WEAK-TYPE AND END-POINT NORM ESTIMATES FOR HARDY OPERATORS

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ABSTRACT. We explicitly calculate the best constants for weak-type and other end-point estimates for the Hardy operator and its adjoint. In particular, we find the right value for decreasing power weights, fixing some previously unclear results.

1. INTRODUCTION

Let f be a locally integrable function on $[0, \infty)$. The classical Hardy operator S acting on f is defined as

$$Sf(x) = \frac{1}{x} \int_0^x f(t) dt$$

and its adjoint operator is given by

$$S^*f(x) = \int_x^\infty \frac{f(t)}{t} dt.$$

For $w(t)$ a non-negative locally integrable function on $(0, \infty)$, usually called a weight, let us consider the weighted L^p -norm for $1 \leq p < \infty$

$$\|f\|_{L^p(w)} = \left(\int_0^\infty |f(t)|^p w(t) dt \right)^{1/p},$$

and, if $p = \infty$,

$$\|f\|_{L^\infty(w)} = \sup_{t>0} |f(t)|w(t).$$

The corresponding weighted weak-type L^p -norm of f is defined as

$$\|f\|_{L^{p,\infty}(w)} = \sup_{\lambda>0} \lambda \left(\int_{\{t>0: |f(t)|>\lambda\}} w(t) dt \right)^{1/p}$$

(even though, for $p = 1$, $L^{1,\infty}(w)$ might not be normable, for simplicity we will use the term norm for all values of $p \geq 1$).

Boundedness of S and S^* on $L^p(t^\alpha)$ (i.e., when $w(t) = t^\alpha$ is a power weight) are the well-known Hardy's inequalities. In recent years, extensions of these estimates to more general weights, different classes of functions (positive or decreasing) and related operators, have been thoroughly considered: In [19] the author finds necessary and sufficient conditions for the boundedness $S : L^p(w) \rightarrow L^p(w)$. The characterization for the restriction of S to decreasing functions in $L^p(w)$ was first obtained in [2] (in terms of the so called B_p condition) and further extended (to the

2010 *Mathematics Subject Classification.* 26D10, 46E30.

Key words and phrases. Hardy operator; dual operator; best constants; decreasing functions.

Both authors have been partially supported by the Spanish Government grant MTM2016-75196-P (MINECO / FEDER, UE) and the Catalan Autonomous Government grant 2017SGR358.

non-diagonal case, the full range of the exponent $p > 0$ or weak-type estimates) in [20, 11, 21] (see also [18] and the references therein).

From the pioneering result of [9] regarding the isometric properties in L^2 of the Hardy operator minus the identity $S - \text{Id}$ (which measures the oscillation of a function and has many applications in harmonic analysis, like Sobolev embeddings or rearrangement estimates of BMO functions), there has also been a great interest in determining the optimal bounds for this difference in other settings. In particular, in [17] the authors found the norm of $S - \text{Id}$ on the cone of decreasing functions in L^p , for $p \in \{2, 3, 4, \dots\}$. This was completed in [7], for $L^p_{\text{dec}}(w)$, when $p \geq 2$ and (essentially) decreasing weights $w \in B_p$ (and also if $p = 1$, then with no restrictions in w). The case $1 < p < 2$, with no weights, was proved in [16]. A new boost to this theory comes from the well-known Iwaniec's conjecture [15] regarding the norm of the Beurling operator B on $L^p(\mathbb{C})$ (see [3, 5] for more information), and using the fact that, for radial functions $F(z) = f(|z|^2)$, since the following equality holds

$$BF(z) = \frac{\bar{z}}{z}(S - \text{Id})f(|z|^2), \quad z \in \mathbb{C},$$

then the norm of B on such radial functions is equivalent to the norm of $S - \text{Id}$ for general functions f . In this case, optimal constants were obtained in [4], if $1 < p \leq 2$, and in [22], for the whole range $1 < p < \infty$ (see also [23] for a recent extension to power weight estimates in $L^p(t^\alpha)$, with $\alpha < p - 1$).

In this work, we are going to consider new weak-type estimates and the end-point case $p = \infty$, for inequalities involving both S and S^* . The main motivation comes from the results in [16] and the crucial technique used to obtain the norm of $S - \text{Id}$ on decreasing functions, which is equivalent to finding the best constant relating the norm of S and S^* on positive functions. This idea was also exploited in [8] in the case of strong-type estimates. In particular, in Section 2 we calculate the weak-type norm of S , for decreasing weights (see Theorem 2.1), and clarify in Remark 2.4 some previously announced results regarding this value. In Section 3, we completely describe the optimal constant for the end-point estimates $p = 1, \infty$, with respect to power weights. Propositions 3.2 and 3.3 deal with the weak-type bounds in $L^1(t^\alpha)$ and Theorem 3.7 establishes the result for $L^\infty(t^\alpha)$.

2. SHARP WEIGHTED WEAK-TYPE ESTIMATES FOR THE HARDY OPERATOR

Although there are very deep results concerning equivalent expressions for the norm of Hardy's operators on weighted $L^p(w)$ spaces [20], the exact value (for either general, positive or decreasing functions) is yet an open problem, except for very restricted situations (like power weights). The case of weak-type estimates is a bit different. In fact, it is proved in [1] that for some $a > 0$ (equivalently, for all),

$$(1) \quad \|S\|_{L^p(w) \rightarrow L^{p,\infty}(w)} \simeq \sup_{r>0} \left(\int_0^r (w(x))^{-1/(p-1)} dx \right)^{1/p'} \left(\int_r^\infty (r/x)^a \frac{w(x)}{x^p} dx \right)^{1/p},$$

with the suitable interpretation for $p = 1$.

For the adjoint of the Hardy operator, the optimal constant is actually obtained: S^* is bounded from $L^p(w)$ into $L^{p,\infty}(w)$ if and only if,

$$(2) \quad B_w = \sup_{r>0} \left(\int_0^r w(x) dx \right)^{1/p} \left(\int_r^\infty x^{-\frac{p}{p-1}} w(x)^{-\frac{1}{p-1}} dx \right)^{1/p'}$$

is finite and

$$(3) \quad \|S^*\|_{L^p(w) \rightarrow L^{p,\infty}(w)} = B_w.$$

Let us now see that we can improve (and simplify) condition (1) under some extra assumption on the weight:

Theorem 2.1. *If w is a decreasing weight and $1 \leq p < \infty$, then*

$$\|S\|_{L^p(w) \rightarrow L^{p,\infty}(w)} = \|S\|_{L^p_{\text{dec}}(w) \rightarrow L^{p,\infty}(w)} = 1.$$

Proof. To prove that $\|S\|_{L^p_{\text{dec}}(w) \rightarrow L^{p,\infty}(w)} \geq 1$, let us consider, for any $c > 1$, the non-increasing function $f_c(t) = c\chi_{(0,1)}(t)$. For such a function,

$$Sf_c(t) = c\chi_{(0,1)}(t) + \frac{c}{t}\chi_{(1,\infty)}(t).$$

In order to get an estimate of the weighted weak-type norm, by homogeneity, it is enough to consider the level set $E_c = \{t > 0 : Sf_c(t) > 1\}$. In fact

$$(4) \quad \sup_{f>0} \sup_{\lambda>0} \frac{\lambda |\{Sf > \lambda\}|^{1/p}}{\|f\|_p} = \sup_{f>0} \frac{|\{Sf > 1\}|^{1/p}}{\|f\|_p}.$$

In this case, $E_c = (0, c)$, and hence, we must estimate the best constant C_w in the inequality

$$\int_{E_c} w(t) dt = \int_0^c w(x) dx \leq C_w c^p \int_0^1 w(x) dx.$$

Therefore, if we denote by $W(t) = \int_0^t w(x) dx$, the primitive of the weight w , and consider $c = 1$, then

$$W(1) \leq C_w W(1);$$

i.e., without any monotonicity assumption on w , $C_w \geq 1$.

To prove that $\|S\|_{L^p(w) \rightarrow L^{p,\infty}(w)} \leq 1$, we observe that, for any non-negative function f on $(0, \infty)$ and $\lambda > 0$, the level set $E_\lambda = \{t > 0 : Sf(t) > \lambda\}$ is open. As before, by an argument of homogeneity (4), we can assume $\lambda = 1$ and then

$E_1 = \bigcup_{k=1}^{\infty} (a_k, b_k)$. Since for any $k \in \mathbb{N}$, $Sf(b_k) = Sf(a_k) = 1$, we have

$$(5) \quad \int_{a_k}^{b_k} f(s) ds = b_k - a_k \text{ and } \int_{a_k}^t f(s) ds \geq t - a_k, \text{ for } t \in (a_k, b_k).$$

Therefore, it is enough to prove that for any interval $(a, b) \in \{(a_k, b_k)\}_k$ the following inequality holds

$$\int_a^b w(t) dt \leq \int_a^b f^p(t) w(t) dt,$$

provided that conditions (5) hold for the interval (a, b) .

Since the weight w is decreasing, using (5), Hardy's Lemma [6, Proposition II.3.6], and Hölder's inequality, we obtain that

$$\int_a^b w(s) ds \leq \int_a^b f(s) w(s) ds \leq \left(\int_a^b f^p(s) w(s) ds \right)^{1/p} \left(\int_a^b w(s) ds \right)^{1/p'}.$$

If $w \in L^1(a, b)$, this gives

$$\int_a^b w(s) ds \leq \int_a^b f^p(s) w(s) ds,$$

as desired. If $w \notin L^1(a, b)$, we use a standard truncation argument, considering the decreasing weight function $\min(n, w)\chi_{(0, n)}$, and let $n \rightarrow \infty$. $\square$

Remark 2.2. As a consequence of Theorem 2.1, we can obtain an improvement of [10, Theorem 4.2], where an estimate of the optimal constant for the Lorentz spaces embedding $\Lambda^p(w) \hookrightarrow \Gamma^{p, \infty}(w)$ is obtained in the whole range $p \in (0, \infty)$. Recall that

$$\|f\|_{\Lambda^p(w)} = \left(\int_0^\infty (f^*(t))^p w(t) dt \right)^{1/p} \quad \text{and} \quad \|f\|_{\Gamma^{p, \infty}(w)} = \sup_{t>0} f^{**}(t) W^{1/p}(t),$$

where f^* is the decreasing rearrangement of f and $f^{**}(t) = S f^*(t)$ is the maximal operator [6]. In particular, for $1 < p < \infty$, the optimal constant in the inequality

$$\sup_{t>0} f^{**}(t) W^{1/p}(t) \leq D_w \left(\int_0^\infty (f^*(t))^p w(t) dt \right)^{1/p}$$

satisfies that

$$D_w \approx \sup_{t>0} \frac{W^{1/p}(t)}{t} \left(\int_0^t \left(\frac{s}{W(s)} \right)^{p'-1} ds \right)^{1/p'}$$

but, if w is a non-increasing weight, we now know that $D_w = 1$.

Concerning the case $p = 1$, it was already proved in [10] that the optimal constant for the embedding $\Lambda^1(w) \hookrightarrow \Gamma^{1, \infty}(w)$ is equal to

$$D_w = \sup_{0 < s \leq t < \infty} \frac{W(t)s}{W(s)t},$$

which, for a decreasing weight w , immediately gives that $D_w = 1$.

The hypothesis of w being decreasing cannot be dropped, as the following counterexamples show. If $p = 1$, let us consider, for $t > 0$, $w(t) = 1 + \chi_{(1, 2)}(t)$. In this case

$$D_w = \frac{3}{2} > 1.$$

For a general $p > 1$, set $w_p(t) = 1 + a\chi_{(1, 2)}(t)$, with $a > 2^p - 2$. Then, with $f(t) = \chi_{(0, 1)}(t)$, and $t = 2$, we have that

$$D_{w_p} \geq \frac{f^{**}(2) \left(\int_0^2 w_p(s) ds \right)^{1/p}}{\left(\int_0^\infty (f^*)^p(s) w_p(s) ds \right)^{1/p}} = \frac{((a+1)2 - a)^{1/p}}{2} = \frac{(a+2)^{1/p}}{2} > 1.$$

Remark 2.3. Similarly to the case of S , it is easy to see that, in general, the boundedness of S^* holds for different classes of weights when this operator is restricted to decreasing functions. For example, in [12, Theorem 3.3], and as a consequence of the duality formula due to E. Sawyer [20], the best constant C_w in the inequality

$$\sup_{r>0} S^* f(r) W(r) \leq C_w \|f\|_{L_{\text{dec}}^1(w)}$$

is

$$\begin{aligned}
 (6) \quad C_w &= \sup_{r>0} \sup_{f \text{ dec}} \frac{\int_0^\infty f(x) \chi_{(r,\infty)}(x) \frac{dx}{x}}{\int_0^\infty f(x) w(x) dx} W(r) = \sup_{r>0} \sup_{s>0} \frac{\int_0^s \chi_{(r,\infty)}(x) \frac{dx}{x}}{W(s)} W(r) \\
 &= \sup_{0<r<s} \frac{W(r)}{W(s)} \log\left(\frac{s}{r}\right) < \infty.
 \end{aligned}$$

Now, observe that condition (2) implies that a weight satisfying (3) cannot be zero in any interval, and (6) holds, for example, for the weight $w_0(x) = 1 - \chi_{(1,2)}(x)$. That is, $S^* : L^1(w_0) \rightarrow L^{1,\infty}(w_0)$, but $S^* : L^1_{\text{dec}}(w_0) \rightarrow L^{1,\infty}(w_0)$.

Remark 2.4. Let us finally consider the case of decreasing power weights $w(t) = t^\alpha$, for which Theorem 2.1 holds. Thus, if $-1 < \alpha \leq 0$, then $\|S\|_{L^1(t^\alpha) \rightarrow L^{1,\infty}(t^\alpha)} = 1$. This result was previously stated in [13, Proposition 3.2], but the proof seems to be a bit confusing, since the upper bound they use, which was proved in [13, Theorem 2.1], is $\|S\|_{L^1(t^\alpha) \rightarrow L^{1,\infty}(t^\alpha)} \leq 1/(1+\alpha)$. There is a similar result in [14, Theorem 1.1] where $1/(1+\alpha)$ is, again, the constant obtained there, which is not optimal.

Observe also that, from (2) and (3), we have that

$$\|S^*\|_{L^1(t^\alpha) \rightarrow L^{1,\infty}(t^\alpha)} = \frac{1}{1+\alpha}.$$

3. END-POINT ESTIMATES FOR POWER WEIGHTS

As we have already mentioned in the Introduction, sharp constants for $S - \text{Id}$ on L^p_{dec} , when $1 < p < 2$, were obtained in [16]. The main technique used to prove this result was to reduce that inequality to an equivalent estimate of the form $\|Sf\|_p \leq C\|S^*f\|_p$, for positive functions $f \geq 0$.

From the results we saw in the previous section, we can now obtain some end-point norm estimates for more general inequalities of the type $\|S\| \lesssim \|S^*\|$ and $\|S^*\| \lesssim \|S\|$, in the case of power weights $w(t) = t^\alpha$, $\alpha \in \mathbb{R}$ (see [23] for the strong-type inequalities in $L^p(t^\alpha)$). Let us first study the weak-type boundedness $\|Sf\|_{L^{1,\infty}(t^\alpha)} \leq C_\alpha \|S^*f\|_{L^1(t^\alpha)}$, for any non-negative function f . In order to establish the result, let us first formulate the following easy, but useful, identities.

Lemma 3.1. *Let $w(t) = t^\alpha$, with $\alpha \in \mathbb{R}$. Then, for any non-negative function $f \geq 0$ and $\alpha > -1$,*

$$\|S^*f\|_{L^1(t^\alpha)} = \frac{1}{\alpha+1} \|f\|_{L^1(t^\alpha)},$$

and for any $\alpha < 0$,

$$\|Sf\|_{L^1(t^\alpha)} = -\frac{1}{\alpha} \|f\|_{L^1(t^\alpha)}.$$

Proof. Both equalities are an easy consequence of Fubini's theorem. First of all, if $\alpha > -1$,

$$\begin{aligned}
 \|S^*f\|_{L^1(t^\alpha)} &= \int_0^\infty t^\alpha \left(\int_t^\infty f(u) \frac{du}{u} \right) dt = \int_0^\infty f(u) \int_0^u t^\alpha dt \frac{du}{u} \\
 &= \int_0^\infty f(u) \frac{u^\alpha}{\alpha+1} du = \frac{1}{\alpha+1} \|f\|_{L^1(t^\alpha)}.
 \end{aligned}$$

Similarly, if $\alpha < 0$,

$$\begin{aligned}\|Sf\|_{L^1(t^\alpha)} &= \int_0^\infty t^\alpha \left(\frac{1}{t} \int_0^t f(u) du \right) dt = \int_0^\infty f(u) \left(\int_u^\infty t^{\alpha-1} dt \right) du \\ &= -\frac{1}{\alpha} \int_0^\infty f(u) u^\alpha du = -\frac{1}{\alpha} \|f\|_{L^1(t^\alpha)}.\end{aligned}$$

□

Before proving the end-point estimates for power weights, we would like to clarify the following notation. Given A and B , operators defined on positive functions $f \geq 0$, and if X and Y are suitable function spaces, when we refer to the least constant $C_{A,B,X,Y}$ in the inequality

$$\|Af\|_Y \leq C_{A,B,X,Y} \|Bf\|_X,$$

we are assuming the following:

- We just consider positive functions $f \not\equiv 0$ for which either $\|Af\|_Y < \infty$ or $\|Bf\|_X < \infty$. We denote this set as $f \in \mathcal{F}(A, B; X, Y)$ and we will always assume $\mathcal{F}(A, B; X, Y) \neq \emptyset$.
- We write $C_{A,B,X,Y} = 0$ if, for all $f \in \mathcal{F}(A, B; X, Y)$, we have that $\|Bf\|_X = \infty$.

Thus,

$$C_{A,B,X,Y} = \sup_{f \in \mathcal{F}(A,B;X,Y)} \frac{\|Af\|_Y}{\|Bf\|_X},$$

with the convention that, for every $c \geq 0$, $\frac{c}{\infty} = 0$ and $\frac{\infty}{c} = \infty$.

Proposition 3.2. *Let $\alpha \in \mathbb{R}$ and C_α be the least constant in the inequality*

$$\|Sf\|_{L^1, \infty(t^\alpha)} \leq C_\alpha \|S^*f\|_{L^1(t^\alpha)},$$

for any positive measurable function. Then:

- (i) *For any $\alpha \leq -1$, $C_\alpha = 0$.*
- (ii) *For any $-1 < \alpha \leq 0$, $C_\alpha = 1 + \alpha$.*
- (iii) *For any $\alpha > 0$, $C_\alpha = \infty$.*

Proof. Part (i) easily follows from the fact that, if $\alpha \leq -1$, $f \geq 0$ and $f \not\equiv 0$, then $\|S^*f\|_{L^1(t^\alpha)} = \infty$. Observe that, for example, $\|Sf\|_{L^1, \infty(t^\alpha)} < \infty$, for $f = \chi_{(1,2)}$, and hence $\mathcal{F}(S, S^*; L^1(t^\alpha), L^1, \infty(t^\alpha)) \neq \emptyset$.

For part (ii), as a consequence of Lemma 3.1, we have that

$$\|Sf\|_{L^1, \infty(t^\alpha)} \leq C_\alpha \|S^*f\|_{L^1(t^\alpha)} = \frac{C_\alpha}{\alpha + 1} \|f\|_{L^1(t^\alpha)},$$

and then, due to the optimal estimate obtained for non-increasing weights in Theorem 2.1,

$$\frac{C_\alpha}{\alpha + 1} = 1.$$

Finally, for part (iii), we just observe that if the inequality is tested for the non-increasing function $f(t) = \chi_{(0,1)}(t)$, we obtain

$$\begin{aligned}
(7) \quad \|Sf\|_{L^{1,\infty}(t^\alpha)} &= \|Sf\|_{\Lambda^{1,\infty}(t^\alpha)} \\
&= \sup_{t>0} \left(\int_0^t s^\alpha ds \right) \frac{1}{t} \int_0^t \chi_{(0,1)}(s) ds = \sup_{t>1} \frac{t^\alpha}{\alpha+1} = \infty,
\end{aligned}$$

and

$$\|S^*f\|_{L^1(t^\alpha)} = \int_0^1 t^\alpha \log\left(\frac{1}{t}\right) dt < \infty.$$

□

Concerning the reverse estimates of the best constant C_α such that

$$\|S^*f\|_{L^{1,\infty}(t^\alpha)} \leq C_\alpha \|Sf\|_{L^1(t^\alpha)},$$

for any non-negative function f , we have the following result.

Proposition 3.3. *Let $\alpha \in \mathbb{R}$ and C_α be the least constant in the inequality*

$$(8) \quad \|S^*f\|_{L^{1,\infty}(t^\alpha)} \leq C_\alpha \|Sf\|_{L^1(t^\alpha)},$$

for any positive measurable function. Then:

- (i) *For any $\alpha \leq -1$, $C_\alpha = \infty$.*
- (ii) *For any $-1 < \alpha < 0$, $C_\alpha = -\alpha/(1+\alpha)$.*
- (iii) *For any $\alpha \geq 0$, $C_\alpha = 0$.*

Proof. Part (i) holds as in (7), since S^*f is a non-increasing function and hence $\|S^*f\|_{L^{1,\infty}(t^\alpha)} = \infty$. If $f = \chi_{(1,2)}$, then $\|Sf\|_{L^1(t^\alpha)} < \infty$ and, as before, we have that $\mathcal{F}(S^*, S; L^1(t^\alpha), L^{1,\infty}(t^\alpha)) \neq \emptyset$.

Similarly, part (iii) follows from the fact that, as in the proof of Lemma 3.1, since $\alpha \geq 0$, then $\|Sf\|_{L^1(t^\alpha)} = \infty$. To finish, with $f = \chi_{(0,1)}$, we easily get that $\|S^*f\|_{L^{1,\infty}(t^\alpha)} < \infty$.

For part (ii) we observe that since S^*f is a non-increasing function, for any positive f , writing $S^*f = g$ in the inequality (8) and taking into account that $S(S^*f) = Sf + S^*f$, we obtain,

$$\|g\|_{L^{1,\infty}(t^\alpha)} = \|g\|_{\Lambda^{1,\infty}(t^\alpha)} \leq C_\alpha \|Sf\|_{L^1(t^\alpha)} = C_\alpha \|Sg - g\|_{L^1(t^\alpha)}.$$

Using that $Sg \geq g$, Lemma 3.1 shows that for $\alpha < 0$,

$$\|Sg - g\|_{L^1(t^\alpha)} = -\frac{\alpha+1}{\alpha} \|g\|_{L^1(t^\alpha)} = -\frac{\alpha+1}{\alpha} \|g\|_{\Lambda^1(t^\alpha)}.$$

Thus, (8) is equivalent to studying the sharp constant D_α in the inequality

$$\|g\|_{\Lambda^{1,\infty}(t^\alpha)} \leq D_\alpha \|g\|_{\Lambda^1(t^\alpha)},$$

with

$$D_\alpha = -C_\alpha \frac{\alpha+1}{\alpha}.$$

But, the embedding

$$\Lambda^{1,\infty}(t^\alpha) \hookrightarrow \Lambda^1(t^\alpha)$$

was characterized in [10, Theorem 3.1], with best constant $D_\alpha = 1$. Hence, $C_\alpha = -\alpha/(1+\alpha)$. □

Concerning the case $p = \infty$, we first observe the following fact that relates the norms of Sf and S^*f with respect to the power weights t^α .

Lemma 3.4. *Let $\alpha \in \mathbb{R}$. Then the inequality*

$$\|S^*f\|_{L^\infty(t^\alpha)} \leq C_\alpha \|Sf\|_{L^\infty(t^\alpha)},$$

holds if and only if

$$\|Sf\|_{L^\infty(t^{1-\alpha})} \leq C_\alpha \|S^*f\|_{L^\infty(t^{1-\alpha})}.$$

Proof. This is just a consequence of a change of variables in the integral defining $Sf(t)$, which shows

$$t^\alpha S^*f(t) = t^\alpha \int_t^\infty f(x) \frac{dx}{x} = t^\alpha \int_0^{1/t} f\left(\frac{1}{r}\right) \frac{dr}{r} = \left(\frac{1}{t}\right)^{1-\alpha} Sg\left(\frac{1}{t}\right),$$

where, for $t > 0$, $g(t) = t^{-1}f(t^{-1})$. Then,

$$\|S^*f\|_{L^\infty(t^\alpha)} = \|Sg\|_{L^\infty(t^{1-\alpha})},$$

and, similarly,

$$\|Sf\|_{L^\infty(t^\alpha)} = \|S^*g\|_{L^\infty(t^{1-\alpha})}.$$

□

This lemma allows us to establish the estimates $\|S\| \lesssim \|S^*\|$ and $\|S^*\| \lesssim \|S\|$ in L^∞ , with respect to power weights, as dual results of each other. So, in this case, we can restrict ourselves to obtain just one of the inequalities.

On the other hand, following the same ideas developed in [16, 8] (see Lemma 3.6), the optimal constant $C_{X,Y}$ for estimates $\|Sf\|_X \leq C_{X,Y} \|f\|_Y$ is equivalent to optimal constant for estimates $\|Sf - f\|_X \leq C_{X,Y} \|f\|_Y$ restricted to decreasing functions, being X and Y quasi-Banach function spaces.

With this purpose in mind, let us establish our first result concerning the optimal norm of the operator $S - \text{Id}$ restricted to decreasing functions in $L^\infty(t^\alpha)$. The case $\alpha = 0$ was already obtained in [8].

Proposition 3.5. *Let $0 \leq \alpha < 1$ and $f \in L^\infty(t^\alpha)$ be a positive and decreasing function defined on $(0, \infty)$. Then, the following sharp estimate holds:*

$$(9) \quad \|Sf - f\|_{L^\infty(t^\alpha)} \leq \frac{1}{1-\alpha} \|f\|_{L^\infty(t^\alpha)}.$$

Proof. First, we observe that for $0 \leq \alpha < 1$, Hardy's inequality holds

$$(10) \quad \|Sf\|_{L^\infty(t^\alpha)} \leq \frac{1}{1-\alpha} \|f\|_{L^\infty(t^\alpha)}.$$

Indeed,

$$\begin{aligned} \sup_{t>0} t^\alpha Sf(t) &= \sup_{t>0} t^{\alpha-1} \int_0^t f(s) ds \\ &\leq \|f\|_{L^\infty(t^\alpha)} \sup_{t>0} t^{\alpha-1} \int_0^t s^{-\alpha} ds = \frac{1}{1-\alpha} \|f\|_{L^\infty(t^\alpha)}. \end{aligned}$$

In fact, the inequality (10) is sharp, as we can easily see taking for every $0 \leq \alpha < 1$, $f_\alpha(t) = t^{-\alpha} \chi_{(0,1)}(t)$ as an optimal function. Since this function is decreasing, we observe that (10) is sharp for every of the three cone of functions in $L^p(t^\alpha)$: arbitrary, positive or positive and decreasing functions.

In order to prove the bound $1/(1-\alpha)$ in (9), we observe that if f is positive and decreasing,

$$|Sf(t) - f(t)| = Sf(t) - f(t) \leq Sf(t).$$

Then, as a consequence of (10)

$$\|Sf - f\|_{L^\infty(t^\alpha)} \leq \|Sf\|_{L^\infty(t^\alpha)} \leq \frac{1}{1-\alpha} \|f\|_{L^\infty(t^\alpha)}.$$

To check the optimality of this bound, let us test the inequality with the function $f_\alpha(t) = t^{-\alpha} \chi_{(0,1)}(t)$, for every $\alpha \in (0, 1)$. Then,

$$Sf_\alpha(t) - f_\alpha(t) = \frac{\alpha}{1-\alpha} t^{-\alpha} \chi_{(0,1)}(t) + \frac{1}{1-\alpha} \frac{1}{t} \chi_{(1,\infty)}(t).$$

Since $0 \leq \alpha < 1$,

$$\|Sf_\alpha - f_\alpha\|_{L^\infty(t^\alpha)} = \max \left\{ \frac{\alpha}{1-\alpha}, \frac{1}{1-\alpha} \right\} = \frac{1}{1-\alpha},$$

and due to the fact that $\|f_\alpha\|_{L^\infty(t^\alpha)} = 1$, this proves that the constant $1/(1-\alpha)$ is sharp in the inequality (9). $\square$

Lemma 3.6. [8, Lemma 1.2] *Let X and Y be two lattice quasi-Banach space over $(0, \infty)$. If for every function f , the following inequality holds*

$$\|Sf - f\|_Y \leq C_{X,Y} \|f\|_X,$$

then

$$\|Sf\|_Y \leq C_{X,Y} \|S^*f\|_X.$$

Using Proposition 3.5 and Lemma 3.6, we can finally obtain the following result:

Theorem 3.7. *Let $\alpha \in \mathbb{R}$.*

(i) *For any positive function, the following estimate is sharp*

$$\|Sf\|_{L^\infty(t^\alpha)} \leq C_\alpha \|S^*f\|_{L^\infty(t^\alpha)},$$

where

$$C_\alpha = \begin{cases} 0, & \text{if } \alpha < 0, \\ \frac{1}{1-\alpha}, & \text{if } 0 \leq \alpha < 1, \\ \infty, & \text{if } \alpha \geq 1. \end{cases}$$

(ii) *For any positive function, the following estimate is sharp*

$$\|S^*f\|_{L^\infty(t^\alpha)} \leq C_\alpha \|Sf\|_{L^\infty(t^\alpha)},$$

where

$$C_\alpha = \begin{cases} \infty, & \text{if } \alpha \leq 0, \\ \frac{1}{\alpha}, & \text{if } 0 < \alpha \leq 1, \\ 0, & \text{if } \alpha > 1. \end{cases}$$

Proof. First of all we observe that, thanks to Lemma 3.4, it is enough to prove condition (i), for $\alpha \in [0, 1)$ and (ii), for $\alpha \notin (0, 1]$. So, assume $\alpha \in [0, 1)$. As a consequence of Lemma 3.6 and (9), we already have that

$$\|Sf\|_{L^\infty(t^\alpha)} \leq \frac{1}{1-\alpha} \|S^*f\|_{L^\infty(t^\alpha)},$$

In order to see that $1/(1-\alpha)$ is optimal, let us consider, for every $0 \leq \alpha < 1$ and $0 < \varepsilon < 1$, the function

$$f_{\alpha,\varepsilon}(t) = \alpha t^{-\alpha} \chi_{(0,1)}(t) + \frac{t}{\varepsilon} \chi_{(1,1+\varepsilon)}(t).$$

Then,

$$S^* f_{\alpha,\varepsilon}(t) = t^{-\alpha} \chi_{(0,1)}(t) + \frac{1+\varepsilon-t}{\varepsilon} \chi_{[1,1+\varepsilon)}(t)$$

and

$$\begin{aligned} S f_{\alpha,\varepsilon}(t) &= \frac{\alpha}{1-\alpha} t^{-\alpha} \chi_{(0,1)}(t) + \frac{1}{t} \left(\frac{t^2-1}{2\varepsilon} + \frac{\alpha}{1-\alpha} \right) \chi_{[1,1+\varepsilon)}(t) \\ &\quad + \frac{1}{t} \left(\frac{\varepsilon+2}{2} + \frac{\alpha}{1-\alpha} \right) \chi_{(1+\varepsilon,\infty)}(t). \end{aligned}$$

Explicit calculations show that, since $0 < \varepsilon < 1/\alpha$, then

$$\begin{aligned} \|S f_{\alpha,\varepsilon}\|_{L^\infty(t^\alpha)} &= \max \left\{ \frac{\alpha}{1-\alpha}, \left(\frac{\varepsilon}{2} + \frac{1}{1-\alpha} \right) (\varepsilon+1)^{\alpha-1} \right\}, \\ \|S^* f_{\alpha,\varepsilon}\|_{L^\infty(t^\alpha)} &= \max \left\{ 1, \sup_{1 \leq t \leq 1+\varepsilon} \frac{1+\varepsilon-t}{\varepsilon} t^\alpha \right\} = 1. \end{aligned}$$

Finally,

$$\frac{1}{1-\alpha} = \lim_{\varepsilon \rightarrow 0^+} \left(\frac{\varepsilon}{2} + \frac{1}{1-\alpha} \right) (\varepsilon+1)^{\alpha-1} = \lim_{\varepsilon \rightarrow 0^+} \frac{\|S f_{\alpha,\varepsilon}\|_{L^\infty(t^\alpha)}}{\|S^* f_{\alpha,\varepsilon}\|_{L^\infty(t^\alpha)}}.$$

In the case $\alpha \leq 0$, let us consider $f_\alpha(t) = t^{-\alpha}$. Then,

$$S^* f_\alpha(t) = \infty \quad \text{and} \quad \|S f_\alpha\|_{L^\infty(t^\alpha)} = \frac{1}{1-\alpha}, \quad \text{for any } t > 0.$$

Hence, the best constant, in the inequality $\|S^* f\|_{L^\infty(t^\alpha)} \leq C_\alpha \|S f\|_{L^\infty(t^\alpha)}$, is $C_\alpha = \infty$.

Finally, in the case $\alpha > 1$, since for every $f \geq 0$ which is not identically equal to zero we have that

$$\|S f\|_{L^\infty(t^\alpha)} = \sup_{t>0} t^\alpha S f(t) = \sup_{t>0} t^{\alpha-1} \int_0^t f(s) ds = \infty,$$

and $\|S^* f\|_{L^\infty(t^\alpha)} < \infty$, with $f = \chi_{(0,1)}$, then the least constant in the inequality

$$\|S^* f\|_{L^\infty(t^\alpha)} \leq C_\alpha \|S f\|_{L^\infty(t^\alpha)},$$

is $C_\alpha = 0$. □

Acknowledgments: We would like to thank the referees for their careful revision, which has greatly improved the final version of this work.

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