}\mathbf{I} + \mathbf{F}_{k-1}\mathbf{Q} - \mathbf{F}_{k-2})\hat{\eta}^{(k-1)} \\ + \dots + (p_1\mathbf{I} + \mathbf{F}_1\mathbf{Q} - \mathbf{F}_0)\dot{\hat{\eta}} + (p_0\mathbf{I} + \mathbf{F}_0\mathbf{Q})\hat{\eta} \\ = -\left(\mathbf{F}_{k-1}\theta^{(k-1)} + \dots + \mathbf{F}_1\dot{\theta} + \mathbf{F}_0\theta\right). \end{aligned} \quad (38)$$

Since the term θ can be piece-wise modeled by an unknown linear exosystem (12) of known order k with characteristic polynomial (17), and the polynomial coefficients can be identified online, then

$$\theta^{(k)} + p_{k-1}\theta^{(k-1)} + \dots + p_1\dot{\theta} + p_0\theta \equiv 0$$

almost everywhere. The matrices $\mathbf{F}_0, \mathbf{F}_1, \dots, \mathbf{F}_{k-1} \in \mathbb{R}^{m \times m}$ are selected to provide $g \rightarrow 0$ asymptotically by enforcing the following equality:

$$\begin{aligned} g(t)^{(k)} + (\mathbf{I} - \mathbf{F}_{k-1})^{-1}(p_{k-1}\mathbf{I} + \mathbf{Q}\mathbf{F}_{k-1} - \mathbf{F}_{k-2})g(t)^{(k-1)} \\ + \dots + (\mathbf{I} - \mathbf{F}_{k-1})^{-1}(p_1\mathbf{I} + \mathbf{F}_1\mathbf{Q} - \mathbf{F}_0)\dot{g}(t) \\ + (\mathbf{I} - \mathbf{F}_{k-1})^{-1}(p_0\mathbf{I} + \mathbf{F}_0\mathbf{Q})g(t) \\ = g(t)^k + C_{k-1}g(t)^{(k-1)} + \dots + C_1\dot{g}(t) + C_0g(t) \end{aligned} \quad (39)$$

where the coefficients $C_0, C_1, \dots, C_{k-1}$ are selected to provide any desired convergence rate of $g(t) \rightarrow 0$.

Matrices $\mathbf{F}_{k-1}, \dots, \mathbf{F}_1, \mathbf{F}_0 \in \mathbb{R}^{m \times m}$ are calculated by equating similar terms on the left and right hand sides of (39) as

$$\begin{aligned} \mathbf{F}_0 &= [\mathbf{I} - \mathbf{F}_{k-1}](C_0\mathbf{Q}^{-1}) - (p_0\mathbf{Q}^{-1}) \\ \mathbf{F}_1 &= [\mathbf{I} - \mathbf{F}_{k-1}](C_1\mathbf{Q}^{-1} + C_0\mathbf{Q}^{-2}) - (p_1\mathbf{Q}^{-1} + p_0\mathbf{Q}^{-2}) \\ &\vdots \\ \mathbf{F}_{k-1} &= [\mathbf{I} - \mathbf{F}_{k-1}]\left(C_{k-1}\mathbf{Q}^{-1} + \dots + C_0\mathbf{Q}^{-(k-1)}\right) \\ &\quad - \left(p_{k-1}\mathbf{Q}^{-1} + \dots + p_0\mathbf{Q}^{-(k-1)}\right) \end{aligned} \quad (40)$$

from where

$$\begin{aligned} \mathbf{F}_{k-1} &= \mathbf{I} - (\mathbf{I} + p_{k-1}\mathbf{Q}^{-1} + \dots + p_0\mathbf{Q}^{-k}) \\ &\quad \times (\mathbf{I} + C_{k-1}\mathbf{Q}^{-1} + \dots + C_0\mathbf{Q}^{-k})^{-1}. \end{aligned} \quad (41)$$

Substituting (40) and (41) into (38) we obtain (19) and (20). As soon as the term $g(t)$ converges to zero, the two linear differential equations

$$\dot{\hat{\eta}} = \mathbf{Q}\hat{\eta} + \theta(\cdot) + g(t), \quad \text{and} \quad \dot{\eta} = \mathbf{Q}\eta + \theta(\cdot)$$

will have an identical structures and the same forcing term $\theta(\cdot)$ which proves the convergence $\hat{\eta} \rightarrow \eta$ as claimed. The theorem is proven.

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Differential Flatness of a Class of n -DOF Planar Manipulators Driven by 1 or 2 Actuators

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Abstract—A fully actuated system can execute any joint trajectory. However, if a system is under-actuated, not all joint trajectories are attainable. In recent years, the authors have actively pursued novel designs of under-actuated robotic arms which are both controllable and feedback linearizable. These robots can perform point-to-point motions in the state space, but potentially can be designed to work with fewer actuators, hence with lower cost. With this same spirit, the technical note investigates the property of *differential flatness* for a class of planar under-actuated open-chain robots having a specific inertia distribution, but driven by only one or two actuators. This technical note addresses the following theoretical question: *What placement of one or two actuators will make an n -DOF planar robot differentially flat if it is designed so that its center of mass always lies at joint 2?*

Index Terms—Differential flatness, robot design, under-actuated robots.

I. INTRODUCTION

Recent statistics from the Robot Industry Association (RIA) show that there are more than one million industrial robots in operation today.

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Robots are used in repetitive point-to-point tasks required for inspection, manufacturing, and assembly. Typically, these robots are fully-actuated, i.e., have as many actuators as the degrees-of-freedom. A large fraction of the cost of a robot is its actuators and drive electronics. Hence, it is beneficial to explore alternative designs that require fewer actuators, but can still perform point-to-point tasks commonly required in manufacturing applications.

A fully-actuated robot can execute any joint trajectory. If some actuators are taken away, the “controllability” of the system may be lost. However, if the system is shown to be “differentially flat,” it can still perform point-to-point maneuvers, despite having fewer actuators ([1], [2]). These systems possess both “controllability” and “feedback linearizability” despite reduced number of actuators. It is important to note that point-to-point maneuverability is an important feature to have in a robot for execution of repetitive industrial tasks. However, these robots still will not be able to track arbitrary trajectories unless they are compatible with under-actuated dynamics.

In general, *all nonlinear dynamic systems do not satisfy the conditions of differential flatness. However, it is possible to achieve this property through design, i.e., by altering the inertia distribution within the robot.* Such a design philosophy can directly impact the *cost* of robots.

A three DOF prototype was recently constructed at the University of Delaware, using this design philosophy and actuated by one or two actuators. *This robot showed satisfactory point-to-point motions in the state space* [3]. This prototype is driven by two actuators and is shown in Fig. 1. In this three link prototype, the center of mass of link 3 is on joint 3 and the center of mass of links 2 and 3 is on joint 2. This was realized by placing the motors at appropriate locations on the links so that these design conditions were met. There are movable counter masses on the last two links which fine tune the location of the center of mass within the links since the design conditions are hard to satisfy in practice. A torsional spring is placed at joint 3. All three joints have incremental encoders to measure the angular positions. The first two joints have dc motors with torque sensors mounted at the output shaft of the gearboxes. Steel shafts a long with roller bearings were used to make low friction and rigid revolute joints. The robot is made out of delrin and weighs 8 Kg. National Instruments’ software LabVIEW and a real-time PXI target is used to implement the controller.

The authors have outlined a set of sufficient conditions under which an under-actuated n -DOF planar arm exhibits differential flatness having a specific inertia distribution given by ([3], [4]): (i) the center of mass of the last link n is made to lie on the joint axis n , (ii) the center of mass of the last two links n and $n - 1$ lies on the joint axis $n - 1$, (iii) this procedure repeats until the center of mass of the last j links, i.e., $n, n - 1, \dots, n - j + 1$ is on the joint axis $n - j + 1$. Such a system is differentially flat if the actuators are placed on the first $n - j + 1$ joints, i.e., $u = [u_1, u_2, \dots, u_{n-j+1}, 0, \dots, 0]^T$ and the last $j - 1$ joints have torsional springs. In such a design, *under-actuation can be as high as $n - 1$* . This result is *significantly different from those in the literature where studies focus on under-actuation by 1 or 2*. The design shown in Fig. 1 is a specific case, where $n = 3$ and $j = 2$.

While the physical prototype and conducted experiments give us confidence that the differentially flat design philosophy is viable, a deeper theoretical question exists which provides the framework for this technical note: *For an n -DOF planar robot, with the center of mass placed on joint 2 through mechanical design, what choices of one or two actuator placements will make the system differentially flat?* Note that there could be up to n choices of one actuator and $n(n - 1)/2$ choices of two actuators.

It is important to point out the differences between our previous paper [3] and the results of this work. The previous paper only outlined a *sufficient condition* for a system under-actuated by $n - 1$ to be differentially flat. In contrast, in this technical note, *necessary and sufficient* conditions are provided for such a system to be differentially flat

Fig. 1. Prototype of a differentially flat 3-DOF planar robot. The center of mass of the third link is at joint three. The center of mass of links two and three is at joint two. The first two joints have dc motors with torque sensors.

under-actuated by $n - 1$. Moreover, in this technical note, we also find conditions when a system is under-actuated by $n - 2$ to be differentially flat.

A. Mathematical Background

Roughly speaking, a dynamic system is said to be differentially flat if and only if there exists m variables (equal in number to the number of inputs, the so-called flat outputs) such that any state variable and any input are functions of these flat outputs and their derivatives up to a certain order. The formal definition of differential flatness in the context of differential algebra was introduced by Fliess and coworkers [1]. In this framework, a system corresponds to a finitely generated differential field extension L/k , and a system is differentially flat if there exists a differential transcendence basis $y = (y_1, \dots, y_m)$ such that any variable of the system is algebraic (not differentially algebraic) over the differential field $k\langle y \rangle$. They also proved the equivalence between differential flatness and dynamic feedback linearization (by endogenous feedback).

Differential flatness has been studied in several mathematical frameworks such as differential algebra [5], finite dimensional differential geometry [6], and infinite dimensional differential geometry [7]. Differentially flat systems include controllable linear systems, static feedback linearizable systems, and systems that are linearizable by prolongations [8]. For all these cases, there does exist a necessary and sufficient condition to check flatness. Flat systems have been summarized in a recent book [2].

A nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i$$

is static feedback linearizable if and only if ([6], [9], [10]) the following distributions have constant rank and are involutive:

$$D_0 = \langle g_1, \dots, g_m \rangle$$

$$D_i = \langle D_{i-1}, ad_f^i g_1, \dots, ad_f^i g_m \rangle \quad i = 1, \dots, n - 2$$

and the rank of D_{n-1} is n . Given a nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i$$

a simple prolongation of this system is

$$\begin{aligned}\dot{x} &= f(x) + \sum_{i=1}^m g_i(x) u_i^0 \\ \dot{u}_i^0 &= u_i^1 \\ &\vdots \\ \dot{u}_i^{k_i-1} &= v_i, \quad \forall i = 1 \dots m\end{aligned}$$

where u_i^j , that correspond to $u_i^{(j)}$, are new state variables $\forall i = 1 \dots m$ $j = 0 \dots k_i - 1$ and the new inputs are v_i . The system is said to be linearizable by prolongations if there exists a simple prolongation of the original system which is static feedback linearizable. We use the terminology simple prolongation to avoid confusion with the general notion of (Cartan) prolongations. Simple prolongations are particular cases of the last ones.

For point-to-point maneuvers of a differentially flat system, the boundary conditions on the states are mapped on to boundary conditions on the flat outputs and their derivatives. A smooth flat output trajectory is then fitted through the end points using interpolation. The states and inputs can be computed from the flat outputs according to the diffeomorphic relation between the state variables and flat variables and their derivatives. Using the *linear chain of integrator* representation of the system, tracking controllers can be designed even in the presence of errors in the initial conditions [2].

II. DESCRIPTION OF THE PROBLEM

The dynamic equations of motion for open-chain robots are given by the following:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u \quad (1)$$

where $M(q)$ is the inertia matrix, $C(q, \dot{q})$ are the Coriolis and Centripetal terms, $g(q)$ are the gravity terms, and u is the vector of joint inputs. It is also well known from Lagrangian structure of the dynamic equations, if the inertia matrix $M(q)$ is independent of q , the Coriolis term $C(q, \dot{q}) = 0$.

As described in Section I, the analysis of this technical note is based on the assumption that the inertia distribution of the robot chain is such that the center of mass of the last $n - 1$ joints is placed *recursively* at joint 2, i.e., the center of mass of link n is at the joint $n - 1$, the center of mass of links n and $n - 1$ is at joint $n - 2$ and so on. This assumption was fulfilled by the design in Fig. 1, for $n = 3$ and $j = 2$, and the actuators were placed within the system such that these design conditions were met naturally. *It is important to note that this special inertia distribution does not affect the geometric workspace of the robot.*

With this assumption, the system dynamic equations in (1) take the following form: (i) the $(n \times n)$ inertia mass matrix $M(q)$ is a constant and has a specific structure given by

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_2 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & a_{n-1} & a_{n-1} & \cdots & a_{n-1} & a_n \\ a_n & a_n & a_n & \cdots & a_n & a_n \end{pmatrix} \quad (2)$$

where a_i are constants for all $i = 1, \dots, n$.

(ii) $C(q, \dot{q})$ vanishes since M is a constant, (iii) $g(q)$ is still nonlinear since the center of mass is on joint 2. In order to preserve controllability of the system, we add a spring in every un-actuated joint except if it occurs on joint 1. The addition of springs provides dynamic coupling between successive joints, which helps preserve controllability

of the robot. In summary, robots have the following dynamic equations of motion:

$$M\ddot{q} = \begin{pmatrix} -mg \sin q_1 + u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad (3)$$

where u_i is substituted by $k_i q_i$ if $u_i = 0$, for all $i \geq 2$. Note that this paper assumes $k_i < 0$.

One can rewrite the dynamic equations in the so-called state space form. If we denote $x = (q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$

$$\dot{x} = \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{2n} \\ M^{-1}b \end{pmatrix} + \sum_{i=1}^n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ M^{-1}e_i \end{pmatrix} u_i \quad (4)$$

where $b = (-mg \sin x_1, k_2 x_2, \dots, k_n x_n)^T \in R^n$, and e_i stands for the i th vector of the canonic base of R^n . Let us remark that only some of the u_i will be nonzero and as stated before, $k_i = 0$ if $u_i \neq 0$. A simple computation shows that

$$M^{-1} = \begin{pmatrix} \lambda_1 & -\lambda_1 & 0 & \cdots & 0 \\ -\lambda_1 & \lambda_1 + \lambda_2 & -\lambda_2 & \cdots & 0 \\ 0 & -\lambda_2 & \lambda_2 + \lambda_3 & \cdots & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{n-1} \\ 0 & 0 & 0 & \cdots & -\lambda_{n-1}\lambda_n \end{pmatrix}$$

where $\lambda_i = 1/a_i - a_{i+1}$, $\forall i < n$, $\lambda_n = -a_{n-1}/a_n$. The drift vector field is

$$f(x) = \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{2n} \\ \lambda_1 G_1(x_1) - \lambda_1 k_2 x_2 \\ -\lambda_1 G_1(x_1) + (\lambda_1 + \lambda_2) k_2 x_2 - \lambda_2 k_3 x_3 \\ -\lambda_2 k_2 x_2 + (\lambda_2 + \lambda_3) k_3 x_3 - \lambda_3 k_4 x_4 \\ \vdots \\ -\lambda_{n-2} k_{n-2} x_{n-2} + (\lambda_{n-2} + \lambda_{n-1}) k_{n-1} x_{n-1} - \lambda_{n-1} k_n x_n \\ -\lambda_{n-1} k_{n-1} x_{n-1} - \lambda_{n-1} \lambda_n k_n x_n \end{pmatrix}$$

with $G_1(x_1) = -mg \sin x_1$. The input vector fields are

$$\begin{aligned} g_i &= -\lambda_{i-1} \frac{\partial}{\partial x_{n+i-1}} + (\lambda_{i-1} + \lambda_i) \frac{\partial}{\partial x_{n+i}} \\ &\quad - \lambda_i \frac{\partial}{\partial x_{n+i+1}} \quad \forall 1 < i < n \\ g_1 &= \lambda_1 \left(\frac{\partial}{\partial x_{n+1}} - \frac{\partial}{\partial x_{n+2}} \right) \\ g_n &= -\lambda_{n-1} \left(\frac{\partial}{\partial x_{2n-1}} + \lambda_n \frac{\partial}{\partial x_{2n}} \right). \end{aligned}$$

The next Lemma will be helpful in proving the main results of the following sections.

Lemma 1:

- 1) $ad_f^2 g_i = -\lambda_{i-1}k_{i-1}g_{i-1} + (\lambda_{i-1} + \lambda_i)k_i g_i - \lambda_i k_{i+1}g_{i+1} \quad \forall 2 < i < n;$
- 2) $ad_f^2 g_1 = \lambda_1 \partial G_1(x_1)/\partial x_1 g_1 - \lambda_1 k_2 g_2;$
- 3) $ad_f^2 g_2 = -\lambda_1 \partial G_1(x_1)/\partial x_1 g_1 + (\lambda_1 + \lambda_2)k_2 g_2 - \lambda_2 k_3 g_3;$
- 4) $ad_f^2 g_n = k_{n-1}g_{n-1} - \lambda_{n-1}\lambda_n k_n g_n.$

Proof: For $2 < i < n - 1$

$$ad_f g_i = -\lambda_{i-1} \frac{\partial}{\partial x_{i-1}} + (\lambda_{i-1} + \lambda_i) \frac{\partial}{\partial x_i} - \lambda_i \frac{\partial}{\partial x_{i+1}}.$$

Therefore

$$ad_f^2 g_i = -\lambda_{i-1} \frac{\partial f}{\partial x_{i-1}} + (\lambda_{i-1} + \lambda_i) \frac{\partial f}{\partial x_i} - \lambda_i \frac{\partial f}{\partial x_{i+1}}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\lambda_{i-1}(-\lambda_{i-2}k_{i-1}) \\ -\lambda_{i-1}(\lambda_{i-2} + \lambda_{i-1})k_{i-1} + (\lambda_{i-1} + \lambda_i)(-\lambda_{i-1}k_i) \\ \lambda_{i-1}^2 k_{i-1} + (\lambda_{i-1} + \lambda_i)^2 k_i + \lambda_i^2 k_{i+1} \\ -\lambda_i k_i(\lambda_{i-1} + \lambda_i) - \lambda_i(\lambda_i + \lambda_{i+1})k_{i+1} \\ \lambda_i \lambda_{i+1} k_{i+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

From this expression, one can write $ad_f^2 g_i = -\lambda_{i-1}k_{i-1}g_{i-1} + (\lambda_{i-1} + \lambda_i)k_i g_i - \lambda_i k_{i+1}g_{i+1}$. The other cases can be proven in a similar manner.

The question that we pose now is the following: Under what choices of inputs is the system static feedback linearizable? If it is not static feedback linearizable, can it be linearized by means of prolongations? We want to remind that here prolongation means a simple prolongation, as defined in Section I-A

III. STATIC FEEDBACK LINEARIZABILITY FOR SINGLE-INPUT SYSTEMS

Theorem 1: The dynamic model 4 with only one input is static feedback linearizable if and only if the input is at the first or the last joint.

Proof: *Sufficiency:* If the input is placed at the last joint, the variable

$$y = \sum_{i=1}^n a_i q_i = \sum_{i=1}^n a_i x_i$$

has relative degree $2n$. Please note that relative degree is the number of times a variable has to be differentiated derivatives for the input to appear explicitly. To see this, differentiate y twice with respect to time and use the structure of the dynamic (3). This procedure is recursively repeated and it can be shown that the output y has a relative degree $2n$ [4], which equals the order of the dynamic system. Therefore, the system is static feedback linearizable, with flat output y . If the input is placed at the first joint, the variable $y = a_n \sum_{i=1}^n q_i = a_n \sum_{i=1}^n x_i$ has relative degree $2n$. This can be shown by exploiting the built-in structure of the dynamic equations [4]. See also [3] for more details of the proof.

Necessity: If the input is placed at any other joint, it can be proved that the system is not static feedback linearizable. In order to see this, consider a system with n links such that the only input is placed at joint i , with $1 < i < n$. Consider the iterative Lie brackets $ad_f g_i, ad_f^2 g_i, \dots, ad_f^{2i-2} g_i$. As stated in Lemma 1

$$ad_f^2 g_i = (-1)\lambda_{i-1}k_{i-1}g_{i-1} + \dots$$

Applying the mentioned Lemma recursively, one gets

$$\begin{aligned} ad_f^4 g_i &= (-1)^2 \lambda_{i-1} \lambda_{i-2} k_{i-1} k_{i-2} g_{i-2} + \dots \\ ad_f^6 g_i &= (-1)^3 \lambda_{i-1} \lambda_{i-2} \lambda_{i-3} k_{i-1} k_{i-2} k_{i-3} g_{i-3} + \dots \\ &\vdots \\ ad_f^{2i-4} g_i &= (-1)^{i-2} \prod_{l=2}^{i-1} \lambda_l k_l g_2 + \dots \end{aligned} \quad (5)$$

Using the structure of the vector field g_2

$$ad_f^{2i-4} g_i = (-1)^{i-1} \lambda_1 \prod_{l=2}^{i-1} \lambda_l k_l \frac{\partial}{\partial x_{n+1}} + \dots$$

In addition $ad_f^{2i-3} g_i =$

$$(-1)^{i-1} \lambda_1 \prod_{l=2}^{i-1} \lambda_l k_l \frac{\partial}{\partial x_1} + \dots$$

and $ad_f^{2i-2} g_i =$

$$(-1)^{i-1} \lambda_{i-1} \lambda_{i-2} \dots \lambda_2 \lambda_1 k_{i-1} k_{i-2} \dots k_2 \frac{\partial G_1(x_1)}{\partial x_1} \frac{\partial}{\partial x_{n+1}} + \dots$$

Thus, the Lie bracket $[ad_f^{2i-3} g_i, ad_f^{2i-2} g_i]$

$$= (\lambda_{i-1} \lambda_{i-2} \dots \lambda_2 \lambda_1 k_{i-1} k_{i-2} \dots k_2)^2 \frac{\partial^2 G_1(x_1)}{\partial x_1^2} \frac{\partial}{\partial x_{n+1}}$$

is a multiple of the vector field $\partial/\partial x_{n+1}$ which clearly does not belong to the distribution

$$D_{2i-2} = \langle g_i, ad_f g_i, \dots, ad_f^{2i-2} g_i \rangle$$

because of the structure of the iterated Lie brackets previously mentioned. Therefore, D_{2i-2} is not involutive and the system is not static feedback linearizable and, hence, it is not differentially flat. Please note that for single input systems, differential flatness is equivalent to static feedback linearization (see, e.g., [11]). $\square$

IV. STATIC FEEDBACK LINEARIZABILITY FOR TWO-INPUT SYSTEMS

We start this section through a motivating example. The results stated in a theorem below can be understood with this example of a five link mechanism with two inputs. We will first show how the choice of location of inputs affects the system's static feedback linearizability. Specifically, we will study a five link mechanism with two inputs, placed at joints 3 and 5. This is an example of a static feedback linearizable system. On the contrary, if the inputs are placed at joints 2 and 5, or joints 2 and 4, it can be seen that the system is not static feedback linearizable. However, these two systems that are not static feedback linearizable can be proven to be linearizable by prolongations.

1) *Example:* A system with 5 links and two inputs, one at joint 3 and the other at joint 5. With the notations given in Section II, the drift and input vector field g_3 are:

$$f = \begin{pmatrix} x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ \lambda_1 G_1(x_1) - \lambda_1 k_2 x_2 \\ -\lambda_1 G_1(x_1) + (\lambda_1 + \lambda_2) k_2 x_2 \\ -\lambda_2 k_2 x_2 - \lambda_3 k_4 x_4 \\ (\lambda_3 + \lambda_4) k_4 x_4 \\ -\lambda_4 k_4 x_4 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\lambda_2 \\ \lambda_2 + \lambda_3 \\ -\lambda_3 \\ 0 \end{pmatrix}, g_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \lambda_5 \end{pmatrix}. \quad (6)$$

In this sequel, we will use the notation

$$g_3 = -\lambda_2 \frac{\partial}{\partial x_7} + (\lambda_2 + \lambda_3) \frac{\partial}{\partial x_8} - \lambda_3 \frac{\partial}{\partial x_9}$$

$$g_5 = \frac{\partial}{\partial x_9} + \lambda_5 \frac{\partial}{\partial x_{10}}.$$

On computing the first order Lie brackets $ad_f g_3, ad_f g_5$ and the second-order Lie brackets $ad_f^2 g_3, ad_f^2 g_5$, the following relationships hold $ad_f^2 g_3 = -\lambda_1 k_2 g_2 - \lambda_3 k_4 g_4$ and $ad_f^2 g_5 = k_4 g_4$. Therefore:

$$D_2 = \langle g_3, g_5, ad_f g_3, ad_f g_5, ad_f^2 g_3, ad_f^2 g_5 \rangle$$

$$= \langle g_3, g_5, ad_f g_3, ad_f g_5, g_2, g_4 \rangle$$

$$D_3 = \langle D_2, ad_f g_2, ad_f g_4 \rangle$$

where $ad_f g_2 = -\lambda_1 \partial/\partial x_1 + (\lambda_1 + \lambda_2) \partial/\partial x_2 - \lambda_2 \partial/\partial x_3$ and $ad_f g_4 = -\lambda_3 \partial/\partial x_3 + (\lambda_3 + \lambda_4) \partial/\partial x_4 - \lambda_4 \partial/\partial x_5$.

Similarly

$$ad_f^2 g_2 = -\lambda_1^2 \frac{\partial G_1(x_1)}{\partial x_1} g_1 + (\lambda_1 + \lambda_2) k_2 g_2$$

while $ad_f^2 g_4$ is a multiple of g_4 : $ad_f^2 g_4 = (\lambda_3 + \lambda_4) g_4$.

Taking into account the above considerations, $D_4 = \langle D_3, g_1 \rangle$. Finally, $D_5 = R^{10}$. Note that all the distributions D_0 to D_5 are involutive and constant rank. Hence, the system is static feedback linearizable. In order to find the change of variables that feedback linearize the system and following the standard computations [6], two differentially independent functions y_1 and y_2 must be obtained. These two functions must also satisfy the following conditions:

$$\nabla y_1 \perp D_4 \quad \nabla y_2 \perp D_2. \quad (7)$$

A solution for the system of partial differential equations arising from (7) is

$$y_1 = \sum_{i=1}^5 a_i x_i \quad y_2 = a_4 \sum_{i=1}^4 x_i + a_5 x_5. \quad (8)$$

On computing their second order time derivatives, one gets

$$\ddot{y}_1 = -mg \sin x_1 \quad \ddot{y}_2 = k_4 x_4$$

from where variables x_1 and x_4 are written as functions of the flat outputs and their derivatives:

$$x_4 = \frac{\ddot{y}_2}{k_4} \quad x_1 = -\arcsin \frac{\ddot{y}_1}{mg}.$$

On the other hand

$$\ddot{y}_1 - (a_1 - a_2) \ddot{x}_1 = k_2 x_2$$

and, therefore

$$x_2 = \frac{1}{k_2} (\ddot{y}_1 - (a_1 - a_2) \ddot{x}_1)$$

$$= -\frac{1}{k_2} \left(\ddot{y}_1 - (a_1 - a_2) \frac{y_1^{(4)} ((mg)^2 - \ddot{y}_1^2) - \ddot{y}_1 y_1^{(3)}}{((mg)^2 - \ddot{y}_1^2)^{3/2}} \right).$$

Fig. 2. Planned flat output trajectories and tracking results for the five DOF case. Even in the presence of initial errors, the system eventually converges to the planned trajectories.

Finally, once x_1, x_2 and x_4 have been written as functions of the flat outputs and their derivatives, (8) constitute a linear system of two equations and two unknowns, namely x_3 and x_5 . This linear system has a unique solution which is the expression of x_3 and x_5 as functions of the flat outputs and their derivatives. Summarizing, the diffeomorphism between the state space $X = (x_1, \dots, x_{10})$ and the flat output space $Y = (y_1, \dots, y_1^{(5)}, y_2, \dots, y_2^{(3)})$ is given by

$$X = \begin{pmatrix} -\arcsin \frac{\ddot{y}_1}{mg} \\ \alpha \\ -\frac{y_1 - y_2}{a_3 - a_4} - \frac{a_1 - a_4}{a_3 - a_4} \arcsin \frac{\ddot{y}_1}{mg} - \frac{a_2 - a_4}{a_3 - a_4} \alpha \\ \frac{\ddot{y}_2}{k_4} \\ \frac{y_2}{a_5} - \frac{a_4}{a_5} \left(\frac{y_1 - y_2}{a_3 - a_4} - \frac{a_3 - a_1}{a_3 - a_4} \arcsin \frac{\ddot{y}_1}{mg} + \frac{a_3 - a_2}{a_3 - a_4} \alpha \right) + \frac{\ddot{y}_2}{k_4} \\ \vdots \\ \dot{x}_5 \end{pmatrix}$$

where

$$\alpha = -\frac{1}{k_2} \left(\ddot{y}_1 - (a_1 - a_2) \frac{y_1^{(4)} ((mg)^2 - \ddot{y}_1^2) - \ddot{y}_1 y_1^{(3)}}{((mg)^2 - \ddot{y}_1^2)^{3/2}} \right)$$

and $\dot{x}_1 \dots \dot{x}_5$ stand for the time derivative of the first five rows.

A system can follow any flat output trajectory because the state trajectories corresponding to any flat output trajectory, by definition, already satisfy the dynamic equations. Once the flat output and its diffeomorphism is established, any state of the system can be converted into the flat output space. A user can specify a set of such states that should be traversed by the robot. Once we have a feasible plan for the flat outputs, a controller can be designed in the flat output space.

We present here control simulations for the five DOF case with only two actuators at joints three and five. The following steps were taken to plan the trajectories: (i) We start with a set of given states, called anchor states, and corresponding time instants that are either user specified or selected appropriately based on the task at hand. These states were transformed into flat outputs and their derivatives, (ii) Then, we collocate flat output trajectories to pass through these anchor states. This collocation can be done using well known methods such as polynomials, Fourier series, splines. We used a Fourier series in the example. (iii) If needed, the flat output trajectories can be parametrized and op-

Fig. 3. Planned joint angle trajectories and tracking results for the five DOF case. Even in the presence of initial errors, the desired final point in the state space is reached.

timized to satisfy auxiliary constraints. (iv) The feedback controllers were derived using the chain of integrator representation, $v_i = y_i^{(r_i)}$, where r_i denotes the relative degree of the i th flat output. Each of the new control inputs are computed using

$$v_i = y_{id}^{(r_i)} + k_1^i (y_{id}^{(r_i-1)} - y_i^{(r_i-1)}) + \dots + k_{r_i-1}^i (y_{id} - y_i). \quad (9)$$

The subscript d in (9) denotes the planned trajectories and $[k_1^i, k_2^i, \dots, k_{r_i-1}^i]^T$ are control gains for the i th output. The actual inputs were then computed using the diffeomorphism. In the example, the control gains were chosen such that all poles lie at -5 . From Figs. 2 and 3, we clearly see that even in presence of initial errors in trajectories the desired final point is reached. Please note that the end state is not an equilibrium point for the system. However, brakes can be used to hold the robot in that state until the robot needs to move to the next point.

Remark: If we consider now the same example with inputs at joints 2 and 5, instead of joints 3 and 5, the system is not static feedback linearizable. The same happens if the inputs are located at joint 2 and 4. However, these two systems can be proven to be linearizable by prolongations. The key results to explain all these cases are summarized in the next four theorems.

Theorem 2: The general system described in (4) with two inputs, one of which is at the first joint, is always static feedback linearizable.

Proof: Due to the second order structure of the system, we only have to check the involutivity of distributions D_{2k} . These distributions are: $D_0 = \langle g_1, g_i \rangle$. According to Lemma 1

$$\begin{aligned} D_2 &= \langle g_1, g_i, \text{ad}_f g_1, \text{ad}_f g_i, \lambda_1 \frac{\partial G(x_1)}{\partial x_1} g_1 - \lambda_1 k_2 g_2, \\ &\quad - \lambda_{i-1} k_{i-1} g_{i-1} + (\lambda_{i-1} + \lambda_i) k_i g_i - \lambda_i k_{i+1} g_{i+1} \rangle \\ &= \langle g_1, g_i, \text{ad}_f g_1, \text{ad}_f g_i, g_2, -\lambda_{i-1} k_{i-1} g_{i-1} - \lambda_i k_{i+1} g_{i+1} \rangle. \end{aligned}$$

Similarly

$$\begin{aligned} D_4 &= \langle g_1, g_2, g_3, \text{ad}_f g_1, \text{ad}_f^3 g_1, g_i, \text{ad}_f g_i, \mu_{i-1}^1 g_{i-1} + \mu_{i+1}^1 g_{i+1}, \\ &\quad \text{ad}_f^3 g_i, \mu_{i-2}^2 g_{i-2} + \mu_{i-1}^2 g_{i-1} + \mu_{i+1}^2 g_{i+1} + \mu_{i+2}^2 g_{i+2} \rangle \end{aligned}$$

where μ_j^i are constants. On carrying on this procedure, it is easy to see that all these distributions are involutive and constant rank since all the vectorfields contained in them are constant. $\square$

TABLE I
SUMMARY OF LINEARIZABLE SYSTEMS WITH 1 ACTUATOR

Only 1 input at first or last joint	Static feedback linearizable
Only 1 input at any other joint	NOT Static feedback linearizable

TABLE II
SUMMARY OF LINEARIZABLE SYSTEMS WITH 2 ACTUATORS

Position of the inputs i, j	
$1 = i < j \leq n$	SFL
$1 < i < j = n, 2i \geq n$	SFL
$1 < i < j = n, 2i < n$	NOT SFL
$1 < i < j = n, 2i < n$	LP
$1 < i < j < n, j = i + 1$	SFL
$1 < i < j < n, j \neq i + 1, 2i \geq n$	SFL
$1 < i < j < n, j \neq i + 1, 2i < n$	NOT SFL
$1 < i < j < n, j \neq i + 1, 2i < n, 2j > n + 1$	LP
$1 < i < j < n, j \neq i + 1, 2i < n, 2j \leq n + 1$	NOT LP

Remark: If a given system is static feedback linearizable, addition of new inputs can lead to a non static feedback linearizable system (this can be easily checked by computing the Lie brackets conditions). Therefore, Theorem 2 is not a direct consequence of Theorem 1.

The following theorems state different conditions under which the system is differentially flat, either by static feedback linearization or using linearization by prolongations. The proofs of these theorems are omitted due to the lack of space. However, the proofs are very similar to the proof of Theorem 2.

Theorem 3: The general system described in (4) with two inputs, one of which is at the last joint and the other at joint i , is static feedback linearizable if and only if $2i \geq n$.

Theorem 4: The general system described in (4) with two inputs at joints i and j (with $i < j$), none of them at first or last joint, is static feedback linearizable if and only if one of the following conditions is met: The inputs are placed at two consecutive joints or $n \leq 2i$.

Theorem 5: The general system described in (4) with two inputs at joints i and j , with $i < j$ and $n > 2i$, is linearizable by prolongations if and only if $2j > n + 1$.

Example 4: As an example, consider the simplest case that does not meet the condition of the last theorem, say, a robot with 7 links ($n = 7$), and two inputs at joints $i = 2$ and $j = 4$. One can compute all the possible prolongations and it can be shown that none of these prolongations lead to a static feedback linearizable system by checking the non-involutivity of the distribution D_6 . Remark that $2j = 8 = n + 1$, which is the case limit in Theorem 5.

Tables I and II summarize the results of the preceding theorems: where SFL stands for static feedback linearizable systems, and LP means linearizable by prolongations.

V. CONCLUSION

This technical note describes the conditions under which a class of an n -DOF planar robot is differentially flat (static or dynamic feedback linearizable using prolongations) using just one or two actuators. For this class, the center of mass of the outward links are placed recursively at joint 2 starting from joint n . Our results show that (i) the system is differentially flat with one actuator, provided that this actuator is placed at the first or the last joint. If two actuators are used (ii) the system is differentially flat if one of the actuators is at the first joint, (iii) the system is differentially flat if one of the actuators is at the last joint and the second actuator is at a joint i , with $2i \geq n$, (iv) If two actuators are used at joints i and j with $i < j$, the system is differentially flat if the actuators are at consecutive joints or $2i \geq n$, (iv) There are additional cases of linearization by prolongation given in Theorem 5.

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Lyapunov Stability of Linear Predictor Feedback for Time-Varying Input Delay

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Abstract—For linear time-invariant systems with a time-varying input delay, an explicit formula for predictor feedback was presented by Nihtila in 1991. In this note we construct a time-varying Lyapunov functional for the closed-loop system and establish exponential stability. The key challenge is the selection of a state for a transport partial differential equation, which has a non-constant propagation speed, and which is the basis of the stability analysis. We illustrate the design and its conditions with several examples. We also develop an observer equivalent of the predictor feedback design, for the case of time-varying sensor delay.

Index Terms—Backstepping, delay systems, distributed parameter systems.

I. INTRODUCTION

Systems with long input delays, even those where the plant is unstable, can be successfully controlled using predictor feedback [4]–[8], [10]–[17], [20], [21], [24]–[26] and methods based on LQ control [2],

[3], [22]. All these results deal with problems where the delay is constant. In the case of systems with *time-varying* delays, various results exist for problems with state delay and no input delay. However, the case of systems with time-varying *input* delay has received very little attention.

A basic idea how to approach problems with time-varying input delay was introduced by Artstein [1], however, the design is not worked out in detail since the case of time-varying delay is considered only for plants that are time-varying, in which case explicit developments are not possible. An explicit state-feedback design for linear time-invariant (LTI) plants with time-varying input delays was presented by Nihtila [19]. (A parameter-adaptive design for a scalar system with known time-varying delay function [18] preceded the author's general result in [19].) In this note we establish exponential stability of the feedback system with the controller from [19]. We do so using an explicit construction of a (strict) Lyapunov functional. Stability is not claimed in [1], [19], where the approach employed in the design uses a transformation that relates infinite-dimensional feedback system with another finite-dimensional system. Such a construction does not yield a Lyapunov functional, though it yields a controller that compensates the time-varying delay.

The challenge in the study of stability under time-varying input delay, as compared to our result for constant input delays [12], is that one has to construct a Lyapunov functional using a backstepping transformation with time-varying kernels, and transforming the actuator state into a transport partial differential equation (PDE) with a convection speed coefficient that varies with both space and time. An additional challenge is how to define the state of the transport PDE modeling the actuator state using the past input signal.

We start in Section II with an intuitive introduction of the predictor feedback under time-varying input delay. Then in Section III we present a stability study. In Section IV we present an observer for the case of a plant with time-varying sensor delay. Finally, in Section V we present several examples, including a numerical example with a scalar unstable plant and with an oscillating time-varying input delay.

II. PREDICTOR FEEDBACK DESIGN WITH TIME-VARYING ACTUATOR DELAY

We consider the system

$$\dot{X}(t) = AX(t) + BU(\phi(t)) \quad (1)$$

where $X \in \mathbb{R}^n$ is the state, U is the control input, and $\phi(t)$ is a continuously differentiable function that incorporates the actuator delay. This function will have to satisfy certain conditions that we shall impose in our development, in particular, that

$$\phi(t) \leq t, \quad \forall t \geq 0. \quad (2)$$

One can alternatively view the function $\phi(t)$ in the more standard form $\phi(t) = t - D(t)$, where $D(t) \geq 0$ is a time-varying delay. However, the formalism involving the function $\phi(t)$ turns out to be more convenient, particularly because the predictor problem requires the inverse function of $\phi(t)$, i.e., $\phi^{-1}(t)$, so we will proceed with the model (1). The invertibility of $\phi(\cdot)$ will be ensured by imposing the following assumption.

Assumption 1: $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuously differentiable function that satisfies

$$\phi'(t) > 0, \quad \forall t \geq 0 \quad (3)$$

and such that

$$\pi_1^* = \frac{1}{\sup_{\vartheta \geq \phi^{-1}(0)} \phi'(\vartheta)} > 0. \quad (4)$$

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