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A characterization of weighted simple games based on pseudoweightings --Manuscript Draft--

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Corresponding Author:	Josep Freixas SPAIN	
Corresponding Author Secondary Information:		
Corresponding Author's Institution:		
Corresponding Author's Secondary Institution:		
First Author:	Josep Freixas, Dr.	
First Author Secondary Information:		
Order of Authors:	Josep Freixas, Dr.	
Order of Authors Secondary Information:		
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A characterization of weighted simple games based on pseudoweightings

Josep Freixas

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Keywords Boolean functions · Threshold functions · Simple games · Weighted games · Pseudoweightings · Optimization

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1 Introduction

Simple games are at the core of voting systems, in them a single alternative, such as a bill or an amendment, is pitted against the status quo, and voters can vote for or against the bill. Simple games are very versatile structures equivalent to monotonic hypergraphs that can also be thought as coherent structures, logic gates or Boolean functions.

J. Freixas

Av. Bases de Manresa, 61-73, 08242 Manresa, Spain. Universitat Politècnica de Catalunya.

Tel.: +34-938777246

E-mail: josep.freixas@upc.edu

Two subclasses of simple games stand out for their mathematical properties and applications to real voting systems. We refer to the class of complete games that contains, in turn, the class of weighted games. Both subclasses have been extensively studied in several different fields and under different denominations. The search for elegant properties to characterize these kinds of games, within the class of simple games, has been and is an important milestone. Each characterization sheds new light on the understanding of these structures and makes it easier to solve related problems and obtain new enumerations of subclasses of games.

A characterization of complete games, swap robustness, establishes that if any two winning coalitions for the game engage in a one-for-one exchange of players, then at least one of the coalitions is still winning after the swap. An elegant characterization of weighted games, trade robustness, [12], establishes that any sequence of winning coalitions can never be rendered losing by arbitrary trades among the coalitions in the sequence. Conspicuous examples of failures of trade robustness can be found in [14]. We refer the interested reader to [15] for background on these characterizations and the connection of trade robustness with the assumability condition of Boolean functions.

The property of swap robustness provides an easy method to show that certain games are not complete. Therefore, the main difficulty lies in knowing if the game is weighted having verified before that it is complete. A refinement of the trade robustness property within the class of complete games, is the invariant-trade robustness property [4]. The combination of these two properties allows in [5], to obtain many weighted game characterizations.

A point-set-additive pseudoweighting for a simple game, also known as a PSA pseudoweighting, differs from a standard weighting for a simple game in that the former assigns weights directly to coalitions (including the one-element coalitions that amount to individuals), while the latter assigns weights directly to individuals (and thus induces a weighting on coalitions).

The PSA pseudoweighting is a weakening of weightedness that characterizes complete simple games. Indeed, [15] prove that a simple game is complete *if and only if* it is a PSA pseudoweighted game. In this paper we consider a strengthening of the PSA pseudoweighting that characterizes weighted simple games. As far as we know this is a new characterization of weighted games.

The paper is organized as follows. Section 2 introduces the basic concepts of simple games that are necessary for the follow up of the paper. Section 3 introduces pregames and proposes a property of simple games based on pseudoweightings. Section 4 is entirely devoted to obtain a characterization of weighted games based on pseudoweightings.

2 Background on simple games

Definition 1 A *simple game* is a pair (N, W) in which $N = \{1, 2, \dots, n\}$ and W is a collection of subsets of N that satisfies: $N \in W$, $\emptyset \notin W$ and, the monotonicity property, if $S \in W$ and $S \subseteq T \subseteq N$ then $T \in W$.

A (monotonic) simple game corresponds to a (monotonic) Boolean function in the field of Boolean algebra.

The set N is called the *grand coalition*. Members of N are called *players* or *voters*, and subsets of N are called *coalitions*, the coalitions that belong to W are called *winning coalitions*, the subfamily of *minimal* winning coalitions is $W^m = \{S \in W : \forall T \subset S \Rightarrow T \notin W\}$, the coalitions in $L = 2^N \setminus W$ are called *losing coalitions*. The subfamily of *maximal* losing coalitions is $L^M = \{S \in L : \forall T \supset S \Rightarrow T \in W\}$. Any of these four sets determines the game. A voter i has *veto* in a simple game whenever $i \in S$ for all $S \in W$.

By $|S|$ we mean the cardinality of a coalition S and use $|S| = s$ whenever there is no confusion. Some studies on real-world examples of simple games can be found in [2, 3, 9–11, 13].

Definition 2 Two simple games (N, W) and (N', W') are *isomorphic* if there exists a one-to-one correspondence $f : N \rightarrow N'$ such that $S \in W$ if and only if $f(S) \in W'$; f is called an *isomorphism* of simple games.

Two isomorphic simple games only differ in the labels, so from now on we will only consider simple games up to isomorphisms. (Monotonic) simple games up to isomorphisms correspond to inequivalent (monotonic) Boolean functions.

One common idea of the most voting systems used in practice is the concept of influence, i.e., that a particular voting system may give one voter more influence than another. The so-called ‘desirability’ relation defined on the set of voters represents a way to make this precise. Isbell already used it in [6].

Definition 3 Let (N, W) be a simple game. Player i is *at least as desirable* as j ($i \succsim j$, in short) in (N, W) if: $S \cup \{j\} \in W \Rightarrow S \cup \{i\} \in W$, for all $S \subseteq N \setminus \{i, j\}$. Players i and j are *equally desirable* ($i \approx j$, in short) in (N, W) if: $i \succsim j$ and $j \succsim i$. Player i is *strictly more desirable* than player j ($i \succ j$, in short) in (N, W) if: $i \succsim j$ and $i \not\approx j$.

Definition 4 A simple game (N, W) is *complete* or *linear* if the desirability relation is a complete preordering.

Complete simple games correspond to 2-monotonic Boolean functions in Boolean algebra.

We also speak of the number of types, t , of voters meaning the number of equivalence classes. This is a fundamental parameter in our study. Thus, a simple game is

either complete or not. Whenever the simple game is complete we will assume in the following and w.l.o.g. that $N_1 > N_2 > \dots > N_t$, which means that $i \succ j$ whenever $i \in N_k$ and $j \in N_{k+1}$ for all $k = 1, \dots, t-1$. Note that the veto players (if any) are the most powerful ones in the system in the sense that they form the strongest equivalence class, N_1 .

Definition 5 A simple game (N, W) is a *weighted game* if there exists a vector of *weights* with non-negative components $w = (w_1, \dots, w_n)$ such that

$$\sum_{i \in S} w_i > \sum_{j \in T} w_j, \quad \text{for all } S \in W \text{ and } T \in L. \quad (1)$$

Weighted games correspond to threshold functions in Boolean algebra. The number of weighted games up to isomorphisms is only known up to $n = 9$. Thus, characterizations or efficient optimizations techniques are needed to solve if a given simple game is weighted. An updated analysis on Boolean optimization techniques is provided in [16].

All weighted games are complete, because $w_i \geq w_j$ implies $i \succeq j$. We have $w_i > w_j$ if $i \succ j$, but not necessarily $w_i = w_j$ if $i \approx j$. In a weighted game a *quota* q may be inserted between $b = \min_{S \in W} \sum_{i \in S} w_i$ and $a = \max_{T \in L} \sum_{j \in T} w_j$, i.e., any $q \in (a, b]$ separates the weights of winning coalitions from the weights of losing coalitions. We use $[q; w_1, \dots, w_n]$ for an arbitrary representation of a weighted game.

There is an infinite number of representations for a weighted game with different weights, because $[q; w_1, \dots, w_n]$ is equivalent to $[c \cdot q; c \cdot w_1, \dots, c \cdot w_n]$ for all $c > 0$ and always exists weighted representations in integers for it. Moreover, every weighted game admits a *normalized weighted representation*, i.e., $w_i = w_j$ whenever $i \approx j$. Then, the weighted game can be presented by $[q; w_1(n_1), \dots, w_t(n_t)]$, where, $w_i(n_i)$ indicates that the weight of the n_i players in N_i is w_i .

Given the equivalence classes of a complete game $N_1 > \dots > N_t$, the vector $\bar{n} = (n_1, n_2, \dots, n_t)$ represents the grand coalition and any vector $\bar{m} = (m_1, m_2, \dots, m_t)$ with $0 \leq m_i \leq n_i$ represents the set of coalitions $\{S : |S \cap N_i| = m_i, \text{ for all } i = 1, \dots, t\}$. Coalitions represented by the *model* $\bar{m}$ are either all winning or all losing. Thus, we write either $\bar{m} \in \bar{W}$ or $\bar{m} \in \bar{L}$. The elements in $\bar{m} \in \bar{W}$ which are minimal componentwise represent minimal winning coalitions and write $\bar{m} \in \bar{W}^m$ for them. Analogously, the elements in $\bar{m} \in \bar{L}$ which are maximal componentwise represent maximal losing coalitions and write $\bar{m} \in \bar{L}^M$.

By $\bar{m} \leq \bar{p}$ we mean $m_i \leq p_i$ for all $i = 1, \dots, t$, and by $\bar{m} < \bar{p}$ we mean $\bar{m} \leq \bar{p}$ and $\bar{m} \neq \bar{p}$. By $\bar{0}$ we mean the t -dimensional vector formed by zeros. The next conditions meet the requirements of monotonic simple games:

- (i) $\bar{n} \in \bar{W}$,
- (ii) $\bar{0} \notin \bar{W}$, and
- (iii) $\bar{m} \leq \bar{p}$ with $\bar{m} \in \bar{W}$ implies $\bar{p} \in \bar{W}$.

As $N_1 > \dots > N_t$ for complete games, we also need to consider the next requirement.

(iv) Let $\bar{m} = (m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_t) \in \bar{W}$ with $m_i < n_i$ and $m_{i+1} > 0$, then $(m_1, \dots, m_{i-1}, m_i + 1, m_{i+1} - 1, \dots, m_t) \in \bar{W}$.

Note that condition (iv) captures the idea that if a player is replaced in a winning coalition by another player being more desirable than her, then the new coalition becomes also winning. Observe that in this argument the condition $N_i > N_{i+1}$ is essential. Thus, the vector $\bar{n}$ and the set $\bar{W}$ fulfilling conditions (i) – (iv) define a complete game by using models instead of coalitions. See [1] for further details on this presentation of complete games. Note that if a complete game has vetoes then the first component for any $\bar{m} \in \bar{W}$ is $m_1 = n_1$.

Another fundamental parameter in our study for each game is r , the number of minimal winning models of the complete game. If we have $r = 1$ for a complete game, and the minimal winning model is $\bar{m} = (m_1, m_2, \dots, m_t)$, then the remaining winning models are vectors $\bar{m}' = (m'_1, m'_2, \dots, m'_t)$ who dominate $\bar{m}$ componentwise, i.e., $m'_i \geq m_i$ for all $i = 1, \dots, t$. If $r > 1$, there is no componentwise relation between two minimal winning models, neither $\bar{m} \geq \bar{m}'$ nor $\bar{m}' \geq \bar{m}$ holds if $\bar{m} \in \bar{W}^m$ and $\bar{m}' \in \bar{W}^m$. In a similar way we can define the parameter s , which is the number of maximal losing models of coalitions.

The most efficient way to define a complete game by means of models is to use $\bar{W}^m$ instead of $\bar{W}$. Then the description of the complete game may be given by $\bar{n}$ and a matrix $\mathcal{M}$ whose r rows are the elements in $\bar{W}^m$ ordered in a lexicographic ordering (to avoid isomorphic presentations of the same game) as follows: the i -th row precedes the $(i + 1)$ -th row if

$$\sum_{j=1}^t (m_{i,j} - m_{i+1,j}) a^{t-j} > 0$$

where here a is the nearest greater power of 10 of the maximum cardinality of the equivalence classes. For instance, if $\max_{1 \leq i \leq t} n_i = 8$, we choose $a = 10$, while if the $\max_{1 \leq i \leq t} n_i = 23$ we choose $a = 100$. Equivalently, the complete game can be given by $\bar{n}$ and a matrix $\mathcal{L}$ with $\alpha_{i,j}$ as entries whose s rows are the elements in $\bar{L}^M$ also lexicographically ordered.

By using $\bar{n}$, $\mathcal{M}$ and $\mathcal{L}$ (this latter matrix can be obtained from $\bar{n}$ and $\mathcal{M}$, or conversely, $\mathcal{M}$ can be obtained from $\bar{n}$ and $\mathcal{L}$) we can determine if a complete game is weighted by solving the following inequalities system, which uses less inequalities than in (1) whenever $t < n$

$$\begin{cases} \sum_{j=1}^t (m_{i,j} - \alpha_{k,j}) \cdot w_j > 0 & i = 1, \dots, r, \quad k = 1, \dots, s \\ w_j > w_{j+1} & j = 1, \dots, t-1 \\ w_t \geq 0 \end{cases} \quad (2)$$

If the inequalities system (2) is consistent and $(w_1, \dots, w_t) \in (\mathbb{N} \cup \{0\})^t$ is a solution for it, then we can choose as a quota the minimum weight of a minimal

winning model, i.e.,

$$q = \min_{1 \leq i \leq r} \sum_{j=1}^t m_{i,j} \cdot w_j.$$

3 Pregames and point-set-additive pseudoweightings

We introduce in this section the notions of pregame and the point-set-additive pseudoweighted representation at level k for a simple game. This latter new notion is fundamental for the characterization obtained in this paper.

Definition 6 A *pregame* is a triple (N, W, L) consisting of a finite set N of players together with disjoint subsets W and L of 2^N satisfying the monotonicity property of pregames: no set in W has a superset in L . Sets in W are called winning coalitions, and sets in L are called losing coalitions. The domain of (N, W, L) is the set $D = W \cup L$.

A simple game is a pregame with domain $D = 2^N$. The pregames with domains $D = \{S \in 2^N : |S| \leq n-3\}$ and $D = \{S \in 2^N : |S| \leq n-2\}$ are respectively called the pregames with $(n-3)$ -domain and with $(n-2)$ -domain. A pregame with a given domain is *weighted* if there exists a vector $(w_1, \dots, w_n)$ that verifies all the inequalities in equation (1) for all the coalitions in the domain.

Definition 7 The simple game (N, W) admits a *point-set-additive pseudoweighting representation at level k* P^kSA (briefly, a P^kSA pseudoweighting for (N, W)) if there exists a function $w : 2^N \rightarrow \mathbb{R}$ for which there exists a quota $q \in \mathbb{R}$ such that for each $j \leq k$:
 $S \cup \{i_1, \dots, i_j\} \in W \Leftrightarrow w(S) + w(i_1) + \dots + w(i_j) \geq q$ whenever $i_l \in N \setminus S$ for all $1 \leq l \leq j$.

Several remarks on Definition 7 are the following:

- As is the case with usual weights (but not with coalitional weights), the condition in Definition 7 can be replaced by another condition that does not mention a quota.
- The condition $w(S \cup \{i_1, \dots, i_j\}) \geq q$ could be added in Definition 7, although is not necessary for the definition.
- The PSA pseudoweighting for (N, W) , i.e., the P^kSA when $k = 1$, characterizes complete games. Indeed, Theorem 3.6.3 in [15] proves that a simple game is complete *if and only if* it is a PSA pseudoweighted game.
- The level k of pseudoweightedness varies between 1 and n . If $k = n$ every simple game (N, W) that bears a P^nSA pseudoweighting for it must be a weighted game.
- If a game (N, W) is P^kSA pseudoweighted for some $k > 1$, then it is a $P^{k-1}SA$ pseudoweighted game.

From the last three remarks we have: given $n > 1$, let $\mathcal{C}_n$ be the class of all complete games with n players, let $\mathcal{W}_n$ be the class of all weighted games with n players and let $\mathcal{P}_n^k$ be the class of all P^kSA pseudoweighted games for n players. Then:

$$\mathcal{W}_n = \mathcal{P}_n^n \subseteq \mathcal{P}_n^{n-1} \subseteq \mathcal{P}_n^{n-2} \subseteq \mathcal{P}_n^{n-3} \subseteq \dots \subseteq \mathcal{P}_n^2 \subseteq \mathcal{P}_n^1 = \mathcal{C}_n. \quad (3)$$

We study in next section whether the condition of weightedness in terms of pseudoweightings for a game can be relaxed to inferior values of k .

Example 1 A system to amend the Canadian Constitution used in the sixties and studied in Kilgour [7] is an example of a complete game with two types of voters with $|N| = 10$ and the elements of N are the ten Canadian provinces, $W^m = \{S \subseteq N : |S| = 7 \text{ and } S \cap \{1, 2\} \neq \emptyset\}$. Then N decomposes into $N_1 = \{1, 2\}$, representing the provinces of Ontario and Quebec, and $N_2 = \{3, \dots, 10\}$. This voting system is complete but not weighted. Indeed,

$$\bar{n} = (2, 8), \quad \mathcal{M} = (1 \ 6), \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} 2 & 4 \\ 0 & 8 \end{pmatrix}$$

Clearly $N_1 > N_2$ and $w_1 + 6w_2 > 2w_1$ and $w_1 + 6w_2 > 8w_2$ lead to a weighted incompatibility. A P^2SA pseudoweighted normalized representation for this game is given by $q = 15$ and the following weights:

Models of coalitions	Weights
$(2, x) \ 0 \leq x \leq 8$	$2x + 5$
$(1, x) \ 0 \leq x \leq 8$	$2x + 3$
$(0, x) \ 0 \leq x \leq 6$	$2x$
$(0, x) \ x = 7, 8$	12

A simplified version of this example is the game with 6 players:

$$\bar{n} = (2, 4), \quad \mathcal{M} = (1 \ 2), \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

which is non-weighted and admits the PSA pseudoweighted representation with quota $q = 7$ and normalized weights: $w(0, 1) = 2$, $w(1, 0) = 3$, $w(0, 3) = w(0, 4) = 4$, and $w(x, y) = 3x + 2y$ for the rest of profiles. It is a simple exercise to check that this game does not admit a P^2SA pseudoweighted representation.

4 A characterization of weighted games

Theorem 1 (N, W) is weighted $\iff (N, W)$ is a $P^{n-3}SA$ pseudoweighted game.

PROOF.

($\Rightarrow$) Let (N, W) be a weighted game. We define $w(S)$ additively for every $S \subseteq N$, representation for which the game is $P^n SA$ pseudoweighted and also $P^k SA$ pseudoweighted for every $k < n$.

($\Leftarrow$) Assume that (N, W) is a $P^{n-3}SA$ pseudoweighted game. Now, consider:

Case 1: $W^m = \{N\}$.

Case 2: $W^m \subseteq \{S : |S| \leq n-2\}$ and $L^M \subseteq \{S : |S| \leq n-2\}$.

Case 3: $W^m \subseteq \{S : |S| \leq n-2\}$ and $L^M \cap \{S : |S| = n-1\} \neq \emptyset$.

Case 4: $W^m \cap \{S : |S| = n-1\} \neq \emptyset$.

Case 1: Clearly the game is weighted and $[n; 1(n)]$ is a representation for it.

Case 2: Let $w : 2^N \rightarrow \mathbb{R}$ be a $P^{n-3}SA$ pseudoweighted representation. We shall see that the restriction of this function to the individual weights endow the game of weighted structure.

Choosing $S = \emptyset$ and $j = n-3$ in Definition 7, changing round the n players of N we deduce that the pregame with $(n-3)$ -domain is weighted.

Additionally, choosing $S = \{1\}$ and $j = n-3$ in Definition 7 changing round the $n-1$ players of $N \setminus \{1\}$, choosing $S = \{2\}$ and $j = n-3$ in Definition 7 changing round the $n-2$ players of $N \setminus \{1, 2\}$, and $S = \{3\}$ and $j = n-3$ in Definition 7 changing round the $n-3$ players of $N \setminus \{1, 2, 3\}$, we obtain that the pregame with $(n-2)$ -domain is weighted.

As the coalitions outside the $(n-2)$ -domain are winning but not minimal and the weights are non-negative, the weighted representation for the pregame with $(n-2)$ -domain is also valid for the game.

Case 3: As the winning coalitions do not necessarily have a greater cardinality than the losing coalitions it must be $t > 2$.

Let $\bar{p} = (p_1, p_2, \dots, p_t) \in \bar{L}^M$ representing coalitions with cardinality $n-1$, then $p_j = n_j - 1$ for some j and $p_i = n_i$ for all $i \neq j$. By monotonicity, this is the unique model of losing coalitions with cardinality $n-1$. For all $\bar{m} \in W^m$ it must be $\bar{m} \not\leq \bar{p}$, which implies $m_j = n_j$.

We claim that $p_1 = n_1 - 1$. In fact, if it was $p_1 = n_1$ then we would have $p_j = n_j - 1$ for some $j > 1$ so that $m_j = n_j$ for all $\bar{m} \in \bar{W}^m$, which would imply that every player in N_j would dominate by the desirability relation to any player in N_{j-1} , which would be a contradiction with $N_{j-1} > N_j$. Thus, $p_1 = n_1 - 1$ and $m_1 = n_1$ for all $\bar{m} \in \bar{W}^m$, and therefore the game has veto players.

The matrices $\mathcal{M} = (m_{ij})$ and $\mathcal{L} = (\alpha_{ij})$ have $r \geq 1$ and $s \geq 1$ rows respectively. If $s = 1$, then $t = 2$ (otherwise $N_3 \geq N_2$) and $\mathcal{M} = (n_1, 0)$ so that the game

is weighted and a representation for it is $[n_1; 1(n_1), 0(n_2)]$. Assume, from now on, $s \geq 2$ and $r \geq 1$. The matrices $\mathcal{M}$ and $\mathcal{L}$ take the form

$$\mathcal{M} = \begin{pmatrix} n_1 & \dots \\ \vdots & \ddots \\ n_1 & \dots \\ n_1 & \dots \end{pmatrix} \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} n_1 & \cdot & \dots & \cdot \\ \vdots & \vdots & \ddots & \cdot \\ n_1 & \cdot & \dots & \cdot \\ n_1 - 1 & n_2 & \dots & n_t \end{pmatrix}$$

where $\bar{\alpha}_s = \bar{p}$, otherwise any other maximal losing model would be dominated componentwise by $\bar{\alpha}_s$, which would be a contradiction by monotonicity. As $W^m \subseteq \{S : |S| \leq n - 2\}$ we deduce that the sum of the components of every row in $\mathcal{M}$ and in every row in $\mathcal{L}$ except the last has cardinality equal or lower than $n - 2$.

As (N, W) admits a $P^{n-3}SA$ representation there is $(w_1, \dots, w_t) \in (\mathbb{N} \cup \{0\})^t$ such that the following system of inequalities is consistent

$$\begin{cases} \sum_{j=1}^t (m_{ij} - \alpha_{kj}) \cdot w_j > 0 & i = 1, \dots, r, \quad k = 1, \dots, s - 1 \\ w_j > w_{j+1} & j = 1, \dots, t - 1 \\ w_t \geq 0 \end{cases} \quad (4)$$

Equivalently, the associated pregame with $(n-2)$ -domain is weighted with $(w_1, \dots, w_t)$ as a solution. Let

$$\min_{1 \leq i \leq r} \sum_{j=1}^t m_{i,j} \cdot w_j - \sum_{j=1}^t \alpha_{s,j} \cdot w_j. \quad (5)$$

be the minimum weight of the minimal winning models minus the weight of the losing model model $\bar{\alpha}_s$.

If the number in (5) is positive, then the game is weighted with the same weights as those in the pregame with $(n-2)$ -domain. If the number in (5) is non-positive, then we only need to increase sufficiently the weight w_1 to make the number in (5) positive. Note that this is always possible since $m_{1,j} = n_1$ for all j and $\alpha_{1,s} = n_1 - 1$.

With this new weight vector the inequalities system (4) is still consistent because $m_{1,j} = n_1$ for all $j = 1, \dots, r$ and $\alpha_{1,k} = n_1$ for all $k = 1, \dots, s - 1$.

Case 4: In this situation it exists a coalition $S \in W^m$ such that $|S| = n - 1$. We consider four subcases:

Subcase 4.1: $W^m = \{S : |S| = n - 1\}$.

Then, $t = 1$ and the game clearly admits the weighted representation $[n - 1; 1(n)]$.

Subcase 4.2: $W^m \subset \{S : |S| = n - 1\}$.

Then the number of equivalence classes is $t = 2$ and the game has $(n_1, n_2 - 1)$ as a unique minimal winning model since $(n_1 - 1, n_2)$ would imply that $N_1 = N_2$ and thus $t = 1$, a contradiction. Then, $(n_1 - 1, n_2)$ is the unique maximal losing model. From all this, the game admits the weighted representation:
 $[2n_1 + n_2 - 1; 2(n_1), 1(n_2)]$.

Subcase 4.3: It exists a coalition $T \in W^m$ such that $|T| < n - 1$ and the game has no veto players.

Then, $t \geq 2$, $r \geq 2$ and $s \geq 2$. The model of minimal winning coalitions of cardinality $n - 1$ is: $(n_1 - 1, n_2, \dots, n_t)$, and, by monotonicity, the models of minimal winning coalitions with a cardinality lower than $n - 1$ are of this type: $(n_1, m_2, \dots, m_t)$ for some values of m_i . The matrices $\mathcal{M}$ and $\mathcal{L}$ take the form

$$\mathcal{M} = \begin{pmatrix} n_1 & \cdot & \dots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ n_1 & \cdot & \dots & \cdot \\ n_1 - 1 & n_2 & \dots & n_t \end{pmatrix} \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} n_1 & \cdot & \dots & \cdot & \cdot \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n_1 & \cdot & \dots & \cdot & \cdot \\ n_1 - 1 & n_2 & \dots & n_{t-1} & n_t - 1 \end{pmatrix}$$

with $(s \geq 1)$ rows for $\mathcal{L}$. If $s = 1$ then the game is weighted. All the coalitions associated to some of the first $r - 1$ rows of $\mathcal{M}$ and to every row of $\mathcal{L}$ have cardinality $n - 2$ or lower. As (N, W) admits a $P^{n-3}SA$ representation it exists a solution $(w_1, \dots, w_t) \in (\mathbb{N} \cup \{0\})^t$ of the following inequalities system:

$$\begin{cases} \sum_{j=1}^t (m_{i,j} - \alpha_{k,j}) \cdot w_j > 0 & i = 1, \dots, r - 1, \quad k = 1, \dots, s \\ w_j > w_{j+1} & j = 1, \dots, t - 1 \\ w_t > 0 \end{cases} \quad (6)$$

Equivalently, the associated pregame with $(n - 2)$ -domain also admits this normalized weighted solution. Consider:

$$\sum_{j=1}^t m_{r,j} \cdot w_j - \max_{1 \leq i \leq s-1} \sum_{j=1}^t \alpha_{i,j} \cdot w_j \quad (7)$$

that is, the weight of the model of minimal winning coalitions with cardinality $n - 1$ minus the maximum weight of the models of maximal losing coalitions.

If the number in (7) is positive, then the game is weighted with the same weights than those obtained for the pregame with $(n - 2)$ -domain. If the number in (7) is non-positive, then we decrease the weight w_1 up to the number in (7) becomes positive. Note that this is always possible since $m_{i,1} = n_1$ for all $i < r$ and $\alpha_{i,1} = n_1$ for all $i < s$. This is a normalized weighted representation of the game.

Subcase 4.4: In this situation it exists a coalition $T \in W^m$ such that $|T| < n - 1$ and the game has veto players.

The existence of this coalition implies $t \geq 3$ and $r \geq 2$ and $s \geq 2$ (if $t = 3$ then $r = 2$). The matrices $\mathcal{M}$ and $\mathcal{L}$ take the form

$$\mathcal{M} = \begin{pmatrix} n_1 & n_2 & \cdot & \dots & \cdot \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_1 & n_2 & \cdot & \dots & \cdot \\ n_1 & n_2 - 1 & n_3 & \dots & n_t \end{pmatrix} \quad \mathcal{L} = \begin{pmatrix} n_1 & \cdot & \cdot & \dots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n_1 & \cdot & \cdot & \dots & \cdot & \cdot \\ n_1 - 1 & n_2 & n_3 & \dots & n_{t-1} & n_t \end{pmatrix}$$

All the models in $\mathcal{M}$ except the last row have at most $n - 2$ players and the same occurs for $\mathcal{L}$.

As (N, W) admits a $P^{n-3}SA$ representation there is a solution $(w_1, \dots, w_t) \in (\mathbb{N} \cup \{0\})^t$ of the following system of inequalities

$$\begin{cases} \sum_{j=1}^t (m_{ij} - \alpha_{kj}) \cdot w_j > 0 & i = 1, \dots, r-1, k = 1, \dots, s-1 \\ w_j > w_{j+1} & j = 1, \dots, t-1 \\ w_t > 0 \end{cases} \quad (8)$$

Equivalently, the associated pregame with $(n - 2)$ -domain is weighted with such representation. Consider:

$$\min_{1 \leq i \leq r} \sum_{j=1}^t m_{i,j} \cdot w_j - \max_{1 \leq i \leq s} \sum_{j=1}^t \alpha_{i,j} \cdot w_j \quad (9)$$

that is, the weight of the models of minimal winning coalitions minus the maximum weight of the models of maximal losing coalitions. If the number in (9) is positive. Then the game is weighted with the same weights of the pregame with $(n - 2)$ -domain. If the number in (9) is non-positive it is because either

$$\min_{1 \leq i \leq r} \sum_{j=1}^t m_{r,j} \cdot w_j - \sum_{j=1}^t \alpha_{s,j} \cdot w_j \leq 0 \quad (10)$$

or

$$\sum_{j=1}^t m_{r,j} \cdot w_j - \max_{1 \leq i \leq s} \sum_{j=1}^t \alpha_{i,j} \cdot w_j \leq 0 \quad (11)$$

but not both simultaneously because the weight of the last row of $\mathcal{M}$ is always greater than the weight of the last row of $\mathcal{L}$ since $w_1 > w_2$. If the inequality (10) holds, then we increase the first weight up to the inequality (10) becomes positive, then the inequality (11) also becomes positive and the game is weighted with such a weighted representation.

If the inequality (11) holds it is because there is some row k of $\mathcal{L}$ that witnesses the inequality (11), such row must verify $\alpha_{k,2} = n_2$, otherwise $\bar{m}_r \geq \bar{\alpha}_j$. Now it suffices to decrease the weight w_2 up to the inequality in (11) becomes positive. In that case also the inequality (10) becomes positive because the last row of $\mathcal{L}$ is dominated by the last row of $\mathcal{M}$ and therefore the game is weighted. $\square$

From this result some inclusions in (3) become equalities for all $n > 3$:

$$\mathcal{W}_n = \mathcal{P}_n^n = \mathcal{P}_n^{n-1} = \mathcal{P}_n^{n-2} = \mathcal{P}_n^{n-3} \subseteq \dots \subseteq \mathcal{P}_n^2 \subseteq \mathcal{P}_n^1 = \mathcal{C}_n. \quad (12)$$

Note that for $n = 4$, the sequence in (12) tells that all complete game is weighted. For studies we have done we conjecture that for n large enough the inclusions in (12) are strict. We encourage future research in studying them in depth.

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