Nonintrusive Stochastic Finite Elements for Crashworthiness with VPS/Pamcrash

M. Rocas^{1,2} · A. García-González^{1,3} · X. Larráyoiz² · P. Díez^{1,3}

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Abstract

Crashworthiness analysis remains an important concern for the design of safety structures. In this context, uncertainties play an essential role in the response of a crash problem with non linear behavior. With this statement at hand, in this work it is presented a review of uncertainty quantification (UQ) techniques, with intrusive and non-intrusive approaches in stochastic finite element methods for crashworthiness. The well-known deterministic finite element solver VPS/Pamcrash is used to illustrate the currently available methods, developing a comparative analysis of these techniques in crashworthiness UQ. Finally, relevant non-intrusive methods are applied to analyze the behavior of a specific quantity of interest in a dynamic crash model.

1 Introduction

Models in Computational Mechanics provide the response of physical systems by solving mathematical problems, depending on specific input parameters. Input data is often subject to uncertainty or lack of knowledge. Thus, uncertainty quantification (UQ) via stochastic analysis is of great importance to assess the credibility of the output [57].

Developments in stochastic analysis are implemented in several physical applications [9, 10, 25, 34, 49, 50, 53]. Specifically, in crash analysis the structured behavior after impact is analyzed to quantify stress, strain and displacements. In automotive industry, the stochastic response is a robust and powerful tool, to quantify the variation of the outputs of the simulations and therefore allows reducing the number of experimental tests.

This paper focuses on summarizing a review of intrusive and non-intrusive methods for stochastic crash analysis. The most relevant methods are applied through the solver

VPS(Pamcrash), an explicit finite element code for transient dynamics, popular in the automotive industry.

Usually, in industry, UQ is developed through classical Monte Carlo [30]. However, this method is computationally unaffordable for simulating a whole vehicle. Nevertheless, alternative approaches are available to reduce computational costs. Here we review those methods considered suitable for crashworthiness UQ, divided in two main groups: (i) Intrusive approaches, where the source code of the used solver VPS requires to implement modifications of the FE formulation, such that Perturbation method based on Taylor series [2, 35], Galerkin Polynomial Chaos (PC) based on orthogonal polynomials [11], Spectral stochastic finite element method based on Karhunen–Loeve expansion, and Polynomial Chaos [23]. (ii) Non-intrusive approaches such as Non-intrusive Polynomial Chaos [15] and variants of Monte Carlo [27], where the source code of the solver requires no modification (VPS is used as a *black box* to run simulations) and only independent post-processing techniques on the FE results are applied.

This paper is structured as follows: Sect. 2 describes a crash benchmark problem used for our analyses. The stochastic FE standard approaches are reviewed, both the intrusive and non-intrusive strategies, in Sects. 3 and 4 respectively. Next, Sect. 5 shows a comparative analysis of the SFEM solvers applied to crashworthiness. Section 6 summarizes the implemented non-intrusive approaches for our industrial benchmark problem. Section 7, illustrates the numerical results obtained using the aforementioned UQ

✉ A. García-González
berto.garcia@upc.edu

¹ Laboratori de Càlcul Numèric, E.T.S. de Ingenieria de Caminos, Universitat Politècnica de Catalunya, Barcelona, Spain

² SEAT, Martorell, Barcelona, Spain

³ International Centre for Numerical Methods in Engineering, CIMNE, Barcelona, Spain

techniques. Finally, with Sect. 8 the manuscript closes with a discussion and conclusions on the strong importance of UQ methods in crashworthiness.

2 Problem Statement

In crashworthiness, a deep knowledge of the input parameters is of critical importance to obtain a robust and credible model. According to [56], the different steps (see Fig. 1 as schematic illustration) involved in a UQ strategy are:

- Step A: defines the mathematical model that describes the physical problem.
- Step B corresponds to the identification and characterization of the input parameters affecting the QoI: geometry, material properties and boundary conditions, finding besides their stochastic or determinist nature.
- Step C analyzes the propagation of the random inputs to the mathematical model. This step is divided in two groups: intrusive and non-intrusive. Intrusive methods reformulate a deterministic scheme to a stochastic scheme. As a consequence, the mathematical model in Step A must be an intrusive stochastic model. Non-intrusive techniques allow to implement an analysis with no modification in the solver, hence, the software is used as a *black box*. Therefore, these techniques focus on the development of pre and post-processing strategy. Usually, non-intrusive approaches are used in the automotive industry since they allow to use deterministic solvers to guaranty a robust response.
- Step D corresponds to the post-processing of the outputs and sensitivity analysis, such that the mean, variance, standard deviation, high-order moments and percentiles.

2.1 Benchmark Crash Problem

This section presents the benchmark crash problem for UQ. Here is considered a simplified model of a structural assembly of the chassis that describes similar behavior the real structural problem. For the sake of time cost, this model is optimal for initial design steps. The crash model is divided in two main parts: *Structure* and the *Impactor*, illustrated in Fig. 2.

The main objective of this dynamic crash problem is to obtain the value of a QoI when the *Impactor* crashes to the *Structure* after one second. Such QoI is chosen to be the local plastic strain average in a specific area of interest, brown color in Fig. 2.

Fig. 2 Crash model. Impactor (red), Structure (grey) and QoI (brown). (Color figure online)

Fig. 1 Uncertainty quantification framework

Fig. 3 Steel *Structure* geometry: plate (orange), vertical profile (blue), horizontal profile (green), where H1, H2 and H3 are the three random thickness parameters corresponding to the three different structural parts of the assembly. (Color figure online)

2.2 Geometry

The *Structure* is considered a manufactured laminated steel sheet by cold folding with three sub-parts. The structure geometry is joined all together with an adhesive bond material used in crash industry. Figure 3 shows the assembly of the three different parts with their corresponding thicknesses H1, H2 and H3. The full structure contains 8609 shell elements, being the *Impactor* a rigid body with 2563 tetrahedral elements. The whole model has a total of $N = 13908$ nodes.

Fig. 4 Adhesive contact area (red) between steel structure parts. (Color figure online)

Fig. 5 Plastic strain of the crash problem for different combination of thicknesses (H1, H2, H3) [mm]

2.3 Deterministic Parameters

We considered as deterministic (with no uncertainty): (i) The material properties of the *Structure* as a specific elasto-plastic steel used in automotive industry. (ii) A tied material used to join the different parts of the *Structure*. Figure 4 illustrates the different areas where such adhesive material is applied with 0.3 [mm] thickness. All these previous properties are directly introduced into the VPS solver, but due to confidentiality, their specific values are not provided. (iii) Besides, a constant *Impactor* velocity is considered as deterministic and equal to 50 [mm/s].

2.4 Stochastic Parameters

According to the definition of the problem handed, the three thicknesses H1, H2 and H3 are considered as stochastic parameters. Their variability and uncertainty is characterized as a result of the aleatoric imperfections produced during the manufacturing processes. Therefore they are collected in a vector of parameters $\mathbf{H}(\theta) = [H_1(\theta), H_2(\theta), H_3(\theta)]$. The explicit dependence as variable θ is used as a notation to denote the stochastic nature of the magnitude [20]. In the following this is particularly useful to quantifies dependences on $\mathbf{x}$ and θ . The random variables $H_i(\theta)$ are assumed to be uncorrelated.

To describe the variability for the three thicknesses $H_i(\theta)$, the Probability Density Functions (PDF) are assumed to be Gaussian distributions $H_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, with

the same means $\mu_1 = \mu_2 = \mu_3 = 1.2$ [mm] and standard deviations $\sigma_1 = \sigma_2 = \sigma_3 = 0.12$ [mm] (corresponding to 10% of deviation).

2.5 Boundary Conditions

The finite element model developed in VPS consists in a dynamic impact on the *Structure* with a constant velocity of an *Impactor* using explicit formulation. For this, following the illustration of Fig. 3, the displacements are fully restricted in the extremes of the horizontal profile of the structure. Being the *Impactor*/vertical profile of the *Structure* (blue color in Fig. 3) contact defined as a surface-surface contact in VPS [45]. Also, the same contact type was defined between vertical profile and horizontal profile (green color in Fig. 3), and the plate (brown color in Fig. 3) with both vertical and horizontal profiles. It is worth noting that both, boundary conditions and material properties, are models tested and validated by the industry.

It is clear in structural design of this type of assemblies, how the variability of the thicknesses will directly affect to the final results of the simulation. Figure 5 shows three different examples of plastic strain for different combinations of H1, H2 and H3.

3 Intrusive SFEM Solvers

This section summarizes the state-of-art of the most significant intrusive SFEM methods applied in crashworthiness UQ analysis.

Intrusive methods reformulate the deterministic finite element matrix scheme into a stochastic model (SFEM) by including the randomness of the variables. Despite the computational complexity, SFEM techniques is used for solving stochastic partial differential equations.

3.1 Formulation and Notation

For the UQ analysis on the proposed benchmark, the commercial software VPS solves the equilibrium equation of transient dynamics [45]

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{F}_{ext}(t), \quad (1)$$

where $\mathbf{M}$ = Mass matrix, $\mathbf{C}$ = damping matrix, $\mathbf{K}$ = Stiffness matrix, $\mathbf{F}_{ext}$ = external force and t = time.

Nevertheless, for the sake of simplicity, a linear static problem is developed next to illustrate the used notation of the different stochastic techniques. The compact equation corresponds to:

$$\mathbf{K}(\theta)\mathbf{U}(\theta) = \mathbf{F}(\theta) \quad (2)$$

where $\mathbf{K}(\theta)$ is the stiffness matrix, $\mathbf{F}(\theta)$ the load vector and $\mathbf{U}(\theta)$ the vector of unknowns, and θ is the vector containing a random discretization of realizations θ_i , for $i = 1, 2, \dots, n_{dis}$, being n_{dis} the number of realizations. The global stiffness matrix $\mathbf{K}(\theta)$ is obtained after assembling the elemental matrices $\mathbf{K}^e$,

$$\mathbf{K}^e = \int_{\Omega_e} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega_e \quad (3)$$

where $\mathbf{D}$ corresponds to the elasticity matrix which depends on the Lamé parameters, and $\mathbf{B}$ is the matrix that relates the components of the stress with the nodal displacements. Assuming that matrix $\mathbf{D}$ has a stochastic behavior, the elasticity matrix is given by:

$$\mathbf{D}(\mathbf{x}, \theta) = R(\mathbf{x}, \theta) \mathbf{D}_0 \quad (4)$$

where $R(\mathbf{x}, \theta)$ is the random field and $\mathbf{D}_0$ is the mean value of the elasticity matrix. The behavior of the random field $R(\mathbf{x}, \theta)$ is described by the mean μ and a fluctuation function $P(\theta)$ such that $R(\mathbf{x}, \theta) = \mu(\mathbf{x}) + P(\mathbf{x}, \theta)$, thus the stiffness matrix becomes:

$$\begin{aligned} \mathbf{K}(\mathbf{x}, \theta) &= \int_{\Omega} R(\mathbf{x}, \theta) \mathbf{B}^T \mathbf{D}_0 \mathbf{B} d\Omega \\ &= \int_{\Omega} (\mu(\mathbf{x}) + P(\mathbf{x}, \theta)) \mathbf{B}^T \mathbf{D}_0 \mathbf{B} d\Omega. \end{aligned} \quad (5)$$

The purpose of the stochastic analysis is to determine reliable statistical information of a QoI response from the solution $\mathbf{U}(\theta)$, which is a random field. Recall that for our dynamic analysis, the QoI is chosen to be the mean of the plastic strain of the area of interest illustrated in Fig. 2.

3.2 Perturbation Method

The Perturbation Method was introduced in 1970 to solve a large number of problems with uncertainty in the inputs. The main idea is to propagate the uncertainty by Taylor series [2, 35], this technique has been used in fields such that structural engineering for solving nonlinear dynamic problems [38, 39].

The stochastic parameters used to construct the stiffness matrix $\mathbf{K}(\theta)$ are varying around their mean. In this approach the stochastic field $R(\mathbf{x}, \theta)$ is discretized into n_{dis} zero-mean random variables $(\theta_i)_{i=1}^{n_{dis}}$. Hence expanding the stiffness matrix $\mathbf{K}(\mathbf{x}, \theta)$ as a Taylor series around their mean $\mathbf{K}_0$ reads [53]:

$$\mathbf{K}(\mathbf{x}, \theta) = \mathbf{K}_0 + \sum_{i=1}^{n_{dis}} \mathbf{K}_i^I \theta_i + \frac{1}{2} \sum_{i=1}^{n_{dis}} \sum_{j=1}^{n_{dis}} \mathbf{K}_{ij}^{II} \theta_i \theta_j + \dots, \quad (6)$$

where $\mathbf{K}_0 = \int_{\Omega} \mu \mathbf{B}^T \mathbf{D}_0 \mathbf{B} d\Omega$ is the mean value of the stiffness matrix. Furthermore, $\mathbf{K}_i^I$, $\mathbf{K}_{ij}^{II}$ are the first and second order derivatives respectively, evaluated at $\theta = 0$, and represent the fluctuation part of the stiffness matrix:

$$\mathbf{K}_i^I = \frac{\partial \mathbf{K}(\mathbf{x}, \theta)}{\partial \theta_i} \Big|_{\theta=0} \quad (7)$$

$$\mathbf{K}_{ij}^{II} = \frac{\partial^2 \mathbf{K}(\mathbf{x}, \theta)}{\partial \theta_i \partial \theta_j} \Big|_{\theta=0} \quad (8)$$

For solving of the system $\mathbf{KU} = \mathbf{F}$, the propagated Taylor expansion of vector $\mathbf{F}$ corresponds to:

$$\mathbf{F}(\theta) = \mathbf{F}_0 + \sum_{i=1}^{n_{dis}} \mathbf{F}_i^I \theta_i + \frac{1}{2} \sum_{i=1}^{n_{dis}} \sum_{j=1}^{n_{dis}} \mathbf{F}_{ij}^{II} \theta_i \theta_j + \dots, \quad (9)$$

and assuming the external forces $\mathbf{F}$ as deterministic (no random behaviour), then the first and second derivatives are $\mathbf{F}_i^I = \mathbf{F}_{ij}^{II} = 0$, thus $\mathbf{F} = \mathbf{F}_0$.

Besides, the propagated Taylor expansion of vectors $\mathbf{U}$ reads

$$\mathbf{U}(\theta) = \mathbf{U}_0 + \sum_{i=1}^{n_{dis}} \mathbf{U}_i^I \theta_i + \frac{1}{2} \sum_{i=1}^{n_{dis}} \sum_{j=1}^{n_{dis}} \mathbf{U}_{ij}^{II} \theta_i \theta_j + \dots \quad (10)$$

where the terms $\mathbf{U}_0$, $\mathbf{U}_i^I$ and $\mathbf{U}_{ij}^{II}$ can be calculated by substituting (10) and the determinist $\mathbf{F}_0$ into (2) and identifying the similar order coefficients on both sides of the equation, is obtained the following iterative scheme:

$$\mathbf{U}_0 = \mathbf{K}_0^{-1} \mathbf{F}_0 \quad (11)$$

$$\mathbf{U}_i^I = -\mathbf{K}_0^{-1} \mathbf{K}_i^I \mathbf{U}_0 \quad (12)$$

$$\mathbf{U}_{ij}^{II} = -\mathbf{K}_0^{-1} (\mathbf{K}_i^I \mathbf{U}_j^I + \mathbf{K}_j^I \mathbf{U}_i^I + \mathbf{K}_{ij}^{II} \mathbf{U}_0) \quad (13)$$

Equation (11) gives the deterministic nodal displacement, equation (12) and (13) give the first and second order perturbations, respectively, of the displacement vector. Additionally, the statistics of $\mathbf{U}(\theta)$ with the mean $\mathbb{E}(\mathbf{U}(\theta))$ and the covariance matrix $\text{Cov}[\mathbf{U}(\theta), \mathbf{U}(\theta)]$ read:

$$\mathbb{E}(\mathbf{U}(\theta)) \approx \mathbf{U}_0 + \frac{1}{2} \sum_{i=1}^{n_{dis}} \sum_{j=1}^{n_{dis}} \mathbf{U}_{ij}^{II} \text{Cov}[\theta_i, \theta_j], \quad (14)$$

$$\text{Cov}[\mathbf{U}(\theta), \mathbf{U}(\theta)] \approx \sum_{i=1}^{n_{dis}} \sum_{j=1}^{n_{dis}} \mathbf{U}_i^I (\mathbf{U}_j^I)^T \text{Cov}[\theta_i, \theta_j]. \quad (15)$$

Increasing the number of terms in the expansion will improve the accuracy of the perturbation method, but affecting the computational cost.

3.3 Galerkin Polynomial Chaos Method

This section is devoted to formulate Galerkin polynomial chaos method for UQ. Having Gaussian random inputs, the main idea of Polynomial Chaos is to propagate the uncertainty by Hermite polynomials [23]. A main condition for using this technique is that the random inputs have to be indenpent variables. This method involves two basic steps:

- Step 1: Implementing the polynomial Hermite expansion into the the stochastic formulation of the model.
- Step 2: Applying Galerkin projection basis to get the polynomial chaos coefficients.

Applying Hermite polynomials for the terms of the equilibrium equation (2), allows to extend $\mathbf{K}(\theta)$, $\mathbf{U}(\theta)$ and $\mathbf{F}(\theta)$ as polynomial expansions in the form:

$$\begin{aligned} \mathbf{K}(\theta) &= \sum_{i=0}^{N_{KC}} \mathbf{K}_i \Psi_i(\theta), \\ \mathbf{U}(\theta) &= \sum_{i=0}^{N_{PC}} \mathbf{U}_i \Psi_i(\theta), \\ \mathbf{F}(\theta) &= \sum_{i=0}^{N_{FC}} \mathbf{F}_i \Psi_i(\theta), \end{aligned} \quad (16)$$

where $\mathbf{K}_i$, $\mathbf{U}_i$, $\mathbf{F}_i$ are the polynomial chaos coefficients (known as Fourier coefficients), $\Psi_i(\theta)$ are orthogonal basis. For the sake of simplicity, the Eq. (16) the polynomial basis corresponds to $\Psi_i(\{H_k(\theta)\}_{k=1}^3)$. The number of terms $N_{KC, PC, FC}$ are defined as

$$N_{KC, PC, FC} + 1 = \frac{(M + d_{KC, PC, FC})!}{M! d_{KC, PC, FC}!}, \quad (17)$$

being d_{KC} , d_{PC} and d_{FC} the order of the polynomial expansions, and M the number of stochastic variables.

To construct an orthogonal polynomial basis is necessary to know the probability density function of the inputs in the called Askey–Wilson scheme [3, 41] to guarantee a good convergence; therefore using Hermite polynomial basis for gaussian distribution, and Legendre polynomials for uniform distribution [64, 65].

In the case of our benchmark problem, three Gaussian input variables for $H_1(\theta)$, $H_2(\theta)$, $H_3(\theta)$, the multivariate Hermite polynomials are created by the tensor product of the univariate polynomials of each random input [18]. Table 1 shows the corresponding relation between multi-indexes and single-indexes for the calculation of the multivariate Hermite polynomial basis of order 2.

As in the Perturbation method, it is assumed that the external force $\mathbf{F}$ is deterministic. Therefore the equilibrium equation 2 takes the form:

Table 1 Multi-index Hermite Polynomials of three dimension

Multi-index	Multi-polynomial	i	$\Psi_j(\theta)$
(0 0 0)	$\Psi_0(H_1)\Psi_0(H_2)\Psi_0(H_3)$	0	1
(1 0 0)	$\Psi_1(H_1)\Psi_0(H_2)\Psi_0(H_3)$	1	H_1
(0 1 0)	$\Psi_0(H_1)\Psi_1(H_2)\Psi_0(H_3)$	2	H_2
(0 0 1)	$\Psi_0(H_1)\Psi_0(H_2)\Psi_1(H_3)$	3	H_3
(2 0 0)	$\Psi_2(H_1)\Psi_0(H_2)\Psi_0(H_3)$	4	$H_1^2 - 1$
(1 1 0)	$\Psi_1(H_1)\Psi_1(H_2)\Psi_0(H_3)$	5	H_1H_2
(1 0 1)	$\Psi_1(H_1)\Psi_0(H_2)\Psi_1(H_3)$	6	H_1H_3
(0 2 0)	$\Psi_0(H_1)\Psi_2(H_2)\Psi_0(H_3)$	7	$H_2^2 - 1$
(0 1 1)	$\Psi_0(H_1)\Psi_1(H_2)\Psi_1(H_3)$	8	H_2H_3
(0 0 2)	$\Psi_0(H_1)\Psi_0(H_2)\Psi_2(H_3)$	9	$H_3^2 - 1$

Note that the first column describes the degree of the univariate polynomials

$$\left(\sum_{i=0}^{N_{KC}} \mathbf{K}_i \Psi_i(\theta) \right) \left(\sum_{j=0}^{N_{PC}} \mathbf{U}_j \Psi_j(\theta) \right) = \mathbf{F} \quad (18)$$

becoming,

$$\sum_{i=0}^{N_{KC}} \sum_{j=0}^{N_{PC}} \mathbf{K}_i \mathbf{U}_j \Psi_i(\theta) \Psi_j(\theta) = \mathbf{F}. \quad (19)$$

The above equation is projected onto n^{th} polynomial basis with $n = (0, 1, \dots, N_{PC})$. To simplify the formulation, It is assumed that $N_{KC} = N_{PC}$, and therefore, making the inner product, it is obtained the expression:

$$\left\langle \sum_{i=0}^{N_{PC}} \sum_{j=0}^{N_{PC}} \mathbf{K}_i \mathbf{U}_j \Psi_i(\theta) \Psi_j(\theta), \Psi_n(\theta) \right\rangle = \langle \mathbf{F}, \Psi_n(\theta) \rangle, \quad (20)$$

renaming $\Psi_i(\theta)$ as Ψ_i to relax the notation, the previous expression can be written such that

$$\sum_{i=0}^{N_{PC}} \sum_{j=0}^{N_{PC}} \mathbf{K}_i \mathbf{U}_j \langle \Psi_i \Psi_j, \Psi_n \rangle = \langle \mathbf{F}, \Psi_n \rangle. \quad (21)$$

Describing the inner product of two random functions, $f(\mathbf{H})$ and $g(\mathbf{H})$, as:

$$\langle f(\mathbf{H}), g(\mathbf{H}) \rangle = \mathbb{E}(f(\mathbf{H})g(\mathbf{H})) = \int_R f(\mathbf{H})g(\mathbf{H})p_H(\mathbf{H})d\mathbf{H}, \quad (22)$$

being $p_H(\mathbf{H})$ is the probability density function of $\mathbf{H}$. Let us introduce the following notation:

$$C_{ijn} = \mathbb{E}[\Psi_i \Psi_j \Psi_n] = \langle \Psi_i \Psi_j, \Psi_n \rangle, \quad (23)$$

$$\mathbf{F}_n = \langle \mathbf{F}, \Psi_n \rangle. \quad (24)$$

Equation (21) can be rewritten as:

$$\left(\sum_{i=0}^{N_{PC}} \sum_{j=0}^{N_{PC}} \mathbf{K}_i C_{ijn} \right) \mathbf{U}_j = \mathbf{F}_n \quad (25)$$

For the sake of simplicity, let us define

$$\mathbf{K}_{jn} = \sum_{i=0}^{N_{PC}} \mathbf{K}_i C_{ijn} \quad (26)$$

Thus the Eq. (25) reads

$$\left(\sum_{j=0}^{N_{PC}} \mathbf{K}_{jn} \right) \mathbf{U}_j = \mathbf{F}_n \quad (27)$$

which means:

$$\begin{pmatrix} \mathbf{K}_{00} & \dots & \mathbf{K}_{0,N_{PC}} \\ \mathbf{K}_{10} & \dots & \mathbf{K}_{1,N_{PC}} \\ \vdots & \ddots & \vdots \\ \mathbf{K}_{N_{PC},0} & \dots & \mathbf{K}_{N_{PC},N_{PC}} \end{pmatrix} \begin{pmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \\ \vdots \\ \mathbf{U}_{N_{PC}} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\mathbf{U}_j$ is a N -dimensional vector (number of nodes nodes), and $\mathbf{K}_{jn}$ is a matrix of size $N \times N$. Therefore, the global linear system will have $N \cdot N_{PC} \times N \cdot N_{PC}$ size. In this case, where $\mathbf{F}$ is deterministic, only $\mathbf{F}_0$ is non-zero. The computational cost to solve the linear system is thus much greater with respect to a deterministic approach [23]. Additionally, the first vector of coefficients $\mathbf{U}_0$ corresponds to the mean of $\mathbf{U}$, that is $\mathbb{E}[\mathbf{U}] = \mathbf{U}_0$.

At this point, for post-processing analysis in polynomial chaos, it is useful to define the covariance matrix of all the components of $\mathbf{U}$, which read:

$$\text{Cov}[\mathbf{U}, \mathbf{U}] = \sum_{i=0}^P \mathbb{E}[\Psi_i^2] \mathbf{U}_i \cdot \mathbf{U}_i^T. \quad (28)$$

3.4 Spectral Stochastic Finite Element Method

The goal of this section is to show the spectral stochastic finite element method developed by Ghanem and Spanos [23]. The essence of this approach is to combine Karhunen–Loève expansion for the stochastic input parameters with Polynomial Chaos for the response variability[57]. In the literature, different developments have been applied combining Karhunen–Loève and PC [13, 22, 23, 44].

Recalling our random inputs $H_i(\theta)$ as the three thicknesses of our crash problem (Sect. 2.4). Each random field $H_i(\mathbf{x}, \theta)$ is discretized expressed as a finite number of uncorrelated random variables, using the truncated Karhunen–Loève decomposition [40, 51, 54].

One main issue in crashworthiness field is the high computational cost required, thus reduction of dimensionality of the problem seems critical for a stochastic approach. The

Fig. 6 Non-intrusive Polynomial Chaos framework

Table 2 Orthogonal array

Simulation number	Input parameter A	Input parameter B	Input parameter C
1	L1	L1	L3
2	L1	L2	L2
3	L1	L3	L1
4	L2	L1	L2
5	L2	L2	L1
6	L2	L3	L3
7	L3	L1	L1
8	L3	L2	L3
9	L3	L3	L2

Karhunen–Loève decomposition allows representing a random field by a sum of mutually uncorrelated (zero-mean) scalar random variables multiplied by deterministic functions of $\mathbf{x}$ [28], namely

$$H(\mathbf{x}, \boldsymbol{\theta}) = \mu(\mathbf{x}) + \sum_{i=1}^{N_{KL}} \sqrt{\lambda_i} \xi_i(\boldsymbol{\theta}) \varphi_i(\mathbf{x}), \quad (29)$$

where $\mu(\mathbf{x})$ is the mean of the random field $H_i(\mathbf{x}, \boldsymbol{\theta})$, λ_i and $\varphi_i(\mathbf{x})$ are the deterministic eigenvalues and eigenfunctions of to the covariance function of the source field $H(\mathbf{x}, \boldsymbol{\theta})$, respectively. The stochasticity of the system is approximated by N_{KL} uncorrelated standard Gaussian random variables $\xi_i(\boldsymbol{\theta})$, with $i = 1, 2, \dots, N_{KL}$. The readers are referred to [22, 23] for deeper theoretical details on Karhunen–Loève technique.

The stiffness matrix $\mathbf{K}$ is computed by Karhunen–Loeve substituting in (3) Eqs. (4) and (29), becoming the elemental matrix $\mathbf{K}^e(\boldsymbol{\theta})$

$$\mathbf{K}^e(\boldsymbol{\theta}) = \mathbf{K}_0^e + \sum_{i=1}^{N_{KL}} \mathbf{K}_i^e \xi_i(\boldsymbol{\theta}), \quad (30)$$

where $\mathbf{K}_0^e$ is the mean value $\int_{\Omega_e} \mu(\mathbf{x}) \mathbf{B}^T \mathbf{D}_0 \mathbf{B} d\Omega_e$ and $\mathbf{K}_i^e$ are deterministic matrices that describe the fluctuation part of the stiffness matrix, given by:

$$\mathbf{K}_i^e = \sqrt{\lambda_i} \int_{\Omega_e} \varphi_i(\mathbf{x}) \mathbf{B}^T \mathbf{D}_0 \mathbf{B} d\Omega_e. \quad (31)$$

Assuming $\mathbf{F}$ is deterministic and expanding the unknown vector $\mathbf{U}(\boldsymbol{\theta})$ by PC expansion, the finite element equilibrium system reads:

$$\left(\sum_{i=0}^{N_{KL}} \mathbf{K}_i \xi_i(\boldsymbol{\theta}) \right) \cdot \left(\sum_{j=0}^{N_{PC}} \mathbf{U}_j \Psi_j(\boldsymbol{\theta}) \right) = \mathbf{F}. \quad (32)$$

Physically, $\mathbf{K}_0$ refers to the mean stiffness, and $\mathbf{K}_i$ to the random fluctuation around the mean. After some algebraic manipulations (analogously to Galerkin PC method), the equilibrium system is given by:

$$\mathbf{K}_{ij} = \sum_{k=0}^{N_{KL}} C_{ijk} \cdot \mathbf{K}_k; \quad k = 0, \dots, N_{PC}, \quad (33)$$

where $C_{ijk} = \mathbb{E}[\xi_i \Psi_j \Psi_k]$, and finally obtaining

Fig. 7 Monte Carlo crash problem scheme

Fig. 8 Monte Carlo method framework

$$\left(\sum_{i=0}^{N_{PC}} \mathbf{K}_{ij} \right) \mathbf{U}_j = \mathbf{F}_n. \quad (34)$$

The dimension of the resulting linear system in this approach depends directly on the number of terms P retained in the PC of the random nodal displacement vector. Since P is multiplied by N (number of degrees of freedom) as shown in Eq. (34), it is obvious that the computational cost required for the solution of this system is much larger than the corresponding deterministic problem. From above expressions, the statistics of the coefficient vector $\mathbf{U} = [\mathbf{U}_0, \mathbf{U}_1, \dots, \mathbf{U}_{N_{PC}}]^T$ are described as in Galerkin PC, since the stochastic response propagation in spectral stochastic finite element method is developed by a PC expansion.

4 Non-intrusive SFEM Solvers

Non-intrusive techniques does not require reformulation of the source code, which facilitates the statistical analyses by direct pre and postprocessing methods. This makes such techniques highly recommended to be applied in a wide range of fields from integrated circuits to computational fluid dynamics [34, 46]. In this section, it is reviewed the most suitable methods for crashworthiness. These approaches will be classified into two groups: Polynomial Chaos expansions and Sampling methods.

4.1 Non-intrusive Polynomial Chaos Expansion

Non-intrusive Polynomial Chaos [15] method is based on the decomposition of a random function $Y(\mathbf{U}(\theta))$ into deterministic and stochastic components in a separable manner.

Thus the $QoI = Y(\mathbf{U}(\theta))$ considered for the computational crash problem is represented as a Polynomial Chaos expansion by the expression:

$$Y = Y(\mathbf{U}(\theta)) = \sum_{n=0}^{N_{PC}} Y_n \cdot \Psi_n(\theta), \quad (35)$$

Fig. 9 Comparison of the various quasi random sequences with respect to MC random points

Fig. 10 Quasi Monte Carlo method framework

where Y_n are the deterministic Fourier coefficients and $\Psi_n(\theta)$ are the random basis functions (orthogonal polynomials chosen in the Askey–Wilson scheme [3]). Recall that for using polynomial chaos, the inputs have to be independent, however, if there are dependencies between them, it is necessary additional methods, for more details the reader is referred to [18].

In the following, an overview of the two main techniques to estimate Fourier Coefficients are described: Pseudo Spectral Projection (quadrature based) and Point Collocation (least square minimisation).

4.1.1 Pseudo Spectral Projection Polynomial Chaos

Pseudo Spectral Projection (or quadrature polynomial chaos) are based on obtaining the coefficients Y_n by quadrature techniques [32, 46]. Each coefficient is calculated by projecting Eq. (35) onto the n th basis, with $n = 0, 1, \dots, N_{PC}$ such that

$$\langle Y, \Psi_n(\theta) \rangle = \left\langle \sum_{n=0}^{N_{PC}} Y_n \Psi_n(\theta), \Psi_n(\theta) \right\rangle. \quad (36)$$

Using the orthogonality properties of the basis functions:

$$\langle Y, \Psi_n(\theta) \rangle = Y_n \langle \Psi_n^2(\theta) \rangle, \quad (37)$$

then,

$$Y_n = \frac{\langle Y, \Psi_n(\theta) \rangle}{\langle \Psi_n^2(\theta) \rangle}. \quad (38)$$

Recalling the definition (22), the previous equation becomes:

$$Y_n = \frac{\mathbb{E}(Y, \Psi_n)}{\mathbb{E}(\Psi_n^2)} = \frac{1}{\mathbb{E}(\Psi_n^2)} \int_{\Omega} Y \Psi_n(\theta) PDF(\theta) d\theta \quad (39)$$

The goal of Pseudo-Spectral Projection is to evaluate the multi-dimensional integral of (39) with numerical quadrature techniques [16, 25, 33, 41, 52], obtaining:

$$Y_n = \frac{1}{\mathbb{E}(\Psi_n^2)} \sum_{k=0}^{K-1} Y(\mathbf{h}_k) \Psi_n(\mathbf{h}_k) w_k, \quad (40)$$

where w_k are weights, $\mathbf{h}_k$ are quadrature points, K is the number of evaluations for the model, determined by $(q+1)^M$, being q the number of quadrature points and M the number of stochastic inputs. $Y(\mathbf{h}_k)$ is the QoI evaluated in $\mathbf{h}_k$, $\Psi_n(\mathbf{h}_k)$ are the basis functions evaluated in $\mathbf{h}_k$. Finally, $\mathbb{E}(\Psi_n^2)$ can be computed analytically for multivariate polynomials [36, 42].

4.1.2 Point Collocation Polynomial Chaos

Point collocation Polynomial Chaos, also known as stochastic response surface or linear regression, is another non-intrusive technique to calculate the Fourier coefficients, Y_n [9, 31]. The goal of this method is to extend the polynomial chaos expansions to be equal to each *black box* evaluation $Y(\mathbf{h}_k)$ at a set of “collocation” sampling points $\mathbf{h}_k$. Hence, the polynomial chaos reads

$$Y(\mathbf{h}_k) = \sum_{n=0}^{N_{PC}} Y_n \cdot \Psi_n(\mathbf{h}_k), k = 0, 1, \dots, K. \quad (41)$$

The corresponding linear system of equations to obtain the Fourier coefficients Y_n reads:

Fig. 11 Multi level Monte Carlo telescoping strategy mesh grid

Fig. 12 Multi level Monte Carlo framework

$$\underbrace{\begin{pmatrix} \Psi_0(\mathbf{h}_0) & \dots & \Psi_{N_{PC}}(\mathbf{h}_0) \\ \Psi_0(\mathbf{h}_1) & \dots & \Psi_{N_{PC}}(\mathbf{h}_1) \\ \vdots & \dots & \vdots \\ \Psi_0(\mathbf{h}_K) & \dots & \Psi_{N_{PC}}(\mathbf{h}_K) \end{pmatrix}}_{\mathbf{M}} \cdot \underbrace{\begin{pmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{N_{PC}} \end{pmatrix}}_{\mathbf{c}} = \underbrace{\begin{pmatrix} Y(\mathbf{h}_0) \\ Y(\mathbf{h}_1) \\ \vdots \\ Y(\mathbf{h}_K) \end{pmatrix}}_{\mathbf{z}}$$

This establishes a system of K equations and N_{PC} unknowns, where generally $K \geq N_{PC}$, being therefore an overdetermined system. The least-square minimisation approach consists in finding a set of coefficients which minimises the mean square error, obtaining the solution

$$\mathbf{c} = (\mathbf{M}^T \mathbf{M})^{-1} \mathbf{M}^T \mathbf{z}. \quad (42)$$

The choice of the point collocation nodes $\mathbf{h}_k$ highly influences computational cost and also the accuracy of the results. Various sampling methods to define $\mathbf{h}_k$ are proposed in the literature such as: Pseudo-Random values, Latin hypercube, Hammersley samples, Halton sequences, and Sobol sequences among others [18, 30, 47, 62].

4.1.3 Post Processing Polynomial Chaos

From the orthonormality of the basis functions, it is easily computed the mean and standard deviation of a polynomial chaos expansion using the coefficients Y_n . The mean of the random solution is given by

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{n=0}^{N_{PC}} Y_n \cdot \Psi_n(\theta)\right] = Y_0, \quad (43)$$

which indicates that the zeroth coefficient of the expansion corresponds to the expected value. Similarly, the variance reads,

$$\sigma^2 = \text{Var}[Y] = \mathbb{E}[(Y - \mathbb{E}(Y))^2] = \sum_{i=1}^P Y_i^2 \langle \Psi_i^2(\theta) \rangle. \quad (44)$$

Additionally, the PC expansion can be used as a response surface. Thus, the probability density function of Y is obtained by evaluating the PC expansion at random points [58]. A framework scheme of the non-intrusive polynomial chaos procedure is shown in Fig. 6.

4.2 Non-intrusive Sampling Techniques

Non intrusive sampling techniques aim at solving a stochastic approach with a more intuitive methodology. However this methods require large number of simulations. Sampling techniques are the most used approaches in science and engineering industry [6, 60, 66]. In this section, we describe the methods of Monte Carlo, Quasi Monte Carlo and Design of Experiment, since we consider such sampling techniques highly suitable for crashworthiness analysis.

Fig. 13 Number of *black box* simulations using **a** Pseudo Spectral Projection Polynomial Chaos, and **b** Point Collocation Polynomial Chaos with $n_f = 1$

4.2.1 Design of Experiments

Computational Design of Experiments is a non stochastic technique used to analyze the variability of the output using a specific set of simulations. The basis of Design of Experiments is the so-called Taguchi method [59] to study the input variability with a small number of simulations. Taguchi is applied in several fields such as automotive engineering, biology, chemistry to optimize the input parameters [21, 26, 61]. The key of this technique is the use of orthogonal arrays by columns. That is, for any pair of columns, all combinations of input levels occur, and they occur an equal number of times [1, 37, 59, 67]. Table 2 shows an example of parameters combination using Taguchi method. It consists on a total 9 simulations to be conducted with three parameters A, B and C (in our case they would be the thicknesses H1, H2 and H3), each one of them at three different levels: L1, L2 and L3. Note that the three levels are three real values of our parameters, For specific details on Taguchi method and orthogonality, the reader is referred to [59].

Fig. 14 200 Hammersley sample points of H1, H2, H3 illustrated in the 3D stochastic space

The steps used in Taguchi methodology [17, 48] are:

- Select of independent variables.
- Select of number of levels for each variable.
- Construct the orthogonal array.
- Conduct the simulations with respect to the array.
- Analyze data.

To post process the outcomes from Taguchi method, a signal-to-noise ratio [5] and an analysis of variance is commonly used to calculate and improve the variability of the samples [12, 17, 59].

4.2.2 Monte Carlo Method

In industry and science, the Monte Carlo (MC) method is a well-know technique for propagating the uncertainty in complex systems [19]. This probabilistic technique generates a finite number of random samples for each stochastic input to evaluate the model as a *black box*. Leading to develop post-process of the set of outputs. Figures 7 and 8 show the main idea of a MC approach for a crash problem and the main steps to implement MC technique.

The input random field is typically parametrized by a large number of random samples, leading to a problem of high dimension. Being the quantity of interest Y , the expected value $\mathbb{E}[Y]$ in the set of samples is defined by:

$$\mathbb{E}[Y] = Y_{MC} = \frac{1}{N_{MCS}} \sum_{i=1}^{N_{MCS}} Y(\mathbf{H}^i(\theta)), \quad (45)$$

where $\mathbf{H}^i(\theta)$ corresponds to the input sample realization, and N_{MCS} the number of Monte Carlo simulations. Hence, it is a vector containing each i -th random parameters that the

Fig. 15 Local plastic strain distribution and evolution for 0s, 0.5s and 1s from left to right images respectively

(a) Mean convergence

(b) Standard deviation convergence

Fig. 16 Mean and standard deviation with respect to the number of simulations (NSIM) obtained by Monte Carlo for crash problem

Table 3 Monte Carlo mean and standard deviation results for the crash problem with respect to the number of simulations

NSIM	1000	1500	2066	2466
Mean	0.0697	0.0699	0.0695	0.0695
Std	0.0234	0.0237	0.0240	0.0239

Fig. 17 Probability density function evolution with different NSIM Monte Carlo approaches

mathematical model uses as a inputs. Given the expected value Y_{MC} , the variance (σ^2) is determined by,

$$\sigma^2[Y] = \frac{1}{N_{MCS}} \sum_{i=1}^{N_{MCS}} (Y(\mathbf{H}^i(\theta)) - Y_{MC})^2. \quad (46)$$

Table 4 Stochastic inputs for each peak on the modes

	H1	H2	H3	Y
Peak 1	1.31	1.47	1.37	0.0294
Peak 2	1.22	1.08	1.12	0.0852

Another important statistical value to analyze is the standard deviation (σ), the square root of the variance. The advantage of the standard deviation is that it is measured in the same units as the data samples, being the variance is measured in squared units.

A disadvantage of MC method is its slow convergence rate, determined by the order $\frac{1}{\sqrt{N_{MCS}}}$. In crashworthiness analysis, each simulation requires a large time to be solved, consequently, the required computational cost to implement this stochastic technique is generally unaffordable. In such cases, it is necessary to find a smart sampling and variance reduction to create variants from the classical MC method, leading to the Multilevel Monte Carlo and Quasi Monte Carlo techniques to improve the computational cost and time.

4.2.3 Quasi Monte Carlo Method

As aforementioned, Quasi Monte Carlo (QMC) method is a variant of the MC method based on reducing the dimension of the input samples $\mathbf{H}^i(\theta)$, $N_{QMCS} < N_{MCS}$, by smarter strategies so that to improve the convergence rate from the classical MC [27] (with a well distributed input points), close to $\frac{1}{N_{QMCS}}$.

(a) Peak 1. $Y=0.0294$

(b) Peak 2. $Y=0.08524$

Fig. 18 Peak cases of the local plastic strain after 1 second

Fig. 19 Coloured mode areas of the quantity of interest PDF

The choice of QMC points are based on low discrepancy sequences, also called quasi-random or sub-random sequences [43]. Sub-random numbers have an advantage over MC random numbers, they cover the domain of interest quickly and evenly. Figure 9 shows a comparative plot with three schemes to generate QMC points in a uniform 2D space. It illustrates: random points, Hammersley, Halton and Sobol sequences [18, 31]. Figure 10 shows the QMC scheme procedure. Analogously to MC, the QMC expected value of the QoI is given by:

$$\mathbf{Y}_{QMC} = \frac{1}{N_{QMCs}} \sum_{i=1}^{N_{QMCs}} Y(\tilde{\mathbf{H}}^i(\theta)) \quad (47)$$

4.2.4 Multi level Monte Carlo Method

Multilevel Monte Carlo method (MLMC) is a variant of MC that has been developed and implemented in different fields [7, 27]. The main key is the use of a hierarchy numerical approach for different levels of accuracy. That means, increasing the level, the model becomes progressively more accurate and computationally costly. The main idea of MLMC is to develop a high number of simulations with less accuracy (giving a sense of average behavior), and low number of simulations with high accuracy (giving a sense of precision) [4]. This levelling method generally provides better results for stochastic analysis, improving computational cost compared to classical MC.

Here the quantity of interest Y is approximated by a calculation of the QoI in a sequence of hierarchical levels $Y_0, \dots, Y_l$ with a different accuracy. Being Y_0 the less accurate level, thus requiring less computing cost. On the contrary, Y_l is the most

accurate and therefore computationally costly [24]. The sense of the multilevel method is then the telescoping sum defined by:

$$\mathbb{E}[Y_{MLMC}] = \mathbb{E}[Y_0] + \sum_{l=1}^L \mathbb{E}[Y_l - Y_{l-1}] \quad (48)$$

where $\mathbb{E}[Y_0]$ it is a low estimator of the QoI and $\sum_{l=1}^L \mathbb{E}[Y_l - Y_{l-1}]$ gives the accuracy. If it is used the MC estimator at each level, the above MLMC expected value $\mathbb{E}[Y_L]$ leads to:

$$\begin{aligned} \mathbb{E}[Y_{MLMC}] &= \frac{1}{N_0} \sum_{n=1}^{N_0} Y_0(\mathbf{H}^n(\theta)) \\ &+ \sum_{l=1}^L \left[\frac{1}{N_l} \sum_{n=1}^{N_l} (Y_l(\mathbf{H}^n(\theta)) - Y_{l-1}(\mathbf{H}^n(\theta))) \right] \end{aligned} \quad (49)$$

where N_0 is the number of simulations in level 0 and N_l is the number of simulations in level l , where $N_0 > N_1 > \dots > N_{l-1} > N_l$. The different levels of accuracy are obtained by two different manners, increasing the time step or refining the mesh grid. In crashworthiness analysis, refining the mesh can be unaffordable, thus a better option is to increase the time step, keeping the mesh fixed. Figure 11 illustrates the MLMC strategy of a 2D geometry with a telescoping increasing mesh. Analogously to MC and QMC, Fig. 12 shows the framework of the main steps to implement MLMC method.

5 Comparison of SFEM Solvers for Crashworthiness

In this section, it is shown a comparative analysis between intrusive and non-intrusive sampling methods to implement an uncertainty quantification approach for crashworthiness stochastic problems.

Intrusive approaches such as Perturbation method, Galerkin Polynomial Chaos and spectral stochastic finite element method described in above sections have important strengths to take into account. Intrusive techniques give the full random responses in the whole time and space domains. Also, in general cases it requires fewer simulation compared with non-intrusive approaches. However, these methodologies are not trivial to implement or even not possible [14]. Furthermore, computational complexity increases with the number of random inputs and the order of the expansion [2]. In the case where the QoI have a non-linear response, high expansion order is required [8].

On the other side, the main advantage of non intrusive methods, Pseudo Spectral Polynomial Chaos, Point Collocation Polynomial Chaos, Monte Carlo and variants (non

(a) 2466 output samples

(b) Frontal view

(c) Vertical view

Fig. 20 Coloured output samples of each mode areas. *Mode 1* (red), *Mode 2* (blue). (Color figure online)

intrusive methods), is the use commercial Softwares to obtain deterministic outputs without inferring in the original code [14].

MC technique is considered the most general and robust method of uncertainty. This approach allows to tackle linear and non-linear problems in a wide range of engineering and science fields [55, 57]. One main drawback lays on the large number of simulations necessary to obtain good results, which in Crashworthiness means unaffordable computational cost. QMC and MLMC present improvements to reduce dimensionality, thus in cases where simulations require a high computing power, its implementation is recommended [27].

Non intrusive PC presents advantages with respect sampling methods in terms of dimensionality, since it allows a reduction of the number of simulations. Unlike MC approach, Non intrusive PC expansions suffer from the curse of dimensionality, the number of terms grow exponentially with the number of random inputs [53, 63] and the polynomial degree. In Pseudo Spectral Projection Polynomial Chaos, the number of simulations increases by $N_{simPS} = (q)^M$, where q is the number of quadrature nodes and M the number of random variables. For Point Collocation Polynomial Chaos approach the number of simulations is defined by $N_{simPC} = n_f \frac{(M+p)!}{M!p!}$, being p the order of the polynomial expansion and n_f a proportional parameter that

(a) Mean convergence

(b) Standard deviation convergence

Fig. 21 Quasi Monte Carlo mean and standard deviation convergence plots for the stochastic crash model

increases the number of collocation points, thus oversampling the number of simulations. If $n_f > 1$ the system is overdetermined and least squares technique is used to reduce the residual error of the response. This proportional parameter is used to improve the response accuracy. Different authors found in the literature oversample the number of simulations to optimize the response [15, 29]. Figure 13 shows the computational cost associated to the Pseudo Spectral Projection and Point Collocation approaches. It is clearly visible the lower computational cost of Point Collocation than Pseudo Spectral Projection with large number of stochastic inputs [18], being similar for a low number of random inputs.

For one or two random variables, the number of simulations between methods have similar computational cost. However, when the number of random variables increases, the computational cost of the numerical quadrature grows significantly [53].

6 Crash Model Implementation

The Benchmark crash problem (Sect. 2.1) is analyzed with different non-intrusive stochastic approaches to quantify the uncertainty. The deterministic finite element simulations were calculated with VPS (Pamcrash) 2015.4 commercial solver, and the stochastic approaches were implemented in Python using *Chaospy* package [18].

As aforementioned, the computational time is mainly the most critical drawback in crashworthiness studies. Thus, to analyse the possible reduction in the number of simulations, the non intrusive stochastic techniques, PC and QMC, are implemented and compared with the results obtained using classical MC.

To obtain a robust solution of the QoI, Monte Carlo method was implemented using 2466 simulations. QMC and Point Collocation Polynomial Chaos methods were

(a) $p = 1$ and $n_f = 1$

(b) $p = 2$ and $n_f = 1$

(c) $p = 3$ and $n_f = 1$

(d) $p = 4$ and $n_f = 1$

Fig. 22 Probability density function evolution of Point Collocation Polynomial Chaos ($n_f = 1$, Hammersley sampling), QMC (Hammersley sampling) and MC (random sampling). Approaches launched with different polynomial order p and NQMC

implemented by using the Hammersley samples [30] to generate the input samples H1, H2, H3, obtaining 330 simulations. Figure 14 illustrates the stochastic space of the three random thickness (H1, H2, H3) using Hammersley technique.

The main objective is to obtain the optimal combination of the two parameters involved in PC for the crash problem: the oversampling n_f and the order of polynomial p . Therefore, different oversampling values $n_f = 1, 2, 3, 4$ at different polynomial orders p were developed.

The deterministic *black box* simulations were launched by parallel computing with two different machines or clusters:

(i) A cluster with 16 CPUs of 3.40 GHz and 252.3 GiB of RAM memory. (ii) A Workstation laptop with 4 CPUs of 3.40 GHz and 32 GiB of RAM. The computational cost for a single simulation was in the order of [20–30] min, whereas all the set of simulations are in the order of weeks.

7 Numerical Results

A first deterministic approach was calculated with constant thicknesses $H1 = H2 = H3 = 1.2$ in [mm], Fig. 15 shows the snapshots of the local plastic strain evolving in time for the

(a) $p = 1$ and $n_f = 2$

(b) $p = 2$ and $n_f = 2$

(c) $p = 3$ and $n_f = 2$

(d) $p = 4$ and $n_f = 2$

Fig. 23 Probability density function evolution of Point Collocation Polynomial Chaos ($n_f = 2$, Hammersley sampling), QMC (Hammersley sampling) and MC (random sampling). Approaches launched with different polynomial order p and NQMC

explicit simulation. The solution for the quantity of interest (plastic strain average on the the area of interest) through this deterministic approach corresponds to $QoI = Y = 0.0798$.

Now Considering the crash problem with thickness variability, next section show the obtained analysis results of using the previously mentioned non-intrusive stochastic approaches.

7.1 Monte Carlo Approach

Monte Carlo model was developed to obtain the corresponding reference values, mean and standard deviation of the QoI area represented in Fig. 16. Table 3 represents The mean and standard deviation values up to 2466 simulations. The MC

models finds convergence to the values of $mean = 0.0695$ and $StD = 0.0239$, which will be considered as reference for further analyses. Accordingly, the obtained probability density functions with different number of simulations are illustrated in Fig. 17, showing a clear bimodal behavior, being *Mode 2* clearly predominant with respect to *Mode 1*. This phenomenon of multi modality usually occurs when small perturbations in the system cause some changes on the output. The prediction of different modes give an important key to understand the stochastic response.

To quantify the bimodality, the two simulations where the quantity of interest Y coincide with each mode peak of Fig. 19 are chosen. Figure 18 shows the two simulations coinciding with each peak.

(a) $p = 1$ and $n_f = 3$

(b) $p = 2$ and $n_f = 3$

(c) $p = 3$ and $n_f = 3$

(d) $p = 4$ and $n_f = 3$

Fig. 24 Probability density function evolution of Point Collocation Polynomial Chaos ($n_f = 3$, Hammersley sampling), QMC (Hammersley sampling) and MC (random sampling). Approaches launched with different polynomial order p and NQMC

In Table 4 it is illustrated the two simulation with its corresponding stochastic inputs H1, H2 and H3. To analyse which parameter or combination of parameters generates this bimodal distribution, in Fig. 19 there are separated the samples that appears at left (red) and right (blue) of the histogram.

Figure 20 shows the stochastic space of the thickness inputs H1, H2, H3 for each simulation according to the previous *mode 1* (red), and *mode 2* (blue). It is observed two differentiated domains regarding the stochastic thickness H3. It seems to be an evidence that parameter H3 has a direct correlation with the QoI. However, H1 and H2

from Fig. 20c shows a mixed distribution of the samples in all the stochastic space, therefore a not clear influence with the output is here visible.

7.2 Quasi Monte Carlo Approach

For the QMC approach with Hammersley technique, Fig. 21 shows the mean and standard deviation convergence plots with respect to the reference values from the MC approach (red line). Similar results with respect to MC were obtained with 330 simulations (recall that for MC a

(a) $p = 1$ and $n_f = 4$

(b) $p = 2$ and $n_f = 4$

(c) $p = 3$ and $n_f = 4$

(d) $p = 4$ and $n_f = 4$

Fig. 25 Probability density function evolution of Point Collocation Polynomial Chaos ($n_f = 4$, Hammersley sampling), QMC (Hammersley sampling) and MC (random sampling). Approaches launched with different polynomial order p and NQMC

total of 2466 simulations were used), leading to a QMC mean of 0.06944 and a standard deviation of 0.02391.

7.3 Non-intrusive Polynomial Chaos Approach

Although QMC estimates good results, the computational cost still remains high because of the number of simulations (more than 300) required for a feasible analysis. This section shows the results and capabilities of the Point Collocation Polynomial Chaos method, implementing different combinations of polynomial orders p and oversampling simulations n_f as mentioned in Sect. 5. Figures 22, 23, 24 and 25 show the results for each

combination cases. It is illustrated the probability density function (blue) from the Monte Carlo reference values, the probability density function of QMC (green) and finally the PC results (red) are illustrated with the same required number of VPS simulations for comparison purposes. According to this illustrations, using polynomial orders higher than four, the accuracy of the results tends to diverge, apart from the case of $n_f = 2$ with $p = 5$ and $p = 6$ (Fig. 26), where the results are better captured. This effect is clearly seen in Fig. 27, illustrating the behavior of the mean and standard deviation, for the different cases described above, increasing the polynomial order.

Fig. 26 Probability density function evolution of Point Collocation Polynomial Chaos ($n_f = 2$, Hammersley sampling), QMC (Hammersley sampling) and MC (random sampling). Approaches launched with different polynomial order p and NQMC

7.4 Computational Cost Comparison

This sub-section shows a comparative analysis of each implemented non intrusive method with respect to the number of simulations required. Figure 28 illustrates the mean and standard deviation of MC, QMC and PC methods with different NSIM. All configurations of PC show less accurate results with respect to QMC approach, that maintains a highly stable results with very low number of simulations.

8 Discussion and Conclusions

In this paper it is presented a review of stochastic methods for solving a crashworthiness problem, using for this the commercial software VPS (Pamcrash). Additionally, a summary of SFEM techniques are also introduced. The classification of solvers for crashworthiness is based on intrusive and non-intrusive approaches for UQ analysis. In this work, a crash model industrial benchmark is studied to determine the effect of three random parameters, the thicknesses, on a quantity of interest (QoI), the plastic strain under a dynamic impact.

SFEM solvers are analyzed and compared, mainly the most suitable UQ approaches, pointing out specifically their competency for solving crash problems with uncertainty inputs. In order to avoid the cumbersome task of an intrusive approach, non-intrusive methods are implemented to the crash problem.

The MC results of the stochastic problem offers a sound framework for uncertainty propagation, however its low efficiency precludes its use for analyses involving crash models. In this case, MC is implemented as a reference method to be compared with other techniques. The results show a bimodal behavior of the QoI with a predominant probability area.

To demonstrate the viability of decreasing the number of simulations, two non-intrusive techniques are studied and compared with MC model: Quasi Monte Carlo and Point Collocation Polynomial Chaos, both implemented for the crash problem. The methodology of QMC is based on deterministic samples, uniformly distributed samples. Compared with the classical MC method, QMC effectively reduces the required number of simulations to achieve a similar accuracy of the results. The so-called Point Collocation Polynomial Chaos (based on a least-square minimization) shows that in order to have a good accuracy of the results it requires three times greater the number of simulations. This means critical low efficiency.

Crashworthiness in industry remains a strongly expensive field of research, nowadays the development of experimental tests remain necessary, even being costly in the economic and the environmental points of view. The use of deterministic numerical crash simulation for analysis and re-design still lacks of computational strength and credibility to be a reliable and substitutive to experimental tests carried out in vehicles. In this way, statistical methods applied to stochastic models are, under the opinion of the authors of this manuscript, a research field worth paying

(a) Mean convergence

(b) Standard deviation convergence

Fig. 27 Mean and standard deviation of the stochastic crash model with Point Collocation Polynomial Chaos

attention to, and able to redirect crash test to a computation environment, with the consequent cost reduction.

With all the aforementioned, to reduce the numerical simulations required to obtain statistically credible results, the study and optimization of non-intrusive

stochastic models are potential research disciplines able (in the future) replace expensive experimental tests for crashworthiness in automotive industry. Nevertheless, not many works were found in literature in this line, here we proposed a benchmark case (typically used in automotive

(a) PC and QMC mean convergence.

(b) PC and QMC standard deviation convergence.

Fig. 28 Mean and standard deviation comparative evolution with respect to the number of simulations. Comparing Point Collocation PC (Changing the Polynomial order p and N_f oversampling parameter) and QMC approaches of the stochastic Crash model

industry) to show the possibilities of UQ methods. The combination of stochastic modelling with data analysis techniques such as principal component analysis can be of great importance to optimize the computation resources required for these analysis.

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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