Chapter 13

The Role of Indivisibles in Mengoli's Quadratures

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Introduction¹

The name of Pietro Mengoli (1626–1686) appears in the University of Bologna registry for the period 1648–1686. He studied with Bonaventura Cavalieri and ultimately succeeded him in the chair of mechanics. He graduated in philosophy in 1650 and 3 years later in canon and civil law. In his first period, he wrote three mathematical books, *Novae quadraturae arithmeticae seu de additione fractionum* (Bologna, 1650), *Via Regia ad Mathematicas per Arithmetica, Algebram Speciosam, & Planimetriam, ornata, maiestati Serenissimae D. Christinae Reginae Suecorum* (Bologna, 1655) and *Geometriae Speciosae Elementa* (Bologna, 1659). He took holy orders in 1660 and was prior at the church of Santa Maria Maddalena in Bologna until his death.²

Mengoli can be included in the group of seventeenth century mathematicians who accepted the new algebraic procedures in their geometrical research. Indeed, in

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² Although Mengoli published nothing between 1660 and 1670, the latter year saw the appearance of two works: *Refrattioni e parallase solare* (Bologna, 1670), *Speculationi di musica* (Bologna, 1670), and later *Circolo* (Bologna, 1672). These reflected Mengoli's new aim of pursuing research not on pure but on mixed mathematics like astronomy, chronology and music. Furthermore, his research was clearly in defence of the Catholic faith. Mengoli went on writing in this line, publishing *Anno* (Bologna, 1675) and *Mese* (Bologna, 1681) on the subject of cosmology and Biblical chronology and *Arithmetica rationalis* (Bologna, 1674) and *Arithmetica realis* (Bologna, 1675) on logic and metaphysics. For more biographical information on Mengoli, see Natucci and Mengoli (1970) and Massa Esteve (2006b).

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his *Geometria* and later in his *Circolo* (1672), he used algebra and geometry in complementary ways in the investigation of quadrature problems. Other mathematicians of the period—such as Fermat, Roberval, Pascal and Wallis—also used methods of quadratures that in some way introduced algebraic elements. Among other things, they attempted to calculate the result, which today would be written $\lim_{p \rightarrow \infty} \frac{1^p + \dots + p^p}{p^{p+1}} = \frac{1}{p+1}$ for t tending to infinity, in order to square the parabolas $y = x^n$, for n any natural number.³ In fact, these quadratures were proved firstly geometrically by Cavalieri, Mengoli's master, in Proposition XXIII of the *Exercitatio quarta*, in Cavalieri's *Exercitationes*.⁴

Mengoli wanted to complete these quadratures, and in his *Geometriae Speciosae Elementa* he computed countless quadratures between 0 and t of mixed-line geometric figures determined by $y = x^n (t - x)^{m-n}$, for natural numbers m and n . Having previously proved these quadratures by the method of indivisibles, he subsequently derived them by using a new arithmetic-algebraic method. However, his principal aim was to square the circle, a goal he achieved by means of his new method in his later work, *Circolo*.

Mengoli's *Geometriae Speciosae Elementa* is a 472-page text in pure mathematics with six *Elementa* whose title: "Elements of Specious Geometry" already indicates the singular use of symbolic language in this work, and in particular Geometry. Mengoli unintentionally created a new field, a "specious geometry" modelled on Viète's "specious algebra", since he was working with "specious" language, that is to say, symbols used to represent not just numbers but also values of any abstract magnitudes. Mengoli, who knew well the work of his master Cavalieri and Archimedes, introduces a new element into his geometry, namely, Viète's algebra *speciosa*, which he quotes repeatedly. Mengoli's method of quadratures was really based on the underlying ideas of the method of indivisibles and Archimedes' method of exhaustion, combined by using algebraic tools suggested by a study of Viète.⁵ At the beginning of this work, in a letter addressed to D. Fernando Riario, Mengoli himself states that his geometry was a combination

³ Information on these subject may be found in the following sources: on Fermat, see Mahoney (1973); on Roberval see Auger (1962), Walker (1986), and Jullien (1996b); on Pascal see Bosmans (1924) and Boyer (1943) and on Wallis see Stedall (2001).

⁴ We may also cite Roberval, who in 1636, in a letter written to Fermat, enunciated the rule for finding the infinite sum of powers, and explained how he employed it for calculating quadratures. Fermat, for his part, stated in a letter to Cavalieri, written before 1644, that he had squared the parabolas, giving both the rule and an example. Ten years later, Pascal arrived, apparently independently, at a similar conclusion in the work *Potestatum numericarum summa*, see Pascal, *Oeuvres de Blaise Pascal* (1954). In 1657, Fermat himself proved the quadratures for a positive rational number n , see Fermat, *Oeuvres*, (1891–1922). Furthermore, Wallis also proved these same quadratures in his *Arithmetica Infinitorum* (1655) using the sum of powers, see Wallis, *Opera* . . . , (1972).

⁵ In fact Mengoli was influenced by Viète's algebra through Hérigone's algebra in his *Cursus Mathematicus* (1634/1637/1642). On a comparative analysis between Viète's specious algebra and Hérigone's algebra see Massa Esteve (2008) and on Hérigone's influence in Mengoli's works see Massa Esteve (2012). On Viète's specious algebra, see Viète, *Opera*, (1970).

of geometries by Cavalieri and Archimedes, obtained by using the tools provided
him by Viète's "specious algebra":

Both geometries, the old form of Archimedes and the new form of indivisibles of my tutor,
Bonaventura Cavalieri, as well as Viète's algebra, are regarded as pleasurable by the
learned. Not through their confusion nor through their mixture, but through their perfect
conjunction, a somewhat new form [of geometry will arise]—our own—which cannot
displease anyone (2012).⁶

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The aim of this chapter is to analyze the explicit and implicit role of the method
of indivisibles in Mengoli's new method of quadratures set forth in his works
Geometria and *Circolo*. In Sect. 1, I analyze the method of indivisibles used
explicitly by Mengoli in his *Geometria*. In Sect. 2, I explore the calculation of
countless quadratures in the *Geometria* and in the *Circolo* with his new arithmetic-
algebraic method, emphasizing the implicit use of indivisibles in the main demon-
stration. In fact, although Mengoli uses a new and original arithmetic-algebraic
method of quadratures, I show that his prior knowledge of the values of quadratures
by the method of indivisibles plays an essential role in achieving what he set out
to do.

Mengoli's First Quadratures. The Method of Indivisibles in Mengoli's *Geometria*

Mengoli developed his algebraic analysis of geometric figures in the *Elementum*
sextum of *Geometria*⁷ and in the *Circolo*. This sixth chapter, entitled *De*
innumerabilibus quadraturis, involves calculating quadratures of plane curves in
the interval $(0, t)$ determined by equations now represented as $y = K x^n \cdot (t-x)^{m-n}$.

The aim of this section is to analyse the method of indivisibles used explicitly by
Mengoli in the introduction of this sixth chapter. In a preliminary calculation, in the
dedicatory letter to Cassini,⁸ Mengoli derived values for the quadratures of these
curves using Cavalieri's method of indivisibles. He outlined that he had determined
these values 12 years before (1647):

Twelve years ago, as a result of the question that Antonio Rocca Reggie [from the ducat of
Reggio] posed to me about the figure described by a line when an ellipse is cut at two points,

⁶ *Ipsae satis amabiles litterarum cultoribus visae sunt utraque Geometria, Archimedis antiqua, & Indivisibilium nova Bonaventura Cavallerij Praeceptoris mei, necnon & Vietae Algebra: quarum non ex confusione, aut mixtione, sed coniunctis perfectionibus, nova quaedam, & propria laboris nostri species, nemini poterit displicere* (Mengoli 1659, pp. 2–3).

⁷ This sixth *Elementum*, with the title *De innumerabilibus quadraturis* contains (besides a letter to Cassini), three triangular tables, 36 definitions, 11 propositions (4 of them he named problems) and lastly, two pages on barycentre.

⁸ Giandomenico Cassini (1625–1712) was a professor of astronomy at the University of Bologna from 1650 to 1669, before moving in the latter year to Paris. On the relation between Cassini and Mengoli see Mengoli, *La corrispondenza*, (1986).

I found countless figures of this kind, which I then squared by the geometry of indivisibles, which however I show after having used this lemma before.⁹

Mengoli wanted to use the method of indivisibles by reproducing a lemma and three quasi-algebraic propositions by Beaugrand,¹⁰ stating that he would use this algebraic technique with indivisibles because the procedure was shorter.¹¹ These propositions by Beaugrand are found in Cavalieri's *Exercitatio quarta*. In the introduction to this part, Cavalieri explains that when he was working on quadratures he told father Nicerone of his discoveries; during a subsequent visit to Paris, Nicerone then passed on this information to Beaugrand. Later Cavalieri learned of Beaugrand's death from Mersenne; Mersenne also told him of the solutions that Beaugrand had found to the proposed quadratures. Cavalieri incorporated these solutions so that they would not be lost.¹² These solutions by Cavalieri-Beaugrand are used by Mengoli to show these quadratures by the method of indivisibles. Thus, the lemma that Mengoli undertakes is similar to Cavalieri's lemma in the *Exercitatio* and comes from Beaugrand, as Mengoli claims:

This lemma, on the other hand, is analogous to that (lemma) by Jean Beaugrand, as explained Bonaventura Cavalieri b. m. my tutor, and I am pleased to imitate him in my exposition [of this demonstration].¹³

The lemma by Cavalieri-Beaugrand (referring to the fifth power) in the *Exercitatio* reads (in modern notation),

$$(t+x)^5 + (t-x)^5 = 2t^5 + 20t^3x^2 + 10tx^4.$$

The lemma by Mengoli (also referring to the fifth power) in the *Geometria* reads (in modern notation),

⁹ *Ante annos duodecim, occasione cuiusdam problematis mihi propositi à D. Io. Antonio Rocca Regiensi, de figura unilinea describenda, quae secaret ellipsim in duobus punctis innumerabiles huiusmodi figuras excogitavi, quas tunc per Geometriam indivisibilium quadrabam, adhibito tamen priùs hoc lemmate* (Mengoli 1659, p. 348). I am unable to identify the question posed by Antonio Rocca (1607–1656), who was a friend and correspondent Cavalieri's and of many other scientists at this time. For more information see Favaro (1983) and Rocca (1785).

¹⁰ Jean de Beaugrand (1595–1640) was also a mathematician; in 1635 he spent an entire year in Italy and visited Cavalieri in Bologna. He published a version of In Artem analyticen Isagoge, which was in fact Viète's work extended with some "scolies" and a mathematical compendium.

¹¹ Mengoli states: "Furthermore, in order to obtain this in a shorter way, we will proceed using Speciosa Algebra"/"Ut autem breviori via id obtineamus, procedemus per Algebram Speciosam", Mengoli, *Geometria*, (1659, p. 349). Furthermore, Cavalieri in the *Exercitatio* after the lemma claims: "But in order to obtain this, the reader who does not ignore these algebraic products will understand that this way is much easier than the Euclidian approach. We have used its longer structure for Propositions 17 and 18." *Ex his ergo Lector harum multiplicationum Algebraicarum non ignarus, intelliget hanc viam multò faciliorem esse quàm Euclidianam, cuius longiorem texturam in Propos 17. & 18. prosecuti sumus* (Cavalieri, 1647, p.286).

¹² Ibid, 243–245. On Cavalieri's chapter see Bosmans (1922).

¹³ *Est autem hoc lemma affine illi, quod recitat Bonaventura Cavallerius b. m. Praeceptor meus ex Io. Beaugrand: quod idcirco in expositione placet imitari* (Mengoli 1659, p. 349).

Fig. 13.1 Mengoli's segment (Mengoli, *Geometria* . . ., 1659, 349)

$$(t-x)^4(t+x) + (t+x)^4(t-x) = 2t^5 + 4t^3x^2 - 6tx^4.$$

The proof of the lemma by Mengoli is similar to that by Cavalieri-Beaugrand in the *Exercitatio quarta*. Like Cavalieri, Mengoli in the *Geometria* divides a segment into two halves $[t]$, AT and TR, and each half into two parts, $[x]$, BT and TC, giving $t+x$, AC and BR, and $t-x$, AB and CR, (Mengoli writes $t+a$ and $t-a$). See Fig. 13.1.

These proofs are made by using letters and products of polynomial, although the letters represent the segments. Let us consider an example; see Fig. 13.2.

After demonstrating the lemma, Mengoli states the results of nine quadratures, and proves three of these results as examples. For example, he derived (in modern notation),

$$\begin{aligned} 6. \int_0^t x.(t-x) dx &= \int_0^t t^2 dx; \quad 12. \int_0^t x.(t-x)^2 dx = \int_0^t t^3 dx; \quad 20. \int_0^t x.(t-x)^3 dx \\ &= \int_0^t t^4 dx \end{aligned}$$

In order to understand how Mengoli uses the method of indivisibles for the quadratures, I analyze the proof of the first of these results. Mengoli defines the same parallelogram that Cavalieri uses in the propositions XXV–XXVI–XXVII in *Exercitatio* with different letters (see Fig. 13.3).

Let AB be a parallelogram, with the diagonal CD. And CD will be halved in E. And the straight lines FG, IH, parallel to the sides of parallelogram AB, will be traced through E. And KLMN and OPQR will be traced equidistant from there in an arbitrary but equal distance from the two. I say that “all the squares” of the parallelogram AB $[(2t)^2]$ are the sextuple of “all the products” (*uniprimas*) $[(t+x)(t-x)]$ of either of the triangles ACD or BCD.¹⁴

Mengoli seeks to show, in modern notation:

$$6. \sum_{ACD} (t+x). \sum_{BCD} (t-x) = \sum_{AB} (2t)^2.$$

Mengoli shows that both members of the equality are equal to 24.

¹⁴ *Esto parallelogrammum AB, cuius diameter CD: dividaturque CD bifariam in E: ducanturque per E, rectae FG, IH, parallelogrammi AB lateribus parallele: ducanturque hinc inde ab E distantes quantumlibet, sed aequaliter, & intra quadratum, duae KLMN, & OPQR. Dico sub triangulis ACD, BCD, omnes sextuplas uniprimas, aequales esse, omnibus secundis potestatibus parallelogrammi AB.* (Mengoli 1659, p. 358).

Fig. 13.2 Mengoli's algebraic proof (Mengoli, *Geometria...*, 1659, 353)

Exemplum 5. in Quadruprimis.

$$\begin{array}{l}
 AB: t^4 - 4t^3a + 6t^2a^2 - 4ta^3 + a^4 \\
 BR: t + a \\
 \hline
 t^5 - 4t^4a + 6t^3a^2 - 4t^2a^3 + ta^4 \\
 - t^4a - 4t^3a^2 + 6t^2a^3 - 4ta^4 + a^5 \\
 \hline
 ABR: t^5 - 3t^4a + 2t^3a^2 - 2t^2a^3 - 3ta^4 + a^5 \\
 \hline
 AC: t^4 + 4t^3a + 6t^2a^2 + 4ta^3 + a^4 \\
 CR: t - a \\
 \hline
 t^5 + 4t^4a + 6t^3a^2 + 4t^2a^3 + ta^4 \\
 - t^4a - 4t^3a^2 - 6t^2a^3 - 4ta^4 - a^5 \\
 \hline
 ACR: t^5 + 3t^4a + 2t^3a^2 - 2t^2a^3 - 3ta^4 - a^5 \\
 t^5 - 3t^4a + 2t^3a^2 + 2t^2a^3 - 3ta^4 + a^5 \\
 \hline
 ABR - ACR: 2t^5 + 4t^3a^2 - 6ta^4.
 \end{array}$$

Fig. 13.3 Mengoli's geometric figure in the dedicatory letter to Cassini (Mengoli, *Geometria...*, 1659, 358)

The demonstration begins by enunciating the corresponding result by means of Beaugrand's lemma.

On the one hand, he applies Beaugrand's lemma in the line KLMN (See Fig. 13.3) and states that:

$$KM \cdot MN = (t + x) \cdot (t - x) = t^2 - x^2 = KL^2 - LM^2.$$

Then he adds all the lines of the parallelogram ACGF, which gives:

$$\sum_{ACFE} (t + x) \cdot \sum_{CEG} (t - x) = \sum_{AE} t^2 - \sum_{IEC} x^2.$$

On the other hand, adding all the lines of the parallelogram FDBG gives:

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$$\sum_{EDBG} (t+x) \cdot \sum_{EFD} (t-x) = \sum_{BE} t^2 - \sum_{DEH} x^2.$$

Thus, if the triangles IEC and DEH and the parallelograms AE and BE are equal, then by adding the two former equalities the result is:

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$$\sum_{ACD} (t+x) \cdot \sum_{BCD} (t-x) = 2 \sum_{AE} t^2 - 2 \sum_{IEC} x^2.$$

Mengoli then assumes that the sum of all the squares of triangle IEC is 1, and on applying a result from proposition XX by Cavalieri¹⁵, he deduces that all the squares of the parallelogram AE have value 3. Therefore, using these values and multiplying by 6, the value of the first member is 24:

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$$6 \sum_{ACD} (t+x) \cdot \sum_{BCD} (t-x) = 6(2 \cdot 3 - 2 \cdot 1) = 24.$$

The point of departure for the demonstration of the second member states that all the squares of the parallelogram AH have value 6, and therefore all the squares of the parallelogram AB are 24. In fact, the parallelogram AB is double AH, and then the square is quadruple. Thus, the second member also has value 24,

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$$\sum_{AH} t^2 = 6; \sum_{AB} (2t)^2 = 4 \cdot 6 = 24.$$

The three proofs by Mengoli are similar to those by Cavalieri-Beaugrand in the *Exercitatio*. All proofs are expressed in rhetorical language and the only figure is the parallelogram in Fig. 13.3. Mengoli bases his proof on results found by Cavalieri in the *Exercitatio*, as though all readers were familiar with them. Although none of these proofs contributed anything new to the method of indivisibles, they show that Mengoli knew this method well. However, it is interesting to note that when he shows the method of indivisibles, he relies on the algebraic ideas of lemma and proofs by Beaugrand.

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Mengoli subsequently conjectured that by adding these results he might obtain a new quadrature like Archimedes,

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¹⁵ Proposition XX of Cavalieri states that: $3 \int_0^t x^2 dx = \int_0^t t^2 dx$, (Cavalieri 1647).

Having demonstrated these [quadratures by indivisibles], I wondered whether I could calculate some other quadrature which would be composed of those found into which any noteworthy quadrature would be resolved in the same way that Archimedes resolved the parabola into triangles.¹⁶

For instance, he indicated in rhetorical language the quadrature obtained by adding,

$$\begin{aligned} \int_0^1 x dx + \int_0^1 x \cdot (1-x) dx + \int_0^1 x \cdot (1-x)^2 dx + \int_0^1 x \cdot (1-x)^3 dx + \dots \\ = 1/2 + 1/6 + 1/12 + 1/20 + \dots = 1 \end{aligned}$$

He stated that he derived the value of this summation from the results obtained by indivisibles and from Proposition 17 in his *Novae Quadraturae Arithmeticae seu de Additione Fractorum*.¹⁷ In Proposition 17, he had proved that,

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1/2 + 1/6 + 1/12 + \dots = 1$$

Subsequently in the *Geometria*, after calculating the value of the quadratures by the method of the indivisibles, Mengoli added up these values in order to obtain a new quadrature:

$$\sum_{n=0}^{\infty} \int_0^1 x (1-x)^n dx = \sum_{n=0}^{\infty} \frac{1}{(n+2) \binom{n+1}{1}} = 1 = \int_0^1 1 dx.$$

He also added up the terms:

$$\begin{aligned} \int_0^1 x^2 dx + \int_0^1 x^2 (1-x) dx + \int_0^1 x^2 (1-x)^2 dx + \dots \\ = 1/3 + 1/12 + 1/30 + \dots = 1/2. \end{aligned}$$

¹⁶ *His demonstratis, cogitabam si possent aliae quadraturae inveniri ex inventis compositae, in quas insignis aliqua resolvatur, quemadmodum in triangula, parabolam Archimedes resolvit* (Mengoli 1659, p. 363). Indeed, Mengoli says that he knew these quadratures by indivisibles in 1647, and in 1650 he published the *Nova*, in which he proves infinite sums. A reading of the preface to the *Nova* makes the relation between these works clear. Mengoli explains the relation between these sums and the calculation of a universal quadrature. See Mengoli (1650) and Giusti (1991).

¹⁷ Mengoli had already published this work, in which he employed infinite series, adding them together and giving them suitable properties. On this subject see Giusti (1991).

In Proposition 8 of book 2 of the *Nova*, Mengoli calculated the sum of the following infinite series:

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)(n+3)} = 1/6 + 1/24 + 1/60 + \dots = \frac{1}{4};$$

If all the terms are multiplied by 2, we obtain 1/3, 1/12, 1/30, ... which added infinitely yield 2/4; that is, 1/2. Expressed in combinatorial numbers, this is:

$$\sum_{n=0}^{\infty} \int_0^1 x^2(1-x)^n dx = \sum_{n=0}^{\infty} \frac{1}{(n+3) \binom{n+2}{2}} = \frac{1}{2} = \int_0^1 x dx.$$

Expressed not in letters but only verbally, Mengoli generalized these sums of series thus:

And in general, I have found that the figure in which the ordinates are all the powers of the abscissae, and successively all the figures in which the ordinates are the product of the same powers of the abscissae and all the possible powers of the remainders, all added together are equal to the figure in which the ordinates are all the powers of the abscissae of the closest lower order.¹⁸

In modern notation and generalizing, the property of these sums would be:

$$\begin{aligned} \sum_{n=0}^{\infty} \int_0^1 x^m(1-x)^n dx &= \frac{1}{(m+1) \binom{m}{0}} + \frac{1}{(m+2) \binom{m+1}{1}} + \frac{1}{(m+3) \binom{m+2}{2}} \\ &+ \dots = \frac{1}{m} = \int_0^1 x^{m-1} dx. \end{aligned}$$

He presented two more examples, but found no new noteworthy quadrature, only relations between quadratures that were already known by means of indivisibles.¹⁹ He therefore proceeded to develop a new and more fruitful method.

However, before developing this new method he acknowledged that he did not publish this research because of the attacks often levelled against quadrature methods:

Meanwhile I left aside this addition that I had made to the Geometry of Indivisibles, because I was afraid of the authority of those who believe that the hypothesis

¹⁸ *Et generaliter inveni, figuram, in qua ordinatae sunt omnes potestates abscissarum, & deinceps omnes figuras, in quibus ordinatae sunt productae sub iisdem potestatibus abscissarum, & sub residuarum potestatibus omnifariam, simul aggregatas, aequales esse figurae, in qua ordinatae, sunt omnes potestates abscissarum ordinis proximè inferioris.* (Mengoli 1659, pp. 363–364).

¹⁹ We can suppose that this “insignis” quadrature he was looking for was the quadrature of the circle. In fact, at the beginning of his *Circolo* (1672), Mengoli stated that he had found the quadrature of the circle in 1660, but he had not published it because, according to him, he only wanted to publish the mathematics he needed to explain natural events.

184 that the infinity of all the lines of a plane figure is the plane figure itself to be false.
 185 Not that I necessarily agreed with them, but rather because I was doubted of it myself,
 186 I checked in my mind whether I could establish new and secure foundations for the
 187 same method of indivisibles or for other methods, which were equivalent.²⁰

188 Mengoli believed that the basis of Cavalieri's method of indivisibles was not
 189 sufficiently sound. He wanted to provide a solid foundation for the application of
 190 this method to square the given figures, the new figures and especially the circle. He
 191 sought to make his procedure for introducing algebra into geometry clear from the
 192 beginning as we analyse in the next section. First of all, using his own system of
 193 coordinates and Viète's symbolic language, he expressed geometric figures by
 194 algebraic expressions. Secondly, he placed these algebraic expressions in a trian-
 195 gular table to compute the already known values of their countless quadratures at a
 196 glance. Thirdly, he used these algebraic expressions as part of a method for the
 197 geometrical construction of ordinates of these geometrical figures. Finally, he used
 198 triangular tables and quasi proportions to produce general demonstrations of quadra-
 199 ture results that he had already determined by indivisibles. It is worth remembering that
 200 in the *Circolo*, by interpolating these triangular tables of quadratures, Mengoli found
 201 new quadratures and an approximation of the number π up to eleven decimal places.

202 **Mengoli's New Method of Quadratures. The Implicit Use** 203 **of Indivisibles in the Main Demonstration**

204 Mengoli was able to compute quadratures using Cavalieri's method of indivisibles,
 205 but he was keen to find another way to verify the values so obtained. Using Viète's
 206 symbolic language, he created new algebraic expressions and constructed triangular
 207 tables and a theory of "quasi proportions". It should be pointed out that the
 208 Euclidean theory of proportions is very important in the *Geometria*. Mengoli
 209 considered Euclid's *Elements* as the book of mathematics *par excellence* and
 210 developed his own theories, the theory of "quasi proportions" and the theory of
 211 logarithmic ratios, using the Euclidean theory of proportions²¹ as a model.

212 In order to understand how Mengoli proved the given quadrature results, I
 213 consider the basic ideas of the theory of "quasi proportions." He put forward this
 214 theory on the notion of "ratio quasi a number", which he clarified thoroughly. He
 215 considered values up to 10 in the ratio $O.a$ to t^2 ; for instance, if $t = 3$, then the ratio

²⁰ *Ipsam interim accessionem, quam Geometriae Indivisibilium feceram, praeterivi: veritus eorum auctoritatem, qui falsum putant suppositum, omnes rectas figurae planae infinitas, ipsam esse figuram planam: non quasi hanc sequens partem; sed illam quasi non prorsus indubiam devitans: tentandi animo, si possem demum eandem indivisibilium methodum, aut aliam equivalentem novis, & indubijs prorsus constituere fundamentis* (Mengoli 1659, p. 364).

²¹ A knowledge of algebraic language enabled Mengoli to extend the Euclidean theory of proportions and create new theories. On the importance of Mengoli's work on the Euclidean theory of proportions, see Massa (2003).

$O.a$ to t^2 is 3–9; if $t = 4$, then the ratio is 6–16; if $t = 5$, then the ratio is 10–25; . . . if $t = 10$, then the ratio is 45–100. He argued that the ratio takes different values as the value of t increases.²² Moreover, these values are eventually nearer to $1/2$ than any other given ratio. Mengoli called this the “ratio quasi $1/2$.” The difference between $1/2$ and the ratio determined when the value of t increases indefinitely is smaller than the difference between $1/2$ and any other given ratio. The “limit” of this succession of ratios, as far as it is thus determinable, is $1/2$, and Mengoli uses the term “ratio quasi $1/2$ ” to denote this limit. The idea of “ratio quasi a number” suggests, though in a somewhat imprecise way, the modern concept of limit.²³

This notion, together with the idea of determinable indeterminate ratio explained above, was used in the definitions of ratio “quasi infinite”, “quasi null”, “quasi equality” and “quasi a number” in the *Elementum tertium*:

1. A determinable indeterminate ratio, which, when determined, can be greater than any given ratio, as far as it is thus determinable, will be called quasi infinite²⁴.
2. And one that can be smaller than any given ratio, as far as it is thus determinable, will be called quasi null.
3. And one that can be smaller than any given ratio greater than equality, and greater than any given ratio smaller than equality, as far as it is thus determinable, will be called quasi equality. Or otherwise, that which can be nearer to equality than any given ratio not equal to equality, as far as it is thus determinable, will be called quasi equality.
4. And one that can be smaller than any ratio larger than a given ratio, and larger than any ratio smaller than the same given ratio, as far as it is thus determinable, will be called quasi equal to this given ratio. Or otherwise one that can be nearer to any given ratio than any other ratio not equal to it, as far as it is thus determinable, will be called quasi equal to the same (given) ratio.
5. And the terms of ratios quasi equal between them will be called quasi proportional.
6. And (the terms) of quasi equality ratios will be called quasi equal.²⁵

²² On these explanations see Massa (1997).

²³ In his Circolo of 1672, Mengoli again uses quasi ratios and explains: *Dissi quasi, e volsi dire, che vadino accostandosi ad essere precisamente tali* Mengoli (1672, p. 49). On Mengoli's Circolo see Massa Esteve-Delshams (2009).

²⁴ To clarify the notion of “ratio quasi infinite” Mengoli in his *Geometria* considered values up to 10 in the ratio $O.a$ to t ; for instance, if $t = 4$, then the ratio is 6 to 4; if $t = 7$ then the ratio is 21 to 7; if $t = 10$ then the ratio is 45 to 10. He argued that the ratio takes increasingly greater values as the value of t increases, so the ratio is quasi infinite. For the ratio quasi null, he considered values up to 10 in the ratio $O.a$ to t^3 .

²⁵ 1. *Ratio indeterminata determinabilis, quae in determinari, potest esse maior, quam data, quaelibet, quatenus ita determinabilis, dicitur, Quasi infinita.* 2. *Et quae potest esse minor, quàm data quaelibet, quatenus ita determinabilis, dicitur, Quasi nulla.* 3. *Et quae potest esse minor, quàm data quaelibet minor inaequalitas; & maior, quàm data quaelibet minor inaequalitas, quatenus ita determinabilis, dicitur, Quasi aequalitas. Vel aliter, quae potest esse propior aequalitati, quàm data quaelibet non aequalitas, quatenus talis, dicitur, Quasi aequalitas.* 4. *Et quae potest esse minor, quàm data quaelibet non maior, proposita quadam ratione; & maior, quàm data quaelibet minor, proposita eadem ratione, quatenus ita determinabilis, dicitur, Quasi eadem ratio. Vel aliter, quae potest esse propior cuidam propositae rationi, quàm data quaelibet alia non eadem, quatenus talis, dicitur, Quasi eadem.* 5. *Et rationum quasi earundem inter se, termini dicentur, Quasi proportionales.* 6. *Et quasi aequalitatum, dicentur, Quasi aequales* (Mengoli 1659, p. 97).

244 In the light of the third definition, the sixth definition can be read as: “And the
 245 terms of ratios that are nearer to equality than any other given ratio other than
 246 equality, as far as these ratios are determinable, will be called quasi equal”. For
 247 calculating quadratures, Mengoli used this interpretation of the definition of quasi
 248 equality ratio. In fact, he considered a “maior inaequalitas” ratio²⁶ and proved that
 249 he could find a number that allowed him to establish a ratio smaller than the given
 250 “maior inaequalitas” ratio.

251 Following the presentation of these six definitions, Mengoli obtained ratios
 252 between all sorts of summations and the number t (recall that these are all
 253 constructed using t , and that these summations have $t-1$ addends with different
 254 exponents). He calculated what these ratios tend toward when t is very large,
 255 obtaining in this way all possible quasi ratios. Specifically, in Theorem 42, Mengoli
 256 demonstrated that

$$(m+n+1) \cdot \binom{m+n}{n} \cdot \sum_{a=1}^{a=t-1} a^m \cdot (t-a)^n$$

257 tends to t^{m+n+1} when t tends to infinity, in the sense that their ratio can be made
 258 arbitrarily close to equality by making t sufficiently large.²⁷ He based this demon-
 259 stration on Theorem 22 and on another theorem that he had previously demon-
 260 strated which established that smaller powers could be ignored as t increases. In
 261 Theorem 22 of *Elementum Secundum in Geometria* he had proved that

$$(m+n+1) \cdot \binom{m+n}{n} \cdot \sum_{a=1}^{a=t-1} a^m \cdot (t-a)^n = t^{m+n+1} - P(t^s)$$

262 Then, in Theorem 41 of *Elementum Tertium*, he demonstrated the following
 263 quasi equality ratio

$$t^{m+n+1} \text{ is quasi equal to } t^{m+n+1} - P(t^s)$$

264 It follows that the left side of the equation given in Theorem 22 is quasi equal to
 265 the first term of Theorem 41:

$$(m+n+1) \cdot \binom{m+n}{n} \cdot \sum_{a=1}^{a=t-1} a^m \cdot (t-a)^n \text{ is quasi equal to } t^{m+n+1}$$

266 This result is used in the calculation of the quadratures, as explained below.

²⁶ The inaequalitas of a ratio denotes a number other than unity, and so ratios minor inaequalitas and maior inaequalitas correspond to numbers smaller and larger than unity, respectively.

²⁷ On this subject, see Massa (1997).

Fig. 13.4 *Tabula Formosa*
(Mengoli, *Geometria* . . . ,
1659, 366)

$$\begin{array}{ccccccc}
 & & & & FO. u. & & \\
 & & & & & FO. a. & FO. r. \\
 & & & & & & FO. r^2. \\
 & & FO. a^2. & & FO. ar. & & \\
 & FO. a^3. & & FO. a^2r. & & FO. ar^2. & FO. r^3.
 \end{array}$$

In the sixth book of his *Geometria*, Mengoli defined his own system of co-ordinates²⁸ and described the geometric figures that he wanted to square as “extended by their ordinates”. He denoted these geometric figures (which he referred to as forms)²⁹ by means of an algebraic expression written as $FO.a^n r^{m-n}$,

which in modern notation can be written as $\int_0^1 x^n (1-x)^{m-n} dx$. In Mengoli’s notation, $FO.a^n r^{m-n}$, “FO.” denotes the form (which we would now call the integral of an expression from 0 to 1), a the abscissa (x) and r the remainder ($1-x$). He called this expression “Form of all products of n abscissae and $m-n$ remainders”. In the singular case $m=n=0$, Mengoli used $FO. u.$ ($\int_0^1 dx$) and called this expression the “form of all rationals”.

Mengoli went on to construct an infinite triangular table (called *Tabula Formosa*, see Fig. 13.4) with the following forms,

The figure at the vertex represents a square of side 1; the two figures in the first row (called by Mengoli the “base of order one”) represent two triangles; the three figures in the second row (the “base of order two”) are determined by the ordinates of a parabola, and so on in the other rows. See Fig. 13.5.

We have seen in the dedicatory letter of the sixth volume from his *Geometria* that Mengoli had already computed the value of these figures by the method of indivisibles. These values are related with the binomial coefficients. Indeed, he multiplied each term $FO. a^n r^{m-n}$ of the *Tabula Formosa*, first by the binomial coefficient $\binom{m}{n}$ and then by the row number plus one unity ($m+1$), thereby obtaining a new table called *Tabula Quadraturarum* (see Fig. 13.6) whose terms take simply the value 1.

In modern notation:

$$(m+1) \binom{m}{n} \int_0^1 x^n (1-x)^{m-n} dx = (m+1) \binom{m}{n} FO.a^n r^{m-n} = 1.$$

²⁸ He defined the abscissa as our x , but in a segment measuring the unit u or t . Mengoli always worked within a finite base in which the abscissa was represented by the letter “ a ” and the remainder was represented by the letter “ $r=t-a$ ” or “ $1-a$ ”, depending on whether the base was a given value t or the unit u , see Massa (2006).

²⁹ The word figure or forma, which dates from the previous century, was identified by measuring the intensity of a given quality; see Massa (2006).

Fig. 13.5 The author's sketches of geometric figures

$FO. u.$
 $FO. 2a.$ $FO. 2r.$
 $FO. 3a^2.$ $FO. 6ar.$ $FO. 3r^2.$
 $FO. 4a^3.$ $FO. 12a^2r.$ $FO. 12ar^2.$ $FO. 4r^3.$

Fig. 13.6 *Tabula Quadraturarum* (Mengoli, *Geometria* . . ., 1659, 366)

291 In order to prove that all terms of the *Tabula Quadraturarum* had value 1, Mengoli
 292 used the theory of quasi proportions, establishing ratios of quasi equality between the
 293 figures or forms. Indeed, in his main demonstration he considered two ratios: the first
 294 one, between a new figure (the “ascribed” figure, explained below) and the figure or
 295 form which he wanted to square, and the second involving this “ascribed” figure and
 296 a square of side 1.³⁰ He showed that these two ratios are quasi equality ratios, and
 297 then used a theorem that he had previously demonstrated, which showed that in quasi
 298 equality ratios with the same antecedents, the consequents of the ratios are also equal.

³⁰ For these demonstrations Mengoli used the definitions from the *Elementum tertium* of quasi equality.

The First Quasi Equality Ratio in Mengoli's Main Demonstration

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For the first quasi equality ratio, Mengoli used Archimedes' definitions of inscribed and circumscribed figures. The inscribed figure is determined by all the greater rectangles included in the figure, while the circumscribed figure is determined by all the smaller rectangles containing the figure. The ascribed figure is determined by all the rectangles built over the ordinates of the divisions of the base. So, the ascribed figure is determined by $t-1$ rectangles when one divides the base into t parts.

33. Figure composed of just as many rectangles as there are ordinates through the points of division and lines adjacent to these ordinates, which will be called "ascribed" of the form.³¹

To get a sense of this, consider the geometric figures on the outside left diagonal of the table *Formosa*, *FO. a^n* (see Fig. 13.7).

The inscribed figure is determined by the rectangles DE and BF; the circumscribed figure is determined by the rectangles AE, CF and DG, and finally the ascribed figure is determined by AE and CF or by DE and BF. In this case, Mengoli demonstrated that the circumscribed figure is larger than the ascribed or inscribed figure by a rectangular quantity determined by the maximum ordinate and one of the equal parts of the base (Proposition 4).

In the preceding example, the inscribed and ascribed figures are identical. This will be true for any curve that is monotonically increasing. In general, the composite rectangles that make up the ascribed figure are sometimes smaller and sometimes larger than the associated curvilinear area elements of the figure. Hence, in general the ascribed figure is larger than the inscribed figure. Such is the case for the entries in the middle of the table *Formosa*, *FO. $a^n r^{m-n}$* (see Fig. 13.8).

The inscribed figure is determined by the rectangles HD, IE and EM; the circumscribed figure is determined by the rectangles AH, CI, DK, ELF and MB; the ascribed figure is determined by the rectangles AH, CI, DK and EM or by the rectangles HD, IE, KF and MB.

In this second example, Mengoli demonstrated that the circumscribed figure is larger than the ascribed figure by a rectangular quantity (the area of the rectangle determined by the maximum ordinate and one of the equal parts of the base). He also proved that the ascribed figure is larger than the inscribed figure, although the difference in size is not greater than this rectangular quantity (Proposition 5). Using the theory of quasi proportions (Proposition 6), Mengoli immediately proved for any figure in the table that the circumscribed and inscribed figures are "quasi equal"; that is to say, he demonstrated that it is possible to find a number of divisions of the base so that the ratio between the circumscribed and the inscribed figures is nearer to equality than is any other given ratio (not equal to equality). With this result he was able to affirm that the ascribed figure, determined by

³¹ 33. *Figura vero ex tot parallelogrammis, quot sunt ordinatae per puncta divisionum, & ad ipsas ordinatas iacentibus composita, dicitur, Adscripta formae* (Mengoli 1659, p. 371).

Fig. 13.7 Mengoli's
geometric figure in
Proposition 4 (Mengoli,
Geometria..., 1659, 379)

Fig. 13.8 Mengoli's
geometric figure in
Proposition 5 (Mengoli,
Geometria..., 1659, 382)

338 rectangles, and the geometric figure or form, determined by ordinates, were quasi
339 equal (Proposition 7).³² Notice that Mengoli's ascribed, inscribed, and
340 circumscribed figures are explicitly determined by a finite number of rectangles.

341 This demonstration follows Archimedes but uses the quasi-ratio method rather
342 than *reductio ad absurdum*. Another difference is that in Archimedes the figure
343 between the inscribed and circumscribed figures is used directly, whereas Mengoli
344 introduced a new figure, the ascribed figure, determined by a finite number of
345 rectangles. The number of rectangles making up the ascribed figure will increase
346 indefinitely. The rectangles of the ascribed figure never actually become the
347 ordinates of the curved figure, and the geometric figure exists independently of
348 the existence of the successive ascribed figures. Mengoli needed the ascribed figure,
349 determined by $t-1$ rectangles, to establish the proportion involving the ratio of the
350 square of side 1 to the ascribed figure and the ratio of one power of t to a summation
351 of $t-1$ powers.

352 In fact, like Newton in Lemma II of the *Principia* Mengoli might well have
353 stated that the ratios between the curvilinear, the inscribed and the circumscribed
354 figures are ratios of equality. However, it is evident that he needed the ascribed
355 figure to be able to establish ratios with finite terms. For Mengoli, the ascribed
356 figure is a tool to clarify the nature of the curved figure, and furthermore to
357 demonstrate in a general way results about the quasi ratio and the value of the
358 quadrature.

³² He used Proposition 67 of *Elementum quintum*, which established ratios of quasi equality between two magnitudes situated between two quasi equals.

The Second Quasi Equality Ratio in Mengoli's Main Demonstration

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360

For the second quasi equality ratio involving the ascribed figure and the square of side 1, Mengoli used the ascribed figure that corresponds to the equation $y = (m + n \cdot x)^n \cdot (m + n + 1) \cdot x^m \cdot (1 - x)^n$. He first established a proportion involving the ratio of the square of side 1 and the ascribed figure, and the ratio of a power of t to a summation of powers:

$$\frac{\text{Square (Side 1)}}{\text{Ascribed figure}} = \frac{t^{m+n+1}}{\binom{m+n}{n} \cdot (m+n+1) \sum_{a=1}^{a=t-1} a^m \cdot (t-a)^n}$$

He then applied the theory of quasi proportions to this proportion. He implicitly assumed that the proportion continues to hold when the number of rectangles on the left side is infinite and the number of addends on the right side is infinite. Since he knew from the theory of quasi proportions that the second ratio is a quasi equality ratio, it follows that the first ratio involving the square and the ascribed figure is also a quasi equality ratio.

I now consider this demonstration in more detail.

Mengoli gave this demonstration in Proposition 8 for the curve corresponding to the expression $FO. 10 a^2 r^3$ from the fifth row of the table of subquadratures, or, alternatively the expression $FO.6 \cdot 10 a^2 r^3$ from the fifth row of the table of quadratures (see Fig. 13.9: as noted below, the proof can be generalised to any entry in these tables). He divided the base of the square into t parts and on these constructed the ordinates of the curved figure and of the square. He also constructed the rectangles of the ascribed figure and of the square of side 1. First, he established a proportion for each rectangle of the ascribed figure and of the square. Notice that as each rectangle has the same base, for each division the ratio of rectangles is the same as the ratio of ordinates; that is,

Rectangle of the square (AQ): rectangle of the ascribed figure (AK) = DQ:DK.

DQ = ordinate of the square; DK = ordinate of the figure.

However, the ordinate of the square is equal to the base of the square. He could then apply the proportion between the base of the square, that is, one, and the ordinate of the geometric figure.

In the case of the first element of the division, I have

$$\begin{aligned} \text{DQ} : \text{DK} &= (1 : 10) \cdot (1 : (1/t))^2 \cdot (1 : (1 - 1/t))^3 = 1 : [10 \cdot 1^2 \cdot (t - 1)^3] / t^5 \\ &= t^5 : 10 \cdot 1^2 \cdot (t - 1)^3 \end{aligned}$$

But rectangle (square) = AQ and rectangle (ascribed) = AK, so that

389

Fig. 13.9 Mengoli's geometric figure in Proposition 8 (It is worth noting that in the *Geometria*, there are only three drawings of these geometrical figures, whereas in the *Circolo* he included no drawings.) (Mengoli, *Geometria*..., 1659, 387)

$$AQ : AK = DQ : DK = t^5 : 10 \cdot 1^2 \cdot (t-1)^3$$

$$AQ : AK = t^5 : 10 \cdot 1^2 \cdot (t-1)^3$$

390 In the case of the second element of the division, I have rectangle (square):
 391 rectangle (ascribed) = $1 : [10 \cdot 2^2 \cdot (t-2)^3] / t^5 = t^5 : 10 \cdot 2^2 \cdot (t-2)^3$, or DR: DL = t^5
 392 : $10 \cdot 2^2 \cdot (t-2)^3$
 393 and so on.

394 On the one side, Mengoli added all the t rectangles in the antecedent to obtain the
 395 square, and added all the $t-1$ rectangles in the consequent to obtain the ascribed
 396 figure. On the other side, in the antecedent, adding t^5 he obtained t^6 and in the
 397 consequent he obtained a finite sum. This yielded

$$\frac{FO.u}{\text{Ascribed } FO \cdot 10a^2r^3} = \frac{t^6}{10 \cdot \sum_{a=1}^{a=t-1} a^2 \cdot (t-a)^3}$$

398 In Proposition 10, Mengoli then stated that “All quadratures on the same base
 399 are equal to each other”³³ and in the demonstration employed the preceding
 400 proportion with both consequents multiplied by 6, that is

$$\frac{FO.u}{\text{Ascribed } FO \cdot 6 \cdot 10a^2r^3} = \frac{t^6}{6 \cdot 10 \cdot \sum_{a=1}^{a=t-1} a^2 \cdot (t-a)^3}$$

401 Because the second ratio is a quasi equality (Theorem 42), the first ratio,
 402 involving the square of side 1 and the ascribed figure, is also a quasi equality

³³ “Theor. 6. Prop. 10. *Omnes quadraturae super eadem basi constitutae, sunt inter se aequales* (Mengoli 1659, p. 389).

ratio. Notice that the justification of this proportion is based on the identification of the algebraic expression and the geometric figure by means of a proportion between segments and quantities.

The proportion derived by Mengoli may be regarded as an attempt to justify the result obtained by Cavalieri's method of indivisibles.³⁴ This proportion can be interpreted as equating a ratio between finite sums of ordinates to a ratio between figures. Mengoli could then apply the quasi proportions, and thus did not have to establish proportions between infinity as Cavalieri did, because he established finite ratios which "tend" to other ratios, that is to say, quasi ratios.

One of the weak points of this demonstration is the step from a ratio of quasi equality between summation of powers and powers (numbers) to a ratio between figures. However, Mengoli had based the theory of quasi proportions on the Euclidean theory of proportions, so for him the former theory was valid for any magnitude, figure or number.

It should be emphasized that this demonstration was independent of the graphical representation of the geometric figure; it does not depend on the degree either, and could be used in all cases where the quasi ratio of the summation of powers was known. It is significant that Mengoli also used the symmetry of triangular tables and the regularity of their rows in order to generalise the proofs. He took it for granted that if a result was true for one row of the table, this result was also true for all rows and there was no need to prove it in the remaining rows.

It is obvious that Mengoli, like Roberval and Wallis, knew the result of these quadratures. However, the latter authors carried out the summations of powers and verified the resulting values only in a few cases. From these results they inferred the general rule and then applied it directly to the quadrature problem by taking limits of ratios between sums of ordinates and areas under curves. Instead, after constructing the theory of quasi proportions to handle these limits, Mengoli gave a proof that provided countless quadratures all at once.

Nevertheless, Mengoli's principal aim was the computation of the quadrature of the circle. In his work *Circolo*, by interpolation, he computed quadratures between 0 and 1 of mixed-line geometric figures determined by $y = x^{n/2} (1-x)^{(m-n)/2}$, for natural numbers m and n . Note that in the special case $m=2$ and $n=1$, the geometric figure is the semicircle of diameter 1. First, he described these interpolated geometric figures and displayed them again in an infinite interpolated triangular table (Interpolated *Tabula Formosa*, see Fig. 13.10).

He then obtained an infinite interpolated triangular table of values of their quadratures, which is nothing less than the interpolated harmonic triangle, and by homology he identified the values of both tables.³⁵ With the help of the properties of a combinatorial triangle, Mengoli was now able to fill the interpolated combinatorial triangle, except for an unknown number " a " which is closely related to the

³⁴ Also according to Malet's interpretation of Gregorie's work, see Malet (1996).

³⁵ On the construction of Mengoli's harmonic triangle and interpolated harmonic triangle see Massa Esteve-Delshams (2009).

$$\begin{aligned}
& FO. u. \\
& FO. a^{1/2}. \quad FO. r^{1/2}. \\
& FO. a. \quad FO. (ar)^{1/2}. \quad FO. r. \\
& FO. a^{3/2}. \quad FO. (a^2r)^{1/2}. \quad FO. (ar^2)^{1/2}. \quad FO. r^{3/2}. \\
& FO. a^2. \quad FO. (a^3r)^{1/2}. \quad FO. ar. \quad FO. (ar^3)^{1/2}. \quad FO. r^2. \\
& FO. a^{5/2}. \quad FO. (a^4r)^{1/2}. \quad FO. (a^3r^2)^{1/2}. \quad FO. (a^2r^3)^{1/2}. \quad FO. (ar^4)^{1/2}. \quad FO. r^{5/2}.
\end{aligned}$$

Fig. 13.10 Interpolated *Tabula Formosa* (Mengoli, *Circolo*, 1672, 7)

quadrature of the circle ($1/2a = \pi/8$). Mengoli obtained successive approximations of the number “ a ” in order to approximate the number π up to eleven decimal places.

Concluding Remarks

The influence of Cavalieri’s work on the thought and work of Mengoli is unquestionable, but it is equally certain that Mengoli did not wish to use the method of his master. Mengoli calculated the first quadratures as a good student by his master’s method of indivisibles. However, unlike Torricelli, Mengoli makes no defence of this method, preferring to withhold publication until he could prove the same quadratures by another method. After Cavalieri died, Mengoli published these quadratures by indivisibles in the opening letter, but following the algebraic method by Beaugrand found in *Exercitationes*. Indeed, Mengoli claims that his purpose was to give solid foundations for a new method of calculating quadratures.

Our study of Mengoli’s work reveals that the basis of his new method of calculating quadratures was not Cavalieri’s method of indivisibles, but rather the triangular tables and the theory of quasi proportions, set out as a development of Viète’s algebra. In this way he created a numerical theory of summations of powers and products of powers and *limits* of these summations which was unrelated to Cavalieri’s *Omnes lineae*. It is not clear why Mengoli did not follow his master’s path; perhaps it was because Cavalieri’s method had received a great deal of criticism, a fact that Mengoli could not ignore. After showing that he was familiar with the method of indivisibles and was able apply this method, Mengoli deliberately pursued research into a new method of calculating quadratures. The knowledge of the values of the quadratures by the method of indivisibles enabled him to create this new method. To this end, he constructed the triangular tables of geometric figures and applied the theory of quasi proportions. Unlike Cavalieri, he never compared two figures through the comparison of lines, nor did he superimpose figures; rather, he established quasi ratios between geometric figures.

But what is the meaning of the statement that a geometric figure is quasi equal to another? Mengoli defined the ascribed, inscribed and circumscribed figures determined by rectangles built on the divisions of the base. He worked at all times with a finite number of divisions. He demonstrated that for any given ratio it is always possible to find a number of divisions of the base so that the ratio between the circumscribed and inscribed figures is nearer to equality than the given ratio. He also demonstrated that as the number of divisions increases, the ascribed figure is quasi equal to the mixed-line figure determined by the ordinates; that is to say, a geometric figure determined by rectangles approximates to a mixed-line figure arbitrarily closely when the number of rectangles increases indefinitely. To a certain extent, this first quasi equality recalls Archimedes' method.

Mengoli also arrived at a second quasi equality by using algebraic procedures. He established a proportion in which the first ratio is between a summation of powers and a power and the second between a unit square and the ascribed figure. This proportion reminds us in some way of Theorem 3 of the method of indivisibles by Cavalieri: "Plane figures constitute between them the same proportion that "all the lines" have" (Cavalieri, *Geometria* . . ., 1635, 209). Although Mengoli adopts a very different approach for the quadratures, the basis of this main demonstration can be compared to the stated proportion by Cavalieri, avoiding the infinite sums and the possible identification between the sum of ordinates and the figure. The step from the geometric figure to its algebraic expression is essential in his demonstration. The Euclidean theory of proportions is once again the link between figure and expression. It enabled Mengoli to operate with segments and to establish ratios and quasi ratios to determine the quadratures of these curves.

The use of the two quasi equalities (the ascribed figure and the square as well as the ascribed and the mixed-line figure) allows us to understand Mengoli's words better when he states that his geometry is a "perfect conjunction" of the geometry of indivisibles, the geometry of Archimedes (method of exhaustion) and the algebra of Viète. Algebraic and geometric methods complement each other, allowing one to obtain new and better results.

Mengoli, like Viète, considered his algebra as a technique in which symbols are used to represent abstract magnitudes. He dealt with species, forms, triangular tables, quasi ratios and logarithmic ratios. However, I argue that the most innovative aspect of his work was his use of letters to work directly with the algebraic expression of the geometric figure. On the one hand, he expressed a figure by an algebraic expression, in which the ordinate of the curve that determines the figure is related to the abscissa by means of a proportion, thus establishing the Euclidean theory of proportions as a link between algebra and geometry. On the other hand, he showed how algebraic expressions could be used to construct the ordinate geometrically at any given point. This allowed him to study geometric figures via their algebraic expressions and to derive known and unknown values for the areas of a large class of curves at once.

Although Mengoli's contributions constituted a step forward in the process of algebraization of mathematics, his principal aim was not to demonstrate the equivalence of algebraic expressions and geometric figures, but rather to develop a new

516 and fruitful algebraic method for solving quadrature problems. One should not
517 forget that Mengoli wished to square the circle by interpolating these tables of
518 quadratures. This investigation appeared in his later publication *Circolo*, in which
519 he studied quadratures of curves determined by equations today represented as
520 $y^p = k \cdot x^m \cdot (1-x)^n$. Mengoli emphasized that these quadratures had never been found
521 before. Indeed, any attempt to calculate quadratures geometrically would have to be
522 done case by case.

523 Thus, from his perfect knowledge of Cavalieri's method of indivisibles, I am
524 able to conclude that Mengoli arrived at an original theory to investigate geometric
525 figures and to determine new quadratures by developing Viète's symbolic language
526 using quasi proportions and interpolating triangular tables.

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