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<https://www.tandfonline.com/doi/full/10.1080/00036811.2021.1892079>.

Published paper:

Liu, Z.; Quintanilla, R.; Wang, Y. On the regularity and stability of three-phase-lag thermoelastic plate. "Applicable analysis: an international journal", 25 Febrer 2021.
doi: [10.1080/00036811.2021.1892079](https://doi.org/10.1080/00036811.2021.1892079)

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On the Regularity and Stability of Three-Phase-Lag Thermoelastic Plate

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Abstract

In this paper we consider the following three-phase-lag thermoelastic plate

$$\begin{cases} \rho \ddot{u} = -\Delta(\mu \Delta u - m\theta), \\ c(\dot{\theta} + \tau_q \ddot{\theta}) = m\Delta(\dot{u} + \tau_q \ddot{u}) + \Delta(k^* \alpha + \tau_\nu^* \theta + k\tau_\theta \dot{\theta}). \end{cases}$$

We obtain analyticity and exponential stability for the associated C_0 -semigroup when the coefficients satisfy $\tau_\nu \geq k^* \tau_q$. The analyticity of the semigroup still holds when $\tau_\nu < k^* \tau_q$. However, instability of solution was proved by the Routh-Hurwitz rule.

Key words: three-phase-lag thermoelastic plate, analyticity, exponential stability.

MSC 2000 35Q35, 35Q30, 35L65, 76N10

1 Introduction

Causality principle is violated by the theory based on the Fourier law when we consider the heat equation

$$c\dot{\theta} = q_{i,i},$$

where c is the thermal capacity and q_i is the heat flux vector and θ is the temperature. To overcome this drawback many scientists have been interested in proposing alternative heat conduction theories. Cattaneo and Maxwell proposition is the first of these alternatives and it leads to a damped hyperbolic equation[1]. It has been extended by Lord and Shulman to a thermoelastic theory[2]. Some other theories have been proposed (see[3–6]).

In the decade of the 1990's several theories were proposed. On one side, Green and Naghdi proposed three thermoelastic theories[7–9] which were based on some axioms of thermoelasticity. On the other side, Tzou[10] proposed an alternative possibility based on an intuitive point of view which generalizes the Cattaneo and Maxwell theory in the case that we consider the Taylor approximations. More recently, Choudhuri[11] considered an extension of these two last theories. This is called three-phase-lag theory and recovers the Green and Naghdi theory when we consider Taylor approximations. This theory has deserved much attention in the last years[13, 14]. Quintanilla and Racke[16] gave the suitable conditions to clarify the stability of the heat equation. Other studies concerning the stability have been considered[17]. It

is also worth recalling that Borgmeyer[18] studied the thermoelastic problem and she obtained several stability results.

In this note we focus on the problem determined for a thermoelastic plate[19] in the case of a three-phase-lag theory. It is worth noting that Choudhuri proposed two particular heat conduction equations, and we here consider the case corresponding to a parabolic heat conducting equation. We will prove that in this case the solutions of the problem are determined by an analytic semigroup. Similar results have been obtained for other heat conduction theories[20, 21]. In addition, we also investigate the stability of this system. In section 2 we will prove the analyticity and exponential stability of the semigroup in the case that certain inequalities for the relaxation parameters hold. Section 3 is devoted to the case that one of the inequalities does not hold. We will prove that the semigroup remains to be analytic, but the stability is lost, which is proved by the Routh-Hurwitz rule.

System of three-phase-lag thermoelastic plate are given by

$$\begin{cases} \rho \ddot{u} = -\Delta(\mu \Delta u + m\theta), \\ c(\dot{\theta} + \tau_q \ddot{\theta}) = m\Delta(\dot{u} + \tau_q \ddot{u}) + \Delta(k^* \alpha + \tau_\nu^* \theta + k\tau_\theta \dot{\theta}). \end{cases} \quad (1.1)$$

Here $\dot{\alpha} = \theta$, and the system coefficients $\rho > 0, \mu > 0, c > 0, k^* > 0, \tau_\theta > 0, \tau_\nu^* = k + k^* \tau_\nu, \tau_\nu > 0, m \neq 0$.

After the change of variable $\hat{u} = u + \tau_q \dot{u}$, we can get

$$\begin{cases} \rho \ddot{u} = -\Delta(\mu \Delta u + m(\theta + \tau_q \dot{\theta})), \\ c(\dot{\theta} + \tau_q \ddot{\theta}) = m\Delta \dot{u} + \Delta(k^* \alpha + \tau_\nu^* \theta + k\tau_\theta \dot{\theta}). \end{cases} \quad (1.3)$$

$$(1.4)$$

We still use notation u instead of $\hat{u}$ hereafter.

Initial condition and boundary condition are

$$u(x, 0) = u^0(x), \dot{u}(x, 0) = v^0(x), \quad (1.5)$$

$$\alpha(x, 0) = \alpha^0(x), \dot{\alpha}(x, 0) = \theta^0(x), \ddot{\alpha}(x, 0) = \eta^0(x), \quad (1.6)$$

$$u = \Delta u = \alpha = 0, \quad \text{on } \partial\Omega, \quad (1.7)$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

2 Analyticity and Exponential Stability for the case $\tau_\nu^* \geq k^* \tau_q$

It is important to identify a proper state space so that the “energy” of the system (1.3)-(1.7) is dissipative. For this purpose, we take the inner product of $\dot{u}$ with (1.3) and $\theta + \tau_q \dot{\theta}$ with (1.4) in $L^2(\Omega)$ to get

$$\frac{1}{2} \frac{d}{dt} \rho \|\dot{u}\|^2 + \frac{1}{2} \frac{d}{dt} \mu \|\Delta u\|^2 = m \langle \nabla(\theta + \tau_q \dot{\theta}), \nabla \dot{u} \rangle \quad (2.1)$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} c \|\theta + \tau_q \dot{\theta}\|^2 + \frac{1}{2} \frac{d}{dt} k^* \|\nabla \alpha\|^2 + \frac{1}{2} \frac{d}{dt} 2k^* \tau_q \langle \nabla \alpha, \nabla \theta \rangle + \frac{1}{2} \frac{d}{dt} (\tau_\nu^* \tau_q + k\tau_\theta) \|\nabla \theta\|^2 \\ &= -m \langle \nabla \dot{u}, \nabla(\theta + \tau_q \dot{\theta}) \rangle - (\tau_\nu^* - k^* \tau_q) \|\nabla \theta\|^2 - k\tau_\theta \tau_q \|\nabla \eta\|^2 \end{aligned} \quad (2.2)$$

with $\eta = \dot{\theta}$. It follows from (2.1) and (2.2) that

$$\frac{1}{2} \frac{dE(t)}{dt} = -(\tau_\nu^* - k^* \tau_q) \|\nabla \theta\|^2 - k\tau_\theta \tau_q \|\nabla \eta\|^2, \quad (2.3)$$

where the “energy” of the system (1.3)-(1.7) is

$$\begin{aligned} E(t) &= \rho \|\dot{u}\|^2 + \mu \|\Delta u\|^2 + c \|\theta + \tau_q \dot{\theta}\|^2 + k^* \|\nabla \alpha\|^2 + 2k^* \tau_q \langle \nabla \alpha, \nabla \theta \rangle + (\tau_\nu^* \tau_q + k\tau_\theta) \|\nabla \theta\|^2 \\ &= \rho \|v\|^2 + \mu \|\Delta u\|^2 + c \|\theta + \tau_q \eta\|^2 + k^* \|\nabla \alpha + \tau_q \nabla \theta\|^2 + (\tau_\nu^* \tau_q + k\tau_\theta - k^* \tau_q^2) \|\nabla \theta\|^2. \end{aligned}$$

where $v = \dot{u}$. Let

$$\mathcal{H} := H_0^1 \cap H^2 \times L^2 \times H_0^1 \times H_0^1 \times L^2.$$

Denoting $U = (u, v, \alpha, \theta, \eta)^T$ and $U^* = (u^*, v^*, \alpha^*, \theta^*, \eta^*)^T$, we can define the inner product

$$\begin{aligned} \langle U, U^* \rangle_{\mathcal{H}} &= \mu \langle \Delta u, \Delta u^* \rangle + \rho \langle v, v^* \rangle + c \langle \theta + \tau_q \eta, \theta^* + \tau_q \eta^* \rangle + k^* \langle \nabla(\alpha + \tau_q \theta), \nabla(\alpha^* + \tau_q \theta^*) \rangle \\ &\quad + (\tau_\nu^* \tau_q + k \tau_\theta - k^* \tau_q^2) \langle \nabla \theta, \nabla \theta^* \rangle, \end{aligned}$$

i.e.,

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 &= \mu \|\Delta u\|^2 + \rho \|v\|^2 + c \|\theta + \tau_q \eta\|^2 + k^* \|\nabla \alpha + \tau_q \nabla \theta\|^2 + (\tau_\nu^* \tau_q + k \tau_\theta - k^* \tau_q^2) \|\nabla \theta\|^2 \\ &= E(t). \end{aligned}$$

We can convert the system into a first-order evolution equation on Hilbert space $\mathcal{H}$,

$$\begin{cases} \frac{dU}{dt} = \mathcal{A}U, \\ U(0) = (u^0, v^0, \alpha^0, \theta^0, \eta^0)^T, \end{cases} \quad (2.4)$$

$$U(0) = (u^0, v^0, \alpha^0, \theta^0, \eta^0)^T, \quad (2.5)$$

where the operator $\mathcal{A}$ is given by

$$\mathcal{A}U = \begin{pmatrix} v \\ -\frac{1}{\rho} \Delta(\mu \Delta u + m(\theta + \tau_q \eta)) \\ \theta \\ \eta \\ \frac{m}{c\tau_q} \Delta v + \frac{1}{c\tau_q} \Delta(k^* \alpha + \tau_\nu^* \theta + k \tau_\theta \eta) - \frac{1}{\tau_q} \eta \end{pmatrix} \quad (2.6)$$

and

$$\begin{aligned} D(\mathcal{A}) = \{U &= (u, v, \alpha, \theta, \eta)^T \in \mathcal{H} \mid \Delta u = 0, \text{ on } \partial\Omega, v \in H_0^1 \cap H^2, \\ &\Delta(k^* \alpha + \tau_\nu^* \theta + k \tau_\theta \eta) \in L^2, \mu \Delta u + m(\theta + \tau_q \eta) \in L^2\}. \end{aligned} \quad (2.7)$$

Theorem 2.1. $\mathcal{A}$ is the infinitesimal generator of a C_0 -semigroup of contractions on the Hilbert space $\mathcal{H}$.

Proof. By (2.3) and $\tau_\nu^* \geq k^* \tau_q$,

$$\operatorname{Re} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = \frac{1}{2} \frac{d}{dt} \|U\|_{\mathcal{H}}^2 = -(\tau_\nu^* - k^* \tau_q) \|\nabla \theta\|^2 - k \tau_\theta \tau_q \|\nabla \eta\|^2 \leq 0. \quad (2.8)$$

Thus, $\mathcal{A}$ is dissipative. It is clear that $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Let $F = (F_1, F_2, F_3, F_4, F_5)^T \in \mathcal{H}$. To solve $\mathcal{A}U = F$, we first have $v = F_1 \in H_0^1 \cap H^2$, $\theta = F_3 \in H_0^1$, $\eta = F_4 \in H_0^1$. Then we have

$$\begin{cases} \Delta(\mu \Delta u + m(\theta + \tau_q \eta)) = -\rho F_2 \in L^2, \\ -\Delta(k^* \alpha + \tau_\nu^* \theta + k \tau_\theta \eta) = -c F_4 - c \tau_q F_5 + m \Delta F_1 \in L^2. \end{cases} \quad (2.9)$$

$$-\Delta(k^* \alpha + \tau_\nu^* \theta + k \tau_\theta \eta) = -c F_4 - c \tau_q F_5 + m \Delta F_1 \in L^2. \quad (2.10)$$

Define a bilinear form $B : ((H_0^1 \cap H^2) \times H_0^1) \times ((H_0^1 \cap H^2) \times H_0^1) \rightarrow \mathbb{C}$

$$((u, \alpha), (\tilde{u}, \tilde{\alpha})) \rightarrow \mu \langle \Delta u, \Delta \tilde{u} \rangle + k^* \langle \nabla \alpha, \nabla \tilde{\alpha} \rangle$$

and a continuous linear function

$$f : (H_0^1 \cap H^2) \times H_0^1 \rightarrow \mathbb{C},$$

$$(\tilde{u}, \tilde{\alpha}) \rightarrow f(\tilde{u}, \tilde{\alpha}) := -\rho \langle F_2, \tilde{u} \rangle + m \langle \nabla(F_3 + \tau_q F_4), \nabla \tilde{u} \rangle - \langle cF_4 + c\tau_q F_5 + m\Delta F_1, \tilde{\alpha} \rangle - \tau_\nu^* \langle \nabla F_3 + k\tau_\theta \nabla F_4, \nabla \tilde{\alpha} \rangle.$$

Then B is a strong coercive sesquilinear form. By the Lax-Milgram Theorem, there exists a unique $(u, \alpha) \in (H_0^1 \cap H^2) \times H_0^1$ satisfying

$$B((u, \alpha), (\tilde{u}, \tilde{\alpha})) = f(\tilde{u}, \tilde{\alpha}), \text{ for } \forall (\tilde{u}, \tilde{\alpha}) \in (H_0^1 \cap H^2) \times H_0^1, \|(u, \alpha)\|_{H^2 \times H^1} \leq C\|F\|_{\mathcal{H}},$$

where C is a positive constant. This solves (2.9)-(2.10), i.e., we obtain $U = (u, v, \alpha, \theta, \eta)^T \in D(\mathcal{A})$ with

$$AU = F \text{ and } \|U\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}$$

which implies $0 \in \rho(\mathcal{A})$. By the modified Lumer-Philips Theorem[22], we conclude that $\mathcal{A}$ generates a C_0 -semigroup of contractions on $\mathcal{H}$. $\square$

Theorem 2.2. *The semigroup $e^{\mathcal{A}t}$ is analytic and exponentially stable.*

Remark 2.1. *It is worth noting that the analyticity of solutions implies the exponential stability of solutions and the impossibility of localization of solutions. That is the only solution which can be identically null after a finite time is the null solution.*

We will use the following theorem to prove Theorem 2.2.

Theorem 2.3. [22] *Let $S(t) = e^{\mathcal{A}t}$ be a C_0 -semigroup of contractions in a Hilbert space $\mathcal{H}$. Suppose that*

$$i\mathbb{R} \subset \rho(\mathcal{A}). \quad (2.11)$$

Then, $S(t)$ is analytic if and only if

$$\overline{\lim}_{|\beta| \rightarrow \infty} \|\beta(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{H}} < \infty \quad (2.12)$$

holds.

Proof. 1. We first check the condition (2.11). Note that, from well-posedness, we have proved that $0 \in \rho(\mathcal{A})$. Then, defining

$$\beta^* := \sup \{ \beta > 0 : (-i\beta, i\beta) \subset \rho(\mathcal{A}) \},$$

Suppose $0 < \beta^* < \infty$, then there exists a unit sequence $U_n = (u_n, v_n, \alpha_n, \theta_n, \eta_n)^T \in D(\mathcal{A})$, such that

$$\|(i\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} \rightarrow 0 \quad \text{when } \beta_n \nearrow \beta^* \text{ (w.l.o.g.)}, \quad (2.13)$$

For convenience, we omit the subscript n .

We can rewrite (2.13) as

$$\begin{cases} i\beta u - v = o(1), & \text{in } H^2, \end{cases} \quad (2.14)$$

$$\begin{cases} i\beta \rho v + \Delta(\mu \Delta u + m(\theta + \tau_q \eta)) = o(1), & \text{in } L^2, \end{cases} \quad (2.15)$$

$$\begin{cases} i\beta \alpha - \theta = o(1), & \text{in } H^1, \end{cases} \quad (2.16)$$

$$\begin{cases} i\beta \theta - \eta = o(1), & \text{in } H^1, \end{cases} \quad (2.17)$$

$$\begin{cases} i\beta \tau_q c \eta - m \Delta v - \Delta(k^* \alpha + \tau_\nu^* \theta + k\tau_\theta \eta) + \tau_q \eta = o(1), & \text{in } L^2. \end{cases} \quad (2.18)$$

Thus by dissipation,

$$\operatorname{Re} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -(\tau_\nu^* - k^* \tau_q) \|\nabla \theta\|^2 - k\tau_\theta \tau_q \|\nabla \eta\|^2 = o(1). \quad (2.19)$$

Since $\tau_\nu^* \geq k^* \tau_q$, from (2.19), we only have $\|\nabla \eta\|^2 = o(1)$. By (2.16)-(2.17), we also get $\|\nabla \theta\|^2 = o(1)$ and $\|\nabla \alpha\|^2 = o(1)$. Therefore, $\|\alpha\|, \|\theta\|, \|\eta\| = o(1)$. Taking the inner product of (2.18) with $\beta^{-1} \Delta u$ in L^2 ,

$$\langle i\tau_q c \eta, \Delta u \rangle - m \langle \beta^{-1} \Delta v, \Delta u \rangle - \beta^{-1} \langle \Delta(k^* \alpha + \tau_\nu^* \theta + k\tau_\theta \eta), \Delta u \rangle + \beta^{-1} \tau_q \langle \eta, \Delta u \rangle = o(1). \quad (2.20)$$

The first and last inner product term in (2.20) is of $o(1)$ due to the boundedness of $\|\Delta u\|$. By (2.14) and (2.15),

$$\begin{aligned}
& -m\langle\beta^{-1}\Delta v, \Delta u\rangle - \beta^{-1}\langle\Delta(k^*\alpha + \tau_\nu^*\theta + k\tau_\theta\eta), \Delta u\rangle \\
& = -im\|\Delta u\|^2 + o(1) - \beta^{-1}\langle(k^*\alpha + \tau_\nu^*\theta + k\tau_\theta\eta), \Delta^2 u\rangle \\
& = -im\|\Delta u\|^2 - \beta^{-1}\mu^{-1}\langle(k^*\alpha + \tau_\nu^*\theta + k\tau_\theta\eta), -i\rho\beta v - m(\Delta\theta + \tau_q\Delta\eta)\rangle + o(1) \\
& = -im\|\Delta u\|^2 - \beta^{-1}\mu^{-1}\langle(k^*\nabla\alpha + \tau_\nu^*\nabla\theta + k\tau_\theta\nabla\eta), -m(\nabla\theta + \tau_q\nabla\eta)\rangle + o(1) \\
& = -im\|\Delta u\|^2 + o(1).
\end{aligned} \tag{2.21}$$

(2.20) now is simplified into

$$\|\Delta u\|^2 = o(1). \tag{2.22}$$

Taking the inner product of (2.15) with v in L^2 ,

$$i\beta\rho\|v\|^2 + \mu\langle\Delta u, \Delta v\rangle - m\langle\nabla(\theta + \tau_q\eta), \nabla v\rangle = o(1). \tag{2.23}$$

By (2.14), (2.22) and dissipation, we can derive

$$\|v\|^2 = o(1). \tag{2.24}$$

We can conclude that $\|U\|_{\mathcal{H}}^2 = o(1)$, which is a contradiction to the assumption $\|U\|_{\mathcal{H}}^2 = 1$. Therefore, $i\mathbb{R} \subseteq \rho(\mathcal{A})$.

2. We now check the condition (2.12).

Assume that (2.12) is false. Then by the uniform boundedness theorem, there exist a sequence $\beta \rightarrow \infty$ and a unit sequence $U = (u, v, \alpha, \theta, \eta)^T \in D(\mathcal{A})$ such that

$$\|(iI - \frac{1}{\beta}\mathcal{A})U\|_{\mathcal{H}} \rightarrow 0. \tag{2.25}$$

We can write (2.25) as

$$\begin{cases} iu - \frac{1}{\beta}v = o(1), & \text{in } H^2, \end{cases} \tag{2.26}$$

$$\begin{cases} i\rho v - \frac{1}{\beta}\Delta(\mu\Delta u + m(\theta + \tau_q\eta)) = o(1), & \text{in } L^2, \end{cases} \tag{2.27}$$

$$\begin{cases} i\alpha - \frac{1}{\beta}\theta = o(1), & \text{in } H^1. \end{cases} \tag{2.28}$$

$$\begin{cases} i\theta - \frac{1}{\beta}\eta = o(1), & \text{in } H^1. \end{cases} \tag{2.29}$$

$$\begin{cases} ic\tau_q\eta - \frac{m}{\beta}\Delta v - \frac{1}{\beta}\Delta(k^*\alpha + \tau_\mu^*\theta + k\tau_\theta\eta) - \frac{\tau_q}{\beta}\eta = o(1), & \text{in } L^2. \end{cases} \tag{2.30}$$

From (2.25) and (2.19),

$$Re\langle(iI - \frac{1}{\beta}\mathcal{A})U, U\rangle_{\mathcal{H}} = -\frac{1}{\beta}Re\langle\mathcal{A}U, U\rangle_{\mathcal{H}} = -\frac{1}{\beta}(\tau_\nu^* - k^*\tau_q)\|\nabla\theta\|^2 - \frac{1}{\beta}k\tau_\theta\tau_q\|\nabla\eta\|^2 = o(1). \tag{2.31}$$

Hence, $\|\beta^{-\frac{1}{2}}\nabla\eta\| = o(1)$, which further leads to

$$\|\nabla\alpha\|^2 = o(1) \quad \text{and} \quad \|\nabla\theta\|^2 = o(1) \tag{2.32}$$

due to (2.28) and (2.29). Taking inner product of (2.30) with η in L^2 , we can obtain

$$ic\tau_q\|\eta\|^2 + m\langle\beta^{-\frac{1}{2}}\nabla v, \beta^{-\frac{1}{2}}\nabla\eta\rangle + \langle\beta^{-\frac{1}{2}}\nabla(k^*\alpha + \tau_\mu^*\theta + k\tau_\theta\eta), \beta^{-\frac{1}{2}}\nabla\eta\rangle - \frac{1}{\beta}\tau_q\|\eta\|^2 = o(1). \tag{2.33}$$

By Interpolation Inequality,

$$\|\beta^{-\frac{1}{2}}\nabla v\|^2 \leq C\beta^{-1}\|\Delta v\|_{L^2}\|v\|_{L^2}$$

for some constant C . By (2.26), we have $\beta^{-1}\|\Delta v\|$ is bounded. Thus $\|\beta^{-\frac{1}{2}}\nabla v\|$ is also bounded. Now (2.33) implies that

$$\|\eta\|^2 = o(1). \quad (2.34)$$

Taking inner product of (2.30) with $-\Delta u$ in L^2 ,

$$\frac{1}{\beta}m\langle\Delta v, \Delta u\rangle + \frac{1}{\beta}\langle\Delta(k^*\alpha + \tau_\mu^*\theta + k\tau_\theta\eta), \Delta u\rangle = o(1). \quad (2.35)$$

Applying (2.26) again, we convert (2.35) into

$$im\|\Delta u\|^2 - \langle\beta^{-\frac{1}{2}}\nabla(k^*\alpha + \tau_\mu^*\theta + k\tau_\theta\eta), \beta^{-\frac{1}{2}}\nabla(\Delta u)\rangle = o(1). \quad (2.36)$$

Taking inner product of (2.27) with Δu in L^2 ,

$$i\rho\langle v, \Delta u\rangle + \mu\|\beta^{-\frac{1}{2}}\nabla(\Delta u)\|^2 + m\langle\beta^{-\frac{1}{2}}\nabla(\theta + \tau_q\eta), \beta^{-\frac{1}{2}}\nabla(\Delta u)\rangle = o(1). \quad (2.37)$$

So we can get $\|\beta^{-\frac{1}{2}}\nabla(\Delta u)\|$ is bounded. Then (2.36) implies

$$\|\Delta u\|^2 = o(1). \quad (2.38)$$

Taking inner product of (2.27) with v in L^2 ,

$$i\rho\|v\|^2 - \frac{1}{\beta}\mu\langle\Delta u, \Delta v\rangle + m\frac{1}{\beta}\langle\nabla(\theta + \tau_q\eta), \nabla v\rangle = o(1). \quad (2.39)$$

By (2.31) and (2.38), we have

$$\|v\|^2 = o(1). \quad (2.40)$$

Therefore, we obtain $\|U\|_{\mathcal{H}}^2 = o(1)$ again. This is a contradiction with the assumption that $\|U\|_{\mathcal{H}}^2 = 1$. $\square$

3 The case of $\tau_\nu^* < k^*\tau_q$

In the previous section, we have proved the analyticity and the exponential decay of the semigroup associated with the system (2.4)-(2.5). The operator $\mathcal{A}$ is dissipative as the coefficients satisfy $\tau_\nu^* \geq k^*\tau_q$. In this section, we will consider the case when $\tau_\nu^* < k^*\tau_q$. The first thing we need to do is to define a new inner product. We consider

$$\begin{aligned} \langle U, U^* \rangle &= \mu\langle\Delta u, \Delta u^*\rangle + \rho\langle v, v^*\rangle + c\langle\theta + \tau_q\eta, \theta^* + \tau_q\eta^*\rangle + k^*\langle\nabla(\alpha + \tau_q\theta), \nabla(\alpha^* + \tau_q\theta^*)\rangle \\ &\quad + (\tau_\nu^*\tau_q + k\tau_\theta - k^*\tau_q^2)\langle\nabla\theta, \nabla\theta^*\rangle + \varepsilon\langle\nabla\theta, \nabla\theta^*\rangle, \end{aligned}$$

where ε is a sufficiently large positive number to guarantee that this inner product which is equivalent to the usual one in our Hilbert space $\mathcal{H}$. We can even define the inner product in the case that

$$\tau_\nu^*\tau_q + k\tau_\theta - k^*\tau_q^2 < 0. \quad (3.1)$$

Our first aim in this section is to prove the analyticity of the semigroup even in the case that (3.1) holds.

Theorem 3.1. *There exists a constant λ large enough to guarantee that the semigroup generated by $\mathcal{A}_\lambda = \mathcal{A} - \lambda I$ is analytic.*

Proof. We note that

$$\begin{aligned} \operatorname{Re}\langle \mathcal{A}_\lambda U, U \rangle &= -\lambda \|\Delta u\|^2 - \lambda \rho \|v\|^2 - (\tau_\nu^* - k^* \tau_q) \|\nabla \theta\|^2 - k \tau_\theta \tau_q \|\nabla \eta\|^2 + \varepsilon \|\nabla \theta\|^2 \\ &\quad - \lambda c \|\theta + \tau_q \eta\|^2 - \lambda k^* \|\nabla \alpha + \tau_q \nabla \theta\|^2 - \lambda (\tau_\nu^* \tau_q + k \tau_\theta - k^* \tau_q^2) \|\nabla \theta\|^2. \end{aligned}$$

It is clear that

$$\begin{aligned} \operatorname{Re}\langle \mathcal{A}_\lambda U, U \rangle &\leq -\lambda \|\Delta u\|^2 - \lambda \rho \|v\|^2 - \left(\lambda (\varepsilon + \tau_\nu^* \tau_q + k \tau_\theta - k^* \tau_q^2) - (\tau_\nu^* - k^* \tau_q) - \frac{\varepsilon \varepsilon_1}{2} \right) \|\nabla \theta\|^2 \\ &\quad - \lambda k^* \|\nabla \alpha + \tau_q \nabla \theta\|^2 - \left(k \tau_\theta \tau_q - \frac{\varepsilon}{2 \varepsilon_1} \right) \|\nabla \theta\|^2 - \lambda c \|\theta + \tau_q \eta\|^2, \end{aligned}$$

where ε_1 is an arbitrary positive constant. We choose ε_1 large enough to guarantee that

$$k \tau_\theta \tau_q - \frac{\varepsilon}{2 \varepsilon_1} > 0$$

and λ large enough to guarantee that

$$\lambda (\varepsilon + \tau_\nu^* \tau_q + k \tau_\theta - k^* \tau_q^2) - (\tau_\nu^* - k^* \tau_q) - \frac{\varepsilon \varepsilon_1}{2} > 0.$$

We obtain that

$$\operatorname{Re}\langle \mathcal{A}_\lambda U, U \rangle \leq -K (\|\Delta u\|^2 + \|v\|^2 + \|\nabla \alpha\|^2 + \|\nabla \theta\|^2 + \|\nabla \eta\|^2),$$

where K is a positive constant. Therefore a similar argument to the one used in the proof of Theorem 2.1 allow us to show the analyticity of $\mathcal{A}_\lambda$. $\square$

Theorem 3.2. *The operator $\mathcal{A}$ generates a quasi-contractive and analytic semigroup.*

Proof. As $\mathcal{A}_\lambda$ is analytic and λI is a bounded perturbation we obtain the analyticity of $\mathcal{A}$. On the other side, $\mathcal{A}$ generates a quasi-contractive semigroup since $\mathcal{A}_\lambda$ generates a semigroup of contraction. $\square$

In general, we cannot expect the exponential stability of the solutions for every domain, but we can conclude the impossibility of localization.

In the remain of this section we want to give examples of domain Ω such that the solutions of the problem (2.4)-(2.5) are unstable. That is, they can be unbounded when the time is increasing. We look for solutions to the system (1.3)-(1.4) of the form

$$\begin{cases} u = A \exp(wt) \phi_n(x), \\ \alpha = B \exp(wt) \phi_n(x), \end{cases}$$

where $\phi_n(x)$ satisfies that $\Delta \phi_n + \lambda_n \phi_n = 0$, $\Delta^2 \phi_n = \lambda_n^2 \phi_n$ and $\phi = \Delta \phi = 0$ on $\partial \Omega$.

We have

$$\begin{cases} (\rho w^2 + \mu \lambda_n^2) A - m w (1 + \tau_q w) \lambda_n B = 0, \\ m w \lambda_n A + (c w^2 (1 + \tau_q w) + k^* \lambda_n + \tau_\nu^* w \lambda_n + k \tau_\theta w^2 \lambda_n) B = 0. \end{cases}$$

If we want solutions for A, B are different from zero, we need that

$$(\rho w^2 + \mu \lambda_n^2) (c w^2 (1 + \tau_q w) + k^* \lambda_n + \tau_\nu^* w \lambda_n + k \tau_\theta w^2 \lambda_n) + m^2 w^2 (1 + \tau_q w) \lambda_n^2 = 0.$$

Therefore, we need that w must be a solution of the equation

$$x^5 + q_1 x^4 + q_2 x^3 + q_3 x^2 + q_4 x + q_5 = 0, \quad (3.2)$$

where

$$\begin{cases} q_1 = \frac{c\rho + \rho k \tau_\theta \lambda_n}{\rho c \tau_q}, \\ q_2 = \frac{c\mu \tau_q \lambda_n^2 + m^2 \tau_q \lambda_n^2 + \rho \tau_\nu^* \lambda_n}{\rho c \tau_q}, \\ q_3 = \frac{\mu k \tau_\theta \lambda_n^3 + c\mu \lambda_n^2 + m^2 \lambda_n^2 + \rho k^* \lambda_n}{\rho c \tau_q}, \\ q_4 = \frac{\mu \tau_\nu^* \lambda_n^3}{\rho c \tau_q}, \\ q_5 = \frac{\mu k^* \lambda_n^3}{\rho c \tau_q}. \end{cases}$$

We want to see that there are solutions to (3.2) such that the real parts are positive. It follows from the Routh-Hurwitz rule that the solutions have negative real part if and only if the main minors of the matrix

$$\begin{pmatrix} q_1 & 1 & 0 & 0 \\ q_3 & q_2 & q_1 & 1 \\ q_5 & q_4 & q_3 & q_2 \\ 0 & 0 & q_5 & q_4 \end{pmatrix}$$

are positive.

Let us consider $q_1 q_2 - q_3$. We have

$$\begin{aligned} & \left(\frac{1}{\tau_q} + \frac{k \tau_\theta \lambda_n}{c \tau_q} \right) \left(\frac{c\mu \tau_q \lambda_n^2 + \rho \tau_\nu^* \lambda_n + m^2 \tau_q \lambda_n^2}{\rho c \tau_q} \right) - \frac{c\mu \lambda_n^2 + \rho k^* \lambda_n + \mu k \tau_\theta \lambda_n^3 + m^2 \lambda_n^2}{\rho c \tau_q} \\ &= \frac{1}{c \tau_q^2} (\tau_\nu^* - k^* \tau_q) \lambda_n + \frac{k \tau_\theta \tau_\nu^*}{c^2 \tau_q^2} \lambda_n^2 + \frac{k \tau_\theta m^2}{\rho c^2 \tau_q} \lambda_n^3. \end{aligned} \quad (3.3)$$

It is clear that there exist domain Ω such that the eigenvalues $\lambda_1, \lambda_2, \dots$ are so small to guarantee that (3.3) is negative since $\tau_\nu^* - k^* \tau_q < 0$. Instability of solutions follows from this. A similar result could be obtained with the other minors. Results of this kind have been proved previously in the context of the dual-phase-lag thermoelasticity[12] and the Moore-Gibson-Thompson thermoelasticity[15].

Acknowledgements. The work of R. Quintanilla was supported by projects “*Análisis Matemático de Problemas de la Termomecánica*” (MTM2016-74934-P) (AEI/FEDER, UE) of the Spanish Ministry of Economy and Competitiveness, and “*Análisis Matemático Aplicado a la Termomecánica*” (Ref. PID2019- 105118GB-I00), (AEI/FEDER, UE) of the Spanish Ministry of Science, Innovation and Universities.

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