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On the approximate problem for the incremental thermoelasticity

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Abstract

In this work, we study an approximate problem arising in the incremental thermoelasticity. The existence of a unique solution is proved applying the theory of linear semigroups. The exponential energy decay is also considered. Then, fully discrete approximations are considered using the finite element method and the implicit Euler scheme. A discrete stability property and a priori error estimates are shown. Finally, numerical simulations are presented to demonstrate the accuracy of the approximations and the behavior of the model. In view of our numerical results the approximate problem could be used only when a certain parameter is small if we compare it with other parameters arising in the problem.

Keywords: Approximate problem, incremental thermoelasticity, decay of solutions, finite elements, error estimates, numerical simulations.

1. Introduction

The study of elastic materials deformed from a nonzero stress and with initial heat flux is usually known as *incremental* problem of thermoelasticity. It

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was firstly considered by several authors for null initial flux (see, for instance, [6, 7, 14]), but Ieşan extended these contributions to include nonzero initial flux. This problem has been considered from several points of view [8, 9, 13]. In particular, we can recall the existence of solutions [16] obtained with the help of the linear semigroups theory. Even more, the incremental problem has been also studied in the case of alternative thermomechanical theories [10, 11, 12]; however, it is worth recalling that the study of the stability of solutions for this problem seems to be a very difficult issue, and the only result that we know is the numerical study proposed in [4], where it was proved the instability of the solutions under suitable data for the problem.

As we consider a very difficult problem to be studied, it seems natural to try to understand the problem under *suitable* assumptions which can simplify the issue. There exist two typical ways to simplify the linear problem of the thermoelasticity. One corresponds to assume the *quasi-static* situation, where it is considered that the acceleration is small enough that it can be negligible. The other one corresponds to the *uncoupled* situation, where the velocity component in the heat equation is negligible. We can find in the literature many ways to justify these assumptions (see [15] and the references cited therein). In most of the engineering applications it is natural to consider several simplifying assumptions without relevant error. Even more, it can be also considered the *approximate* problem of thermoelasticity where both simplifications are imposed at the same time. In fact, we can recall that Boley and Weiner [1, Chapter 2] and Carlson (see [3, p. 311]) proposed this situation for the linear thermoelasticity, giving substantial justifications to this kind of approximations.

As the incremental proposition in the case of the thermoelasticity seems to be a difficult problem to be studied, it is natural to try to understand simplified versions of the situation and to study the approximate proposition of the incremental system. That is, to consider the *approximate* version of the incremental problem. We mean to consider the system obtained when we neglect the acceleration of the displacement for the motion system and the velocity term for the heat equation. We believe that this could be a way to try to obtain a first

knowledge about the behavior of the solutions for the incremental problem of the thermoelasticity. In this paper, we will try to make a first contribution in this line.

The paper is structured in the following way. In the next section, we state the basic equations of the model and we prove the existence of a unique solution. Later, we study the energy decay in Section 3. Then, in Section 4 we introduce a fully discrete approximation by using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. A discrete stability property and a priori error estimates are proved. Finally, some numerical simulations are performed to demonstrate the accuracy of the approximation and the behavior of the solution. In particular, we will see that the solutions to the exact problem blows up whenever a certain coefficient is assumed large enough.

2. Basic Equations

We shall denote by B a domain in $\mathbb{R}^d$, for $d = 1, 2, 3$, with boundary smooth enough to guarantee the use of the divergence theorem.

The equations governing the problem of the incremental thermoelasticity are given by the evolution equations

$$\rho \ddot{u}_i = t_{ij,j}, \quad \rho T \dot{\eta} = q_{k,k},$$

where

$$t_{ij} = C_{ijkl} u_{k,l} + \beta_{ij} \theta, \quad \rho \eta = c \theta - \beta_{ij} u_{i,j}, \quad q_k = h_{ijk} u_{i,j} + a_k \theta + k_{ik} \theta_{,i}.$$

Here, u_i is the incremental displacement vector, t_{ij} is the incremental stress tensor, η is the incremental entropy, q_i is the incremental heat flux vector, T is the temperature in the primary configuration, θ is the incremental temperature, C_{ijkl} is the incremental elasticity tensor, β_{ij} is the incremental coupling term, ρ is the mass density, c is the thermal capacity, k_{ij} is the incremental thermal conductivity tensor and h_{ijk}, a_k are two tensors which are typical of

the incremental theory. It is worth recalling that the major symmetry

$$C_{ijkl} = C_{klij}$$

is satisfied.

In the context of the incremental problem it is usual to assume that $\rho(\mathbf{x})$, $c(\mathbf{x})$ and $T(\mathbf{x})$ are functions strictly positive in the domain B that we consider.

We also assume that the elasticity tensor is definite positive. That is, there exists a positive constant C_1 such that

$$\int_B C_{ijkl} u_{i,j} u_{k,l} dv \geq C_1 \int_B u_{i,j} u_{i,j} dv, \quad (1)$$

for every smooth field vector u_i vanishing on the boundary.

The thermal conductivity tensor is positive definite. That is, there exists a positive constant K_1 such that

$$\frac{1}{T} k_{ij} \xi_i \xi_j \geq K_1 \xi_i \xi_i, \quad (2)$$

for every vector ξ_i .

If we substitute the constitutive equations into the evolution equations we obtain the system:

$$\rho \ddot{u}_i = (C_{ijkl} u_{k,l} + \beta_{ij} \theta)_{,j}, \quad T \dot{\theta} = \beta_{i,j} \dot{u}_{i,j} + (h_{ijk} u_{i,j} + a_k \theta + k_{ik} \theta_{,i})_{,k}.$$

As we are going to consider the *approximate problem*, we drop the terms concerning the acceleration and the velocity and so, we obtain the system:

$$(C_{ijkl} u_{k,l} + \beta_{ij} \theta)_{,j} = 0, \quad T \dot{\theta} = (h_{ijk} u_{i,j} + a_k \theta + k_{ik} \theta_{,i})_{,k}. \quad (3)$$

We will study this problem when we assume the boundary conditions:

$$u_i(\mathbf{x}, t) = \theta(\mathbf{x}, t) = 0 \quad \text{for a.e. } \mathbf{x} \in \partial B, \quad t \geq 0, \quad (4)$$

and the initial condition:

$$\theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}) \quad \text{for a.e. } \mathbf{x} \in B. \quad (5)$$

The proposed system is very *abstract* in the sense that we cannot identify the precise values of the tensors because they depend of many aspects. For this

reason, it would be useful to consider a particular (and perhaps the easiest) case for the problem. Therefore, we want to recall the situation proposed in [13, page 41], when the primary state is identical with the initial body and we consider that the body is subjected to a constant stress and the nonuniform temperature:

$$\zeta = T_1 + T_2 x_1, \quad T_1 > 0, \quad T_2 > 0.$$

In the case of an isotropic and homogeneous material we can find the system of equations that we can find in [13, page 41, system (2.1.80)]. Moreover, if we look for one-dimensional solutions of the form:

$$u_1 = u(x_1), \quad u_2 = u_3 = 0, \quad \theta = \theta(x_1),$$

then we see that u, θ satisfy the system

$$0 = \mu u_{xx} + \beta \theta_x, \quad \zeta \dot{\theta} = \hbar u_{xx} + m \theta_x + \kappa \theta_{xx}, \quad (6)$$

where we have dropped the index “1” and $\mu > 0, \beta, \hbar, m$ and $\kappa > 0$ are constants. When we study this last system, we need to consider the problem in an interval $[0, L]$ and we assume the boundary conditions:

$$u(0, t) = u(L, t) = \theta(0, t) = \theta(L, t) = 0 \quad \text{for } t > 0, \quad (7)$$

and the initial condition:

$$\theta(x, 0) = \theta^0(x) \quad \text{for } x \in (0, L). \quad (8)$$

Now, we will give an existence and uniqueness result of solutions to the multidimensional problem determined by system (3), boundary conditions (4) and initial conditions (5).

To this end, it will be relevant to note that, for every $\theta(\mathbf{x}, t)$ in $L^2(B)$, we can define a function $\mathbf{u}(\mathbf{x}, t) \in [W^{2,2}(B)]^d \cap [W_0^{1,2}(B)]^d$ such that $(C_{ijkl} u_{k,l})_{,j} = -(\beta_{ij} \theta)_{,j}$ and we denote by $\mathbf{u} = \mathcal{L}\theta$. In the case that we consider the operator

$$\mathcal{H}_s \theta = (h_{irs}(\mathcal{L}\theta)_{i,r}),$$

we see that it defines a bounded operator from $L^2(B)$ to $[L^2(B)]^d$. To be precise, we can find a constant $C(\epsilon)$ depending on an arbitrary parameter ϵ , which is assumed to be as small as we want, such that

$$\int_B \mathcal{H}_i \theta \theta_{,i} dv \leq C \int_B \theta^2 dv + \epsilon \int_B k_{ij} \theta_{,i} \theta_{,j} dv.$$

Therefore, it is possible to note that we can define a semigroup quasi-contractive in the Hilbert space $L^2(B)$ defined by the operator:¹

$$\mathcal{A}\theta = \frac{1}{T}(\mathcal{H}_i \theta + a_i \theta + k_{ij} \theta_{,j})_{,i},$$

and our problem can be written as

$$\dot{\theta} = \mathcal{A}\theta, \quad \theta(0) = \theta^0.$$

It is clear that the domain of this operator is $W^{2,2}(B) \cap W_0^{1,2}(B)$. It is also obvious that there exists a positive constant K such that

$$\langle \mathcal{A}\theta, \theta \rangle \leq K \|\theta\|^2.$$

At the same time, it can be proved the existence of a positive constant ξ such that $\xi I - \mathcal{A}$ is surjective.

Thus, we can obtain $\theta(\mathbf{x}, t)$ for every initial condition in the Hilbert space $L^2(B)$ and so, we can also find $\mathbf{u}(\mathbf{x}, t)$ solving the respective equation for the displacement. This solution for the displacement belongs to $[W^{2,2}(B)]^d \cap [W_0^{1,2}(B)]^d$. Therefore, the existence and uniqueness of solutions to our problem is proved.

3. Exponential decay

In this section, we prove the exponential decay of the solutions under suitable conditions on the constitutive tensors. In fact, apart of the conditions proposed

¹In fact, it can be proved that the semigroup is analytic, but we do not pay attention to this fact in this work.

previously we assume that there exist three constants λ_i , for $i = 1, 2, 3$, such that

$$\begin{aligned} \int_B (\lambda_1 C_{ijkl} u_{i,j} u_{k,l} + \lambda_1 \beta_{ij} u_{i,j} \theta + h_{irs} u_{r,s} \theta_{,i} + a_i \theta_{,i} \theta + k_{ij} \theta_{,i} \theta_{,j}) dv \\ \geq \int_B (\lambda_2 \theta_{,i} \theta_{,i} - \lambda_3 \theta^2) dv, \end{aligned} \quad (9)$$

with $\lambda_2 \lambda^* - \lambda_3 > 0$, where λ^* is the first eigenvalue of the Laplacian operator with null Dirichlet boundary conditions. That is,

$$\int_B |\nabla \phi|^2 dv \geq \lambda^* \int_B |\phi|^2 dv$$

for every function ϕ vanishing at the boundary.

In this situation, we can define the function

$$E(t) = \frac{1}{2} \int_B T \theta^2 dv.$$

We see that

$$E'(t) = - \int_B (\lambda_1 C_{ijkl} u_{i,j} u_{k,l} + \lambda_1 \beta_{ij} u_{i,j} \theta + h_{irs} u_{r,s} \theta_{,i} + a_i \theta_{,i} \theta + k_{ij} \theta_{,i} \theta_{,j}) dv.$$

In view of assumption (9) we find that

$$E'(t) \leq -(\lambda_2 \lambda^* - \lambda_3) \int_B \theta^2 dv,$$

and so, it follows that there exists a positive constant η such that

$$E'(t) + \eta E(t) \leq 0 \quad \forall t \geq 0.$$

This inequality implies that $E(t) \leq E(0)e^{-\eta t}$ and therefore, we obtain the exponential decay for the temperature.

To prove the decay for the displacement we note that

$$\int_B C_{ijkl} u_{i,j} u_{k,l} dv = - \int_B \beta_{ij} u_{i,j} \theta dv \leq \frac{1}{2} \int_B C_{ijkl} u_{i,j} u_{k,l} dv + C^* \int_B \theta^2 dv.$$

It then follows that

$$\int_B C_{ijkl} u_{i,j} u_{k,l} dv \leq 2C^* \int_B \theta^2 dv \leq C^{**} E(t),$$

for a positive constant C^{**} .

If we denote

$$E^*(t) = \int_B C_{ijkl} u_{i,j} u_{k,l} dv,$$

then we have $E^*(t) \leq C^{**} E(0) e^{-\eta t}$, which implies that the $W^{1,2}$ -norm of the displacement also decays in an exponential way.

Remark 3.1. *The previous analysis can be adapted straightforwardly if we change condition (9), replacing the term “ $a_i \theta_{,i} \theta$ ” by “ $-a_{i,i} \theta^2/2$ ”.*

Remark 3.2. *The previous energy decay has been proved in the general case; however, in the one-dimensional case it can be seen in a direct way. First, we note that we can reduce our system to a unique equation:*

$$\zeta \dot{\theta} = \left(-\frac{\hbar \beta}{\mu} + m\right) \theta_x + \kappa \theta_{xx}.$$

If we define now the function

$$E(t) = \frac{1}{2} \int_0^L \zeta \theta^2(t) dx,$$

we see that

$$E'(t) = -\kappa \int_0^L \theta_x^2(t) dx.$$

The decay can be concluded directly after the use of the Poincaré inequality. Again, the exponential decay for u can be easily obtained.

Remark 3.3. *In fact, in the case that we include the gradient of the velocity in the heat equation² and we impose Neumann boundary conditions for the displacement, we would obtain the equation:*

$$\zeta \dot{\theta} + \frac{\beta^2}{\mu} \dot{\theta} = \left(m - \frac{\hbar \beta}{\mu}\right) \theta_x + \kappa \theta_{xx}.$$

It is clear that we can adapt in a very easy way the results about existence and exponential decay of solutions.

²It corresponds to the usual quasi-static problem.

4. Numerical analysis of a fully discrete scheme

In this section, we numerically analyze a variational formulation of problem (3)-(5). So, we assume that the time interval is fixed as $[0, \tau]$, where τ denotes the final time.

By using boundary conditions (4), denoting by $Y = L^2(B)$, $H = [L^2(B)]^d$, $E = W_0^{1,2}(B)$ and $V = [W_0^{1,2}(B)]^d$, thermomechanical problem (3)-(5) leads to the following variational formulation, for a.e. $t \in [0, \tau]$ and $\mathbf{w} = (w_i)_{i=1}^d \in V$, $r \in E$,

$$(C_{ijkl}u_{k,l}(t), w_{i,j})_Y = -(\beta_{ij}\theta(t), w_{i,j})_Y, \quad (10)$$

$$(T\dot{\theta}(t), r)_Y + (k_{ij}\theta_{,i}(t), r_{,j})_Y = -(h_{ijk}u_{i,j}(t), r_{,k})_Y - (a_k\theta(t), r_{,k})_Y, \quad (11)$$

where $\theta(0) = \theta^0$ and we have used the notation $(\cdot, \cdot)_Y$ for the usual scalar product in Y .

We note that from the previous section, and using well-known arguments on the theory of linear semigroups, we can conclude the regularity on the solution to the previous variational problem:

$$\mathbf{u} \in C([0, T]; V), \quad \theta \in C^1([0, T]; Y) \cap C([0, T]; E).$$

For the spatial approximation of the above problem, we assume the domain $\overline{B}$ is polyhedral and we denote by $\mathcal{T}^h$ a regular triangulation in the sense of [5]. Thus, we construct the finite dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by

$$\begin{aligned} E^h &= \{r^h \in C(\overline{B}) \cap E; r|_{Tr}^h \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h\}, \\ V^h &= \{\mathbf{w}^h \in [C(\overline{B})]^d \cap V; \mathbf{w}|_{Tr}^h \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h\}, \end{aligned}$$

where $P_1(Tr)$ represents the space of polynomials of degree less or equal to one in the element Tr , i.e. the finite element spaces V^h and E^h are composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial condition θ^{0h} is given by

$$\theta^{0h} = \mathcal{P}_{E^h}^h \theta^0,$$

where $\mathcal{P}_{E^h}^h$ is the classical finite element interpolation operator over E^h (see, e.g., [5]).

In order to provide the time discretization of the previously given variational problem, we consider a uniform partition of the time interval $[0, \tau]$, denoted by $0 = t_0 < t_1 < \dots < t_N = \tau$, with constant step size $k = \tau/N$ and nodes $t_n = nk$ for $n = 0, 1, \dots, N$.

Therefore, using the well-known implicit Euler scheme, the fully discrete approximation is the following.

Find the discrete displacement field $\mathbf{u}^{hk} = \{\mathbf{u}_n^{hk}\}_{n=1}^N \subset V^h$ and the discrete temperature $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset E^h$ such that $\theta_0^{hk} = \theta^{0h}$, and, for $n = 1, \dots, N$ and for all $\mathbf{w}^h \in V^h$ and $r^h \in E^h$,

$$(C_{ijkl}(\mathbf{u}_n^{hk})_{k,l}, w_{i,j}^h)_Y = -(\beta_{ij}\theta_n^{hk}, w_{i,j}^h)_Y, \quad (12)$$

$$(T(\theta_n^{hk} - \theta_{n-1}^{hk})/k, r^h)_Y + (k_{ij}(\theta_n^{hk})_{i,j}, r_{,j}^h)_Y = -(h_{ijk}(\mathbf{u}_n^{hk})_{i,j}, r_{,k}^h)_Y - (a_k\theta_n^{hk}, r_{,k}^h)_Y. \quad (13)$$

We note that the above fully discrete problem leads to an uncoupled linear system which is solved by using Cholesky's method. Moreover, thanks to conditions (1)-(2), the existence of a unique discrete solution is guaranteed.

First, we have the following discrete stability property.

Theorem 4.1. *Under assumptions (1)-(2), it follows that there exists a positive constant C , independent of the discretization parameters h and k , such that*

$$\|\mathbf{u}_n^{hk}\|_V^2 + \|\theta_n^{hk}\|_Y^2 \leq C,$$

where we used the notation $\|\cdot\|_V$ to represent the norm in V .

Proof. Taking the test function $\mathbf{w}^h = \mathbf{u}_n^{hk}$ as a test function in discrete variational equation (12) we find that

$$(C_{ijkl}(\mathbf{u}_n^{hk})_{k,l}, (\mathbf{u}_n^{hk})_{i,j})_Y = -(\beta_{ij}\theta_n^{hk}, (\mathbf{u}_n^{hk})_{i,j})_Y,$$

and, keeping in mind condition (1), applying the following arithmetic geometric inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad (14)$$

with $\epsilon > 0$ is small enough, we have

$$\|\mathbf{u}_n^{hk}\|_V^2 \leq C\|\theta_n^{hk}\|_Y^2.$$

Now, taking as a test function $r^h = \theta_n^{hk}$ in discrete variational equation (13) we obtain

$$\begin{aligned} (T(\theta_n^{hk} - \theta_{n-1}^{hk})/k, \theta_n^{hk})_Y + (k_{ij}(\theta_n^{hk})_{,i}, (\theta_n^{hk})_{,j})_Y = & -(h_{ijk}(\mathbf{u}_n^{hk})_{i,j}, (\theta_n^{hk})_{,k})_Y \\ & - (a_k \theta_n^{hk}, (\theta_n^{hk})_{,k})_Y, \end{aligned}$$

and therefore, using again Cauchy's inequality (14), condition (2) and keeping in mind that

$$(T(\theta_n^{hk} - \theta_{n-1}^{hk})/k, \theta_n^{hk})_Y \geq \frac{C}{2k} \{ \|\theta_n^{hk}\|_Y^2 - \|\theta_{n-1}^{hk}\|_Y^2 \},$$

it leads to the following estimates on the discrete temperature:

$$\frac{1}{k} \{ \|\theta_n^{hk}\|_Y^2 - \|\theta_{n-1}^{hk}\|_Y^2 \} \leq C \left(\|\theta_n^{hk}\|_Y^2 + \|\mathbf{u}_n^{hk}\|_V^2 \right).$$

Multiplying the above estimates by k and summing up to n we have

$$\|\theta_n^{hk}\|_Y^2 \leq Ck \sum_{j=1}^n \left(\|\theta_j^{hk}\|_Y^2 + \|\mathbf{u}_j^{hk}\|_V^2 \right) + \|\theta^{0h}\|_Y^2.$$

Combining the previous estimates on the discrete displacement and temperature it follows that

$$\|\theta_n^{hk}\|_Y^2 + \|\mathbf{u}_n^{hk}\|_V^2 \leq Ck \sum_{j=1}^n \left(\|\theta_j^{hk}\|_Y^2 + \|\mathbf{u}_j^{hk}\|_V^2 \right) + C\|\theta^{0h}\|_Y^2.$$

Finally, using a discrete version of Gronwall's inequality (see, for instance, [2]), we conclude the discrete stability property.

In the rest of this section, we will prove some a priori error estimates on the numerical errors $\mathbf{u}_n - \mathbf{u}_n^{hk}$ and $\theta_n - \theta_n^{hk}$.

Theorem 4.2. *Under assumptions (1)-(2), it follows that there exists a positive constant C , independent of the discretization parameters h and k , such that*

$$\begin{aligned} \max_{0 \leq n \leq N} \{ \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 \} & \leq Ck \sum_{j=1}^N \left(\|\dot{\theta}_j - (\theta_j - \theta_{j-1})/k\|_Y^2 \right. \\ & \quad \left. + \|\theta_j - r_j^h\|_E^2 + \|\mathbf{u}_j - \mathbf{w}_j^h\|_V^2 \right) + \frac{C}{k} \sum_{j=1}^{N-1} \|\theta_j - r_j^h - (\theta_{j+1} - r_{j+1}^h)\|_Y^2 \\ & \quad + C \max_{0 \leq n \leq N} \|\theta_n - r_n^h\|_Y^2 + C\|\theta^0 - \theta^{0h}\|_Y^2, \end{aligned}$$

where the (artificial) initial conditions for the displacements $\mathbf{u}_0$ and $\mathbf{u}_0^{hk}$ are the respective solutions to the following problems:

$$\begin{aligned} (C_{ijkl}(\mathbf{u}_0)_{k,l}, w_{i,j})_Y &= -(\beta_{ij}\theta_0, w_{i,j})_Y \quad \forall \mathbf{w} \in V, \\ (C_{ijkl}(\mathbf{u}_n^{hk})_{k,l}, w_{i,j}^h)_Y &= -(\beta_{ij}\theta_n^{hk}, w_{i,j}^h)_Y \quad \forall \mathbf{w}^h \in V^h. \end{aligned}$$

Proof. First, we obtain the error estimates for the displacement field. Thus, we subtract variational equation (10) at time $t = t_n$ for a test function $\mathbf{w} = \mathbf{w}^h \in V^h$ and discrete variational equation (12) to obtain

$$(C_{ijkl}(\mathbf{u}_n - \mathbf{u}_n^{hk})_{k,l}, w_{i,j}^h)_Y = -(\beta_{ij}(\theta_n - \theta_n^{hk}), w_{i,j}^h)_Y,$$

and therefore, for all $\mathbf{w}^h \in V^h$ we find that

$$\begin{aligned} (C_{ijkl}(\mathbf{u}_n - \mathbf{u}_n^{hk})_{k,l}, (\mathbf{u}_n - \mathbf{u}_n^{hk})_{i,j})_Y &+ (\beta_{ij}(\theta_n - \theta_n^{hk}), (\mathbf{u}_n - \mathbf{u}_n^{hk})_{i,j})_Y \\ &= (C_{ijkl}(\mathbf{u}_n - \mathbf{u}_n^{hk})_{k,l}, (\mathbf{u}_n - \mathbf{w}^h)_{i,j})_Y + (\beta_{ij}(\theta_n - \theta_n^{hk}), (\mathbf{u}_n - \mathbf{w}^h)_{i,j})_Y. \end{aligned}$$

Using assumption (1) and applying several times Cauchy's inequality (14) we find that

$$\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 \leq C(\|\mathbf{u}_n - \mathbf{w}^h\|_V^2 + \|\theta_n - \theta_n^{hk}\|_Y^2). \quad (15)$$

Now, we derive the error estimates for the temperature field. So, subtracting variational equation (11) at time $t = t_n$ for a test function $r = r^h \in E^h$ and discrete variational equation (13) we have

$$\begin{aligned} (T(\dot{\theta}_n - (\theta_n^{hk} - \theta_{n-1}^{hk})/k), r^h)_Y &+ (k_{ij}(\theta_n - \theta_n^{hk})_{,i}, r_{,j}^h)_Y \\ &+ (h_{ijk}(\mathbf{u}_n - \mathbf{u}_n^{hk})_{i,j}, r_{,k}^h)_Y - (a_k(\theta_n - \theta_n^{hk}), r_{,k}^h)_Y = 0, \end{aligned}$$

and therefore we obtain, for all $r^h \in E^h$,

$$\begin{aligned} (T(\theta_n - \theta_{n-1} - (\theta_n^{hk} - \theta_{n-1}^{hk}))/k, \theta_n - \theta_n^{hk})_Y &+ (k_{ij}(\theta_n - \theta_n^{hk})_{,i}, (\theta_n - \theta_n^{hk})_{,j})_Y \\ &+ (h_{ijk}(\mathbf{u}_n - \mathbf{u}_n^{hk})_{i,j}, (\theta_n - \theta_n^{hk})_{,k})_Y - (a_k(\theta_n - \theta_n^{hk}), (\theta_n - \theta_n^{hk})_{,k})_Y \\ &= (T(\theta_n - \theta_{n-1} - (\theta_n^{hk} - \theta_{n-1}^{hk}))/k, \theta_n - r^h)_Y + (k_{ij}(\theta_n - \theta_n^{hk})_{,i}, (\theta_n - r^h)_{,j})_Y \\ &\quad - (a_k(\theta_n - \theta_n^{hk}), (\theta_n - r^h)_{,k})_Y + (h_{ijk}(\mathbf{u}_n - \mathbf{u}_n^{hk})_{i,j}, (\theta_n - r^h)_{,k})_Y. \end{aligned}$$

Taking into account that

$$\begin{aligned} (T(\dot{\theta}_n - (\theta_n^{hk} - \theta_{n-1}^{hk})/k), \theta_n - \theta_n^{hk})_Y &= (T(\dot{\theta}_n - (\theta_n - \theta_{n-1})/k), \theta_n - \theta_n^{hk})_Y \\ &\quad + (T(\theta_n - \theta_{n-1} - (\theta_n^{hk} - \theta_{n-1}^{hk}))/k, \theta_n - \theta_n^{hk})_Y \\ &\geq (T(\dot{\theta}_n - (\theta_n - \theta_{n-1})/k), \theta_n - \theta_n^{hk})_Y + \frac{C}{2k} \left\{ \|\theta_n - \theta_n^{hk}\|_Y^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_Y^2 \right\}, \end{aligned}$$

applying several times Cauchy's inequality (14) and using condition (2), we have, for all $r^h \in E^h$,

$$\frac{1}{k} \left\{ \|\theta_n - \theta_n^{hk}\|_Y^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_Y^2 \right\} \leq C \left(\|\dot{\theta}_n - (\theta_n - \theta_{n-1})/k\|_Y^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 + \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 + \|\theta_n - r^h\|_E^2 + (T(\theta_n - \theta_{n-1} - (\theta_n^{hk} - \theta_{n-1}^{hk}))/k, \theta_n - r^h)_Y \right).$$

Multiplying these estimates by k and summing up to n we find that

$$\begin{aligned} \|\theta_n - \theta_n^{hk}\|_Y^2 &\leq Ck \sum_{j=1}^n \left(\|\dot{\theta}_j - (\theta_j - \theta_{j-1})/k\|_Y^2 + \|\theta_j - \theta_j^{hk}\|_Y^2 + \|\mathbf{u}_j - \mathbf{u}_j^{hk}\|_V^2 \right. \\ &\quad \left. + \|\theta_j - r_j^h\|_E^2 + (T(\theta_j - \theta_{j-1} - (\theta_j^{hk} - \theta_{j-1}^{hk}))/k, \theta_j - r_j^h)_Y \right) \\ &\quad + C\|\theta^0 - \theta^{0h}\|_Y^2 \quad \forall \{r_j^h\}_{j=1}^n \subset E^h. \end{aligned} \quad (16)$$

Combining estimates (15) and (16) we have, for all $\mathbf{w}_n^h \in V^h$ and $\{r_j^h\}_{j=1}^n \subset E^h$,

$$\begin{aligned} \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 &\leq Ck \sum_{j=1}^n \left(\|\dot{\theta}_j - (\theta_j - \theta_{j-1})/k\|_Y^2 + \|\theta_j - \theta_j^{hk}\|_Y^2 \right. \\ &\quad \left. + \|\mathbf{u}_j - \mathbf{u}_j^{hk}\|_V^2 + \|\theta_j - r_j^h\|_E^2 + (T(\theta_j - \theta_{j-1} - (\theta_j^{hk} - \theta_{j-1}^{hk}))/k, \theta_j - r_j^h)_Y \right. \\ &\quad \left. + \|\mathbf{u}_n - \mathbf{w}_n^h\|_V^2 \right) + C\|\theta^0 - \theta^{0h}\|_Y^2. \end{aligned}$$

It is straightforward to prove the following estimates:

$$\begin{aligned} k \sum_{j=1}^n (T(\theta_j - \theta_{j-1} - (\theta_j^{hk} - \theta_{j-1}^{hk}))/k, \theta_j - r_j^h)_Y &= (T(\theta_n - \theta_n^{hk}), \theta_n - r_n^h)_Y \\ &\quad + (T(\theta^0 - \theta^{0h}), \theta_1 - r_1^h)_Y + \sum_{j=1}^{n-1} (T(\theta_j - \theta_j^{hk}), \theta_j - r_j^h - (\theta_{j+1} - r_{j+1}^h))_Y. \end{aligned}$$

Finally, applying a discrete version of Gronwall's inequality (see, again, [2]), from the above estimates we conclude the desired a priori error estimates.

The estimates provided in Theorem 4.2 can be used to obtain the convergence order of the approximations given by problem (12)-(13). As an example, if we assume the additional regularity on the solution to problem (10)-(11):

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, \tau; [W^{2,2}(B)]^2), \\ \theta &\in W^{2,2}(0, \tau; Y) \cap L^\infty(0, \tau; W^{2,2}(B)) \cap W^{1,2}(0, \tau; W^{1,2}(B)), \end{aligned} \quad (17)$$

then we have the following result which states the linear convergence of the approximations given by problem (12)-(13).

Corollary 4.3. *Let the assumptions of Theorem 4.2 hold. If we assume the additional regularity (17) then there exists a positive constant C , independent of the discretization parameters h and k , such that*

$$\max_{0 \leq n \leq N} \left\{ \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\|_Y \right\} \leq C(h + k).$$

5. Numerical results

In this section, we describe some numerical results to show the accuracy of the finite element approximation and a comparison with respect to the original incremental thermoelastic problem.

The numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical one-dimensional run (with parameters $h = k = 0.01$) took about 0.59 seconds of CPU time.

5.1. First example: numerical convergence

As an academical example, in order to show the accuracy of the approximations we solve problem (6)-(8) we use the data:

$$\begin{aligned} \tau = 1, \quad L = 1, \quad T_1 = 1, \quad T_2 = 1, \quad \hbar = 1, \quad \mu = 1, \quad \kappa = 1, \quad \beta = 1, \\ m = 1, \end{aligned}$$

the initial condition, for $x \in (0, 1)$,

$$\theta^0(x) = x(x - 1),$$

the boundary conditions, for $t \in (0, 1)$ and $x = 0, 1$,

$$u(x, t) = \theta(x, t) = 0,$$

and the (artificial) supply terms, for $(x, t) \in (0, 1) \times (0, 1)$,

$$\begin{aligned} F_1(x, t) &= e^t((2x - 1) - 2), \\ F_2(x, t) &= e^t(x(x - 1) + x^2(x - 1) - 4 - (2x - 1)). \end{aligned}$$

With the previous data and conditions, we can calculate the exact solution to problem (6)-(8) and it has the following form, for $(x, t) \in [0, 1] \times [0, 1]$:

$$u(x, t) = \theta(x, t) = e^t x(x - 1).$$

Thus, the approximation errors estimated by

$$\max_{0 \leq n \leq N} \left\{ \|u_n - u_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\|_V \right\}$$

are shown in Table 1 for several values of the discretization parameters h and k . Moreover, the evolution of the error depending on the parameter $h + k$ is plotted in Figure 1. We notice that the convergence of the algorithm is clearly proved, and the linear convergence, stated in Corollary 4.3, is achieved.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^2$	0.212270	0.212361	0.212421	0.212441	0.212452	0.212458	0.212460
$1/2^3$	0.102204	0.102264	0.102316	0.102336	0.102346	0.102352	0.102355
$1/2^4$	0.050212	0.050153	0.050176	0.050193	0.050203	0.050209	0.050211
$1/2^5$	0.025273	0.024928	0.024848	0.024851	0.024858	0.024863	0.024865
$1/2^6$	0.013105	0.012632	0.012406	0.012372	0.012369	0.012371	0.012373
$1/2^7$	0.007139	0.006556	0.006274	0.006198	0.006175	0.006171	0.006172
$1/2^8$	0.004338	0.003572	0.003228	0.003137	0.003098	0.003084	0.003083
$1/2^9$	0.003177	0.002171	0.001724	0.001614	0.001569	0.001546	0.001541
$1/2^{10}$	0.002786	0.001591	0.001003	0.000862	0.000807	0.000780	0.000773
$1/2^{11}$	0.002676	0.001396	0.000688	0.000501	0.000431	0.000399	0.000390
$1/2^{12}$	0.002647	0.001341	0.000575	0.000344	0.000251	0.000210	0.000199

Table 1: Example 1: Numerical errors for some h and k .

If we assume now that there are not supply terms and we use the same data as in the previous example, taking the discretization parameters $h = k = 0.001$, the evolution in time of the discrete energy E_n^{hk} given by

$$E_n^{hk} = \frac{1}{2} \int_0^1 \zeta(\theta_n^{hk})^2 dx$$

Figure 1: Example 1: Asymptotic constant error.

is plotted in Figure 2 (in both natural and semi-log scales). As can be seen, it converges to zero and an exponential decay seems to be achieved.

Figure 2: Evolution in time of the discrete energy (natural and semi-log scales).

5.2. Comparison with the incremental thermoelastic problem

In this final subsection, our aim is to provide the comparison with the results obtained in the original incremental thermoelastic problem. So, we recall that

the one-dimensional version is written as follows:

$$\begin{aligned}
\rho \ddot{u} &= \mu u_{xx} + \beta \theta_x \quad \text{in } (0, 1) \times (0, 1), \\
\zeta \dot{\theta} &= \beta^* \dot{u}_x + \hbar u_{xx} + m \theta_x + \kappa \theta_{xx} \quad \text{in } (0, 1) \times (0, 1), \\
u(0, t) &= u(L, t) = \theta(0, t) = \theta(L, t) = 0 \quad \text{for } t > 0, \\
u(x, 0) &= u^0(x), \quad v(x, 0) = v^0(x), \quad \theta(x, 0) = \theta^0(x) \quad \text{for } x \in (0, L),
\end{aligned}$$

where ρ is the density of the material, coefficient β^* satisfies $\beta\beta^* > 0$ and u^0 and v^0 are initial conditions for the displacement field.

The above problem has a similar variational form as (10)-(11) and it is approximated by the following fully discrete problem (see also (12)-(13)).

Find the discrete velocity field $v^{hk} = \{v_n^{hk}\}_{n=1}^N \subset V^h$ and the discrete temperature $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset E^h$ such that $\theta_0^{hk} = \theta^{0h}$, $u_0^{hk} = u^{0h}$ and $v_0^{hk} = v^{0h}$, and, for $n = 1, \dots, N$ and for all $w^h \in V^h$ and $r^h \in E^h$,

$$\begin{aligned}
\rho((v_n^{hk} - v_{n-1}^{hk})/k, w^h)_Y + \mu((u_n^{hk})_x, w_x^h)_Y &= -(\beta \theta_n^{hk}, w_x^h)_Y, \\
(T(\theta_n^{hk} - \theta_{n-1}^{hk})/k, r^h)_Y + \kappa((\theta_n^{hk})_x, r_x^h)_Y &= -(\hbar(u_n^{hk})_x, r_x^h)_Y \\
&\quad - (m \theta_n^{hk}, r_x^h)_Y + \beta^*((v_n^{hk})_x, r^h)_Y,
\end{aligned}$$

where the discrete initial conditions u^{0h} and v^{0h} are defined as

$$u^{0h} = P_{V^h} u^0, \quad v^{0h} = P_{V^h} v^0,$$

and the discrete displacement field u_n^{hk} is given by $u_n^{hk} = k v_n^{hk} + u_{n-1}^{hk}$.

Taking the discretization parameters $h = 10^{-3}$ and $k = 10^{-5}$, we solve the previous discrete problem with the following data:

$$\begin{aligned}
\tau &= 1, \quad L = 1, \quad T_1 = 0.1, \quad T_2 = 0.1, \quad \hbar = 1, \quad \mu = 2, \quad \kappa = 10, \quad \beta = 1, \\
m &= 1, \quad \beta^* = 1, \quad \rho = 0.1,
\end{aligned}$$

the initial conditions, for $x \in (0, 1)$,

$$u^0(x) = v^0(x) = \theta^0(x) = x(x - 1),$$

and the boundary conditions, for $t \in (0, 1)$ and $x = 0, 1$,

$$u(x, t) = \theta(x, t) = 0.$$

Therefore, in Figure 3 we plot the evolution in time of the discrete energy E_n^{hk} given now by

$$E_n^{hk} = \frac{1}{2} \int_0^1 \zeta(\theta_n^{hk})^2 + \mu(u_n^{hk})_x + \rho(v_n^{hk})^2 dx$$

for some values of parameter $\hbar$ ($= 1, 50, 100, 200$). As we can see, an exponential decay seems to be achieved although for the greater values $\hbar = 100, 200$ some oscillations appear.

Figure 3: Evolution in time of the discrete energy for values $\hbar = 1$ (upper left), $\hbar = 10$ (upper right), $\hbar = 100$ (down left) and $\hbar = 200$ (down right).

However, if we consider higher values of parameter $\hbar$ the behavior is com-

pletely different. In Figure 4 we show the evolution in time of the discrete energy for another values of parameter $\hbar$ ($= 500, 700, 1000, 2000$). As can be clearly seen, for values $\hbar = 500, 700$ big oscillations are found. Moreover, if we use the higher values $\hbar = 1000, 2000$ the solution clearly blows up.

Figure 4: Evolution in time of the discrete energy for values $\hbar = 500$ (upper left), $\hbar = 700$ (upper right), $\hbar = 1000$ (down left) and $\hbar = 2000$ (down right).

The results of this subsection suggest that the approximate problem can be considered to study the incremental theory whenever tensor h_{ijk} is sufficiently small; however, the behavior of the solutions is very different in the case that this tensor is large compared with the elasticity tensor and the thermal conduc-

tivity. Of course, to clarify this claim is a relevant question in the study of the incremental thermoelasticity.

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