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**Almost Every Set in Exponential Time
is P-Bi-Immune**

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Report LSI-91-46

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Abstract

A set A is P -bi-immune if neither A nor its complement has an infinite subset in P . We investigate here the abundance of P -bi-immune languages in linear-exponential time (E). We prove that the class of P -bi-immune sets has measure 1 in E . This implies that “almost” every language in E is P -bi-immune. A bit further, we show that every p -random (pseudorandom) language is E -bi-immune. Regarding the existence of P -bi-immune sets in NP , we show that if NP does not have measure 0 in E , then NP contains a P -bi-immune set. Another consequence is that the class of $\leq_m^p$ -complete languages for E has measure 0 in E .

In contrast, it is shown that in E , and even in REC , the class of P -bi-immune languages lacks the property of Baire (the Baire category analogue of Lebesgue measurability).

* This work was supported by a Spanish Government grant FPI PN90. It was in part done while visiting Iowa State University, supported by that grant.

Introduction

A set A is P -bi-immune if neither A nor its complement has an infinite subset in P .

The notion of immunity was first introduced by Post [P44] in recursive function theory. Flajolet and Steyaert transformed it into the complexity theoretic setting [FS74a, FS74b].

Several characterizations of the concept of P -bi-immune set were presented in [BS85]. An infinite set X is a complexity core for a recursive set A iff for every Turing machine M that accepts A and every polynomial p there are at most finitely many $x \in X$ such that the time of M on x is bounded by $p(|x|)$. Balcázar and Schöning relate this concept with P -bi-immunity by showing that a recursive set A is P -bi-immune if and only if Σ^* is a complexity core for A .

Meyer [M77] showed that there is a P -bi-immune set in E (linear-exponential time), this result is reported by Berman and Hartmanis ([Be76], [HB77]) while proving that there are no sparse complete sets for E (the fact that the set he constructed is P -bi-immune is observed in [KM81]). An easy consequence of the results in [GHS91] is that there exists a $DTIME(2^{cn})$ -bi-immune set in E , for all $c > 0$.

Regarding the existence of P -immune sets in NP it has been shown that there exist, in certain relativizations, P -immune sets in NP [Ba85, BG81, Ha83, HM83, SB84], and E -immune sets in NP [GH83].

We address here the question of whether P -bi-immunity defines a “large” or a “small” class within linear-exponential time. To answer this, we will use the approach of resource-bounded measure introduced by Lutz in [L90, L91]. This method is motivated by the fact that Lebesgue classical measure is useless when dealing with recursive classes. This is because, as the class of recursive languages is countable, it has measure 0 (it is “small”). Resource-bounded measure limits the resources that can be used to compute the measure of a class, so that it is possible to have “large” (measure 1) recursive classes.

We will prove here that the class of P -bi-immune languages has p -measure 1. This implies that almost every language in E is P -bi-immune, and so it extends the previously mentioned Berman’s result [Be76] (in fact, we see that for any $c > 0$, almost every language in E is $DTIME(2^{cn})$ -bi-immune). Furthermore, we show that every p -random (pseudorandom) language is E -bi-immune.

We also obtain generalizations of this result, such as: E -bi-immunity defines a measure 1 class within E_2 (polynomial exponential time). This implies that almost every language in E_2 is E -bi-immune.

We then have a sufficient condition for the existence of P -bi-immune sets in NP : if NP does not have measure 0 in E , then NP contains a P -bi-immune set. It also holds that if NP does not have measure 0 in E_2 then NP contains an E -bi-immune set. Recall that E_2 is the smallest deterministic-time defined class known to contain NP .

The relationship of P -bi-immunity with $\leq_p^m$ is from [Be76] and it leads to a simple proof of the result that the class of $\leq_p^m$ -complete sets for exponential time has measure 0 in E (this result has also been proved with different techniques in [JL91]).

In contrast, we show here that regarding resource-bounded category ([L90]), the class of P -bi-immune sets does not have the property of Baire (the Baire category analogue of Lebesgue measurability). This gives us a natural example that witnesses the differences between category and measure. We also contrast pseudogenericity and pseudorandomness: a pseudorandom language is necessarily P -bi-immune, while a pseudogeneric language can be either P -bi-immune or have an infinite subset in P .

The third and fourth sections of this paper, dealing respectively with resource-bounded measure and resource-bounded category, can be read independently.

Preliminaries

First, we review the notion of immunity.

Definition 1: Let C be a class of languages, and L be a language. We say that L is C -immune iff L does not have an infinite subset that belongs to C .

Definition 2: Let C be a class of languages, and L be a language. We say that L is C -bi-immune iff both L and the complement of L are C -immune.

The symmetric difference of two sets A and B , denoted $A \Delta B$, is defined by $A \Delta B = (A \cup B) - (A \cap B)$.

We will use the alphabet $\Sigma = \{0, 1\}$. We denote with λ the empty string.

Let $s_0, s_1, s_2, \dots$ be the standard enumeration of the strings in Σ^* in alphabetical order.

From now on we will use the bitstring α of a language L to denote it, where α is defined as follows:

$$\alpha \in \Sigma^\infty \text{ and } \alpha[i] = 1 \text{ iff } s_i \text{ belongs to } L.$$

We let $\mathbf{D} = \{m2^{-n} \mid m, n \in \mathbf{N}\}$ be the set of *nonnegative dyadic rationals*.

Let *all* be the class of all functions $f : \Sigma^* \rightarrow \Sigma^*$, and *rec* be the class of recursive functions in *all*.

Let $\mathcal{P}(\Sigma^*)$ be the class of all languages over Σ , RE be the class of recursively enumerable languages, and REC be the class of recursive languages.

For each $i \in \mathbb{N}$ we define a class G_i of subexponential functions from $\mathbb{N}$ to $\mathbb{N}$ as follows [L90]:

$$\begin{aligned} G_0 &= \{f \mid (\exists k)f(n) \leq kn \text{ almost everywhere} \} \\ G_{i+1} &= \{f \mid (\exists g \in G_i)f(n) \leq 2^{g(\log n)} \text{ almost everywhere} \}. \end{aligned}$$

Using this hierarchy we define, for $i \geq 1$, the following uniform complexity classes:

$$\begin{aligned} E_i &= DTIME(2^{G_{i-1}}) = \bigcup_{g \in G_{i-1}} \{DTIME(2^g)\}, \\ E_iSPACE &= DSPACE(2^{G_{i-1}}) = \bigcup_{g \in G_{i-1}} \{DSPACE(2^g)\}, \\ p_i &= \{f \in rec \mid f \text{ is computable in } G_i \text{ time} \}, \\ p_ispace &= \{f \in rec \mid f \text{ is computable in } G_i \text{ space} \}. \end{aligned}$$

Note that $E_1 = \bigcup_c DTIME(2^{c^n})$, $E_2 = \bigcup_c DTIME(2^{n^c})$, p_1 is the class of polynomial time computable functions, and p_1space is the class of polynomial space computable functions. We will use $p, pspace, E$, and $ESPACE$ for p_1, p_1space, E_1 , and E_1SPACE .

We will use Γ in the rest of the paper to refer to a class of functions.

Next, we can associate with each class of functions Γ a class of languages $R(\Gamma)$ as follows:

Definition 3: $f \in \Gamma$ is a constructor iff $\forall w \in \Sigma^*, w \sqsubseteq f(w)$.

Definition 4: Let h be a constructor in Γ , then $R(h)$ is the only element in Σ^∞ such that $\forall i \ h^i(\lambda) \sqsubseteq R(h)$.

Definition 5: $R(\Gamma)$ is the class of languages $\{R(h) \mid h \text{ constructor in } \Gamma\}$.

Intuitively, $R(\Gamma)$ is the class of the languages built with the constructor functions in Γ .

From the classes of functions we mentioned, we obtain well-known classes [L90]:

$$\begin{aligned} R(p_i) &= E_i, \\ R(p_ispace) &= E_iSPACE, \\ R(rec) &= REC, \\ R(all) &= \mathcal{P}(\Sigma^*). \end{aligned}$$

P-bi-immunity and resource-bounded measure

In this our main section we prove that the class of P -bi-immune languages has p -measure 1, which implies that it has measure 1 in E . At the end of the section we explain some consequences of this result for $\leq_m^P$ -complete languages and for the existence of P -bi-immune sets in NP .

The formulation of resource-bounded measure that we will be using here has been recently introduced by Lutz [L91]. It is more powerful than the one in [L90], in the sense that there are more measurable classes in this new formulation.

First we give some definitions.

Definition 6: Let $w \in \Sigma^*$. C_w denotes the class of languages $\{x \in \Sigma^\infty \mid w \sqsubseteq x\}$.

Definition 7: A density function is a function $d : \Sigma^* \rightarrow [0, \infty)$ satisfying

$$d(w) \geq \frac{d(w0) + d(w1)}{2}$$

for all $w \in \Sigma^*$. The global value of a density function d is $d(\lambda)$. The set covered by a density function d is

$$S[d] = \bigcup_{\substack{w \in \Sigma^* \\ d(w) \geq 1}} C_w.$$

Definition 8: A density function d covers a set $X \subset \Sigma^\infty$ if $X \subseteq S[d]$.

Consider the random experiment in which a sequence $x \in \Sigma^\infty$ is chosen by using an independent toss of a fair coin to decide each bit of x . Definition 7 implies that $\Pr[x \in S[d]] \leq d(\lambda)$. Intuitively, we will regard a density function d as a verification that $\Pr[x \in X] \leq d(\lambda)$ for all sets $X \subseteq S[d]$.

Definition 9: An n -dimensional density system (n - DS) is a function

$$d : \mathbb{N}^n \times \Sigma^* \rightarrow [0, \infty)$$

such that $d_{\vec{k}}$ is a density function for every $\vec{k} \in \mathbb{N}^n$.

Definition 10: An n - DS d is Γ -computable iff there is a function $\hat{d} \in \Gamma$, such that

$$\hat{d} : \mathbb{N}^{n+1} \times \Sigma^* \rightarrow \mathbb{D}, \quad |\hat{d}_{\vec{k},r}(x) - d_{\vec{k}}(x)| \leq 2^{-r}$$

for all $\vec{k} \in \mathbb{N}^n, r \in \mathbb{N}$, and $x \in \Sigma^*$.

We now come to the key idea of resource bounded measure theory.

Definition 11: A set $X \subset \Sigma^\infty$ has Γ -measure 0 iff there exists a Γ -computable 1 - DS d such that, for all $k \in \mathbb{N}$, d_k covers X with global value $d_k(\lambda) \leq 2^{-k}$.

Thus a set X has Γ -measure 0 if Γ provides sufficient computational resources to compute uniformly good approximations to a system of density functions that cover X with rapidly vanishing global value.

Definition 12: A set X has Γ - measure 1 iff the complement of X (denoted X^c) has Γ -measure 0.

Definition 13: A language L is Γ -random iff it belongs to every Γ -measure 1 class.

Definition 14: A set X has measure 0 in $R(\Gamma)$ iff $X \cap R(\Gamma)$ has Γ -measure 0.

Definition 15: A set X has measure 1 in $R(\Gamma)$ iff X^c has measure 0 in $R(\Gamma)$.

The following lemma from [L91] states that a special kind of union of Γ -measure 0 sets has measure 0.

Definition 16: A set X is a Γ -union of the Γ -measure 0 sets $X_0, X_1, X_2, \dots$ iff $X = \bigcup_{j=0}^{\infty} X_j$ and there exists a Γ -computable 2-DS d such that each d_j testifies that X_j has Γ -measure 0.

Lemma 1: If X is a Γ -union of Γ -measure 0 sets, then X has Γ -measure 0.

Using this lemma we can prove our main theorem.

Theorem 1: The class of P -bi-immune languages has p -measure 1.

Proof: Let Y be the class of non- P -bi-immune languages. By Definition 12 we have to prove that Y is a p -measure 0 class.

We can write Y as a union of all the classes A_n and B_n , where:

$$A_n = \{L \mid L(M_n) \subseteq L \text{ and } |L(M_n)| = \infty\}, \text{ and}$$

$$B_n = \{L \mid L(M_n) \subseteq L^c \text{ and } |L(M_n)| = \infty\}.$$

Here $\{M_n\}$ is a feasible enumeration of the polynomial time Turing machines whose time on input x is strictly bounded by $2^{|x|}$. It suffices to enumerate only those polynomial time TM's whose time is bounded by $2^{|x|}$ for all inputs x , because we are only interested in the existence of infinite subsets in P .

Now we are going to build a p -computable 2-DS d , such that d_{2n-1} witnesses that A_n is a p -measure 0 set and d_{2n} witnesses that B_n is a p -measure 0 set.

We build d_{2n-1} so that $d_{2n-1,k}(x) \geq 1$ if and only if the following holds: there are more than k elements of $L(M_n)$ in the language denoted by $x0^\infty$ and $L(M_n)$ is included in the language denoted by $x1^\infty$.

$$\begin{aligned} d_{2n-1,k}(\lambda) &= 2^{-k} \\ d_{2n-1,k}(x0) &= 0 && \text{if } s_{|x|} \in L(M_n) \\ &= d(x) && \text{otherwise} \end{aligned}$$

$$d_{2n-1,k}(x1) = \begin{array}{ll} 2 * d(x) & \text{if } s_{|x|} \in L(M_n) \\ d(x) & \text{otherwise.} \end{array}$$

So it is clear that $A_n \subseteq S[d_{2n-1,k}]$ for all n and k .

We can build d_{2n} in a similar way, but this time having $d_{2n,k}(x) \geq 1$ if and only if there are more than k elements of $L(M_n)$ in the complement of the language denoted by $x1^\infty$ and $L(M_n)$ is included in complement of the language denoted by $x0^\infty$.

It is clear that $d \in p$, so d is trivially p -computable. Applying Lemma 1, we have that Y itself has p -measure 0. $\square$

Note that, by the same proof, the class of $DTIME(2^{cn})$ -bi-immune languages has p -measure 1, for any $c > 0$.

Using the same technique we obtain:

Theorem 2: The class of E_{i-1} -bi-immune sets has p_i -measure 1.

Proof: Take $\{M_n\}$ a presentation of E_{i-1} . Use a time bound s for all the machines, where $E_{i-1} \subseteq DTIME(o(s))$ and $DTIME(s) \subseteq E_i$. $\square$

And as corollaries we obtain:

Corollary 1: The class of P -bi-immune sets has measure 1 in E and the class of E -bi-immune sets has measure 1 in E_2 .

So almost every set in E has Σ^* as a complexity core.

Note that we now have here a simple proof of the following result. (The result for E was also proven by Juedes and Lutz [JL91], using different techniques.)

Corollary 2: The class of $\leq_m^p$ -complete languages for E has measure 0 in E . The class of $\leq_m^p$ -complete languages for NP has measure 0 in E_2 . The class of $\leq_m^p$ -complete languages for NE has measure 0 in $E_i, \forall i \geq 2$.

Proof: As proven in [Be76], every $\leq_m^p$ -complete set for E is non- P -bi-immune, and every $\leq_m^p$ -complete set for NP or NE is non- E -bi-immune. $\square$

It is not known if there is an i such that $NE \subseteq E_i$.

An important consequence dealing with the existence of P -bi-immune sets in NP states:

Corollary 3: If NP does not have measure 0 in E then NP contains a P -bi-immune set. If NP does not have measure 0 in E_2 then NP contains an E -bi-immune set.

Regarding pseudo-randomness we have the following corollary:

Corollary 4: Every p -random language is E -bi-immune and every p_i -random language is E_i -bi-immune.

Proof: From the remark that follows Theorem 1 we have that the class of $DTIME(2^{cn})$ -bi-immune sets has p -measure 1, for every $c > 0$. Since p -random languages belong to every p -measure 1 class, they are E -bi-immune. $\square$

P-bi-immunity and resource-bounded category

In this section we prove that in E , and even in REC , the class of P -bi-immune languages lacks the property of Baire (the Baire category analogue of Lebesgue measurability). The main part of the proof, establishing that the class of P -bi-immune languages is neither meager nor comeager in E (or in REC), comes first. The consequence for the property of Baire is explained at the end of this section.

For a formal definition of the notion of category and other related concepts in this section, see [L90]. Here we will just indicate that a category Γ is a class of functions closed under certain operations that include composition, concatenation and pairing.

Some examples of categories are *all*, *rec*, p_i for $i \geq 0$ and $p_i\text{space}$ for $i \geq 0$.

Informally, a Banach-Mazur game is an infinite game in which two players construct a language L by taking turns extending an initial bitstring of L . There is a distinguished class of languages X such that player I wins if $L \in X$; player II wins otherwise.

Definition 17: Let X be a class of languages, let Γ_1 and Γ_2 be two categories. A Banach-Mazur game $G[X; \Gamma_1, \Gamma_2]$ is a game with two players I and II such that player I has chosen a constructor $g \in \Gamma_1$ and player II has chosen a constructor $h \in \Gamma_2$. They play as follows:

Starting from $w := \lambda$

Repeat indefinitely

I plays: $w := g(w)$

II plays: $w := h(w)$.

We denote as $R(g, h)$ the language built in this Banach-Mazur game. (As they play infinitely, they build an element of Σ^∞).

Definition 18: A winning strategy for player II in the game $G[X; \Gamma_1, \Gamma_2]$ is a constructor $h \in \Gamma_2$ such that for every constructor $g \in \Gamma_1$, $R(g, h) \notin X$.

Now we can define Γ -meager classes.

Definition 19: Let X be a class of languages. X is Γ -meager iff player II has a winning strategy for $G[X; all, \Gamma]$.

Intuitively, a class X is “ Γ -small” iff, with the resources provided by Γ , you can avoid X .

We remind the reader that this is not the original definition of Γ -meager in [L90], but a characterization obtained from [L90] and [F91] for categories Γ such that $p \subseteq \Gamma$.

We define the other related concepts.

Definition 20: Let X be a class of languages. X is Γ -co-meager iff X^c is Γ -meager.

Definition 21: Let L be a language. L is Γ -generic iff L belongs to every Γ -co-meager class.

We finish these definitions with the concept of a class being meager in another one.

Definition 22: Let X be a class of languages. X is meager in $R(\Gamma)$ iff $X \cap R(\Gamma)$ is Γ -meager.

Note that in order to measure classes within $R(\Gamma)$ we can not use more resources than Γ because we would obtain that $R(\Gamma)$ itself is meager.

The concepts of co-meager in $R(\Gamma)$ and generic in $R(\Gamma)$ are defined analogously.

The following lemma states a basic property of meager sets, and is proved in [L90].

Lemma 2: A subset of a set meager in $R(\Gamma)$ is meager in $R(\Gamma)$. A finite union of sets meager in $R(\Gamma)$ is meager in $R(\Gamma)$.

In this section we will denote with X the class of P -bi-immune languages and with Y the class of non- P -immune languages.

Let us start by showing that the class of P -bi-immune languages is neither meager nor co-meager in E .

Theorem 3: The class of P -bi-immune languages is not meager in E .

Proof: We have to see that there is no winning strategy for player II in the game $G[X \cap E; all, p]$, that is to say: for every constructor $h \in p$, there exists a constructor $g \in all$, such that $R(g, h) \in X \cap E$.

So given h we build $g \in all$ performing a diagonalization of the sets in P out of $R(g, h)$ and its complement. The condition of the language $R(g, h)$ being in E just slows down the diagonalization a bit.

Given $h \in p$, there exists an increasing polynomial q such that for all x , $|h(x)| \leq q(|x|)$.

Let $\{M_n\}$ be a feasible enumeration of the polynomial time Turing machines as in Theorem 1. Recall that each M_n runs in time $\leq 2^{|x|}$ on input x .

Let g be the function computed by the algorithm of Figure 1.

For input α we consider the machines from $M_1, M_2, \dots, M_{|\alpha|}$ that do not have witnesses of $L(M_n) \not\subseteq R(g, h)$ and $L(M_n) \not\subseteq R(g, h)^c$ yet. We find the first of them, M_j , that can be discarded using a string from $s_{|\alpha|}, \dots, s_{q(|\alpha|)-1}$ and we then advance a step in the diagonalization. Next we make sure that player II's next turn is not going to jeopardize the opportunity for diagonalizing against machines $M_1, \dots, M_{j-1}$. For this we keep a list of the machines from $M_1, \dots, M_{j-1}$ that have not been discarded yet and explore it exhaustively until we ensure that there is no possible diagonalization using strings $s_{|\alpha|}, \dots, s_{q(|\alpha|)-1}$.

$R(g, h)$ is P -bi-immune by the construction of g .

The language built in this game, $R(g, h)$, is in E because to see if $x \in R(g, h)$ it is enough to play the game until you obtain a string of length $2^{|x|+1} - 1$. For this we have to evaluate a maximum of $2^{|x|+1} - 2$ machines on $q(2^{|x|+1} - 2)$ inputs of length up to $\lceil \log(q(2^{|x|+1} - 2)) \rceil$, and to compute h for up to $2^{|x|+1} - 2$ inputs of length up to $2^{|x|+1} - 2$. So the total time is bounded by $2^{O(|x|)}$. This is why, even though $g \notin p$, $R(g, h) \in E$. $\square$

Theorem 4: The class of P -immune languages is not co-meager in E .

Proof: We have to see that there is no winning strategy for player II in the game $G[Y \cap E; all, p]$, that is to say: for every constructor $h \in p$, there exists a constructor $g \in all$, such that $R(g, h) \in Y \cap E$.

So given h , we have to build g that puts a set in P inside $R(g, h)$.

For $h \in p$, there exists c with $|h(x)| < 2^{|x|}$ for all x such that $|x| \geq c$.

We define the sequence $\{a_n\}$:

$$\begin{aligned} a_0 &= 0 \\ a_1 &= c \\ a_n &= 2^{a_{n-1}}, \quad n > 1. \end{aligned}$$

The set in P we are going to include in $R(g, h)$ is $L = \{0^{a_n} \mid n \geq 0\}$.

Algorithm for g :

Begin

Input α

Calculate n such that $a_{n-1} \leq |\alpha| < a_n$

$\alpha := \alpha 1^{a_n - |\alpha|}$

output α

End.

To see if $x \in R(g, h)$ it is enough to compute g and h for $2^{|x|}$ inputs of length up to $2^{|x|}$ and to calculate n such that $a_n \leq |x| < a_{n+1}$ if $x \in \{0\}^*$. As the exponential function is time constructible, $R(g, h) \in E$. $\square$

Using essentially the same techniques we have the following results:

Theorem 5: The class of P -immune languages is not co-meager in E_i , $\forall i \geq 1$.

Proof: The same as Theorem 4. $\square$

Theorem 6: The class of E_{i-1} -bi-immune languages is not meager in E_i , $\forall i > 1$.

Proof: Take $\{M_n\}$ a presentation of E_{i-1} . Use a time bound s for all the machines, where $E_{i-1} \subseteq DTIME(o(s))$ and $DTIME(s) \subseteq E_i$. $\square$

For space bounds we can do something similar:

Theorem 7: The class of $E_{i-1}SPACE$ -bi-immune languages is not meager in E_iSPACE , $\forall i > 1$.

Theorem 8: The class of P -immune languages is not co-meager in E_iSPACE , $\forall i \geq 1$.

And for REC we obtain:

Theorem 9: The class of $\mathcal{C}$ -bi-immune languages is not meager in REC for any recursively presentable class $\mathcal{C}$.

Theorem 10: The class of P -immune languages is no co-meager in REC .

Lutz (personal communication) has pointed out that these results imply that the class of P -bi-immune languages lacks the property of Baire in E (and classes up to REC). For the sake of completeness, we introduce the resource-bounded property of Baire and prove the zero-one law for Baire category that supports this inference.

Classically, a set X has the property of Baire if and only if there is an open set G such that $X \Delta G$ is meager. (This is the Baire category analogue of the fact that a set X is Lebesgue measurable if and only if there is an F_σ set -equivalently, a G_δ set- H such that $X \Delta H$ has measure 0). The extension of this notion to complexity classes is natural.

Definition 23: A class X is open in $R(\Gamma)$ iff $\exists h \in \Gamma$ such that $X \cap R(\Gamma) = (\bigcup_k C_{h(0^k)}) \cap R(\Gamma)$. A class X is closed in $R(\Gamma)$ iff it is the complement of an open class in $R(\Gamma)$.

Definition 24: A class X has the property of Baire in $R(\Gamma)$ iff $X = G \Delta Q$, where G is open in $R(\Gamma)$ and Q is meager in $R(\Gamma)$.

Definition 25: A class X of languages is *closed under finite variations* if for all languages L and L' , if $L \in X$ and $|L \Delta L'| < \infty$, then $L' \in X$.

The following result is a straightforward generalization of Theorem 21.4 in [O80], which is the Baire category analogue of the Kolmogorov zero-one law for measure.

Lemma 3: If X is a class of languages that is closed under finite variations and has the property of Baire in $R(\Gamma)$, then X is either meager in $R(\Gamma)$ or co-meager in $R(\Gamma)$.

Proof: Assume that X is closed under finite variations, has the property of Baire in $R(\Gamma)$, and is not meager in $R(\Gamma)$. It suffices to prove that X is co-meager in $R(\Gamma)$.

Since X has the property of Baire in $R(\Gamma)$, there is a set G that is open in $R(\Gamma)$ such that $X \Delta G$ is meager in $R(\Gamma)$. Since X is not meager in $R(\Gamma)$, $G \neq \emptyset$. Thus there exists $w \in \Sigma^*$ such that $C_w \cap R(\Gamma) \subseteq G \cap R(\Gamma)$.

Since $X \Delta G$ is meager in $R(\Delta)$, there is a winning strategy h for player II in the game $G[(X \Delta G) \cap R(\Gamma); all, \Gamma]$. Define $\hat{h} : \Sigma^* \rightarrow \Sigma^*$ as follows. If $w \sqsubseteq x$, then $\hat{h}(x) = h(x)$. If $w \not\sqsubseteq x$, then $\hat{h}(x)$ is the first string $y \in \Sigma^*$ such that $x \sqsubseteq y$ and $C_w \cap C_y = \emptyset$.

We claim that $\hat{h}$ is a winning strategy for player II in the game

$$G[X^c \cap C_w \cap R(\Gamma); all, \Gamma].$$

It is clear that $\hat{h} \in \Gamma$. To see that $\hat{h}$ wins, let g be an arbitrary strategy for player I. We have two cases:

(i) If $w \sqsubseteq g(\lambda)$, then $R(g, \hat{h}) \notin (X \Delta G) \cap R(\Gamma)$ by our choice of h . On the other hand, $R(g, \hat{h}) \in C_{g(\lambda)} \subseteq C_w \subseteq G$, so we must have $R(g, \hat{h}) \notin X^c \cap R(\Gamma)$.

(ii) If $w \not\sqsubseteq g(\lambda)$, then $R(g, \hat{h}) \notin C_w$.

Each of (i) and (ii) implies that $R(g, \hat{h}) \notin X^c \cap C_w \cap R(\Gamma)$, so $\hat{h}$ is indeed a winning strategy for player II.

We have now shown that $X^c \cap C_w$ is meager in $R(\Gamma)$. Since X is closed under finite variations, it follows by Lemma 2 that

$$X^c = \bigcup_{z \in \Sigma^{|\Sigma|}} (X^c \cap C_z)$$

is meager in $R(\Gamma)$. This completes the proof. $\square$

The following result thus summarizes the results of this section.

Theorem 11: The class of P -bi-immune languages does not have the property of Baire in E , $E_i (i \geq 1)$, $E_iSPACE (i \geq 1)$, or REC .

Proof: This follows immediately from Theorems 3, 4, 5, 6, 7, 8, 9, 10 and Lemma 3.

□

Since the class of RE -bi-immune sets is *all-co-meager*, we have completed the study of P -bi-immunity.

From these results and using Lemma 3.9 from [F91] we remark that we can not assume anything about the immunity of a pseudo-generic language:

- There exists a p_i -generic language in E_i -bi-immune $\cap E_{i+1}$.
- There exists a p_i -generic language in non- P -immune $\cap E_{i+1}$.
- There exists a p_i space-generic language in E_iSPACE -bi-immune $\cap E_{i+1}SPACE$.
- There exists a p_i space-generic language in non- P -bi-immune $\cap E_{i+1}SPACE$.

Conclusion

P -bi-immunity is a natural example of a class lacking the Property of Baire. Since we have seen that it has p -measure 0, we notice the different behavior of measure and category.

Also we see that every pseudo-random language is P -bi-immune, whereas pseudo-generic languages can be either P -bi-immune or non- P -immune.

Acknowledgments

I would like to thank Jack Lutz for many helpful ideas and discussions, as well as for showing me the preliminary version of [L91]. José Luis Balcázar proposed this problem and Stephen Fenner remarked that Theorem 10 was also true.

Begin

Input α

$\alpha := \alpha 0$; $list := empty$

For $n := 1$ to $|\alpha|$ do

If $\nexists j < |\alpha|$ with $s_j \in L(M_n)$ and $\alpha[j] = 0$

$\{s_j \text{ would be a witness of } L(M_n) \not\subseteq R(g, h).\}$

then

If $\exists |\alpha| \leq j < q(|\alpha|)$ with $s_j \in L(M_n)$ then $\{we \text{ advance in the diagonalization.}\}$

$\alpha := \alpha 0^{j-|\alpha|+1}$ for the minimum such j ; exit for

else $list := list \cup \{(n, 1)\}$

$\{we \text{ take note that we have not yet ensured that } L(M_n) \not\subseteq R(g, h).\}$

If $\nexists j < |\alpha|$ with $s_j \in L(M_n)$ and $\alpha[j] = 1$

$\{s_j \text{ would be a witness of } L(M_n) \not\subseteq R(g, h)^c.\}$

then

If $\exists |\alpha| \leq j < q(|\alpha|)$ with $s_j \in L(M_n)$ then $\{we \text{ advance in the diagonalization.}\}$

$\alpha := \alpha 0^{j-|\alpha|} 1$ for the minimum such j ; exit for

else $list := list \cup \{(n, 0)\}$

$\{we \text{ take note that we have not yet ensured that } L(M_n) \not\subseteq R(g, h)^c.\}$

end for

$\{next \text{ we make sure that we do not lose any opportunity for diagonalizing in player II's turn}\}$

$max := \max\{n/(n, 0) \in list \text{ or } (n, 1) \in list\}$; $n := 1$

While $n \leq max$ do

If $(n, 1) \in list$ and $\exists |\alpha| \leq j < q(|\alpha|)$ with $s_j \in L(M_n)$ then

$\alpha := \alpha 0^{j-|\alpha|+1}$ for the minimum such j ;

$list := list - \{(n, 1)\}$; $n := 0$

If $(n, 0) \in list$ and $\exists |\alpha| \leq j < q(|\alpha|)$ with $s_j \in L(M_n)$ then

$\alpha := \alpha 0^{j-|\alpha|} 1$ for the minimum such j ;

$list := list - \{(n, 0)\}$; $n := 0$

$n := n + 1$

end while

output α

End.

Figure 1. Algorithm for g in proof of Theorem 3.

References

- [Ba85] J.L.Balcázar, *Simplicity, Relativizations, and Nondeterminism*, SIAM J. Comp. 14, pp.148-157, 1985.
- [Be76] L.Berman, *On the Structure of Complete Sets: Almost Everywhere Complexity and Infinitely Often Speed-up*, Proc. 17th IEEE Symp. Foundations of Computer Science, pp.76-80, 1976.
- [BG81] C.H.Bennet and J.Gill, *Relative to a Random Oracle A , $P^A \neq NP^A \neq co - NP^A$ with Probability 1*, SIAM J. Comp. 10, pp.96-113, 1981.
- [BS85] J.L.Balcázar and U.Schöning, *Bi-Immune Sets for Complexity Classes*, Math. Systems Theory 18, pp.1-10, 1985.
- [F91] S.A.Fenner, *Notions of Resource-Bounded Category and Genericity*, Proc. sixth annual conf. Structure in Complexity Theory, pp.196-212, 1991.
- [FS74a] P.Flajolet and J.M.Steyaert, *On Sets Having Only Hard Subsets*, *Automata, Languages and Programming*, Lecture Notes in Computer Science 14, pp.446-457, Springer-Verlag, 1974.
- [FS74b] P.Flajolet and J.M.Steyaert, *Une Généralisation de la Notion d'Ensemble Immune*, RAIRO Informatique Théorique 8, pp.37-48, 1974.
- [GH83] W.I.Gasarch and S.Homer, *Relativizations Comparing NP and Exponential Time*, Information and Control 58, pp.88-100, 1983.
- [GHS91] J.G.Geske, D.T.Huynh, and J.I.Seiferas, *A Note on Almost-Everywhere-Complex Sets and Separating Deterministic-Time-Complexity Classes*, Information and Comp. 92, pp.97-104, 1991.
- [Ha83] J.Hartmanis, *Generalized Kolmogorov Complexity and the Structure of Feasible Computations*, Proc. 24th IEEE Symp. Foundations of Computer Science, pp.439-445, 1983.
- [HB77] J.Hartmanis and L.Berman, *On Isomorphisms and density of NP and other complete sets*, SIAM J. Comp. 6, pp.305-332, 1977.
- [HM83] S.Homer and W.Maass, *Oracle Dependent Properties of the Lattice of NP Sets*, Theoret. Comput. Sci. 24, pp.279-289, 1983.
- [JL91] D.W.Juedes and J.H.Lutz, *The complexity and distribution of hard problems*, in preparation.
- [KM81] K.Ko and D.Moore, *Completeness, Approximation and Density*, SIAM J. Comp. 10, pp.787-796, 1981.

- [L90] J.H.Lutz, *Category and Measure in Complexity Classes*, SIAM J. Comp. V19, N6, pp.1100-1131, 1990.
- [L91] J.H.Lutz, *Almost Everywhere High Nonuniform Complexity*, J. of Computer and System Sciences, to appear.
- [M77] A.R.Meyer, reported in [HB77].
- [O80] J.C.Oxtoby, *Measure and Category*, Graduate Texts in Mathematics, Vol. 2, Springer-Verlag, 1980.
- [P44] E.L.Post, *Recursively Enumerable Sets of Integers and their Decision Problems*, Bull. Amer. Math. Soc. 50, pp.284-316, 1944.
- [SB84] U.Schöning and R.V.Book, *Immunity, Relativizations, and Nondeterminism*, SIAM J. Comp. 13, pp.329-337, 1984.