UPCommons

Portal del coneixement obert de la UPC

<http://upcommons.upc.edu/e-prints>

© 2021 ACM. This is the Accepted Manuscript version of the work.
The definitive Version of Record was published in *ACM
COMMUNICATIONS IN COMPUTER ALGEBRA*,
<http://dx.doi.org/10.1145/3511528.3511533>.

Pfeifle, J. Large final polynomials from integer programming. "ACM
COMMUNICATIONS IN COMPUTER ALGEBRA", Setembre 2021, vol.
55, núm. 3, p. 82-86. DOI [10.1145/3511528.3511533](https://doi.org/10.1145/3511528.3511533)

URL d'aquest document a UPCommons E-prints:

<https://upcommons.upc.edu/handle/2117/362955>

Large final polynomials from integer programming

Julian Pfeifle

Department de Matemàtica Aplicada
Universitat Politècnica de Catalunya
Barcelona, Spain

julian.pfeifle@upc.edu

May 5, 2021

Abstract

We introduce a new method for finding a non-realizability certificate of a simplicial sphere Σ . It enables us to prove for the first time the non-realizability of a balanced 2-neighborly 3-sphere by Zheng, a family of highly neighborly centrally symmetric spheres by Novik and Zheng, and several combinatorial prisms introduced by Criado and Santos.

The method, implemented in the `polymake` framework, uses integer programming to find a monomial combination of classical 3-term Plücker relations that must be positive in any realization of Σ ; but since this combination should also vanish identically, the realization cannot exist.

Previous approaches by Firsching, implemented using SCIP, and by Gouveia, Macchia and Wiebe, implemented using Singular and Macaulay2, are not able to process these examples.

1 Introduction

A *simplicial $(d - 1)$ -sphere* Σ is a simplicial complex homeomorphic to a $(d - 1)$ -dimensional sphere. We say that Σ is *non-realizable* if there does not exist a (necessarily simplicial) d -polytope whose boundary complex is isomorphic to Σ .

For example, the following list of 19 facets defines a 3-sphere Σ on 8 vertices with 38 triangles. The signs define an orientation of Σ , so that the following sum generates the top homology class:

$$\begin{aligned} &+[0123] -[0124] +[0135] -[0146] +[0157] -[0167] -[0234] +[0345] -[0456] +[0567] +[1237] \\ &-[1246] -[1267] +[1357] -[2347] +[2456] -[2457] -[2567] +[3457] \end{aligned}$$

How do we prove that this 3-sphere is non-realizable? In this case, the venerable *Plücker relations* suffice. These are polynomial relations that are satisfied by the determinants of any $d + 1$ points of a putative realization of Σ in d -space. The most basic ones are the *three-term Plücker relations*

$$\Gamma(S|ijkl) = [Sij][Sk l] - [Sik][Sj l] + [Sil][Sj k] \quad (1)$$

which are valid for any subset $S \subset \{1, 2, \dots, n\}$ of size $(d - 1)$, and any four indices $i, j, k, l \in \{1, 2, \dots, n\} \setminus S$. A typical 3-term Plücker relation in our example is

$$0 = \Gamma(045|1267) = [04512][04567] - [04516][04527] + [04517][04526].$$

By permuting the entries inside these determinants, we can change their signs — even permutations will leave the sign unchanged, while odd permutations will flip it. A particularly advantageous way of changing

the signs is

$$\begin{aligned} 0 &= \Gamma(045|1267) = (-1)[01425] \cdot (-1)[05674] - (+1)[04651] \cdot (-1)[24750] + (+1)[01574] \cdot (+1)[24560] \\ &= [01425][05674] + [04651][24750] + [01574][24560]. \end{aligned}$$

The advantage of rewriting $\Gamma(045|1267)$ in this way is that now *all determinants are guaranteed to be positive!* For example, $[01425] > 0$ because $[01425] = -[01245]$, and $[01245]$ is the “signed slack” of the point x_5 with respect to the facet $[0124]$ in the supposed convex realization of Σ ; but the orientation of 0124 in Σ is negative by the above list. The other determinants can be similarly checked to be positive. But this expresses zero as a positive combination of positive numbers, which is impossible; therefore, this *final polynomial* shows that there is no convex realization of Σ .

2 The non-realizability of Zheng’s 3-sphere

A $(d - 1)$ -dimensional simplicial complex is *balanced* if its 1-skeleton is *d-colorable* in the graph-theoretic sense, i.e., its vertices can be colored with d colors in such a way that the endpoints of all edges receive different colors. Moreover, a $(d - 1)$ -dimensional balanced simplicial complex Σ is *balanced k-neighborly* if each k -subset of the vertex set that contains at most one vertex of each color class is actually a face of Σ .

One would like to have a balanced analogue of the celebrated Upper Bound Theorem by McMullen and Stanley. For this, balanced analogues of the extremal examples should in particular even *exist*, which entails constructing infinite families of balanced k -neighborly simplicial spheres. In [Zhe20a], Zheng constructs a minimal candidate for such an extremal example. More precisely, she shows that (i) there is no balanced 2-neighborly homology 3-sphere on 12 vertices; (ii) there is no balanced 2-neighborly homology 4-sphere on 15 vertices; (iii) but taking suspensions over a certain balanced 2-neighborly simplicial 3-sphere Z yields a balanced 2-neighborly homology $(3 + m)$ -sphere on $16 + 2m$ vertices for every $m \geq 0$.

An orientation of the facets of her example Z is given by the following list:

$$\begin{aligned} &-[048c] + [048e] + [049c] - [049d] + [04ad] - [04ae] + [059d] - [059f] - [05ad] + [05ae] - [05be] \\ &+ [05bf] + [068c] - [068e] - [069c] + [069e] - [079e] + [079f] + [07be] - [07bf] + [148c] - [148e] \\ &+ [14ae] - [14af] - [14bc] + [14bf] - [158c] + [158d] - [159d] + [159f] + [15bc] - [15bf] + [168e] \\ &- [168f] - [16ae] + [16af] - [178d] + [178f] + [179d] - [179f] - [24ad] + [24af] + [24bd] - [24bf] \\ &+ [258c] - [258d] - [25ac] + [25ad] - [268c] + [268d] + [269c] - [269e] + [26ae] - [26af] - [26bd] \\ &+ [26bf] - [279c] + [279e] + [27ac] - [27ae] - [349c] + [349d] + [34bc] - [34bd] + [35ac] - [35ae] \\ &- [35bc] + [35be] - [368d] + [368f] + [36bd] - [36bf] + [378d] - [378f] + [379c] - [379d] - [37ac] \\ &+ [37ae] - [37be] + [37bf] \end{aligned}$$

Intriguingly, if Z were polytopal, it would also furnish the first known example of a 4-polytope with complete 4-partite graph $K_{4,4,4,4}$ other than the 4-dimensional cross-polytope. But:

Theorem 1 *Zheng’s balanced sphere Z is not polytopal.*

Proof. We prove the non-realizability of Z in a similar way as before, but we allow Grassmann–Plücker relations in which we do not have full control over the signs. For example,

$$\begin{aligned} 0 &= \Gamma(18f|56bd) = [18f56][18fbd] - [18f5b][18f6d] + [18f5d][18f6b] \\ &= [16f85][18bdf]^? + [15fb8][16f8d] + [158df][16f8b], \end{aligned}$$

where all (black) signs are known to be positive in any realization of Z , but the (red) sign with a question mark can be either positive or negative because no four-element subset of $[18bdf]^?$ appears in the list of facets of Z . To balance this uncertainty, we look for another Grassmann–Plücker relation that involves $[18bdf]^?$. A candidate is

$$\begin{aligned} 0 &= \Gamma(1bf|48de) = [1bf48][1bfde] - [1bf4d][1bf8e] + [1bf4e][1bf8d] \\ &= [14bf8][1bdef]^? + [14bfd][18bef]^? - [14bfe][18bdf]^?. \end{aligned}$$

We can eliminate the indeterminate sign $[18\mathbf{bdf}]^?$ by forming the polynomial combination

$$[14\mathbf{bfe}] \cdot \Gamma(18\mathbf{f}|56\mathbf{bd}) + [16\mathbf{f85}] \cdot \Gamma(1\mathbf{bf}|48\mathbf{de}),$$

but we now have to contend with two additional indeterminate signs to worry about.

Somewhat surprisingly, we are able to bring this process to a closure by forming the following polynomial combination of Grassmann–Plücker relations:

$$\begin{aligned} & [36\mathbf{fb5}] \left([36\mathbf{fb4}] \left([14\mathbf{bf3}] \left([16\mathbf{f85}] ([14\mathbf{bfd}] (-\Gamma(18\mathbf{f}|46\mathbf{be})) \right. \right. \right. \\ & \quad \left. \left. \left. + [16\mathbf{f84}] \Gamma(1\mathbf{bf}|48\mathbf{de}) \right) \right. \right. \\ & \quad \left. \left. + [16\mathbf{f84}] [14\mathbf{bfe}] \Gamma(18\mathbf{f}|56\mathbf{bd}) \right) \right. \\ & \quad \left. + [16\mathbf{f84}] [14\mathbf{bf8}] [16\mathbf{f85}] \Gamma(1\mathbf{bf}|34\mathbf{de}) \right) \\ & \quad \left. + [16\mathbf{f84}] [14\mathbf{bf8}] [14\mathbf{bfe}] [16\mathbf{f85}] (-\Gamma(3\mathbf{bf}|146\mathbf{d})) \right) \\ & + [16\mathbf{f84}] [14\mathbf{bf8}] [14\mathbf{bfd}] [16\mathbf{f85}] [36\mathbf{fb4}] \Gamma(3\mathbf{bf}|156\mathbf{e}) . \end{aligned}$$

It corresponds to the *positive Plücker tree* in Figure 1. Substituting the appropriately normalized Plücker

Figure 1: Plücker tree proving the non-realizability of Zheng's 3-sphere

polynomials and simplifying yields the following certificate for the non-realizability of Z :

$$\begin{aligned} & [14\mathbf{bf3}] [16\mathbf{f84}] [14\mathbf{bf8}] [14\mathbf{bfe}] [16\mathbf{f85}] [36\mathbf{fb5}] [36\mathbf{fb4}] + [14\mathbf{bf3}] [16\mathbf{f84}] [14\mathbf{bfe}] [15\mathbf{fb8}] [16\mathbf{f8d}] [36\mathbf{fb4}] [36\mathbf{fb5}] \\ & + [14\mathbf{bf3}] [16\mathbf{f84}] [14\mathbf{bfe}] [158\mathbf{df}] [16\mathbf{f8b}] [36\mathbf{fb4}] [36\mathbf{fb5}] + [14\mathbf{bf3}] [14\mathbf{bf8}] [14\mathbf{bfd}] [16\mathbf{f85}] [16\mathbf{f8e}] [36\mathbf{fb4}] [36\mathbf{fb5}] \\ & + [14\mathbf{bf3}] [14\mathbf{e8f}] [14\mathbf{bfd}] [16\mathbf{f85}] [16\mathbf{f8b}] [36\mathbf{fb4}] [36\mathbf{fb5}] + [15\mathbf{fb3}] [16\mathbf{f84}] [14\mathbf{bf8}] [14\mathbf{bfd}] [16\mathbf{f85}] [36\mathbf{fb4}] [36\mathbf{fb5}] \\ & + [36\mathbf{fb1}] [16\mathbf{f84}] [14\mathbf{bf8}] [14\mathbf{bfd}] [16\mathbf{f85}] [36\mathbf{fb4}] [35\mathbf{bef}] + [36\mathbf{fb1}] [16\mathbf{f84}] [14\mathbf{bf8}] [14\mathbf{bfe}] [16\mathbf{f85}] [34\mathbf{dbf}] [36\mathbf{fb5}] . \end{aligned}$$

Arranging the Grassmann–Plücker polynomials into a tree, i.e., a graph without cycles, guarantees that the final polynomial does not depend on the order in which the certificate is multiplied out; showing this is a short exercise.

3 Finding Positive Plücker Trees

We use integer programming to find such certificates. We say that a three-term Grassmann–Plücker relation as in (1) *has no adjacent indeterminate solids* if no two determinants that are multiplied together have indeterminate sign. To set up the variables for our integer program, we first define the *GP graph* in which we search for our certificate. Its nodes are

$$V_{\Sigma} = \{ \pm \Gamma(S|ijkl) : \Gamma(S|ijkl) \text{ has no two adjacent indeterminate solids} \},$$

and the edges are colored with the set $\mathfrak{S}(\Sigma)$ of normal forms of solids of Σ . There can be multiple edges between two nodes, but they have different colors: two nodes $\Gamma, \Gamma' \in V_{\Sigma}$ are joined with an edge $(\bar{\pi}^?, \{\Gamma, \Gamma'\})$

colored $\bar{\pi}^? \in \mathfrak{S}(\Sigma)$ in $E_\Sigma \subseteq \mathfrak{S}(\Sigma) \times \binom{V_\Sigma}{2}$ iff they share a solid π such that $\sigma_i = -\sigma'_i$, where σ_i, σ'_i are the canonical signs of the terms containing π in Γ, Γ' .

We now distill a tree from this graph by solving the following integer program on the integer indicator variables $\{x_\Gamma : \Gamma \in V_\Sigma\}$ and $\{x_e : e = (\bar{\pi}^?, \{\Gamma, \Gamma'\}) \in E_\Sigma\}$:

$$\min \sum_{\Gamma \in V_\Sigma} x_\Gamma \quad \text{s.t.} \quad 2 \sum_{\bar{\pi}^?: e=(\bar{\pi}^?, \{\Gamma, \Gamma'\}) \in E_\Sigma} x_e \leq x_\Gamma + x_{\Gamma'} \quad \text{for each } \{\Gamma, \Gamma'\} \in \binom{V_\Sigma}{2} \quad (2)$$

$$\sum_{\bar{\pi}^?: e=(\bar{\pi}^?, \{\Gamma, \Gamma'\}) \in E_\Sigma} x_e \leq 1 \quad \text{for each } \{\Gamma, \Gamma'\} \in \binom{V_\Sigma}{2}, \quad (3)$$

$$1 + \sum_{e \in E_\Sigma} x_e = \sum_{\Gamma \in V_\Sigma} x_\Gamma \quad (4)$$

$$\sum_{\Gamma': e=(\bar{\pi}^?, \{\Gamma, \Gamma'\}) \in E_\Sigma} x_e = x_\Gamma \quad \text{for all } \Gamma \in V_\Sigma, \text{ for all } \bar{\pi}^? \in \Gamma \quad (5)$$

These inequalities for these variables $x_\Gamma, x_e \in \{0, 1\}$ have the following interpretation:

- (2) ensures that both endpoints of an edge present in the solution are present;
- (3) ensures that at most one edge is selected between two selected nodes;
- (4) forces the solution to be a tree with at least one node;
- (5) ensures that if a node Γ with an indeterminate sign $\bar{\pi}^?$ is present in the solution, then there is exactly one edge of that color incident to Γ .

For Zheng's example Z , there are 99 456 eligible GP polynomials and 6 892 320 edges in the GP graph. The integer program takes 10 seconds to set up from scratch in `polymake 3.4.3` [GJ00], and about 4 hours and 20 GB of memory to solve on 8 threads using `gurobi 9.1.2`.

3.1 Topological Prismatoids

In [CS19], Criado and Santos introduced *topological prismatoids*, a combinatorial abstraction of the geometric prismatoids used by Santos [San12] to construct counterexamples to the Hirsch conjecture. Criado and Santos construct four combinatorially distinct non- d -step topological 4-dimensional prismatoids on 14 vertices, referred to as #1039, #1963, #2669 and #3513, which imply the existence of 8-dimensional spheres on 18 vertices whose combinatorial diameter exceeds the Hirsch bound. In [CS19], the question of polytopality of these combinatorial prismatoids was left open.

Theorem 2 *The topological prismatoids #1039, #1963, #2669 and #3513 are not polytopal.* $\square$

3.2 Novik and Zheng's centrally symmetric neighborly d -spheres

In [NZ19], Novik and Zheng give several constructions of centrally symmetric, highly neighborly d -spheres. They are based on a family Δ_n^d of $\text{cs-}\lceil \frac{d}{2} \rceil$ -neighborly combinatorial d -spheres on $2n \geq 2d + 2$ vertices, which arise as the case $i = \lceil \frac{d}{2} \rceil$ of an inductively constructed family $\Delta_n^{d,i}$ of $\text{cs-}i$ -neighborly combinatorial d -spheres. Each of *those* contains a certain combinatorial d -ball $B_n^{d,i-1}$, which is the only part that gets deleted in a step of the inductive construction. For $d = 3$, Novik and Zheng's family $\{\Delta_n^3 : n \geq 4\}$ is precisely Jockusch's family from [Joc95].

Theorem 3 *For $n \geq 6$, no member Δ_n^4 of Novik and Zheng's family is realizable.* $\square$

Theorem 4 (with [Zhe20b]) *For $n - 2 \geq d \geq 3$, no member Δ_n^d of Novik and Zheng's family is realizable.*

References

- [CS19] Francisco Criado and Francisco Santos. Topological Prisms and Small Simplicial Spheres of Large Diameter. *Experimental Mathematics*, 2019. <https://doi.org/10.1080/10586458.2019.1641766>.
- [GJ00] Evgenij Gawrilow and Michael Joswig. `polymake`: a framework for analyzing convex polytopes. In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 43–73. Birkhäuser, Basel, 2000.
- [Joc95] William Jockusch. An infinite family of nearly neighborly centrally symmetric 3-spheres. *J. Comb. Theory, Ser. A*, 72(2):318–321, 1995.
- [NZ19] Isabella Novik and Hailun Zheng. Highly neighborly centrally symmetric spheres, 2019.
- [San12] Francisco Santos. A counterexample to the Hirsch conjecture. *Ann. Math. (2)*, 176(1):383–412, 2012.
- [Zhe20a] Hailun Zheng. Ear decomposition and balanced neighborly simplicial manifolds. *Electron. J. Comb.*, 27(1):research paper p1.10, 17, 2020.
- [Zhe20b] Hailun Zheng. personal communication, 2020.